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On V-regular Semigroups

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Abstract

A regular semigroup S is V-regular if $V(ab) \subseteq V(b)V(a)$ for all $a, b \in S$. A characterization of a V-regular semigroup is given. Congruences on V-regular semigroups are described in terms of certain congruence pairs.

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1 Introduction and Preliminaries

A regular semigroup S is called *V-regular* if $V(ab) \subseteq V(b)V(a)$ for all $a, b \in S$. This concept was introduced by Onstad [8]. This class of semigroups is dual to orthodox semigroups, namely, regular semigroups satisfy that $V(b)V(a) \subseteq V(ab)$ for all elements a, b in the semigroup. Properties of V-regular semigroups were given by Nambooripad and Pastijn in [7].

Congruences on regular semigroups have been explored extensively. The kernel-trace approach is an effective tool for handling congruences on regular semigroups, which had been investigated in the previous literature, such as Crvenković and Dolinka [1], Feigenbaum [2], Gomes [3, 4], Imaoka [6], Pastijn and Petrich [9], Petrich [10], Scheiblich [11], Trotter [12, 13] and the author [14].

The purpose of this paper is to give a characterization of a V-regular semigroup, and to describe congruences on V-regular semigroups in terms of certain congruence pairs.

For standard terminology and notation in semigroup theory see Howie [5].

If S is a regular semigroup, $a \in S$, then $V(a)$ denotes the set of inverses of a in S. The set of idempotents of S is denoted by $E(S)$. On $E(S)$ we shall consider the *natural partial order* ω given by

$$e\omega f \Leftrightarrow ef = fe = e.$$

For $e, f \in E(S)$,

$$S(e, f) = fV(ef)e$$

is the *sandwich set* of e and f.

The following simple statements will be applied without further mention: for $e, f \in E(S)$,

$$e\mathcal{L}f \Rightarrow S(e, f) = \{f\},$$
$$e\mathcal{R}f \Rightarrow S(e, f) = \{e\}.$$

If ρ is a congruence on S and $h \in S(e, f)$, then $h\rho \in S(e\rho, f\rho)$.

Let τ be a relation on S. The congruence generated by τ is denoted by τ^*. If γ is an equivalence on S, then γ^0 is the *greatest congruence on* S *contained in* γ. $\mathcal{C}(S)$ is the *lattice of congruences* on S.

Lemma 1.1. [7] *A regular semigroup S is V-regular if and only if the partial band $(E(S), \circ)$ determined by S satisfies the following:*

(1) $\omega\mathcal{L} = \mathcal{L}\omega$;

(2) $\omega\mathcal{R} = \mathcal{R}\omega$;

(3) *for all* $e, f \in E(S), h \in S(e, f)$ *there exist* $e_1, f_2 \in E(S)$ *such that* $e_1\mathcal{L}e, f_2\mathcal{R}f$, *and* $h = f_2e_1$.

Lemma 1.2. [5] *Let S is a regular semigroup, $\rho \in \mathcal{C}(S)$. If $a\rho \in E(S/\rho)$, then there exists $e \in E(S)$ such that $a\rho = e\rho$.*

Lemma 1.3. *Let S be a V-regular semigroup, $\rho \in \mathcal{C}(S), a\rho \in E(S/\rho), x\rho \in S/\rho$. If $(a\rho)\mathcal{R}(x\rho)$ in S/ρ, then there exists $e \in E(S)$ such that $a\rho = e\rho$ and $e\mathcal{R}x$.*

Proof. By Lemma 1.2, there exists $f \in E(S)$ such that $a\rho = f\rho$. Let $g \in E(S)$ be such that $g\mathcal{R}x$. Then $(g\rho)\mathcal{R}(x\rho)$. Since $a\rho = f\rho$ and $(a\rho)\mathcal{R}(x\rho)$, we have $(f\rho)\mathcal{R}(g\rho)$. Let $h \in S(f, g)$. Then $h\rho \in S(f\rho, g\rho)$, and so $h\rho = f\rho$. Notice that $hg \in E(S), h\mathcal{R}(hg)\omega g$, it follows from Lemma 1.1 that there exists $e \in E(S)$ such that $h\omega e\mathcal{R}g$. Since $g\mathcal{R}x$, $e\mathcal{R}x$. Now $(h\rho)\omega(e\rho)\mathcal{R}(g\rho)$ implies that $(f\rho)\omega(e\rho)\mathcal{R}(f\rho)$. Hence $a\rho = f\rho = f\rho \cdot e\rho = e\rho$. □

Corollary 1.4 *Let S be a V-regular semigroup, $\rho \in \mathcal{C}(S), e, f \in E(S)$. If $(e\rho)\mathcal{R}(f\rho)$, then there exist $g, h \in E(S)$ such that $g\mathcal{R}f, h\mathcal{R}e, g\rho = e\rho$ and $h\rho = f\rho$.*

Remark The dual results of Lemma 1.3 and Corollary 1.4 hold.

2 Main Results

The theorem below give a characterization of a V-regular semigroup.

Theorem 2.1. *A regular semigroup S is V-regular if and only if for all $a, b \in S, (ab)' \in V(ab)$ there exist $e_1, e_2, f_1, f_2 \in E(S)$ such that $b(ab)'a = f_2e_1, e_1\mathcal{L}a\mathcal{R}e_2, f_1\mathcal{L}b\mathcal{R}f_2, ab(ab)'\omega e_2$ and $(ab)'ab\omega f_1$.*

Proof. $\Rightarrow$. Since S is V-regular, for all $a, b \in S, (ab)' \in V(ab)$ there exist $a' \in V(a), b' \in V(b)$ such that $(ab)' = b'a'$. Let

$$e_1 = a'a, f_1 = b'b, e_2 = aa', f_2 = bb'.$$

Then $e_1, e_2, f_1, f_2 \in E(S)$ and

$$b(ab)'a = bb'a'a = f_2e_1,\ e_1 = a'a\mathcal{L}a\mathcal{R}aa' = e_2,\ f_1 = b'b\mathcal{L}b\mathcal{R}bb' = f_2.$$

Now

$$(ab)(ab)'e_2 = (ab)(ab)'aa' = (ab)(b'a'aa') = (ab)b'a' = (ab)(ab)'$$

and

$$e_2(ab)(ab)' = (aa')(ab)(ab)' = (aa'a)b(ab)' = (ab)(ab)'.$$

It follows that $(ab)(ab)'\omega e_2$.

Similarly, $(ab)'ab\omega f_1$.

$\Leftarrow$. Let a, b satisfy the condition stated in the theorem. Now $e_1\mathcal{L}a\mathcal{R}e_2, f_1\mathcal{L}b\mathcal{R}f_2$ imply that there exist

$$a' \in V(a) \cap (L_{e_2} \cap R_{e_1}), b' \in V(b) \cap (L_{f_2} \cap R_{f_1})$$

such that

$$a'a = e_1, aa' = e_2, b'b = f_1, bb' = f_2.$$

Since $b(ab)'a = f_2e_1$, we have that

$$b'a' = (b'f_2)(e_1a') = b'(f_2e_1)a' = b'b(ab)'aa'.$$

Thus

$$\begin{aligned}(b'a')(ab)(b'a') &= (b'b(ab)'aa')(ab)(b'b(ab)'aa') \\ &= b'b(ab)'ab(ab)'aa' \\ &= b'b(ab)'aa' = b'a'\end{aligned}$$

and

$$(ab)(b'a')(ab) = ab(b'b(ab)'aa')ab = ab(ab)'ab = ab,$$

that is, $b'a' \in V(ab)$.
Also

$$\begin{aligned}(b'a')(ab) &= (b'b(ab)'aa')ab = b'b(ab)'ab \\ &= f_1(ab)'ab \\ &= (ab)'ab \qquad\qquad (\text{since } (ab)'ab\,\omega f_1)\end{aligned}$$

and

$$\begin{aligned}(ab)(b'a') &= (ab)(b'b(ab)'aa') = ab(ab)'aa' \\ &= (ab)(ab)'e_2 \\ &= (ab)(ab)' \qquad\qquad (\text{since } (ab)(ab)'\omega e_2).\end{aligned}$$

It follows that

$$(ab)' = (ab)'(ab)(ab)' = (b'a')(ab)(ab)' = (b'a')(ab)(b'a') = b'a'.$$

Therefore, S is V-regular. □

Theorem 2.2. *Let S be a V-regular semigroup, $\rho \in \mathcal{C}(S), a, b \in S$. If $a\rho b$, then for any $a' \in V(a)$ there exists $b' \in V(b)$ such that $a'\rho b'$.*

Proof. Let $a' \in V(a)$. Then $a'\rho \in V(a\rho)$. Since $a\rho b$, we have that $a'\rho \in V(a\rho) = V(b\rho)$. Let $f\rho = b\rho \cdot a'\rho, f'\rho = a'\rho \cdot b\rho$. Then

$$(f\rho)\mathcal{R}(b\rho), (f'\rho)\mathcal{L}(b\rho), f\rho, f'\rho \in E(S/\rho).$$

By Lemma 1.3 and its dual, there exist $e, e' \in E(S)$ such that $e\mathcal{R}b\mathcal{L}e', f\rho = e\rho$ and $f'\rho = e'\rho$.

Take $b' \in V(b) \cap L_e \cap R_{e'}$. Then $b'\rho \in L_{e\rho} \cap R_{e'\rho}$. Hence

$$\begin{aligned}b'\rho &= e'\rho \cdot b'\rho \cdot e\rho = f'\rho \cdot b'\rho \cdot f\rho = a'\rho b\rho \cdot b'\rho \cdot b\rho a'\rho \\ &= a'\rho \cdot b\rho b'\rho b\rho \cdot a'\rho = a'\rho \cdot b\rho \cdot a'\rho = a'\rho \cdot a\rho \cdot a'\rho = a'\rho,\end{aligned}$$

that is, $a'\rho b'$. □

To provide a characterization of congruences on V-regular semigroups in terms of certain congruence pairs, we need the following results.

Lemma 2.3. *Let S be a V-regular semigroup, $\rho \in \mathcal{C}(S)$ with $\tau = \mathrm{tr}\,\rho$.*
(1) $(e\rho)\mathcal{R}(f\rho)$ in $S/\rho \Leftrightarrow e(\tau\mathcal{R})f$ in $S \Leftrightarrow e(\mathcal{R}\tau)f$ in S $(e, f \in E(S))$;
(2) $\mathcal{R}\tau\mathcal{R}\tau\mathcal{R} = \mathcal{R}\tau\mathcal{R}$.

Proof. (1) Let $e, f \in E(S)$ be such that $(e\rho)\mathcal{R}(f\rho)$ in S/ρ. By Corollary 1.4, there exist $g, h \in E(S)$ such that

$$g\mathcal{R}f, h\mathcal{R}e, g\rho = e\rho, h\rho = f\rho.$$

Thus $e\rho g\mathcal{R}f, e\mathcal{R}h\rho f$, whence $e(\tau\mathcal{R})f, e(\mathcal{R}\tau)f$.

If conversely $e(\tau\mathcal{R})f$, then exists $g \in E(S)$ such that $e\tau g\mathcal{R}f$, and so $(e\rho) = (g\rho)\mathcal{R}(f\rho)$.

Similarly, $e(\mathcal{R}\tau)f$ implies that $(e\rho)\mathcal{R}(f\rho)$.

(2) Obviously, $\mathcal{R}\tau\mathcal{R}\tau\mathcal{R} \supseteq \mathcal{R}\tau\mathcal{R}$.

If $a(\mathcal{R}\tau\mathcal{R}\tau\mathcal{R})b$ for $a, b \in S$, then by [9, Lemma 2.6 (ii)] we have $(a\rho)\mathcal{R}(b\rho)$ in S/ρ. Hence for $a' \in V(a), b' \in V(b)$, we have

$$(aa'\rho)\mathcal{R}(a\rho)\mathcal{R}(b\rho)\mathcal{R}(bb'\rho).$$

Since $aa', bb' \in E(S)$, by part (1) we have $(aa')\tau\mathcal{R}(bb')$ and thus

$$a\mathcal{R}(aa')\tau\mathcal{R}(bb')\mathcal{R}b,$$

whence $a(\mathcal{R}\tau\mathcal{R})b$. □

An equivalence τ on the set $E(S)$ of idempotents of a regular semigroup S is normal if $\tau = \operatorname{tr}\tau^*$ [9]. It follows from Lemma 2.3 [9] that an equivalence τ on $E(S)$ is normal if and only if τ is the trace of a congruence on S.

Let K be a subset of a regular semigroup S. A congruence ρ on S saturates K if $a \in K$ implies $a\rho \subseteq K$. The greatest congruence on S which saturates K is denoted by π_K. Recall from Result 1.5 [9] that for $a, b \in S, a\pi_K b$ if and only if

$$xay \in K \Leftrightarrow xby \in K \ (x, y \in S^1),$$

and $\pi_K = \theta_K^0$, where the equivalence relation θ_K on S is defined by

$$a\theta_K b \Leftrightarrow a, b \in K \text{ or } a, b \in S \setminus K.$$

A subset K of a regular semigroup S is normal if $K = \ker\pi_K$ [9]. Recall from [9] that a subset K of S is normal if and only if K is the kernel of a congruence on S.

The pair (K, τ) is a congruence pair for a regular semigroup S (see [9]) if

(i) K is a normal subset of S,

(ii) τ is a normal equivalence on $E(S)$,

(iii) $K \subseteq \ker(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0$,

(vi) $\tau \subseteq \operatorname{tr}\pi_K$.

In such a case $\rho_{(K,\tau)}$ is defined by

$$\rho_{(K,\tau)} = \pi_K \cap (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0.$$

Note that

$$\rho_{(K,\tau)} = (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \theta_K \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0.$$

When S is a V-regular semigroup it follows from Lemma 2.3 (2) and its dual result that

$$\rho_{(K,\tau)} = (\mathcal{L}\tau\mathcal{L} \cap \theta_K \cap \mathcal{R}\tau\mathcal{R})^0.$$

The characterization of congruences on a V-regular semigroup in terms of congruence pairs follows from [9, Theorem 2.13].

Theorem 2.4. *If* (K, τ) *is a congruence pair for a V-regular semigroup* S*, then* $\rho_{(K,\tau)}$ *is the unique congruence on* S *such that* $\ker \rho_{(K,\tau)} = K$ *and* $\operatorname{tr} \rho_{(K,\tau)} = \tau$*. Conversely, if* ρ *is a congruence on* S*, then* $(\ker \rho,\ \operatorname{tr} \rho)$ *is a congruence pair for* S *and* $\rho = \rho_{(\ker \rho,\, \operatorname{tr} \rho)}$*.*

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