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ON VARIANTS OF A SEMIGROUP

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If S is a (multiplicative) semigroup and $a \in S$, the binary operation $\circ$ defined on the set S by $x \circ y = x\,a\,y$ is associative and the resulting semigroup is called a variant of S. We study the congruence α defined on S by saying that two elements are α-related if and only if they determine the same variant of S. Certain quotients of variants are used to provide an arbitrary semigroup with a generalised local structure. The variant formulation of Nambooripad's partial order on a regular semigroup is used to show that the order possesses a certain property (involving $\mathcal{D}$-equivalence).

If S is a (multiplicative) semigroup and $a \in S$, the binary operation $\circ$ defined on the set S by $x \circ y = x\,a\,y$ is associative; the resulting semigroup is denoted (S,a) and called a variant of S [4]. In this paper we investigate the congruence α defined on a semigroup S by saying that two elements of S are α-related if and only if they determine the same variant of S. We consider also, for $a \in S$, a congruence δ^a on (S,a), and show that the quotients $(S,a)/\delta^a$ generalise (up to isomorphism) to an arbitrary semigroup the local

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subsemigroups in a semigroup with idempotents. Finally, Nombooripad's partial order on a regular semigroup (in its variant formulation [4,1]) is considered and shown to possess a certain 'local' property.

In Section 2 we show that, for an arbitrary semigroup with an idempotent, the α-class of each idempotent is an ideal extension of a rectangular sub-band of S by a semigroup U satisfying $U^3 = 0$. If S is regular then the α-class of an idempotent e is a rectangular sub-band of S contained in $V(e) \cap E(S)$ (the containment being strict, in general); in particular, α is idempotent-determined here.

The congruence δ^a is defined in Section 3. (It was introduced in the context of sandwich semigroups by Symons [10] and was further studied in [7].) We see that δ^a is contained in the congruence α on (S,a) and that these two congruences coincide if a is regular in S. The quotient semigroups $(S,a)/\delta^a$ $(a \in S)$ are considered: it is shown that $(S,e)/\delta^e \cong eSe$ when $e \in E(S)$ and that, if a and b are $\mathcal{D}$-equivalent in S, then $(S,a)/\delta^a \cong (S,b)/\delta^b$.

In the final section we consider the subsets $\downarrow x$ of a regular semigroup S defined by $\downarrow x = \{s \in S: s \leq x\}$, where $\leq$ denotes Nambooripad's partial order on S. Using the variant formulation of $\leq$, we show that, if x and y are $\mathcal{D}$-equivalent elements of S then $\downarrow x$ is order-isomorphic to $\downarrow y$.

1. Preliminaries.

The notation of [5,2] will be used throughout.

We first recall some ideas and results from [4,1]. If $(S,.)$ is a semigroup and $a \in S$, the variant (S,a) of S is the semigroup obtained by taking the set S under the binary operation $\circ$ defined by $x \circ y = x\,a\,y$ $(x,y \in S)$ [4]. We adhere to the convention that, if it is stated or implied that S (or a subset of it) is a semigroup, then the multiplication in question will be that in (or inherited from) $(S,.)$.

Let a be an element of a semigroup S. By a pre-inverse of a we mean an element $b \in S$ satisfying $a\,b\,a = a$ [4]. We shall denote the set of preinverses[inverses] of a by $\mathrm{Pre}(a)$ $[V(a)]$.

By a midldentity in a semigroup S we mean an element u with the property that $x\,u\,y = xy$ for all $x,y \in S$. If u is a mididentity in S then clearly the variant (S,u) coincides with S.

Nambooripad's partial order $\leq$ on a regular semigroup S is defined in [9]. We shall use the following equivalent formulation of it [4, Theorem 5.1], where $E(S,a)$ denotes the set of idempotents of (S,a):

$$x \leq y \iff \begin{cases} \text{there exists } a \in S \text{ with } x,y \in E(S,a) \\ \qquad \text{and } x = x \circ y = y \circ x\,. \end{cases}$$

The following lemma shows that, in order to determine whether or not the statement $x \leq y$ is true in S, we may choose any pre-inverse y' of y and calculate in (S,y').

LEMMA 1.1 [1]. *Let* x,y *be elements of a regular semigroup and let* $y' \in Pre(y)$. *Then*

$$x \leq y \iff (x \in E(S,y') \text{ with } x = x\circ y = y\circ x \text{ in } (S,y')).$$

We note that this partial order $\leq$ on a regular semigroup S extends the usual partial order on $E(S)$.

For a congruence ρ on a regular semigroup S we shall need the following definitions: ρ is said to be strictly compatible [9] if

$$(\forall\, x,y \in S)\quad x\;\rho\;y \text{ and } x \leq y \Rightarrow x = y\,,$$

and to be idempotent-determined [3] if the ρ-class of each idempotent consists entirely of idempotents.

By the local subsemigroups of a semigroup S we mean the subsemigroups of S of the form eSe $(e \in E(S))$ [6].

LEMMA 1.2 [8]. *If* e *and* f *are* $\mathcal{D}$*-equivalent idempotents in a semigroup* S *then* $eSe \cong fSf$.

We will close this section with an example, constructed by McAlister [6] for use in a context somewhat different from the present one. First we need to describe a certain type of regular semigroup.

Let S be a regular semigroup, let I, Λ be sets and let P be a $\Lambda \times I$ matrix over S. Then the set of all triples $(i,s,\lambda) \in I \times S \times \Lambda$ is a semigroup under the multiplication

$$(i,s,\lambda)(j,t,\mu) = (i,sp_{\lambda j}t,\mu)\,.$$

This semigroup is not regular, in general, but the set of regular elements in it forms a regular semigroup. This latter semigroup is denoted by $RM(S; I,\Lambda; P)$ and termed a regular Rees matrix semigroup over S [6].

EXAMPLE 1.3 [6]. Let S be the chain semilattice $\{1,a,b,0\}$ with $1 > a > b > 0$. Let $I = \Lambda = \{1,2\}$ and let P be the 2×2 matrix $\begin{pmatrix} 1 & a \\ b & 0 \end{pmatrix}$. Then $RM(S; I,\Lambda; P)$ contains precisely eleven elements, namely

$$\begin{array}{llll} (1,1,1), & & & \\ (1,a,1), & (2,a,1), & & \\ (1,b,1), & (1,b,2), & (2,b,1), & (2,b,2), \\ (1,0,1), & (1,0,2), & (2,0,1), & (2,0,2). \end{array}$$

The element $(2,b,2)$ is the only non-idempotent in the semigroup.

2. The congruence α.

Let S be a semigroup. The relation α defined on S by

$$(x,y) \in \alpha \iff s\,x\,t = s\,y\,t \quad \text{for all} \quad s,t \in S$$

is a congruence on S, as is readily verified. Clearly, two elements x,y are α-related in S precisely when the variants (S,x) and (S,y) coincide.

When two or more semigroups are being discussed we may write $\alpha(S)$ instead of α in order to avoid confusion; also, we will denote the congruence α on (S,a) by $\alpha(S,a)$.

If a, b are two regular elements in S that are α-related then the set of mididentities [idempotent mididentities] in (S,a) coincides with the set of mididentities [idempotent mididentities] in (S,b). Thus $\mathrm{Pre}(a) = \mathrm{Pre}(b)$ $[V(a) = V(b)]$ by [4, Lemma 3.1]. It follows that α is the equality relation on an inverse semigroup.

Two α-related idempotents in a semigroup S must be mutually inverse, as is easily proved. Suppose again that a, b are regular elements that are α-related in S, and let $x \in \mathrm{Pre}(a) = \mathrm{Pre}(b)$. Then $(ax, bx) \in \alpha$, since α is a congruence. But ax, $bx \in E(S)$ and so these elements are mutually inverse. We now have $a \mathcal{R} ax$, $ax \mathcal{D} bx$, $bx \mathcal{R} b$. It follows that $a \mathcal{D} b$. In particular, if S is a regular

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semigroup then $\alpha \subseteq \mathcal{D}$.

When S is a monoid, α is clearly the equality relation 1_S on S ; a stronger result in the same vein, however, is the following.

LEMMA 2.1. *Let a and b be regular elements of a semigroup S. Then $\alpha \cap (aSb \times aSb)$ is the equality relation on aSb.*

Proof. Let $a' \in \mathrm{Pre}(a)$, $b' \in \mathrm{Pre}(b)$. Then, for $x,y \in S$,

$$(a\,x\,b,\ a\,y\,b) \in \alpha \Rightarrow a\,a'(a\,x\,b)b'b = a\,a'(a\,y\,b)b'b$$
$$\Rightarrow a\,x\,b = a\,y\,b\ .$$

The result follows.

LEMMA 2.2. *Let S be a semigroup. Then $\alpha \cap \mathcal{H} = 1_S$.*

Proof. Let $(x,y) \in \alpha \cap \mathcal{H}$ $(x,y \in S)$ and suppose that $x \neq y$. Then $x = ys$, $y = tx$ for some $s,t \in S$. So $txs = x$, $tys = y$. But $txs = tys$ since $(x,y) \in \alpha$, giving $x = y$, a contradiction. This proves the lemma.

LEMMA 2.3. *Let S be a regular semigroup and let $x, y, z \in S$ be such that $x \leq z$ and $y \leq z$. Then $(x,y) \in \alpha \Rightarrow x = y$.*

Proof. Suppose $(x,y) \in \alpha$ and let $z' \in \mathrm{Pre}(z)$. Then, by Lemma 1.1, $x = x\,z'z = z\,z'x$, so $x = z\,z'x\,z'z$. Similarly $y = z\,z'y\,z'z$. Since $(x,y) \in \alpha$, we have $x = y$, as required.

We immediately have

COROLLARY 2.4. *For a regular semigroup the congruence α is strictly compatible.*

Let S be a regular semigroup and let $e \in E(S)$. Then Corollary 2.4 and [9, Theorem 2.8] tell us that the α-class $e\alpha$ is a completely simple subsemigroup of S; further, by Lemma 2.2, $e\alpha$ has trivial $\mathcal{H}$-classes and so is a rectangular sub-band of S. In particular, α is idempotent-determined.

In the next result we take an arbitrary semigroup S containing an idempotent and improve on the results stated in the previous paragraph.

We recall [2, Section 4.4] that if I is an ideal of a semigroup T then T is said to be an <u>ideal extension of</u> I <u>by the</u> (<u>Rees quotient</u>) <u>semigroup</u> T/I.

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For any semigroup S, let Reg(S) denote the set of regular elements of S.

THEOREM 2.5. *Let* S *be a semigroup and let* $e \in E(S)$. *Write* $T = e\alpha$, $I = e\alpha \cap Reg(S)$. *Then* I *is a rectangular sub-band of* S *and* T *is an ideal extension of* I *by a semigroup* U *satisfying* $U^3 = 0$.

Proof. We note at the outset that T is a subsemigroup of S, being a congruence class of an idempotent. Suppose now that $x \in I$ and that $x' \in$ Pre(x). Then $(x,x^2) \in \alpha$, so $xx'.x.x'x = xx'.x^2.x'x$, that is $x = x^2$. Thus $I \subseteq E(S)$.

Now let $x,y \in I$. Then $(xy)^2 = xyxy = xyyy = xy$, so that xy is regular and hence belongs to I. Thus I is a subsemigroup of S. Further, for $x,y \in I$, we have $xyx = xxx = x$, so I is a rectangular band [5, Chapter IV, Proposition 3.2].

If $x \in T$, $y \in I$ then we argue as above to get that $(xy)^2 = xy$, $(yx)^2 = yx$, so that xy, $yx \in I$. Thus I is an ideal of T. Finally, if $x, y, z \in T$, then

$$(xyz)^2 = xyzxyz = xe^4z = xez = xyz,$$

so that $xyz \in I$. This shows that the Rees quotient semigroup $U = T/1$ satisfies $U^3 = 0$. The theorem is now proved.

The next result follows from Theorem 2.5 and the fact that α-equivalent idempotents are mutually inverse.

COROLLARY 2.6. *Let* S *be a regular semigroup and let* $e \in E(S)$. *Then the congruence class* $e\alpha$ *is a rectangular sub-band of* S *contained in* $V(e) \cap E(S)$.

The containment in the statement of Corollary 2.6 is strict in general: in the semigroup of Example 1.3 the idempotents $(1,a,1)$ and $(2,a,1)$ are mutually inverse but are not α-related, since, for example,

$$(1,b,2)(1,a,1)(1,b,2) = (1,b,2),\ (1,b,2)(2,a,1)(1,b,2) = (1,0,2).$$

In fact this semigroup has just one non-trivial α-class, namely

$$\{(1,0,1),\ (1,0,2),\ (2,0,1),\ (2,0,2)\}.$$

The next result shows that α-equivalence of regular elements is

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closely linked to that of related idempotents. The proof is straightforward and is omitted; it uses the fact, noted earlier, that if two regular elements a, b in a semigroup are α-equivalent then $\mathrm{Pre}(a) = \mathrm{Pre}(b)$.

THEOREM 2.7. *Let* a, b *be regular elements in a semigroup. Then* $(a,b) \in \alpha \iff$ *there exists* $x \in \mathrm{Pre}(a) \cap \mathrm{Pre}(b)$ *such that* $(ax,bx) \in \alpha$ *and* $(xa,xb) \in \alpha$.

3. A generalization of local structure.

Let S be a semigroup. For each $a \in S$ we define a relation δ^a on the set S by the rule

$$x\, \delta^a\, y \iff a\,x\,a = a\,y\,a \,.$$

This relation was one of three congruences introduced in the context of sandwich semigroups (where it was denoted d) by Symons [10]; it was studied further in [7].

LEMMA 3.1. *Let* S *be a semigroup and let* $a \in S$. *Then*

(i) δ^a *is a congruence on* (S,a) *and* $\delta^a \subseteq \alpha(S,a)$,

(ii) if a *is regular in* S *then* $\delta^a = \alpha(S,a)$.

Proof. (i) Clearly δ^a is an equivalence relation on the set S. Suppose $x\delta^a y$ $(x,y \in S)$ and let $z \in S$. Then $a\,z\,a\,x\,a = a\,z\,a\,y\,a$, that is $a(z \circ x)a = a(z \circ y)a$, where $\circ$ denotes multiplication in (S,a) . So $z \circ x\ \delta^a\ z \circ y$. Similarly $x \circ z\ \delta^a\ y \circ z$. Thus δ^a is a congruence on (S,a) . Further suppose $x\ \delta^a\ y$ $(x,y \in S)$. Then, if $s, t \in S$,

$$s \circ x \circ t = s\,a\,x\,a\,t = s\,a\,y\,a\,t = s \circ y \circ t \,,$$

so $(x,y) \in \alpha(S,a)$. This proves (i).

(ii) Now let a be a regular in S . Let $x,y \in S$ be such that $(x,y) \in \alpha(S,a)$. Then, for all $s,t \in S$, $s \circ x \circ t = s \circ y \circ t$, that is $s(a\,x\,a)t = s(a\,y\,a)t$. Thus $(a\,x\,a,\ a\,y\,a) \in \alpha(S)$, so $a\,x\,a = a\,y\,a$ by Lemma 2.1. Thus $\alpha(S,a) \subseteq \delta^a$, and hence $\alpha(S,a) = \delta^a$, by part (i). This completes the proof.

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For S a semigroup, the quotients $(S,a)/\delta^a$ $(a \in S)$ provide a generalisation of (semigroups isomorphic to) the local subsemigroups of S, as the following lemma shows.

LEMMA 3.2. *Let S be a semigroup and let $e \in E(S)$. Then $(S,e)/\delta^e \cong eSe$*

Proof. The mapping $\psi : S \to eSe$ defined by $x\psi = exe$ is a homomorphism from (S,e) onto eSe, and $\psi \circ \psi^{-1} = \delta^e$. The result follows.

THEOREM 3.3. *Let S be a semigroup and let a, b be $\mathcal{D}$-related elements of S. Then $(S,a)/\delta^a \cong (S,b)/\delta^b$.*

Proof. Since $a \,\mathcal{D}\, b$ in S we can find $c \in S$ such that $a \,\mathcal{R}\, c$, $c \,\mathcal{L}\, b$. Then there exist elements $s, s', t, t' \in S^1$ such that

(1) $$as = c,\ cs' = a,\ tc = b,\ t'b = c.$$

Then

(2) $$ass' = t'ta = t'bs' = a,\quad bs's = tt'b = b.$$

We may now define a mapping $\theta : (S,a)/\delta^a \to (S,b)/\delta^b$ by the rule $(x\,\delta^a)\theta = (s'xt')\delta^b$. For suppose that $x\,\delta^a = y\,\delta^a$ $(x,y \in S)$. Then $axa = aya$ and, using (1), we get

$$\begin{aligned} b(s'xt')b &= tcs'xc = taxas = tayas \\ &= tcs'yc = b(s'yt')b\,. \end{aligned}$$

This shows that the mapping θ is well-defined.

Similarly the rule $(x\,\delta^b)\phi = (sxt)\delta^a$ $(x \in S)$ defines a mapping $\phi : (S,b)/\delta^b \to (S,a)/\delta^a$. Now, for $x \in S$,

$$(x\,\delta^a)\theta\phi = [(s'xt')\delta^b]\phi = (ss'xt't)\delta^a\,.$$

But $a(ss'xt't)a = axa$, by (2), and so $(x\,\delta^a)\theta\phi = x\,\delta^a$. Similarly we may show, using (2), that $(x\,\delta^b)\phi\theta = x\,\delta^b$ for $x \in S$, and so θ, ϕ are mutually inverse bijections.

Finally, for $x, y \in S$, consider the product $(x\,\delta^a) \circ (y\,\delta^a)$ in $(S,a)/\delta^a$. We have

$$(x\,\delta^a)\circ(y\,\delta^a) = (x\circ y)\delta^a \quad \text{(where } \circ \text{ is multiplication in } (S,a))$$
$$= (x\,a\,y)\delta^a\ ,$$

and so

$$[(x\,\delta^a)\circ(y\,\delta^a)]\theta = (s'\,x\,a\,y\,t')\delta^b\ .$$

Then, in $(S,b)/\delta^b$,

$$[(x\,\delta^a)\theta]\circ[(y\,\delta^a)\theta] = [(s'\,x\,t')\delta^b]\circ[(s'\,y\,t')\delta^b]$$
$$= [(s'\,x\,t')\circ(s'\,y\,t')]\delta^b \quad \text{(where } \circ \text{ is multiplication in } (S,b))$$
$$= (s'\,x\,t'\,b\,s'\,y\,t')\delta^b = (s'\,x\,a\,y\,t')\delta^b \quad \text{(using (2))}.$$

Thus $[(x\,\delta^a)\circ(y\,\delta^a)]\theta = [(x\,\delta^a)\theta]\circ[(y\,\delta^a)\theta]$, and so θ is an isomorphism. This proves the result.

The following is an obvious consequence of Theorem 3.3 and Lemma 3.2.

COROLLARY 3.4. *Let a be a regular element of a semigroup S and let $e \in E(S)$ be such that $e\ \mathcal{D}\ a$ in S . Then $(S,a)/\delta^a \cong eSe$.*

We note that Corollary 3.4 implies Lemma 1.2.

THEOREM 3.5. *Let S be a semigroup and let $a \in S$. Then, for $x \in S$, $x\,\delta^a$ is regular in $(S,a)/\delta^a \Longleftrightarrow axa$ is regular in S; consequently, $(S,a)/\delta^a$ is regular $\Longleftrightarrow$ $aSa \subseteq Reg(S)$.*

Proof. We use $\circ$ to denote the operation in (S,a) and also that in $(S,a)/\delta^a$. Let $x \in S$. Then

$$x\,\delta^a \text{ is regular in } (S,a)/\delta^a \Longleftrightarrow (\exists\, y \in S)(x\,\delta^a = (x\,\delta^a)\circ(y\,\delta^a)\circ(x\,\delta^a))$$
$$\Longleftrightarrow (\exists\, y \in S)((x, x\circ y\circ x) \in \delta^a)$$
$$\Longleftrightarrow (\exists\, y \in S)(a\,x\,a = a\,x\,a\,y\,a\,x\,a)$$
$$\Longleftrightarrow a\,x\,a \text{ is regular in } S\ ,$$

proving the first assertion. The second assertion follows immediately.

COROLLARY 3.6. *In a regular semigroup S each quotient $(S,a)/\delta^a$ $(a \in S)$ is a regular monoid.*

This is a consequence of Corollary 3.4 and Theorem 3.5; alternatively, it follows from Corollary 3.4 and the well-known fact that the

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local subsemigroups of a regular semigroup are regular.

The relations $\alpha(S,a)$ and δ^a coincide when a is a regular element in a semigroup S (by Lemma 3.1 (ii)). We will frame the final results of this section in terms of $\alpha(S,a)$ rather than δ^a.

COROLLARY 3.7. *(i) Let* S *be a monoid with identity element* 1. *Then, for all* $a \in D_1$, $(S,a)/\alpha(S,a) \cong S$.

(ii) If u *is an idempotent mididentity in a semigroup* S *then* $S/\alpha(S) \cong uSu$.

Proof. (i) follows from Corollary 3.4; to prove (ii) we use Lemma 3.2 and note that, for a mididentity u in a semigroup S, the semigroups (S,u) and S coincide, so $\alpha(S,u) = \alpha(S)$.

Note. Let S be the full transformation semigroup $T(X)$ on a set X. Then Symons [10, Theorem 1.7] has shown that, for $\theta \in S$, $(S,\theta)/\delta^\theta \cong T(X\theta)$. It follows from this (and known properties of $T(X)$) that, for $\theta,\phi \in S$,

$$(S,\theta)/\delta^\theta \cong (S,\phi)/\delta^\phi \iff \theta \,\mathcal{D}\, \phi \text{ in } S.$$

(see also [7, Theorem 3.2].)

In an arbitrary regular semigroup S, however, we may have $(S,a)/\delta^a$ and $(S,b)/\delta^b$ isomorphic $(a,b \in S)$ without a and b being $\mathcal{D}$-related. For example, let E be a uniform semilattice (that is a semilattice with the property that $Ee \cong Ef$ for all $e,f \in E$) with $|E| > 1$. Then for all $e,f \in E$ we have $eEe \cong fEf$, that is $(E,e)/\delta^e \cong (E,f)/\delta^f$ (by Lemma 3.2). However, no two distinct elements of E are $\mathcal{D}$-related.

4. Nambooripad's order.

Let S be a regular semigroup and let $\leq$ denote Nambooripad's partial order on S. For $x \in S$ write

$$\downarrow x = \{s \in S : s \leq x\},$$

and, for $A,B \subseteq S$, write $A \cong B$ to mean that A and B are order-isomorphic under $\leq$.

The following lemma is an easy consequence of the results Proposition 1.2(d) and Corollary 1.3 of [9]; alternatively our Lemma 1.1 can be used to prove it.

LEMMA 4.1. *Let* S *be a regular semigroup and let* $e \in E(S)$. *Then* ${\downarrow}e = E(eSe)$.

LEMMA 4.2. *Let* a *be an element of a regular semigroup* S, *let* $a' \in Pre(a)$ *and let* $e = aa'$. *Then* ${\downarrow}a \cong {\downarrow}e$.

Proof. We have a mapping $\phi\colon {\downarrow}a \to {\downarrow}e$ defined by the rule $x\phi = xa'$ $(x \in {\downarrow}a)$. To check that ϕ maps ${\downarrow}a$ into ${\downarrow}e$, suppose that $x \leq a$. Then, by Lemma 1.1, we can work in (S, a') to get

$$x = x \circ x = x \circ a = a \circ x ,$$

that is $$x = xa'x = xa'a = aa'x.$$

Then $(xa')^2 = xa'$, that is $x\phi \in E(S)$. Also,

$$xa'.aa' = aa'.xa' = xa',$$

so that $x\phi \leq e$. Thus ϕ does indeed map ${\downarrow}a$ into ${\downarrow}e$.

Similarly we may show that the rule $f\psi = fa$ $(f \in {\downarrow}e)$ defines a mapping $\psi\colon {\downarrow}e \to {\downarrow}a$. Further, if $x \in {\downarrow}a$, $x\phi\psi = xa'a = x$, and if $f \in {\downarrow}e$, $f\psi\phi = faa' = fe = f$, and so ϕ, ψ are mutually inverse bijections.

Suppose next that $x, y \in {\downarrow}a$ with $x \leq y$. Thus $x \leq y \leq a$. Since $y \leq a$ we have, by Lemma 1.1, $y \in E(S, a')$, that is $ya'y = y$. So $a' \in$ Pre(y) and, by Lemma 1.1 again, we may express the inequaltiy $x \leq y$ in (S, a') . We thus have

$$x = xa'x = xa'y = ya'x .$$

So $(xa')(ya') = (ya')(xa') = xa'$, that is $x\phi \leq y\phi$.

Finally, suppose that $f \leq g$ $(f, g \in {\downarrow}e)$. Then, calculating in (S, a') , we get

$$(fa) \circ (fa) = faa'fa = fefa = fa,$$

and, similarly, $(ga) \circ (ga) = ga$. Also,

$$(fa) \circ (ga) = faa'ga = fega = fa,$$

and, similarly, we have $(ga) \circ (fa) = fa$. Thus $f\psi \leq g\psi$.

We have thus shown that ϕ is an order-isomorphism from $\downarrow a$ to $\downarrow e$ and the lemma is proved.

We can now state the main result of this section.

THEOREM 4.3. *If a and b are two $\mathcal{D}$-equivalent elements of a regular semigroup then $\downarrow a \cong \downarrow b$.*

Proof. Let S be a regular semigroup and let $a\,\mathcal{D}\,b$ $(a,b \in S)$. Let $e = aa'$, $f = bb'$ $(a' \in \mathrm{Pre}(a),\ b' \in \mathrm{Pre}(b))$. Then a, b, e, f are all $\mathcal{D}$-related in S. The subsemigroups eSe and fSf are isomorphic and so, under the ordering of idempotents, $E(eSe)$ is order-isomorphic to $E(fSf)$. Thus

$$\begin{aligned} \downarrow a &\cong \downarrow e && \text{by Lemma 4.2} \\ &= E(eSe) && \text{by Lemma 4.1} \\ &\cong E(fSf) && \\ &= \downarrow f && \text{by Lemma 4.1} \\ &\cong \downarrow b && \text{by Lemma 4.2.} \end{aligned}$$

This proves the result.

References

[1] T. S. Blyth and J. B. Hickey, "*RP*-dominated regular semigroups", *Proc. Roy. Soc. Edinburgh Sect. A* 99 (1984), 185-191.

[2] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, (Vol. 1, Math. Surveys No. 7, Amer. Math. Soc. Providence, R.I., 1961).

[3] D. G. Green, "Extensions of a semilattice by an inverse semigroup", *Bull. Austral. Math. Soc.* 9 (1973), 21-31.

[4] J. B. Hickey, "Semigroups under a sandwich operation", *Proc. Edinburgh Math. Soc.* 26 (1983), 371-382.

[5] J. M. Howie, *An introduction to semigroup theory*, (Academic Press, 1976).

[6] D. B. McAlister, "Regular Rees matrix semigroups and regular Dubreil-Jacotin semigroups", *J. Austral. Math. Soc. (A)* 31 (1981), 325-336.

[7] K. D. Magill, Jr., P. R. Misra and U. B. Tewari, "Symons' *d*-congruence on sandwich semigroups", *Czechoslovak Math. J.* 33 (1983), 221-236.

[8] W. D. Munn, "The idempotent-separating congruences on a regular 0-bisimple semigroup", *Proc. Edinburgh Math. Soc. (2)* 15 (1967), 233-240.

[9] K. S. S. Nambooripad, "The natural partial order on a regular semigroup", *Proc. Edinburgh Math. Soc. (2)* 23 (1980), 249-260.

[10] J. S. V. Symons, "On a generalization of the transformation semigroup", *J. Austral. Math. Soc. (A)* 19 (1975), 47-61.

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