On Vector Bimeasures (*).

KARI YLINEN (Turku, Finland)

Summary. – *See Introduction.*

1. – Introduction.

The notion of a bimeasure in a bilinear functional sense has been studied by M. MORSE and W. TRANSUE in a series of papers (cf. e.g. [12-16]) and E. THOMAS [20]. The bimeasures considered by these authors are (separately or, equivalently, jointly) continuous scalar valued bilinear forms defined on the Cartesian product of the spaces $\mathcal{K}(S_1)$ and $\mathcal{K}(S_2)$ of continuous functions with compact support on the locally compact Hausdorff spaces S_1 and S_2 ($\mathcal{K}(S_i)$ being equipped with the usual locally convex inductive limit topology). The approach in the present article, where (vector) bimeasures are defined as (vector valued) separately σ-additive functions on the Cartesian product of two σ-algebras, is motivated by a desire to find an analogue of the Riesz (-Markov-Kakutani) representation theorem and its vector generalization (cf. [1], [6]) which says that the weakly compact operators from $C_0(S)$ to X (where X is a Banach space, S is a locally compact Hausdorff space, and $C_0(S)$ is the space of continuous scalar functions on S vanishing at infinity, equipped with the supremum norm) are via integration in a bijective correspondence with the regular X-valued Borel vector measures on S. In the scalar case a Riesz type representation theorem in some form seems to be part of the folklore of the subject (see e.g. E. Thomas's review of [5] in *Math. Reviews*, **46**, no. 9285), though we haven't seen any proof in the generality involving bounded bilinear forms on $C_0(S_1)\times C_0(S_2)$ for arbitrary locally compact Hausdorff spaces S_1 and S_2. In this connection it may be observed that the early history of the representation of bilinear forms (see [7]) is closely related (even temporally) to F. Riesz's pioneering work on the representation of linear forms.

Our main representation theorem is Theorem 6.9. That result resembles the vector generalization of the Riesz representation theorem. The counterpart of a weakly compact operator is here a bounded bilinear operator $B: C_0(S_1)\times C_0(S_2)\to X$ whose canonical extension (i.e. the unique separately weak*-to-weak* continuous

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extension $B_e: C_0(S_1)'' \times C_0(S_2)'' \to X''$ of B; such an extension is shown to always exist) maps $C_0(S_1)'' \times C_0(S_2)''$ into X. These bilinear operators are shown to be via integration (this term is made precise in Section 5) in a bijective correspondence with the mappings $\beta: \mathfrak{B}_1 \times \mathfrak{B}_2 \to X$ ($\mathfrak{B}_j$ is the Borel σ-algebra of S_j) for which $\beta(E, \cdot): \mathfrak{B}_2 \to X$ and $\beta(\cdot, F): \mathfrak{B}_1 \to X$ are regular vector measures for all $E \in \mathfrak{B}_1$, $F \in \mathfrak{B}_2$. Such a mapping β is called a (separately regular) *vector bimeasure*.

In general, a vector bimeasure in our terminology is a separately σ-additive mapping $\beta: \Sigma_1 \times \Sigma_2 \to X$ where Σ_1 and Σ_2 are σ-algebras. In Section 5 we develop a theory of integration of a pair of functions with respect to a vector bimeasure in much greater generality than would be necessary for the representation theorem (where the consideration of continuous functions vanishing at infinity suffices). The semivariation of a vector bimeasure is discussed in Section 4. In Section 3 we deal with some aspects of the theory of vector measures needed in the sequel.

Once we have the correspondence between the separately regular vector bimeasures $\beta: \mathfrak{B}_1 \times \mathfrak{B}_2 \to X$ and the class of bilinear operators $B: C_0(S_1) \times C_0(S_2) \to X$ specified above, another approach to the integration of pairs of functions, more closely reminiscent of the method of Morse and Transue, suggests itself. In Section 7 it is shown, roughly, that a technique based on the use of the vector measures corresponding to the weakly compact operators

$$B(\cdot, k): C_0(S_1) \to X \qquad \text{and} \qquad B(h, \cdot): C_0(S_2) \to X$$

for $k \in C_0(S_2)$, $h \in C_0(S_1)$, in place of the vector measures $\beta(\cdot, F)$ and $\beta(E, \cdot)$, $F \in \mathfrak{B}_2$, $E \in \mathfrak{B}_1$, leads to a theory equivalent to the one developed in Section 5 if and only if X does not contain any isomorphic copy of c_0.

2. – Preliminaries and notation.

The notation introduced here will remain fixed throughout the paper. The scalar field can be either $\boldsymbol{R}$ or $\boldsymbol{C}$; we use (consistently) the common notation $\boldsymbol{K}$ for both. We let X always be a Banach space over $\boldsymbol{K}$. The normed dual of X is denoted by X', and we write $x'(x) = \langle x', x\rangle = \langle x, x'\rangle$ for $x \in X$, $x' \in X'$. By definition, $\sigma(X, X')$ is the weak topology on X and $\sigma(X', X)$ is the weak* topology on X'. The norm in X and in X' is denoted by $|\cdot|$, and we write e.g. $X_1' = \{x' \in X' : |x'| \leq 1\}$. The notation $|\cdot|$ is also used for the norm of a bounded linear or bilinear operator; thus e.g. $|B| = \sup\{|B(y, z)| : y \in Y_1, z \in Z_1\}$, if Y and Z are also normed spaces and $B: Y \times Z \to X$ is a bounded bilinear operator. The adjoint of a bounded linear operator T is denoted by T'.

In most questions pertaining to integration theory we follow [6]. For $j = 1, 2$, S_j (resp. S) is a non-empty set, Σ_j (resp. Σ) is a σ-algebra of subsets of S_j (resp. S). A σ-additive (or, equivalently, weakly σ-additive [6, p. 318]) set function $\mu: \Sigma \to X$ is called a *vector measure*. The *semivariation* (in the sense of [6, p. 320]) of a vector measure $\mu: \Sigma \to X$ is denoted by $\|\mu\|$; $\|\mu\|$ is a bounded nonnegative-valued σ-subadditive function on Σ. If here $X = \boldsymbol{K}$, μ is called a *measure*, and $\|\mu\|$ agrees with the total variation of μ. We usually denote the total variation of a measure $\mu: \Sigma \to \boldsymbol{K}$ by $v(\mu)$ and its value for a set $E \in \Sigma$ by $v(\mu, E)$. We let $ca(S, \Sigma)$ denote the Banach space [6, p. 161] of the measures $\mu: \Sigma \to \boldsymbol{K}$ equipped with the norm $|\mu| = v(\mu, S)$ The μ-measurability of a function $f: \Sigma \to \boldsymbol{K}$ for $\mu \in ca(S, \Sigma)$ is defined in [6, p. 106] and its Σ-measurability in [6, p. 240]. Recall that f is μ-measurable if and only if it is Σ^*-measurable, where Σ^* is the Lebesgue extension of Σ relative to μ [6, p. 148]. If $E \subset S$, $\chi_E: S \to \{0, 1\}$ denotes the characteristic function of E.

3. – Measurability and integrability with respect to vector measures.

In this section we present some material on vector measures needed in the study of vector bimeasures. Throughout, $\mu: \Sigma \to X$ is a vector measure. For each $x' \in X'$, the measure $E \mapsto \langle x', \mu(E)\rangle$, $E \in \Sigma$, is denoted by $x'\mu$. An examination of the steps leading to Corollary 2.4 in [1, p. 294] yields the version of that result appearing in part (*b*) of the following lemma. Part (*a*) is well known and easy to prove.

LEMMA 3.1. – (*a*) $\|\mu\|(E) = \sup_{x' \in X_1'} v(x'\mu, E)$ *for all* $E \in \Sigma$. (*b*) *There exists a positive measure* $\lambda \in ca(S, \Sigma)$ *such that*

(1) $\lambda(E) \leqslant \|\mu\|(E)$ *for all* $E \in \Sigma$, *and*

(2) $\lim_{\lambda(E) \to 0} \|\mu\|(E) = 0$.

Let $D \subset X_1'$ *be such that for a fixed constant* $C > 0$ *we have* $\sup\{|\langle x', x\rangle| : x' \in D\} \geqslant C|x|$ *for all* $x \in X$. *Then a measure satisfying* (1) *and* (2) *can be chosen to be the sum of an absolutely convergent series* $\sum_{n=1}^{\infty} a_n v(x_n'\mu)$ *where* a_n *is a positive number and* $x_n' \in D$ *for all* $n \in \boldsymbol{N}$.

The expressions *μ-null set*, *μ-almost everywhere* (abbreviated *μ-a.e.*) and *μ-measurable function* will have the same meanings as in [6, p. 322]. The following result is analogous to Proposition 2.17 in [20, p. 95].

THEOREM 3.2. – *Let* $D \subset X_1'$ *be as in Lemma* 3.1. *Then a function* $f: S \to \boldsymbol{K}$ *is μ-measurable if and only if f is $x'\mu$-measurable for every* $x' \in D$.

PROOF. – Let $a_n > 0$, $x_n' \in D$ for $n \in \boldsymbol{N}$ and the measure $\lambda = \sum_{n=1}^{\infty} a_n v(x_n'\mu)$ be as in Lemma 3.1. Denote by Σ^* (resp. $\Sigma_{x'}^*$) the Lebesgue extension of Σ relative to λ (resp. $x'\mu$) [6, p. 143]. We show that

$$\Sigma^* = \bigcap_{x' \in D} \Sigma_{x'}^* . \tag{1}$$

Choose $A \in \bigcap_{x' \in D} \Sigma^*_{x'} \subset \bigcap_{n=1}^{\infty} \Sigma^*_{x'_n}$. For each $n \in \mathbf{N}$ there are sets $E_n \in \Sigma$, $N_n \subset S$ and $M_n \in \Sigma$ such that $A = E_n \cup N_n$, $N_n \subset M_n$ and $v(x'_n\mu, M_n) = 0$ [6, p. 142]. Write $E = \bigcup_{n=1}^{\infty} E_n$, $N = \bigcap_{n=1}^{\infty} N_n$, $M = \bigcap_{n=1}^{\infty} M_n$. Then $A = E \cup N$, $E \in \Sigma$, $N \subset M$, and $M \in \Sigma$. Since $v(x'_n\mu, M) \leqslant v(x'_n\mu, M_n) = 0$ for all $n \in \mathbf{N}$, $\lambda(M) = \sum_{n=1}^{\infty} a_n v(x'_n\mu, M) = 0$. Thus $A \in \Sigma^*$. Conversely, let $B \in \Sigma^*$, so that $B = F \cup P$, where $F \in \Sigma$ and $P \subset R$ for some $R \in \Sigma$ with $\lambda(R) = 0$. Then $v(x'\mu, R) \leqslant \|\mu\|(R) = 0$ for all $x' \in D$, and so $B \in \bigcap_{x' \in D} \Sigma^*_{x'}$. Since the μ-measurability of f is equivalent to its Σ^*-measurability, and the $x'\mu$-measurability of f is equivalent to its $\Sigma^*_{x'}$-measurability, the assertion follows from (1).

A Σ-measurable function $f: S \to \mathbf{K}$ which assumes only a finite number of values is called a *Σ-simple function*. The definition of the integral $\int_E f\, d\mu$, $E \in \Sigma$, of a Σ-simple function f is obvious [6, p. 322]. The following definition is used e.g. in [6, p. 323].

DEFINITION 3.3. – A function $f: S \to \mathbf{K}$ is said to be *μ-integrable*, if there is a sequence (f_n) of Σ-simple functions converging to f μ-a.e. and such that the sequence $\left(\int_E f_n\, d\mu\right)$ is (norm) convergent in X for each $E \in \Sigma$. We then write

$$\lim_{n \to \infty} \int_E f_n\, d\mu = \int_E f\, d\mu = \int_E f(s)\mu(ds)\,, \qquad E \in \Sigma\,.$$

REMARK 3.4. – Every μ-integrable function $f: S \to \mathbf{K}$ is μ-measurable [6, p. 150]. The integral is an unambiguously defined element of X [6, p. 323]. We shall use without explicit mention the well-known fact that, in case $X = \mathbf{K}$, Definition 3.3 is equivalent to Definition 17 in [6, p. 112]. A proof could be given by using, for one direction, Corollary 3 in [6, p. 145], and Egoroff's theorem and the Vitali-Hahn-Saks theorem for the other.

The following theorem is closely related to some results in [11], but we give a complete proof, because in our case f need not be Σ-measurable.

THEOREM 3.5. – *Let $D \subset X'_1$ and $C > 0$ be as in Lemma 3.1. A function $f: S \to \mathbf{K}$ is μ-integrable if and only if the following two conditions hold:*

(i) *f is $x'\mu$-integrable for each $x' \in D$;*

(ii) $\lim_{n \to \infty} \sup_{x' \in D} \int_{E_n} |f(s)| v(x'\mu, ds) = 0$ *whenever the sets $E_n \in \Sigma$ satisfy $E_{n+1} \subset E_n$, $n \in \mathbf{N}$, and $\bigcap_{n=1}^{\infty} E_n = \emptyset$.*

PROOF. – Suppose first that f is μ-integrable. Clearly, (i) holds [6, p. 324]. Write $\nu(E) = \int_E f\, d\mu$, $E \in \Sigma$. Then $\nu: \Sigma \to X$ is a vector measure [6, p. 323]. If the sets E_n

are as in (ii), then

$$\sup_{x'\in D}\int_{E_n}|f(s)|v(x'\mu, ds) = \sup_{x'\in D} v(x'\nu, E_n) \leqslant \|\nu\|(E_n)\to 0 ,$$

as $n\to\infty$ (see [6, p. 114] and use Lemma 3.1 or e.g. Theorem 1.3 in [11]). Conversely, assume (i) and (ii). Since f is μ-measurable by Remark 3.4 and Theorem 3.2, there is a Σ-measurable function $f_0: S\to \boldsymbol{K}$ which agrees with f μ-a.e. (see e.g. [19, p. 145]); in particular, (i) and (ii) hold for f_0. Denote $E_n = \{s\in S: |f_0(s)|\geqslant n\}\in\Sigma$, $n\in \boldsymbol{N}$. For each $n\in\boldsymbol{N}$ there is a Σ-simple function $f_n: S\to\boldsymbol{K}$ satisfying $f_n(s)=0$ for $s\in E_n$ and $|f_n(s)|\leqslant|f_0(s)|$, $|f_n(s)-f_0(s)|<1/n$ for $s\in S\setminus E_n$. Then $\lim_{n\to\infty} f_n(s) = f_0(s)$, $s\in S$. If $x'\in D$, we have for all $E\in\Sigma$,

$$\left|\left\langle \int_{EE_m}(f_n - f_m)\,d\mu, x'\right\rangle\right| \leqslant 2\int_{EE_m}|f_0(s)|v(x'\mu, ds) ,$$

and so

$$\left|\int_{EE_m}(f_n - f_m)\,d\mu\right| \leqslant \frac{2}{C}\sup_{x'\in D}\int_{EE_m}|f_0(s)|\,v(x'\mu, ds) .$$

If e.g. $n\geqslant m$, $E_n\subset E_m$, so that

$$|f_n(s)-f_m(s)|\leqslant|f_n(s)-f_0(s)| + |f_0(s)-f_m(s)| < \frac{1}{n}+\frac{1}{m}, \qquad s\in S\setminus E_m .$$

Thus

$$\left|\int_E f_n\,d\mu - \int_E f_m\,d\mu\right| \leqslant \left|\int_{EE_m}(f_n - f_m)\,d\mu\right| + \\ + \left|\int_{E\setminus E_m}(f_n - f_m)\,d\mu\right| \leqslant \frac{2}{C}\sup_{x'\in D}\int_{E_m}|f_0(s)|\,v(x'\mu, ds) + \|\mu\|(S)\left(\frac{1}{n}+\frac{1}{m}\right).$$

It follows that $\left(\int_E f_n\,d\mu\right)$ is a Cauchy sequence in X. By definition f_0, and hence f, is μ-integrable.

The following corollary is essentially (i.e. modulo Theorem 3.2) contained in Theorem 2.4 in [11].

COROLLARY 3.6. – *A function $f: S\to\boldsymbol{K}$ is μ-integrable if and only if the following two conditions are satisfied:*

(i) *f is $x'\mu$-integrable for each $x'\in X'$;*

(ii) *for each $E\in\Sigma$ there is $\nu(E)\in X$ (clearly unique) such that $\langle\nu(E), x'\rangle = \int_E f\,dx'\mu$ for all $x'\in X'$.*

If this is the case, then $\nu(E) = \int_E f\,d\mu$, $E\in\Sigma$.

PROOF. – Assume (i) and (ii). Since $\nu\colon \Sigma \to X$ is weakly, hence strongly, σ-additive, condition (ii) of Theorem 3.5 holds when $D = X_1'$ (see Theorem 20 (*a*) in [6, p. 114] and use Lemma 3.1), so that f is μ-integrable. The converse part and the equation $\nu(E) = \int_E f\,d\mu$ follow from Theorem 8 (*f*) in [6, p. 324].

COROLLARY 3.7. – *If a function* $f\colon S \to \mathbf{K}$ *is integrable with respect to each of the vector measures* $\mu\colon \Sigma \to X$ *and* $\nu\colon \Sigma \to X$, *then* f *is* $(\mu + \nu)$*-integrable, and* $\int_E f\,d(\mu + \nu) = \int_E f\,d\mu + \int_E f\,d\nu$ *for all* $E \in \Sigma$.

PROOF. – The above corollary reduces the proof to the standard scalar case.

4. – The Banach space of vector bimeasures.

Recall that Σ_j is a σ-algebra of subsets of $S_j \neq \emptyset$, $j = 1, 2$.

DEFINITION 4.1. – If $\beta\colon \Sigma_1 \times \Sigma_2 \to X$ is separately σ-additive, i.e. if $\beta(E, \cdot)$ and $\beta(\cdot, F)$ are vector measures for all $E \in \Sigma_1$, $F \in \Sigma_2$, then β is called a *vector bimeasure*. In case $X = \mathbf{K}$, β is simply called a *bimeasure*.

We shall define a norm in the space of vector bimeasures, and for that purpose we introduce the notion of semivariation. In Section 6 we need the concept also for separately (finitely) additive mappings, so the definition is formulated in that generality.

DEFINITION 4.2. – A partition $(E_k)_{k=1}^m$ of a set $E \in \Sigma_j$ is called a Σ_j*-partition*, if $E_k \in \Sigma_j$ for all $k = 1, \dots, m$. Let $\beta\colon \Sigma_1 \times \Sigma_2 \to X$ be separately additive. For $E \in \Sigma_1$, $F \in \Sigma_2$, we let $\|\beta\|(E, F)$ denote the supremum of the numbers $\left|\sum_{k=1}^m \sum_{p=1}^n a_k b_p \beta(E_k, F_p)\right|$ where always $(E_k)_{k=1}^m$ is a Σ_1-partition of E, $(F_p)_{p=1}^n$ is a Σ_2-partition of F, and $a_k, b_p \in \mathbf{K}$, $|a_k| \leq 1$, $|b_p| \leq 1$ for $k = 1, \dots, m$, $p = 1, \dots, n$. The extended real valued function $(E, F) \mapsto \|\beta\|(E, F)$ on $\Sigma_1 \times \Sigma_2$ is called the *semivariation* of β.

LEMMA 4.3. – *If* β *is as in the above definition, and* $D \subset X_1'$ *is such that* $|x| = \sup_{x' \in D} |\langle x, x'\rangle|$ *for all* $x \in X$, *then* $\|\beta\|(E, F) = \sup_{x' \in D} \|x' \circ \beta\|(E, F)$ *for all* $E \in \Sigma_1$, $F \in \Sigma_2$.

PROOF. – If $(E_k)_{k=1}^m$, $(F_p)_{p=1}^n$, and a_k, b_p are as in Definition 4.2, we have

$$\sup_{x' \in D} \left|\sum_{k=1}^m \sum_{p=1}^n a_k b_p x' \circ \beta(E_k, F_p)\right| = \left|\sum_{k=1}^m \sum_{p=1}^n a_k b_p \beta(E_k, F_p)\right|.$$

Thus the assertion follows from elementary properties of the supremum.

In the rest of this section we confine our attention to separately σ-additive functions, although the separately finitely additive case could be treated in an analogous way.

THEOREM 4.4. – *Let $\beta: \Sigma_1 \times \Sigma_2 \to \boldsymbol{K}$ be a bimeasure, and denote $\beta_1(E) = \beta(E, \cdot)$ for each $E \in \Sigma_1$, and $\beta_2(F) = \beta(\cdot, F)$ for each $F \in \Sigma_2$. Then the mappings $\beta_1: \Sigma_1 \to ca(S_2, \Sigma_2)$ and $\beta_2: \Sigma_2 \to ca(S_1, \Sigma_1)$ are vector measures, and $\|\beta\|(S_1, S_2) = \|\beta_1\|(S_1) = \|\beta_2\|(S_2)$.*

PROOF. – To show that e.g. β_1 is a vector measure, let $(E_k)_{k=1}^{\infty}$ be a sequence of pairwise disjoint members of Σ_1. Since $\lim_{n\to\infty} \beta\left(\bigcup_{k=1}^{n} E_k, F\right)$ exists for all $F \in \Sigma_2$, the measures $\beta_1\left(\bigcup_{k=1}^{n} E_k\right)$, $n \in \boldsymbol{N}$, form a bounded sequence in $ca(S_2, \Sigma_2)$ (see Theorem 8 in [6, p. 309] and Lemma 5 in [6, p. 97]). From Theorem 5 in [6, p. 308] it follows that the sequence of the measures $\beta_1\left(\bigcup_{k=1}^{n} E_k\right)$ converges weakly in $ca(S_2, \Sigma_2)$ to $\beta_1\left(\bigcup_{k=1}^{\infty} E_k\right)$. Thus β_1 is weakly, hence strongly, σ-additive. Let now $(E_k)_{k=1}^{m}$ be a Σ_1-partition of S_1 and $(F_p)_{p=1}^{n}$ a Σ_2-partition of S_2, and suppose $a_k, b_p \in \boldsymbol{K}$, $|a_k| \le 1$, $|b_p| \le 1$, $k = 1, \ldots, m$, $p = 1, \ldots, n$. Then

$$\left|\sum_{p=1}^{n} b_p\left(\left(\sum_{k=1}^{m} a_k \beta_1(E_k)\right)(F_p)\right)\right| = \left|\sum_{k=1}^{m} \sum_{p=1}^{n} a_k b_p \beta(E_k, F_p)\right|.$$

The supremum of all numbers obtainable in this way as the left hand side of this equation is easily seen to be $\|\beta_1\|(S_1)$, and so $\|\beta_1\|(S_1) = \|\beta\|(S_1, S_2)$. The equality $\|\beta\|(S_1, S_2) = \|\beta_2\|(S_2)$ is proved similarly.

THEOREM 4.5. – (*a*) *For any vector bimeasure $\beta: \Sigma_1 \times \Sigma_2 \to X$*

$$\sup\{|\beta(E, F)|: E \in \Sigma_1, F \in \Sigma_2\} \le \|\beta\|(S_1, S_2) \le 16 \sup\{|\beta(E, F)|: E \in \Sigma_1, F \in \Sigma_2\} < \infty.$$

(*b*) *The set of vector bimeasures $\beta: \Sigma_1 \times \Sigma_2 \to X$ is a Banach space with respect to the pointwise operations and the norm $\beta \mapsto \|\beta\|(S_1, S_2)$.*

PROOF. – (*a*) The first inequality follows at once from the definition. By the preceding theorem and Lemma 4 in [6, p. 320],

$$\|x' \circ \beta\|(S_1, S_2) = \|(x' \circ \beta)_1\|(S_1) \le 4 \sup_{E \in \Sigma_1} v\left((x' \circ \beta)_1(E), S_2\right) \le$$
$$\le 4 \sup_{E \in \Sigma_1}\left(4 \sup_{F \in \Sigma_2} |\langle x', \beta(E, F)\rangle|\right) \le 16 \sup\{|\beta(E, F)|: E \in \Sigma_1, F \in \Sigma_2\}$$

for all $x' \in X_1'$, and so the second inequality follows from Lemma 4.3. Finally $\|x' \circ \beta\|(S_1, S_2) = \|(x' \circ \beta)_1\|(S_1)$ is finite for all $x' \in X'$ by Theorem 4.4 and Lemma 4 in [6, p. 320], i.e. $\sup\{|\langle x', \beta(E, F)\rangle|: E \in \Sigma_1, F \in \Sigma_2\} < \infty$. Alternatively, this follows directly from Theorem 8 in [6, p. 309]. By the uniform boundedness principle $\sup\{|\beta(E, F)|: E \in \Sigma_1, F \in \Sigma_2\} < \infty$.

(*b*) It is clear that the vector bimeasures form a linear space. It is quickly verified that $\|\cdot\|(S_1, S_2)$ (which is finite by (*a*)) is a norm. To prove the completeness of the space, let (β_n) be a Cauchy sequence of vector bimeasures. Then $\beta(E, F) = \lim_{n\to\infty} \beta_n(E, F)$ exists for all $E \in \Sigma_1$, $F \in \Sigma_2$, and the convergence is uniform on $\Sigma_1 \times \Sigma_2$. A standard argument then shows that β is separately σ-additive, and by (*a*), $\lim_{n\to\infty} \|\beta_n - \beta\|(S_1, S_2) = 0$.

5. – Integration with respect to a vector bimeasure.

Throughout this section, $\beta: \Sigma_1 \times \Sigma_2 \to X$ is a vector bimeasure. Before considering the integration of a pair of functions with respect to β we prove an auxiliary result.

LEMMA 5.1. – *Let $f: S_1 \to \boldsymbol{K}$ be $\beta(\cdot, F)$-integrable for every $F \in \Sigma_2$. Then the set function $F \mapsto \int_E f\, d\beta(\cdot, F)$, $F \in \Sigma_2$, is σ-additive for every $E \in \Sigma_1$.*

PROOF. – Let us first treat the special case where $X = \boldsymbol{K}$. Consider the $ca(S_2, \Sigma_2)$-valued vector measure β_1, i.e. $E \mapsto \beta(E, \cdot)$ (cf. Theorem 4.4). Let $\lambda \in ca(S_1, \Sigma_1)$ be a positive measure satisfying the conditions (1) and (2) of Lemma 3.1 relative to β_1. Each $F \in \Sigma_2$ determines a bounded linear functional χ'_F on $ca(S_2, \Sigma_2)$ by $\langle \nu, \chi'_F \rangle = \nu(F)$, and the set $D = \{\chi'_F : F \in \Sigma_2\}$ has the property $\sup_{\varphi \in D} |\langle \nu, \varphi \rangle| \geq \frac{1}{4} v(\nu, S_2)$ for all $\nu \in ca(S_2, \Sigma_2)$ [6, p. 97]. Since f is β_1-measurable by Theorem 3.2, i.e. f is Σ_1^*-measurable where Σ_1^* is the Lebesgue extension of Σ_1 relative to λ, there is a Σ_1-measurable function $f_0: S_1 \to \boldsymbol{K}$ which agrees with f β_1-a.e. (and hence $\beta(\cdot, F)$-a.e. for each $F \in \Sigma_2$) [19, p. 145]. There is a sequence (f_n) of Σ_1-simple functions such that $|f_n(s)| \leq |f_0(s)|$ and $\lim_{n\to\infty} f_n(s) = f_0(s)$ for all $s \in S_1$. As f_0 is $\beta(\cdot, F)$-integrable for each $F \in \Sigma_2$,

$$\lim_{n\to\infty} \int_E f_n\, d\beta(\cdot, F) = \int_E f_0\, d\beta(\cdot, F) = \int_E f\, d\beta(\cdot, F) \qquad \text{for all } F \in \Sigma_2$$

by the Lebesgue dominated convergence theorem [6, p. 151]. Since the set function $F \mapsto \int_E f_n\, d\beta(\cdot, F)$, $F \in \Sigma_2$, is σ-additive for each $n \in \boldsymbol{N}$, Corollary 4 in [6, p. 160] shows that $F \mapsto \int_E f\, d\beta(\cdot, F)$ is σ-additive on Σ_2. In the case of a general Banach space X the above discussion proves for every $x' \in X'$ the σ-additivity of the set function $F \mapsto \left\langle \int_E f\, d\beta(\cdot, F), x' \right\rangle = \int_E f\, d(x' \circ \beta(\cdot, F))$, and so the assertion follows from Theorem 1 in [6, p. 318].

The following definition is inspired by [15, p. 482] and the definition in [20, p. 145] (see, however, Remark 7.3).

DEFINITION 5.2. – The pair (f, g) of functions $f: S_1 \to \boldsymbol{K}$ and $g: S_2 \to \boldsymbol{K}$ is said to be integrable with respect to the vector bimeasure: $\beta: \Sigma_1 \times \Sigma_2 \to X$ (or β-integrable for short) if the following three conditions hold:

(i) f is $\beta(\cdot, F)$-integrable for all $F \in \Sigma_2$, and g is $\beta(E, \cdot)$-integrable for all $E \in \Sigma_1$ (so that one obtains the vector measures ${}_f\beta(S_1, \cdot): \Sigma_2 \to X$ and $\beta_g(\cdot, S_2)$: $\Sigma_1 \to X$ defined by ${}_f\beta(S_1, F) = \int_{S_1} f\, d\beta(\cdot, F)$, $\beta_g(E, S_2) = \int_{S_2} g\, d\beta(E, \cdot)$, cf. Lemma 5.1);

(ii) f is $\beta_g(\cdot, S_2)$-integrable and g is ${}_f\beta(S_1, \cdot)$-integrable;

(iii) $\int_{S_1} f\, d\beta_g(\cdot, S_2) = \int_{S_2} g\, d_f\beta(S_1, \cdot)$.

If these conditions hold, each side of the equation in (iii) will be denoted by $\int (f, g)\, d\beta$.

EXAMPLE 5.3. – This example shows that in the above definition (iii) does not follow from (i) and (ii). Choose $S_1 = S_2 = \boldsymbol{N} = \{1, 2, 3, \ldots\}$, and let $\Sigma_1 = \Sigma_2$ be the set of all subsets of $\boldsymbol{N}$. Let (a_k) be a sequence of positive numbers with $\sum_{k=1}^{\infty} a_k < \infty$ Construct inductively a function $f: S_1 \to \boldsymbol{R}$ for which

$$f(1)a_1 = c_1 > 0 , \qquad f(k)a_k - f(k-1)a_{k-1} = c_k > 0$$

for $k \geqslant 2$ and $\sum_{k=1}^{\infty} c_k < \infty$. Then choose a decreasing positive sequence (b_k) such that $\sum_{k=1}^{\infty} f(k)a_k b_k < \infty$ (note that $f(k) > 0$), and define the function $g: S_2 \to \boldsymbol{R}$ by $g(n) = 1/b_n$. Define

$$\beta(\{n\}, \{n\}) = a_n b_n , \qquad \beta(\{n\}, \{n+1\}) = -a_n b_{n+1}$$

for $n \in \boldsymbol{N}$, and $\beta(\{m\}, \{n\}) = 0$ if $m \neq n \neq m+1$. Since $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\beta(\{m\}, \{n\})| < \infty$, it follows from well-known properties of summable families of numbers that the function $(E, F) \mapsto \beta(E, F) = \sum_{m \in E} \sum_{n \in F} \beta(\{m\}, \{n\})$ on $\Sigma_1 \times \Sigma_2$ is a bimeasure. We have

$$\int_{S_1} f\, d\beta(\cdot, \{n\}) = \sum_{m=1}^{\infty} f(m)\beta(\{m\}, \{n\}) = b_n(f(n)a_n - f(n-1)a_{n-1}) = b_n c_n \quad \text{if } n \geqslant 2$$

and $\int_{S_1} f\, d\beta(\cdot, \{1\}) = f(1)a_1 b_1 = b_1 c_1$. Since

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m)\beta(\{m\}, \{n\})| = \sum_{m=1}^{\infty} f(m)a_m(b_m + b_{m+1}) \leqslant 2 \sum_{m=1}^{\infty} f(m)a_m b_m < \infty ,$$

we can again use a property of summable families (i.e. the discrete version of

Fubini's theorem) to show that for any $F \in \Sigma_2$,

$$\sum_{n\in F} \sum_{m=1}^{\infty} f(m)\beta(\{m\}, \{n\}) = \sum_{m=1}^{\infty} f(m) \sum_{n\in F} \beta(\{m\}, \{n\})$$

(absolute convergence). Thus f is $\beta(\cdot, F)$-integrable and

$$\int_{S_1} f\, d\beta(\cdot, F) = \sum_{n\in F} \sum_{m=1}^{\infty} f(m)\beta(\{m\}, \{n\}) = \sum_{n\in F} b_n c_n .$$

Since

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |g(n)\beta(\{m\}, \{n\})| = a_1 + \sum_{n=2}^{\infty} (a_n + a_{n-1}) < \infty ,$$

we see similarly that g is $\beta(E, \cdot)$-integrable for every $E \in \Sigma_1$, and

$$\int_{S_2} g\, d\beta(E, \cdot) = \sum_{m\in E} \sum_{n=1}^{\infty} g(n)\beta(\{m\}, \{n\}) = \sum_{m\in E} a_m(g(m)b_m - g(m+1)b_{m+1}) = 0$$

for all $E \in \Sigma_1$.

Since

$$\int_{S_1} f\, d\beta_g(\cdot, S_2) = 0 \quad \text{and} \quad \int_{S_2} g\, d_f\beta(S_1, \cdot) = \sum_{n=1}^{\infty} g(n) b_n c_n = \sum_{n=1}^{\infty} c_n > 0 ,$$

conditions (i) and (ii) of Definition 5.2 hold in this example, but (iii) does not.

The next result is an easy consequence of Theorem 8 in [6, p. 323] and Corollary 3.7.

THEOREM 5.4. – (*a*) *If* $a_j, b_j \in \boldsymbol{K}$ *for* $j = 1, 2$, *and if* $f_j\colon S_1 \to \boldsymbol{K}$, $g_j\colon S_2 \to \boldsymbol{K}$ *are functions such that the pairs* (f_i, g_j) *for* $i = 1, 2$, $j = 1, 2$, *are* β*-integrable, then the pair* $(a_1 f_1 + a_2 f_2, b_1 g_1 + b_2 g_2)$ *is* β*-integrable, and*

$$\int (a_1 f_1 + a_2 f_2, b_1 g_1 + b_2 g_2)\, d\beta = \sum_{i=1}^{2} \sum_{j=1}^{2} a_i b_j \int (f_i, g_j)\, d\beta .$$

(*b*) *Let* Y *be a Banach space and* $T\colon X \to Y$ *a bounded linear operator. Then* $T\circ\beta$ *is a vector bimeasure, and if the pair* (f, g) *is* β*-integrable, it is* $T\circ\beta$*-integrable, and* $\int (f, g)\, dT\circ\beta = T\left(\int (f, g)\, d\beta\right)$.

LEMMA 5.5. – *Let* $X = \boldsymbol{K}$, *and let* $f\colon S_1 \to \boldsymbol{K}$ *be* $\beta(\cdot, F)$*-integrable for every* $F \in \Sigma_2$. *Then* f *is* β_1*-integrable* (*cf. Theorem* 4.4).

PROOF. – We define ${}_f\beta(E, F) = \int_E f\, d\beta(\cdot, F)$ for $E \in \Sigma_1$, $F \in \Sigma_2$. Then ${}_f\beta$ is a bimeasure by Lemma 5.1 and the σ-additivity of the indefinite integral (cf. e.g.

Theorem 20 (*a*), (*b*) in [6, p. 114]). Thus ν where $\nu(E) = {}_f\beta(E, \cdot)$, $E \in \Sigma_1$, is a $ca(S_2, \Sigma_2)$-valued vector measure defined on Σ_1 (Theorem 4.4). We let $\chi'_F \in ca(S_2, \Sigma_2)'$ for $F \in \Sigma_2$ be as in the proof of Lemma 5.1 and write $D = \{\chi'_F\colon F \in \Sigma_2\}$. Let (E_n) be any decreasing sequence of members of Σ_1 such that $\bigcap_{n=1}^{\infty} E_n = \emptyset$. Then

$$\sup_{F\in\Sigma_2} \int_{E_n} |f(s)| v(\chi'_F \circ \beta_1, ds) = \sup_{F\in\Sigma_2} v({}_f\beta(\cdot, F), E_n)$$

by Theorem 20 (*a*) in [6, p. 114]. But $\sup_{F\in\Sigma_2} v({}_f\beta(\cdot, F), E_n) \leqslant \|\nu\|(E_n) \to 0$ as $n \to \infty$ (Lemma 3.1.). Since $\sup_{F\in\Sigma_2} |\langle\lambda, \chi'_F\rangle| \geqslant \frac{1}{4} v(\lambda, S_2)$ for all $\lambda \in ca(S_2, \Sigma_2)$ [6, p. 97], Theorem 3.5 shows that f is β_1-integrable.

The following theorem is analogous to part of the proposition in [20, p. 145].

THEOREM 5.6. – *For any function* $f\colon S_1 \to \boldsymbol{K}$ *the following three conditions are equivalent*:

(i) f *is* $\beta(\cdot, F)$*-integrable for all* $F \in \Sigma_2$;

(ii) *the pair* (f, χ_F) *is* β*-integrable for all* $F \in \Sigma_2$;

(iii) *the pair* (f, g) *is* β*-integrable for every bounded* Σ_2*-measurable function* $g\colon S_2 \to \boldsymbol{K}$.

PROOF. – It is easily verified that (i) implies (ii). Assume now (ii). Let $g\colon S_2 \to \boldsymbol{K}$ be bounded and Σ_2-measurable. Fix $x' \in X'$ and $E_0 \in \Sigma_1$, and consider the bimeasure $x'\beta = x' \circ \beta$. By Theorem 8 (*f*) in [6, p. 324] and Lemma 5.5, $f\chi_{E_0}$ is integrable with respect to the $ca(S_2, \Sigma_2)$-valued vector measure $x'\beta_1$ $(= (x'\beta)_1)$. Let $g' \in ca(S_2, \Sigma_2)'$ be defined by $\langle g', \lambda\rangle = \int_{S_2} g\, d\lambda$. As $f\chi_{E_0}$ is $x'\beta_1$-integrable, it is also integrable with respect to the scalar measure $g' \circ x'\beta_1$ $\left(\text{i.e. } E \mapsto \int_{S_2} g\, d(x' \circ \beta(E, \cdot)) = \langle x', \beta_g(E, S_2)\rangle\right)$. We have

$$(1) \qquad \int_{E_0} f d(g' \circ x'\beta_1) = \left\langle g', \int_{E_0} f\, dx'\beta_1 \right\rangle = \int_{S_2} g\, d\big({}_{f\chi_{E_0}}x'\beta(S_1, \cdot)\big) = \left\langle x', \int_{S_2} g\, d_{f\chi_{E_0}}\beta(S_1, \cdot) \right\rangle .$$

For any $E \in \Sigma_1$, we denote by ${}_f\beta(E, \cdot)$ the vector measure $F \mapsto \int_E f\, d\beta(\cdot, F)$, $F \in \Sigma_2$ (cf. Lemma 5.1), and define $\nu(E) = \int_{S_2} g\, d_f\beta(E, \cdot) \in X$ (cf. Theorem 8 (*c*) in [6, p. 323]). Then we get by (1),

$$\langle x', \nu(E)\rangle = \left\langle x', \int_{S_2} g\, d\big({}_{f\chi_E}\beta(S_1, \cdot)\big) \right\rangle = \int_E f\, d(g' \circ x'\beta_1) = \int_E f\, d(x' \circ \beta_g(\cdot, S_2))$$

for all $E \in \Sigma_1$. From Corollary 3.6 it thus follows that f is $\beta_g(\cdot, S_2)$-integrable and $\int_{S_1} f\, d\beta_g(\cdot, S_2) = \nu(S_1) = \int_{S_2} g\, d_f\beta(S_1, \cdot)$, i.e. (iii) holds. As (iii) obviously implies (i), the theorem is proved.

COROLLARY 5.7. – *If* $f\colon S_1 \to \mathbf{K}$ *is bounded and* Σ_1*-measurable, and* $g\colon S_2 \to \mathbf{K}$ *is bounded and* Σ_2*-measurable, then the pair* (f, g) *is* β*-integrable, and*

$$\left|\int (f, g)\, d\beta\right| \leqslant \|\beta\|(S_1, S_2) \sup_{s\in S_1} |f(s)| \sup_{t\in S_2} |g(t)|\,.$$

PROOF. – The β-integrability of the pair (f, g) follows at once from Theorem 8 (*c*) in [6, p. 323] combined with the above theorem. (A direct elementary argument could also be given.) Clearly, we may assume that f and g are bounded in absolute value by 1. Then

$$\left|\int (f, g)\, d\beta\right| = \left|\int_{S_1} f\, d\beta_g(\cdot, S_2)\right| \leqslant \|\beta_g(\cdot, S_2)\|(S_1)$$

by Theorem 8 (*c*) in [6, p. 323]. Let $(E_j)_{j=1}^k$ be a Σ_1-partition of S_1, and $a_j \in \mathbf{K}$, $|a_j| \leqslant 1$, $j = 1, \ldots, k$. There is a sequence (g_n) of Σ_2-simple functions bounded by 1, converging uniformly to g. Since

$$\left|\sum_{j=1}^{k} a_j \int_{S_2} g\, d\beta(E_j, \cdot)\right| = \lim_{n\to\infty} \left|\sum_{j=1}^{k} a_j \int_{S_2} g_n\, d\beta(E_j, \cdot)\right| \leqslant \|\beta\|(S_1, S_2)\,,$$

we have $\|\beta_g(\cdot, S_2)\| \leqslant \|\beta\|(S_1, S_2)$.

6. – A Riesz type representation theorem.

Throughout this section, S_1 and S_2 will be locally compact Hausdorff spaces. For $j = 1, 2$, $\mathcal{B}_j$ will denote the Borel σ-algebra of S_j, i.e. $\mathcal{B}_j$ is the σ-algebra generated by the open subsets of S_j. In the following discussion S will stand for S_j and $\mathcal{B}$ for $\mathcal{B}_j$. An additive set function $\mu\colon \mathcal{B} \to X$ is said to be *regular*, if for every $A \in \mathcal{B}$ and every $\varepsilon > 0$ there exist a compact set C and an open set U such that $C \subset A \subset U$ and $|\mu(E)| < \varepsilon$ for all $E \in \mathcal{B}$, $E \subset U \setminus C$. Let $rca(S, \mathcal{B}, X)$ denote the vector space of the regular vector measures $\mu\colon \mathcal{B} \to X$. We write simply $rca(S, \mathcal{B}, \mathbf{K}) = rca(S, \mathcal{B})$.

According to the Riesz (-Markov-Kakutani) representation theorem the mapping $\mu \mapsto \varphi_\mu$ where for $\mu \in rca(S, \mathcal{B})$ $\varphi_\mu \in C_0(S)'$ is defined by the formula $\int_E f\, d\mu = \langle f, \varphi_\mu\rangle$, $f \in C_0(S)$, is an isometric isomorphism from the closed linear subspace $rca(S, \mathcal{B})$ of $ca(S, \mathcal{B})$ onto $C_0(S)'$ (cf. e.g. [19, p. 131]). To simplify notation we often identify μ and φ_μ; we then use the common notation $M(S)$ for both of the spaces $rca(S, \mathcal{B})$ and $C_0(S)'$.

A vector measure $\mu\colon \mathcal{B} \to X$ is known to be regular if (and obviously only if) $x'\mu \in rca(S, \mathcal{B})$ for all $x' \in X'$ (cf. e.g. [9, p. 263] or [11, p. 159]). The proof of Corollary 2 in [9, p. 263] shows that for the regularity of μ it is in fact sufficient that $x'\mu \in rca(S, \mathfrak{B})$ for all x' in some subset D of X' which separates the points of X. The more elementary technique used in [11] would yield the regularity of μ in case $D \subset X_1'$ is such that for some constant $C > 0$ $\sup_{x' \in D} |\langle x', x\rangle| \geqslant C|x|$ for all $x \in X$ and $x'\mu \in rca(S, \mathfrak{B})$ for all $x' \in D$. This generality will suffice for our purposes.

The next result can (essentially) be found e.g. in [6, p. 493]; clearly the proof given there also works in the case of a locally compact space S and $C_0(S)$.

LEMMA 6.1. – *For $\mu \in rca(S, \mathfrak{B}, X)$ and $f \in C_0(S)$ write $T_\mu f = \int_S f\, d\mu$. Then $\mu \mapsto T_\mu$ is a linear bijection from $rca(S, \mathfrak{B}, X)$ onto the set of all weakly compact linear operators $T: C_0(S) \to X$. Moreover, $|T_\mu| = \|\mu\|(S)$ for all $\mu \in rca(S, \mathfrak{B}, X)$.*

REMARK 6.2. – We shall always regard X canonically as a closed linear subspace of X''. Let $T: C_0(S) \to X$ be a bounded linear operator and $T'': M(S)' \to X''$ its second adjoint. For $E \in \mathfrak{B}$ we write $\mu(E) = T''\chi_E' \in X''$ where $\chi_E' \in M(S)'$ is defined by $\langle \chi_E', \lambda\rangle = \lambda(E)$, $\lambda \in M(S)$. Then T is weakly compact if and only if $\mu(E) \in X$ for all $E \in \mathfrak{B}$, and in this case $\mu \in rca(S, \mathfrak{B}, X)$ and $T = T_\mu$ [6, p. 493]. It was observed in [3, p. 154] that the σ-additivity of the set function $\mu: \mathfrak{B} \to X''$ already implies the weak compactness of T. Here is a proof of this fact. Since $\lim_{n\to\infty} \mu(E_n) = 0$, i.e. $T'x'(E_n) = \langle \mu(E_n), x'\rangle \to 0$ uniformly in $x' \in X_1'$, for any decreasing sequence (E_n) of members of $\mathfrak{B}$ satisfying $\bigcap_{n=1}^{\infty} E_n = \emptyset$, T' is weakly compact (see Theorem 1 in [6, p. 305]), and so is T [6, p. 485].

We intend to establish in the context of vector bimeasures an analogue of Lemma 6.1. First we need to consider extensions of certain bilinear operators.

LEMMA 6.3. – *Let $B: C_0(S_1) \times C_0(S_2) \to X$ be a bounded bilinear operator. There is a unique bilinear operator $B_e: C_0(S_1)'' \times C_0(S_2)'' \to X''$ which is an extension of B (when each of the Banach spaces $C_0(S_1)$, $C_0(S_2)$ and X is canonically embedded in its bidual) and separately continuous when each of the spaces $C_0(S_1)''$, $C_0(S_2)''$ and X'' is equipped with its weak* topology. The norms of B and B_e are the same, and B_e depends linearly on B.*

PROOF. – According to a well-known result due to Grothendieck [8], every bounded linear operator from $C_0(S_1)$ to $C_0(S_2)'$ is weakly compact. (One way of seeing this is to combine Theorem 6 in [6, p. 494] with Theorem 4 in [6, p. 308] and Proposition 5.1 in [20, p. 135].) Thus, for each $x' \in X'$, $x' \circ B$ has a unique separately weak* continuous extension $B_e^{x'}: C_0(S_1)'' \times C_0(S_2)'' \to \boldsymbol{K}$, and $|B_e^{x'}| = |x' \circ B|$ (see e.g. [21, p. 365]). We define $(B_e(u, v))(x') = B_e^{x'}(u, v)$ for $u \in C_0(S_1)''$, $v \in C_0(S_2)''$, $x' \in X'$. Then $B_e(u, v) \in X''$. In fact, $B_e^{ax'+by'} = aB_e^{x'} + bB_e^{y'}$ for $x', y' \in X'$, $a, b \in \boldsymbol{K}$, because $aB_e^{x'} + bB_e^{y'}$ is separately weak* continuous and extends $(ax' + by') \circ B$. Thus $B_e(u, v)$: $X' \to \boldsymbol{K}$ is linear. Since $|(B_e(u, v))(x')| = |B_e^{x'}(u, v)| \leqslant |x'|\,|B|\,|u|\,|v|$, $B_e(u, v)$ is continuous. This also shows that $|B_e| \leqslant |B|$, so that $|B_e| = |B|$. The separate weak*-to-weak* continuity of B_e is obvious, and so is its uniqueness, because $C_0(S_j)$ is weak* dense in $C_0(S_j)''$. Finally, if $B_1, B_2\colon C_0(S_1)\times C_0(S_2)\to X$ are bilinear and bounded, and $a, b\in \boldsymbol{K}$, $a(B_1)_e+b(B_2)_e$ is a separately weak*-to-weak* continuous extension of aB_1+bB_2, and so $a(B_1)_e+b(B_2)_e=(aB_1+bB_2)_e$.

DEFINITION 6.4. – If B and B_e are as in the preceding lemma, B_e is called the *canonical extension* of B.

We shall make use of the canonical extension in proving representation theorems for bounded bilinear operators from $C_0(S_1)\times C_0(S_2)$ to X. As a preliminary step, let us treat the scalar case.

NOTATION. – If $f\colon S_j\to \boldsymbol{K}$ is a bounded Borel (i.e., $\mathcal{B}_j$-measurable) function, $f'\in M(S_j)'$ will denote the functional defined by $\langle f', \lambda\rangle=\int_{S_j} f\,d\lambda$, $\lambda\in M(S_j)$.

LEMMA 6.5. – *If $b\colon C_0(S_1)\times C_0(S_2)\to \boldsymbol{K}$ is a bounded bilinear form, there is a unique function $\beta_b\colon \mathcal{B}_1\times\mathcal{B}_2\to\boldsymbol{K}$ satisfying the following two conditions:*

(i) $\beta_b(E,\cdot)\in rca(S_2,\mathcal{B}_2)$ *for all* $E\in\mathcal{B}_1$ *and* $\beta_b(\cdot,F)\in rca(S_1,\mathcal{B}_1)$ *for all* $F\in\mathcal{B}_2$;

(ii) $\int(h,k)\,d\beta_b=b(h,k)$ *for all* $h\in C_0(S_1)$, $k\in C_0(S_2)$.

If $b_e\colon M(S_1)'\times M(S_2)'\to\boldsymbol{K}$ is the canonical extension of b and $f\colon S_1\to\boldsymbol{K}$, $g\colon S_2\to\boldsymbol{K}$ are bounded Borel functions, we have $b_e(f',g')=\int(f,g)\,d\beta_b$. The norm of b equals $\|\beta_b\|(S_1,S_2)$. Conversely, if $\beta\colon\mathcal{B}_1\times\mathcal{B}_2\to\boldsymbol{K}$ satisfies (i), *there is a unique bounded bilinear form $b\colon C_0(S_1)\times C_0(S_2)\to\boldsymbol{K}$ such that $\beta=\beta_b$.*

PROOF. – We define β_b by the formula $\beta_b(E,F)=b_e(\chi'_E,\chi'_F)$, $E\in\mathcal{B}_1$, $F\in\mathcal{B}_2$. From the separate weak* continuity of b_e it follows that e.g. for all bounded Borel functions $f\colon S_1\to\boldsymbol{K}$ the functional $b_e(f',\cdot)\colon M(S_2)'\to\boldsymbol{K}$ is the canonical image of some $\lambda_f\in M(S_2)$ [6, p. 421]. In particular, (i) holds. Let $f\colon S_1\to\boldsymbol{K}$ and $g\colon S_2\to\boldsymbol{K}$ be bounded Borel functions. The pair (f,g) is β_b-integrable by Corollary 5.7. The measure $F\mapsto\int_{S_1} f\,d\beta_b(\cdot,F)$, $F\in\mathcal{B}_2$, is just λ_f (as can be seen by approximating f uniformly by $\mathcal{B}_1$-simple functions). By definition we thus have $\int(f,g)\,d\beta_b=\int_{S_2} g\,d\lambda_f=$ $=b_e(f',g')$. In particular, (ii) holds. We now show that if any $\beta\colon\mathcal{B}_1\times\mathcal{B}_2\to\boldsymbol{K}$ in place of β_b satisfies (i) and (ii), then $\|\beta\|(S_1,S_2)=|b|$; this will also prove (by an obvious linearity argument) the uniqueness of β_b. Consider the vector measure $\beta_1\colon\mathcal{B}_1\to ca(S_2,\mathcal{B}_2)$ defined by $(\beta_1(E))(F)=\beta(E,F)$ (cf. Theorem 4.4). The values of β_1 lie in $rca(S_2,\mathcal{B}_2)$, and it is regular (see the discussion preceding Lemma 6.1, and [6, p. 97]). Therefore, $\|\beta_1\|(S_1)=\sup\left\{\left|\int_{S_1} h\,d\beta_1\right|: h\in C_0(S_1),\ |h|\leq 1\right\}$ by Lemma 6.1. Since

$$\int_{S_1} h\,d\beta_1\in rca(S_2,\mathcal{B}_2)\qquad\text{for all } h\in C_0(S_1),$$

we have

$$\left|\int_{S_1} h\, d\beta_1\right| = \sup\left\{\left|\int_{S_2} k\, d\left(\int_{S_1} h\, d\beta_1\right)\right| : k \in C_0(S_2),\ |k| \leqslant 1\right\} = \sup_{|k|\leqslant 1} |b(h, k)|\,,$$

since

$$\int_{S_2} k\, d\left(\int_{S_1} h\, d\beta_1\right) = \int (h, k)\, d\beta = b(h, k)\,.$$

Thus

$$\|\beta\|(S_1, S_2) = \|\beta_1\|(S_1) = \sup_{|h|\leqslant 1}\ \sup_{|k|\leqslant 1} |b(h, k)| = |b|$$

by Theorem 4.4. As to the last assertion, observe that $(h, k) \mapsto \int (h, k)\, d\beta$ is a bounded bilinear form on $C_0(S_1) \times C_0(S_2)$ (cf. Theorem 4.5 and Corollary 5.7).

THEOREM 6.6. – *Let $B: C_0(S_1) \times C_0(S_2) \to X$ be a bounded bilinear operator. There exists a unique mapping $\beta_B: \mathcal{B}_1 \times \mathcal{B}_2 \to X''$ satisfying the following two conditions:*

(i) *if $x' \in X'$, then the function $F \mapsto \langle x', \beta_B(E, F)\rangle$ is in $rca(S_2, \mathcal{B}_2)$ for all $E \in \mathcal{B}_1$, and the function $E \mapsto \langle x', \beta_B(E, F)\rangle$ is in $rca(S_1, \mathcal{B}_1)$ for all $F \in \mathcal{B}_2$;*

(ii) *for all $x' \in X'$, $h \in C_0(S_1)$ and $k \in C_0(S_2)$, $\langle B(h, k), x'\rangle$ equals the integral of the pair (h, k) with respect to the bimeasure $(E, F) \mapsto \langle x', \beta_B(E, F)\rangle$, $E \in \mathcal{B}_1$, $F \in \mathcal{B}_2$. We have*

$$\beta_B(E, F) = B_e(\chi'_E, \chi'_F) \qquad \text{for } E \in \mathcal{B}_1\,,\ F \in \mathcal{B}_2\,, \tag{1}$$

and $\|\beta_B\|(S_1, S_2) = |B|$.

PROOF. – Define β_B by (1). As for any $x' \in X'$ the canonical extension of $x' \circ B$ is $x' \circ (B_e)$ (with the interpretation $x' \in X'''$), (i), (ii) and the uniqueness statement follow from Lemma 6.5. Since $|x' \circ B| = \|x' \circ \beta_B\|(S_1, S_2)$ (Lemma 6.5), $|B| = \|\beta\| \cdot (S_1, S_2)$ by Lemma 4.3.

We are going to characterize those bounded bilinear operators $B: C_0(S_1) \times C_0(S_2) \to X$ for which $\beta_B(\mathcal{B}_1 \times \mathcal{B}_2) \subset X$. Let us prepare the proof of that result with a lemma.

LEMMA 6.7. – *Let $B: C_0(S_1) \times C_0(S_2) \to X$ be a bounded bilinear operator and B_e its canonical extension. Let $\varphi \in C_0(S_1)''$ be such that $B_e(\varphi, k') \in X$ for all $k \in C_0(S_2)$. Then $B_e(\varphi, \cdot): C_0(S_2)'' \to X''$ is the second adjoint of the operator $k \mapsto B_e(\varphi, k')$ from $C_0(S_2)$ to X.*

PROOF. – Both $B_e(\varphi, \cdot)$ and the second adjoint of $k \mapsto B_e(\varphi, k')$ are continuous from $\sigma(C_0(S_2)'', C_0(S_2)')$ to $\sigma(X'', X')$. As they agree on the canonical image of $C_0(S_2)$, which is weak* dense in $C_0(S_2)''$, they are the same.

THEOREM 6.8. – *Let* $B: C_0(S_1)\times C_0(S_2)\to X$ *and* $\beta_B: \mathcal{B}_1\times\mathcal{B}_2\to X''$ *be as in Theorem* 6.6. *The following six conditions are equivalent*:

(i) $\beta_B(E,\cdot)$ *and* $\beta_B(\cdot, F)$ *are regular for all* $E\in\mathcal{B}_1$, $F\in\mathcal{B}_2$;

(ii) $\beta_B(E,\cdot)$ *and* $\beta_B(\cdot, F)$ *are* σ-*additive for all* $E\in\mathcal{B}_1$, $F\in\mathcal{B}_2$;

(iii) $\beta_B(E, F)\in X$ *for all* $E\in\mathcal{B}_1$, $F\in\mathcal{B}_2$;

(iv) $B_e(f', g')\in X$ *for all bounded Borel functions* $f: S_1\to\boldsymbol{K}$, $g: S_2\to\boldsymbol{K}$;

(v) $B_e(f', g')\in X$ *whenever* $f\in C_0(S_1)\cup\{\chi_E: E\subset S_1$ *is open*$\}$ *and* $g\in C_0(S_2)\cup$ $\cup\{\chi_F: F\subset S_2$ *is open*$\}$;

(vi) $B_e(C_0(S_1)''\times C_0(S_2)'')\subset X$.

PROOF. – Since every additive and regular $\mu: \mathcal{B}_j\to X''$ is σ-additive (see e.g. the proof of Theorem 13 in [6, p. 138], or [4, p. 510]), (i) implies (ii). Assume (ii). For each $h\in C_0(S_1)$ there is a sequence (f_n) of $\mathcal{B}_1$-simple functions converging uniformly to h. Each function $F\mapsto B_e(f_n', \chi_F')$ is σ-additive, and the sequence of these functions converges uniformly on $\mathcal{B}_2$ to the function $F\mapsto B_e(h', \chi_F')$, which is therefore σ-additive, too. But Lemma 6.7 shows that $B_e(h', \chi_F') = B(h, \cdot)''(\chi_F')$, and so $B_e(h', \chi_F')\in X$ for all $F\in\mathcal{B}_2$ (see Remark 6.2). Using Lemma 6.7 again we see that if $F\in\mathcal{B}_2$, the second adjoint of the operator $h\mapsto B_e(h', \chi_F')$, $h\in C_0(S_1)$, has the value $B_e(\chi_E', \chi_F')$ for all $E\in\mathcal{B}_1$. Since $E\mapsto B_e(\chi_E', \chi_F')$ is σ-additive, (iii) holds (Remark 6.2). As each bounded Borel function on S_j can be approximated uniformly by $\mathcal{B}_j$-simple functions, (iii) implies (iv). Clearly, (iv) implies (v). Assume now (v). To show that then (vi) holds, we shall use a result of Grothendieck which says that a bounded linear operator $T: C_0(S_j)\to X$ is weakly compact if (and only if) $T''(\chi_E)\subset X$ for all open sets $E\subset S_j$ (see [8, pp. 160-161]). For a fixed function $h\in C_0(S_1)$, $B_e(h', \cdot): M(S_2)'\to X''$ is the second adjoint of $B(h, \cdot)$ (Lemma 6.7). Since $B_e(h', \chi_F')\in X$ for all open sets $F\subset S_2$, $B(h, \cdot)$ is weakly compact, and so $B_e(h', \psi)\in X$ for all $\psi\in M(S_2)'$ [6, p. 482]. For a fixed functional $\psi\in M(S_2)'$, the operator $h\mapsto B_e\cdot$ $\cdot(h', \psi)$ from $C_0(S_1)$ to X has $B_e(\cdot, \psi): M(S_1)'\to X$ as its second adjoint (Lemma 6.7). If $g: S_2\to\boldsymbol{K}$ is in $C_0(S_k)$ or if g is the characteristic function of an open set, $B_e(\chi_E', g')\in X$ for every open set $E\subset S_1$, so that the operator $h\mapsto B_e(h', g')$ is weakly compact by Grothendieck's theorem. Thus $B_e(\varphi, g')\in X$ for all $\varphi\in M(S_1)'$ [6, p. 482]. Fix now $\varphi\in M(S_1)'$. By Lemma 6.7, $B_e(\varphi, \cdot)$ is the second adjoint of the operator $k\mapsto B_e(\varphi, k')$ from $C_0(S_2)$ to X, and since $B_e(\varphi, \chi_F')\in X$ for all open sets $F\subset S_2$, the latter operator is weakly compact by Grothendieck's theorem, and so $B_e(\varphi, \psi)\in X$ for all $\psi\in M(S_2)'$, i.e., (vi) holds. Clearly, (vi) implies (iii). In view of Theorem 6.6 and the Orlicz-Pettis theorem [6, p. 318], (iii) implies (ii). Theorem 6.6 combined with the discussion preceding Lemma 6.1 shows that (ii) implies (i).

We are now ready to prove our main representation theorem. A vector bimeasure $\beta: \mathcal{B}_1\times\mathcal{B}_2\to X$ is said to be *separately regular* if $\beta(E,\cdot)\in rca(S_2, \mathcal{B}_2, X)$ for each $E\in\mathcal{B}_1$ and $\beta(\cdot, F)\in rca(S_1, \mathcal{B}_1, X)$ for all $F\in\mathcal{B}_2$. We denote by $srca(S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$ the set of the separately regular vector bimeasures $\beta: \mathcal{B}_1\times\mathcal{B}_2\to X$. It is easy to verify that $srca(S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$ is a closed linear subspace of the Banach space of all vector bimeasures $\beta: \mathcal{B}_1 \times \mathcal{B}_2 \to X$ equipped with the semivariation norm (Section 4).

THEOREM 6.9. – *Denote* $B_\beta(h, k) = \int (h, k)\, d\beta$ *for all* $\beta \in srca(S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$ *and* $h \in C_0(S_1)$, $k \in C_0(S_2)$. *Then the mapping* $\beta \mapsto B_\beta$ *is an isometric linear bijection from* $srca(S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$ *onto the Banach space of those bounded bilinear operators* $B: C_0(S_1) \times C_0(S_2) \to X$ *for which* $B_e(C_0(S_1)'' \times C_0(S_2)'') \subset X$. *Moreover,* $(B_\beta)_e(f', g') = \int (f, g)\, d\beta$ *for all bounded Borel functions* $f: S_1 \to \boldsymbol{K}$, $g: S_2 \to \boldsymbol{K}$ *and* $\beta \in srca(S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$. *In particular,* $(B_\beta)_e(\chi'_E, \chi'_F) = \beta(E, F)$ *for* $E \in \mathcal{B}_1$, $F \in \mathcal{B}_2$.

PROOF. – Suppose $\beta \in srca(S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$. Then $B_\beta: C_0(S_1) \times C_0(S_2) \to X$ is a bounded bilinear operator by Theorem 5.4 (or a direct elementary argument), Corollary 5.7 and Theorem 4.5. Let $f: S_1 \to \boldsymbol{K}$ and $g: S_2 \to \boldsymbol{K}$ be bounded Borel functions. If $x' \in X'$, $(x' \circ B_\beta)_e(f', g') = \int (f, g) d(x' \circ \beta)$ by Lemma 6.5, because $x' \circ B_\beta \cdot (h, k) = \int (h, k)\, d(x' \circ \beta)$ for $h \in C_0(S_1)$, $k \in C_0(S_2)$ (Theorem 5.4 (*b*)). As $(u, v) \mapsto \langle x', (B_\beta)_e(u, v)\rangle$ is the canonical extension of $x' \circ B_\beta$, it follows that

$$\langle x', (B_\beta)_e(f', g')\rangle = (x' \circ B_\beta)_e(f', g') = \left\langle x', \int (f, g)\, d\beta \right\rangle$$

(Theorem 5.4 (*b*)), i.e. $(B_\beta)_e(f', g') = \int (f, g)\, d\beta$ $(\in X)$. Thus Theorem 6.8 shows that $(B_\beta)_e(C_0(S_1)'' \times C_0(S_2)'') \subset X$. It is easily verified that the mapping $\beta \mapsto B_\beta$, $\beta \in srca \cdot (S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$, is linear. Since $(B_\beta)_e(\chi'_E, \chi'_F) = \beta(E, F)$ for all $E \in \mathcal{B}_1$, $F \in \mathcal{B}_2$, it is injective. Let now $B: C_0(S_1) \times C_0(S_2) \to X$ be any bounded bilinear operator satisfying the six equivalent conditions in Theorem 6.8. If $\beta = \beta_B: \mathcal{B}_1 \times \mathcal{B}_2 \to X \subset X''$ is defined as in Theorem 6.6 (so that $\beta \in srca(S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$ by Theorem 6.8), then by Theorem 6.6 $B = B_\beta$ (because $\langle x', \int (h, k)\, d\beta\rangle = \int (h, k)\, d(x' \circ \beta) = \langle x', B(h, k)\rangle$, $x' \in X$, $h \in C_0(S_1)$, $k \in C_0(S_2)$), and $\|\beta\|(S_1, S_2) = |B|$.

REMARK 6.10. – If $B: C_0(S_1) \times C_0(S_2) \to X$ is a bounded bilinear operator such that $B_e(C_0(S_1)'' \times C_0(S_2)'') \subset X$, it is clear from Lemma 6.7 that $B(\cdot, k): C_0(S_1) \to X$ and $B(h, \cdot): C_0(S_2) \to X$ are weakly compact operators for all $k \in C_0(S_2)$, $h \in C_0(S_1)$ (see [6, p. 482]). The weak compactness of all these operators does not in turn, however, imply that $B_e(C_0(S_1)'' \times C_0(S_2)'') \subset X$. For example, denote as usual $c_0 = C_0(\boldsymbol{N})$ and define $B: c_0 \times c_0 \to c_0$ by pointwise multiplication. It is clear that when l^∞, the set of all bounded sequences, is in the usual way identified with c_0'', $B_e(f, g) = fg$ for all $f, g \in l^\infty$. In fact, the bilinear operator $(f, g) \mapsto fg$ extends B and is obviously separately weak*-to-weak* continuous. Thus $B_e(c_0'' \times c_0'') = l^\infty$, although all the operators $B(\cdot, k): c_0 \to c_0$ and $B(h, \cdot): c_0 \to c_0$ for $h, k \in c_0$ are even compact.

We conclude this section with a theorem which one would expect to be true of a satisfactory bilinear analogue of a weakly compact operator. We first prove a lemma.

LEMMA 6.11. – *Let* $\beta \in srca(S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$ *and let* $B = B_\beta: C_0(S_1) \times C_0(S_2) \to X$ *be defined as in Theorem* 6.9. *Then the sets* $U = \{\beta(E, F): E \in \mathcal{B}_1, F \in \mathcal{B}_2\}$ *and* $V = \{B_\beta(h, k): h \in C_0(S_1), k \in C_0(S_2)\}$ *span the same closed linear subspace of* X.

PROOF. – Let Y_1 be the closed linear hull of U and Y_2 that of V. Fix $k \in S_0(S_2)$. We have $B_e(\chi'_E, k') = B(\cdot, k)''(\chi'_E)$ for all $E \in \mathcal{B}_1$ by Lemma 6.7. As $(B\cdot, k)''$: $C_0(S_1)'' \to X$ is continuous from $\sigma(C_0(S_1)'', C_0(S_1)')$ to $\sigma(X, X')$ (note that $B(\cdot, k)$ is weakly compact), and $C_0(S_1)$ is weak* dense in $C_0(S_1)''$, $B_e(\chi'_E, k') \in Y_2$ for all $E \in \mathcal{B}_1$, because Y_2 is $\sigma(X, X')$-closed [6, p. 422]. Fix now $E \in \mathcal{B}_1$. By Lemma 6.7 again, $B_e(\chi'_E, \cdot)$ is the second adjoint of the weakly compact [6, p. 482] operator $k \mapsto B_e \cdot \cdot (\chi'_E, k')$, and so a similar reasoning shows that $B_e(\chi'_E, \chi'_F) \in Y_2$ for all $F \in \mathcal{B}_2$. Thus $Y_1 \subset Y_2$. Conversely, it follows from Definitions 5.2 and 3.3 that each $x \in V$ can be approximated in norm by linear combinations of elements from U.

REMARK 6.12. – The method used in the first part of the above proof also shows that if $B: C_0(S_1) \times C_0(S_2) \to X$ is a bounded bilinear operator whose range is contained in a closed linear subspace Y of X, and such that $B_e(C_0(S_1)'' \times C_0(S_2)'') \subset X$, then $B_e(C_0(S_1)'' \times C_0(S_2)'') \subset Y$.

THEOREM 6.13. – *Let Y be a closed linear subspace of X and* $B: C_0(S_1) \times C_0(S_2) \to X$ *a bounded bilinear operator whose range is contained in Y. Then the six equivalent conditions in Theorem* 6.9 *hold for B if and only if they hold for B regarded as a mapping into Y.*

PROOF. – This is an easy consequence of Theorem 6.9 and the above lemma.

7. – Remarks on integration with respect to bounded bilinear operators.

We retain the notational conventions of the previous section. Let $\beta: \mathcal{B}_1 \times \mathcal{B}_2 \to X$ be a separately regular vector bimeasure and $B = B_\beta: C_0(S_1) \times C_0(S_2) \to X$ the bounded bilinear operator corresponding to it as in Theorem 6.9. The purpose of this section is to make some comments on the possibility of replacing condition (i) in Definition 5.2 by a condition which involves the weakly compact operators $B(\cdot, k)$ and $B(h, \cdot)$ for $k \in C_0(S_2)$, $h \in C_0(S_1)$, and the corresponding regular vector measures in place of $\beta(\cdot, F)$ and $\beta(E, \cdot)$ for $E \in \mathcal{B}_1$, $F \in \mathcal{B}_2$. We shall first deal with the scalar case.

LEMMA 7.1. – *Let* $b: C_0(S_1) \times C_0(S_2) \to \boldsymbol{K}$ *be a bounded bilinear form and* β *the separately regular bimeasure defined by* $\beta(E, F) = b_e(\chi'_E, \chi'_F)$, $E \in \mathcal{B}_1$, $F \in \mathcal{B}_2$ (*cf. Lemma* 6.5). *For a function* $f: S_1 \to \boldsymbol{K}$ *the following three conditions are equivalent*:

(i) f *is* $\beta(\cdot, F)$*-integrable for all* $F \in \mathcal{B}_2$;

(ii) f *is integrable with respect to the vector measure* $\beta_1: \mathcal{B}_1 \to M(S_2)$ (*cf. Theorem* 4.4);

(iii) f *is integrable with respect to* $b(\cdot, k) \in M(S_1)$ *for all* $k \in C_0(S_2)$.

PROOF. – By Lemma 5.5, (ii) follows from (i) (note that $M(S_2) = rca(S_2, \mathcal{B}_2)$ is a closed subspace of $ca(S_2, \mathcal{B}_2)$). Clearly, (ii) implies both (i) and (iii) (observe that $b(\cdot, k)$ is just $k' \circ \beta_1$, because β_1 is regular by the discussion preceding Lemma 6.1, and $\int_{S_1} h\, d(k' \circ \beta_1) = \langle \int_{S_1} h\, d\beta_1, k \rangle = b(h, k)$ for $h \in C_0(S_1)$ by Lemma 6.5). Assume now (iii). We shall prove (ii). This could be done by using some considerations in [20, pp. 132-133, 144], but we give a direct argument. For $n \in \boldsymbol{N}$, we define $f_n = f\chi_{A_n}$ where $A_n = \{s \in S_1 : |f(s)| \leq n\}$. Then f_n is $b(\cdot, k)$-measurable (i.e. measurable with respect to the Lebesgue extension of $\mathcal{B}_1$ relative to $b(\cdot, k)$), hence $b(\cdot, k)$-integrable [6, p. 117], for all $k \in C_0(S_2)$, and so are $f_n h$ and fh for all $h \in C_0(S_1)$. From the Lebesgue dominated convergence theorem [6, p. 151] it follows that

$$\lim_{n\to\infty} \int_{S_1} f_n h\, db(\cdot, k) = \int_{S_1} fh\, db(\cdot, k) .$$

Since $(h, k) \mapsto \int_{S_1} f_n h\, db(\cdot, k)$ is for every $n \in \boldsymbol{N}$ a bounded bilinear form on $C_0(S_1) \times C_0(S_2)$, the uniform boundedness principle can be used to show that the bilinear form $b_f : C_0(S_1) \times C_0(S_2) \to \boldsymbol{K}$ defined by $b_f(h, k) = \int_{S_1} fh\, db(\cdot, k)$ is bounded. Let $T : C_0(S_1) \to C_0(S_2)'$ be the bounded bilinear operator defined by $\langle Th, k \rangle = b_f(h, k)$. As T is weakly compact (see the proof of Lemma 6.3), there is by Lemma 6.1 a regular vector measure $\nu : \mathcal{B}_1 \to C_0(S_2)'$ such that $b_f(h, k) = \langle \int_{S_1} h\, d\nu, k \rangle$, $h \in C_0(S_1)$, $k \in C_0(S_2)$. Writing $\lambda_k(E) = \int_E f\, db(\cdot, k)$ we have $k' \circ \nu(E) = \lambda_k(E)$ for all $E \in \mathcal{B}_1$, $k \in C_0(S_2)$, because λ_k, being $b(\cdot, k)$-continuous [6, p. 114], is a regular measure, and $\int_{S_1} h d(k' \circ \nu) = b_f(h, k) = \int_{S_1} h\, d\lambda_k$ for all $h \in C_0(S_1)$ [6, p. 180]. Since

$$\int_{E_n} |f(s)| v(k' \circ \beta_1, ds) = \int_{E_n} |f(s)| v(b(\cdot, k), ds) = v(k' \circ \nu, E_n)$$

by Theorem 20 (*a*) in [6, p. 114] (recall that $b(\cdot, k)$ is identified with $k' \circ \beta_1$), we have

$$\sup \left\{ \int_{E_n} |f(s)| v(k' \circ \beta_1, ds) : k \in C_0(S_2),\ |k| \leq 1 \right\} \leq \|\nu\|(E_n) \to 0$$

as $n \to \infty$ for every decreasing sequence (E_n) of members of $\mathcal{B}_1$ with $\bigcap_{n=1}^{\infty} E_n = \emptyset$, and so (ii) follows from Theorem 3.5.

The following theorem is an easy consequence of the above lemma. In preparation, observe that if e.g. $f : S_1 \to \boldsymbol{K}$ is $b(\cdot, k)$-integrable for all $k \in C_0(S_2)$, the linear functional $b(f, \cdot)$ for which

$$b(f, k) = \int_{S_1} f\, db(\cdot, k) , \qquad k \in C_0(S_2) ,$$

belongs to $M(S_2)$; by Lemma 7.1 and its proof we have actually

$$b(f, k) = \left\langle \int_{S_1} f \, d\beta_1, k \right\rangle, \qquad k \in C_0(S_2).$$

We define $b(\cdot, g)$ analogously.

THEOREM 7.2. – *Let $b: C_0(S_1) \times C_0(S_2) \to \boldsymbol{K}$ and $\beta: \mathcal{B}_1 \times \mathcal{B}_2 \to \boldsymbol{K}$ be as in Lemma 7.1. The pair (f, g) of functions $f: S_1 \to \boldsymbol{K}$, $g: S_2 \to \boldsymbol{K}$ is β-integrable if and only if the following three conditions hold:*

(i) *f is $b(\cdot, k)$-integrable for all $k \in C_0(S_2)$, and g is $b(h, \cdot)$-integrable for all $h \in C_0(S_1)$;*

(ii) *f is $b(\cdot, g)$-integrable and g is $b(f, \cdot)$ integrable;*

(iii) $\int_{S_1} f \, db(\cdot, g) = \int_{S_2} g \, db(f, \cdot)$.

If this is the case, both sides in (iii) *are equal to $\int (f, g) \, d\beta$.*

REMARK 7.3. – Lemma 7.1 becomes false, if in (iii) $C_0(S_2)$ is replaced by $\mathcal{K}(S_2)$, the set of continuous functions with compact support. For example, let $S_1 = S_2$ be the half-open interval $]0, 1]$, and define $b: C_0(S_1) \times C_0(S_2) \to \boldsymbol{R}$ by $b(h, k) = \int_0^1 h(x) \cdot$ $\cdot k(x) \, dx$. The function $f: S_1 \to \boldsymbol{R}$, $f(x) = x^{-2}$, is $b(\cdot, k)$-integrable for all $k \in \mathcal{K}(S_2)$, but not $b(\cdot, k)$-integrable, if $k(x) = x$, $x \in]0, 1]$. Moreover, if $g(x) = x^2$, the pair (f, g) is b-integrable in the sense of [15, p. 482] and [20, p. 145], but not β-integrable in the sense of our definition (in this case $\beta(E, F) = m(E \cap F)$, where m is the Lebesgue measure on $]0, 1]$). In [17, p. 23], a pair (f, g) of functions satisfying the conditions (i), (ii) and (iii) of Theorem 7.2 is said to be «strongly» integrable with respect to b.

We now turn to the question, to what extent analogues of Lemma 7.1 and Theorem 7.2 are true in the vector case. Part of Lemma 7.1 can be easily generalized without any restriction on the Banach space X. In the next lemma we assume that $\beta \in srca(S_1, \mathcal{B}_1; S_2, \mathcal{B}_2; X)$ and let $B: C_0(S_1) \times C_0(S_2) \to X$ be the bounded bilinear operator defined by $B(h, k) = \int (h, k) \, d\beta$. By Theorem 6.9 and Remark 6.10 the operators $B(\cdot, k): C_0(S_1) \to X$ and $B(h, \cdot): C_0(S_2) \to X$ are weakly compact, $k \in C_0(S_2)$, $h \in C_0(S_1)$.

LEMMA 7.4. – *If the function $f: S_1 \to \boldsymbol{K}$ is $\beta(\cdot, F)$-integrable for all $F \in \mathcal{B}_2$, then for every $k \in C_0(S_2)$ f is integrable with respect to the regular vector measure on $\mathcal{B}_1$ corresponding to $B(\cdot, k)$ by Lemma* 6.1.

PROOF. – By Theorem 5.6, f is integrable with respect to the vector measure $E \mapsto \int_{S_2} k \, d\beta(E, \cdot)$, $E \in \mathcal{B}_1$. But this is just the regular vector measure corresponding to $B(\cdot, k)$, because the value of the latter for $E \in \mathcal{B}_1$ is $B(\cdot, k)''(\chi'_E)$, which equals $B_e(\chi'_E, k')$ by Lemma 6.7, and clearly $B_e(\chi'_e, k') = \int_{S_2} k \, d\beta(E, \cdot)$.

Following E. THOMAS [20, p. 135] we say that a Banach space X is *weakly Σ-complete*, if for every sequence (x_n) of elements of X such that $\sum_{n=1}^{\infty} |\langle x_n, x'\rangle| < \infty$ for all $x' \in X'$ there is $x \in X$ such that $\langle x, x'\rangle = \sum_{n=1}^{\infty} \langle x_n, x'\rangle$ for all $x' \in X'$ $\left(\text{or in view of the Orlicz-Pettis theorem, equivalently, } x = \sum_{n=1}^{\infty} x_n\right)$. In the next lemma we collect some well-known characterizations of weakly Σ-complete Banach spaces.

LEMMA 7.5. – *For a Banach space X the following four conditions are equivalent:*

(i) *X is weakly Σ-complete;*

(ii) *X does not contain any isomorphic copy of c_0;*

(iii) *for every locally compact Hausdorff space S every bounded linear operator $T: C_0(S) \to X$ is weakly compact;*

(iv) *for every set $S \neq \emptyset$, every σ-algebra Σ of subsets of S and every vector measure $\mu: \Sigma \to X$ the following holds: if $f: S \to \mathbf{K}$ is a function which is $x'\mu$-integrable for all $x' \in X'$, then f is μ-integrable.*

PROOF. – The equivalence of (i) and (ii) is due to C. BESSAGA and A. PEŁCZYŃSK' (see [2, p. 160]). As to the equivalence of (i) and (iii), see [18, p. 219] and [20 pp. 135-136]. Assume now (i). To prove (iv), one may observe that if f is $x'\mu$-integrable for all $x' \in X'$, then some Σ-measurable function agrees with f μ-a.e. (see Theorem 3.2 and [19, p. 145]); then apply Theorem 1 in [10, p. 31] and Corollary 3.6. Finally, assume (iv). We prove (ii). Suppose, to the contrary, that there is a linear injection $\alpha: c_0 \to X$ which is a homeomorphism onto its range. Take $S = \mathbf{N}$ and let Σ be the set of all subsets of $\mathbf{N}$. Define $f(n) = n$ and $g(n) = n^{-1}$ for $n \in \mathbf{N}$. Clearly, the set function $\mu: \Sigma \to X$ defined by $\mu(A) = \alpha(\chi_A g)$ is an X-valued vector measure. As the dual of c_0 is l^1, it is easy to verify that f is $x'\mu$-integrable for all $x' \in X'$. But f is not μ-integrable. This contradiction proves (ii).

COROLLARY 7.6. – *A Banach space X is weakly Σ-complete, if and only if for all locally compact Hausdorff spaces S_1 and S_2 every bounded bilinear operator $B: C_0(S_1) \times \times C_0(S_2) \to X$ satisfies the condition $B_e(C_0(S_1)'' \times C_0(S_2)'') \subset X$.*

PROOF. – Let X be weakly Σ-complete. By Lemma 6.7 and the weak compactness of $B(\cdot, k): C_0(S_1) \to X$ (Lemma 7.5) $B_e(\varphi, k') \in X$ for all $k \in C_0(S_2)$, $\varphi \in C_0(S_1)''$ [6, p. 482]. Fix $\varphi \in C_0(S_1)''$. Again by Lemma 6.7 and the weak compactness of the operator $k \mapsto B_e(\varphi, k')$ from $C_0(S_2)$ to X, $B_e(\varphi, \psi) \in X$ for all $\psi \in C_0(S_2)''$. Suppose, conversely, that X is not weakly Σ-complete. Then there is a linear injection $\alpha: c_0 \to X$ which is a homeomorphism onto its range. Let $B: c_0 \times c_0 \to c_0$ be the bilinear operator considered in Remark 6.10. Using Theorem 6.13 we see that $\alpha \circ B$ does not satisfy the condition $(\alpha \circ B)_e(C_0(S_1)'' \times C_0(S_2)'') \subset X$.

In the next theorem we make the assumption that X is weakly Σ-complete. By identifying a bounded operator (which is weakly compact by Lemma 7.5) from $C_0(S_j)$ to X with the corresponding regular vector measure (Lemma 6.1), and defining $B(\cdot, g)$ and $B(f, \cdot)$ in a natural way, we can formulate a generalization of Theorem 7.2. (Specifically, if e.g. $f\colon S_1 \to \boldsymbol{K}$ is $B(\cdot, k)$-integrable for all $k \in C_0(S_2)$, the discussion preceding Theorem 7.2 shows that for all $x' \in X'$ the functional

$$k \mapsto \left\langle x', \int_{S_1} f\, dB(\cdot, k)\right\rangle = \int_{S_1} f\, d(x' \cdot B(\cdot, k))$$

on $C_0(S_2)$ is bounded, and so by the uniform boundedness principle $k \mapsto \int_{S_1} f\, dB(\cdot, k)$ is a bounded linear operator from $C_0(S_2)$ to X. We denote this operator by $B(f, \cdot)$ and define $B(\cdot, g)$ similarly.)

THEOREM 7.7. – *Let X be a weakly Σ-complete Banach space. Suppose $\beta \in srca\cdot$ $\cdot(S_1, \mathfrak{B}_1; S_2, \mathfrak{B}_2; X)$ and let $B = B_\beta\colon C_0(S_1) \times C_0(S_2) \to X$ be defined as in Theorem 6.9. The pair (f, g) of functions $f\colon S_1 \to \boldsymbol{K}$, $g\colon S_2 \to \boldsymbol{K}$ is β-integrable if and only if the following three conditions hold:*

(i) *f is $B(\cdot, k)$-integrable for all $k \in C_0(S_2)$, and g is $B(h, \cdot)$-integrable for all $h \in C_0(S_1)$;*

(ii) *f is $B(\cdot, g)$-integrable and g is $B(f, \cdot)$-integrable;*

(iii) $\int_{S_1} f\, dB(\cdot, g) = \int_{S_2} g\, dB(f, \cdot)$.

If this is the case, both sides of (iii) *are equal to* $\int (f, g)\, d\beta$.

PROOF. – Since $x' \circ \beta(E, F) = (x' \circ B)_e(\chi'_E, \chi'_F)$ for all $x' \in X'$, $E \in \mathfrak{B}_1$, $F \in \mathfrak{B}_2$, the theorem is an easy consequence of Theorem 7.2 and Lemma 7.5.

We conclude by showing that the restriction made on X in the above theorem is the right one.

THEOREM 7.8. – *For a Banach space X the following two conditions are equivalent:*

(i) *X is weakly Σ-complete;*

(ii) *for all locally compact Hausdorff spaces S_1 and S_2, and for all bounded bilinear operators $B\colon C_0(S_1) \times C_0(S_2) \to X$ satisfying the six equivalent conditions in Theorem* 6.8 *the following holds: if for all $k \in C_0(S_2)$ $f\colon S_1 \to \boldsymbol{K}$ is integrable with respect to the regular vector measure on $\mathfrak{B}_1$ corresponding to the weakly compact operator $B(\cdot, k)\colon C_0(S_1) \to X$, then f is $\beta_B(\cdot, F)$-integrable for every $F \in \mathfrak{B}_2$, where β_B is defined by $\beta_B(E, F) = B_e(\chi'_E, \chi'_F)$, $E \in \mathfrak{B}_1$, $F \in \mathfrak{B}_2$.*

PROOF. – By Theorem 7.6, (i) implies (ii). Suppose now that X is not weakly Σ-complete. We shall construct a bounded bilinear operator $B\colon c_0 \times c_0 \to c_0$ satisfying the conditions in Theorem 6.8 and such that the statement in (ii) is not true of B.

Since X contains an isomorphic copy of c_0 (Lemma 7.5), it will then, in view of Lemma 6.11 and Theorem 6.13, be clear that (ii) does not hold for X. Let $\varphi: \boldsymbol{N} \to \boldsymbol{K}$ be defined by $\varphi(n) = n^{-1}$. If $h, k \in c_0 = C_0(\boldsymbol{N})$, we define $B(h, k) \in c_0$ as the pointwise product φhk. When l^∞ is identified in the usual way with the bidual of c_0, it is seen as in Remark 6.10 that $B_e(f, g) = \varphi fg$ for all $f, g \in l^\infty$. Thus $B_e(c_0'' \times c_0'') \subset c_0$. Now, if $\psi(n) = n$, $n \in \boldsymbol{N}$, it is easily verified that for all $k \in c_0$ ψ is integrable with respect to the vector measure corresponding to $B(\cdot, k)$, but ψ is not integrable with respect to $E \mapsto B_e(\chi_E', \chi_F')$.

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