ON VECTOR LYAPUNOV FUNCTIONS[1]

S. G. DEO[2]

ABSTRACT. It has been proved that the use of a vector Lyapunov function is more advantageous in certain situations rather than a scalar function. Moreover, each function needs to satisfy less rigid requirements. In this paper a new situation has been considered where vector Lyapunov functions play a further useful role. For this purpose, a new type of stability, namely, strict partial stability has been defined. The principal tool employed is the second method of Lyapunov and a comparision theorem of a more general type.

1. **Introduction.** In an interesting paper Bellman [1] has raised a question whether it might be more convenient, in certain situations, to use a vector Lyapunov function rather than a scalar function. The answer is positive and it has been pointed out that the application of several Lyapunov functions is indeed useful [4], [5]. The object of the present paper is to extend this idea further.

Lyapunov's principal theorems give sufficient conditions for the stability, asymptotic stability and instability of systems. During the last few years these stability concepts have been refined and further generalized in several directions. In particular, the concept of partial stability has been studied by Corduneanu [3]. This type of stability is useful from the practical point of view since in many situations one is interested in the behaviour of some variables only. This idea suggests that it is possible to consider the behaviour of the remaining variables and thus this approach leads to the concept of strict partial (s.p.) stability.

While proving the main results in [3], a differential inequality has been employed to study the partial stability properties. This technique involves estimation of a function, satisfying differential inequality, by the maximal solution of a related differential system. Such differential inequalities have been used in considerable details to solve a number of problems in the analysis of systems and the most comprehensive text for such a literature is [4].

Following the above approach, it is believed that the other type of extremal solution, namely the minimax solution, has not yet been exploited to study the properties of differential systems. The existence of this type of critical solution has been proved by Burton and Whyburn [2] under certain monotonic conditions. We employ below the second method of Lyapunov and the minimax solution of the related system to study the strict partial stability properties of differential systems.

2. **Notation and definitions.** Let J denote the interval $0 \leq t < \infty$ and R^n—the n-dimensional Euclidean space. Let $H_\tau = \{z \in R^n, \ \|z\| < \tau, \ \tau > 0\}$ where $\|\cdot\|$ denotes a suitable norm in R^n. Consider a differential system

$$\text{(2.1)} \qquad\qquad\qquad \dot{z} = Z(t, z)$$

or a more explicit form

$$\text{(2.2)} \qquad\qquad \dot{x} = X(t, z), \qquad \dot{y} = Y(t, z),$$

where $x, X \in R^p$, $y, Y \in R^q$, $p + q = n$ and Z is a real n-vector function defined and continuous on $J \times H_\tau$. Suppose that the function Z is sufficiently smooth in order that, for every $(t_0, z_0) \in J \times H_\tau$, there exists a solution $z(t; t_0, z_0) = (x(t; t_0, z_0), y(t; t_0, z_0))$ depending continuously on t_0, z_0 and equal to z_0 at t_0. We do not assume the uniqueness of solutions of the system (2.1). Let $Z(t, 0) = 0$, for every $t \in J$, so that (2.1) admits the trivial solution.

We say that a real valued function $a(r)$ belongs to class K if it is defined, continuous and strictly increasing on $0 \leq r < \infty$ and if it vanishes at $r = 0 : a(0) = 0$. Our interest lies in the following stability properties. The trivial solution is said to be:

(P_1) *s.p. equistable* if given $0 < \epsilon < \tau$ and $t_0 \in J$, there exists a function $\delta = \delta(t_0, \epsilon)$ which is continuous in t_0 for each ϵ and a $t_1 > t_0$ such that $\|x(t; t_0, z_0)\| < \epsilon$ for $t \geq t_0$ and $\|y(t_1; t_0, z_0)\| \geq \epsilon$ whenever $\|z_0\| \leq \delta$;

(P_2) *s.p. uniformly stable* if the δ in (P_1) is independent of t_0;

(P_3) *s.p. quasi-equi-asymptotically stable* if, for each $0 < \epsilon < \tau$, $t_0 \in J$, there exist positive numbers $\delta_0 = \delta_0(t_0)$, $T = T(t_0, \epsilon)$ and $t_1 > t_0$ such that $\|x(t; t_0, z_0)\| < \epsilon$ for $t \geq t_0 + T$ and $\|y(t_1; t_0, z_0)\| \geq \epsilon$ whenever $\|z_0\| \leq \delta_0$;

(P_4) *s.p. quasi-uniformly asymptotically stable* if the numbers δ_0 and T in (P_3) are independent of t_0;

(P_5) *s.p. equi-asymptotically stable* if (P_1) and (P_3) hold simultaneously;

(P_6) *s.p. uniformly asymptotically stable* if (P_2) and (P_4) hold simultaneously.

3. **Related system.** Let $\lambda(t, u)$ be a 2-vector function defined and continuous on the product space $J \times R^2$. Let $\lambda_1(t, u_1, u_2)$ be monotone decreasing in u_2 for each (t, u_1) and $\lambda_2(t, u_1, u_2)$ be monotone decreasing in u_1 for each (t, u_2). Consider the differential system

(3.1) $\dot{u} = \lambda(t, u), \quad u(t_0; t_0, u_0) = u_0; \quad (t \geqq t_0).$

We note that the system (3.1), in general, may not possess the maximal and minimal solutions since the existence of these extremal solutions need that $\lambda(t, u)$ should be quasi-monotonic in u [2]. However, it has another critical solution known as minimax (1 max-1 min) solution. It has been proved in [2] that under the given monotonicity conditions the minimax solution for the system (3.1) exists.

Further, let $S = J \times D$ where D is an open set in H_r containing the origin and let $v = [v_1, v_2]$ be such that

(3.2) $v_1 \in C[J \times H_r, R^+] \quad \text{and} \quad v_2 \in C[\overline{S}, R^+];$

where $\overline{S}$ denotes the closure of S;

(3.3) $v_1(t, z) \to 0$ as $\|z\| \to 0$ for each $t \in J$, it is locally Lipschitzian in z;

(3.4) $v_2(t, z) = 0$ for $z \in \overline{S} - S$, v_2 is bounded on $\overline{S}$ and locally Lipschitzian in z.

Define, for $(t, z) \in \overline{S}$,

(3.5) $v^*(t, z) = \limsup_{h \to 0^+} \frac{1}{h} \big[v(t + h, z + hZ(t, z)) - v(t, z) \big].$

The principal tool employed to prove the main results is the following lemma.

LEMMA. *Let the vector function* $\lambda(t, u)$ *in* (3.1) *be such that* $\lambda_2(t, u) \geqq 0$. *Let the function* $v(t, z)$ *satisfy conditions* (3.2), (3.3) *and* (3.4). *Suppose that* $u(t; t_0, u_0)$ *is the minimax solution of* (3.1) *existing for* $t \geqq t_0$ *and the function* $v^*(t, z)$ *of* (3.5) *satisfies the inequalities*

(3.6) $(-1)^{i+1} v_i^*(t, z) \leqq (-1)^{i+1} \lambda_i(t, v(t, z)) \quad (i = 1, 2)$

for $(t, z) \in S$. *Let* $z(t; t_0, z_0)$ *be any solution of* (2.1) *such that*

(3.7) $(-1)^{i+1} v_i(t_0, z_0) \leqq (-1)^{i+1} u_{i0}, \quad (t_0, z_0) \in S.$

Then

(3.8) $(-1)^{i+1} v_i(t, z(t; t_0, z_0)) \leqq (-1)^{i+1} u_i(t; t_0, u_0) \quad (i = 1, 2)$

for $(t, z) \in S$.

REMARK. The proof of this lemma can be formulated by following the argument of Theorem 1 of [2]. The details are omitted. However, it is important to note that the solution $z(t; t_0, z_0)$ belongs to S under the given conditions. For, choose $(t_0, z_0) \in S$, $z_0 \neq 0$; then, due to (3.6),

$$v_2^*(t, z) \geqq \lambda_2(t, v_1(t, z), v_2(t, z)) \geqq 0.$$

Further, since v_2 is Lipschitzian by hypothesis, we have

(3.9) $$v_2(t, z(t; t_0, z_0)) \geqq v_2(t_0, z_0) > 0.$$

Further, since $v_2(t, z) = 0$ for all $(t, z) \in \overline{S} - S$, it follows from the inequality (3.9) that $(t, z(t; t_0, z_0)) \in S$ for $t \geqq t_0$.

Now, let $u(t; t_0, u_0)$ be any solution of (3.1). Corresponding to properties (P_1) to (P_6) we can formulate properties (P_1^*) to (P_6^*) for the system (3.1). For example, corresponding to (P_1), we have:

(P_1^*) given $0 < \epsilon < \tau$ and $t_0 \in J$, there exists a function $d = d(t_0, \epsilon)$, which is continuous in t_0 for each ϵ and a $t_1 > t_0$ such that $u_1(t; t_0, u_0) < \epsilon$ for $t \geqq t_0$ and for arbitrary small $u_{20} > 0$, $u_2(t; t_0, u_0)$ is either unbounded or indeterminate whenever $u_{10} \leqq d$ and $u_{20} > 0$.

4. **Main results.** We now establish the following theorems which give sufficient conditions for properties (P_1) to (P_6) to hold in terms of vector Lyapunov functions.

THEOREM 1. *Let the assumptions of the lemma hold and let the vector function v described above be such that, for $(t, z) \in J \times H_\tau$,*

(4.1) $$b(\|x\|) \leqq v_1(t, z), \qquad b \in K.$$

Then the properties (P_1^*), (P_3^*) and (P_5^*) imply the corresponding properties (P_1), (P_3), and (P_5) respectively.

PROOF. Let $0 < \epsilon < \tau$. Since (P_1^*) holds, given $b(\epsilon) > 0$, $t_0 \in J$, there exists a positive function $d = d(t_0, \epsilon)$, which is continuous in t_0 for each ϵ, such that

(4.2) $$u_1(t; t_0, u_0) < b(\epsilon), \qquad t \geqq t_0,$$

whenever

(4.3) $$u_{10} \leqq d \quad \text{and} \quad u_{20} > 0.$$

Since the assumptions of the lemma hold, we have, for $(t, z(t; t_0, z_0)) \in S$,

(4.4) $$v_1(t, z(t; t_0, z_0)) \leqq u_1(t; t_0, u_0)$$

whenever

(4.5) $v_1(t_0, z_0) \leqq u_{10};$ $(t_0, z_0) \in S.$

We choose $v(t_0, z_0) = u_0$; then the fact that $v_1(t, z)$ is continuous and $v_1(t, z) \to 0$ as $\|z\| \to 0$ for each t yields, in view of (4.3) and (4.5), that there exists a $\delta = \delta(t_0, \epsilon)$, which is continuous in t_0 for each ϵ, such that the inequalities

$$\|z_0\| \leqq \delta, \qquad v_1(t_0, z_0) \leqq d,$$

hold simultaneously. We now show that with this δ the property (P_1) holds. For, otherwise, there exists a $\bar{t} > t_0$ for which

$$\|x(\bar{t}; t_0, z_0)\| = \epsilon \quad \text{and} \quad \|x(t; t_0, z_0)\| \leqq \epsilon, \qquad t_0 \leqq t \leqq \bar{t},$$

whenever $\|z_0\| \leqq \delta$. But then we find

$$b(\epsilon) \leqq v_1(\bar{t}, z(\bar{t}; t_0, z_0)) \leqq u_1(\bar{t}; t_0, u_0) < b(\epsilon)$$

by virtue of (4.1), (4.2) and (4.4), which is impossible. Also, we have, for $(t, z) \in S$,

(4.6) $v_2(t, z(t; t_0, z_0)) \geqq u_2(t; t_0, u_0),$

by (3.8). Further, since $v_2(t, z)$ is bounded by assumption, according to property (P_1^*), the inequality (4.6) cannot hold if we assume that the trivial solution of (2.1) is stable. Thus, the trivial solution is stable with respect to x-components but it is unstable with respect to y-components. This proves that the property (P_1) holds. To prove that (P_3^*) implies (P_3), let $b(\epsilon) > 0$, $t_0 \in J$, be given. Then there exist positive numbers $d_0 = d_0(t_0)$ and $T = T(t_0, \epsilon)$ such that

(4.7) $u_1(t; t_0, u_0) < b(\epsilon), \qquad t \geqq t_0 + T,$

whenever $u_{10} \leqq d$ and $u_{20} > 0$.

As in the previous case, we choose a number δ_0. Now, suppose that there is a divergent sequence $\{t_k\}$, $t_k \geqq t_0 + T$, such that $\|x(t_k; t_0, z_0)\| = \epsilon$ whenever $\|z_0\| \leqq \delta_0$. Then, in view of (4.1), (4.4) and (4.7), we have

$$b(\epsilon) \leqq v_1(t_k, z(t_k; t_0, z_0)) \leqq u_1(t_k; t_0, u_0) < b(\epsilon)$$

which is impossible. This proves that $\|x(t; t_0, z_0)\| < \epsilon$ for $t \geqq t_0 + T$. As in the previous case, we can show that the trivial solution is unstable with respect to y-components. This proves that (P_3^*) holds.

Now, it is easy to conclude that (P_5^*) implies property (P_5).

THEOREM 2. *Let the assumptions of the lemma hold and let the vector function v described above be such that, for $(t, z) \in J \times H_r$,*

(4.8) $b(\|x\|) \leqq v_1(t, z) \leqq a(\|z\|), \qquad a, b \in K.$

Then the properties (P_2^*), (P_4^*) *and* (P_6^*) *imply the corresponding properties* (P_2), (P_4) *and* (P_6) *respectively.*

PROOF. The proof is similar to that of Theorem 1 except that the function $a \in K$ assumed in (4.8) helps to determine δ in (P_1), δ_0 and T in (P_2) respectively, independent of t_0. The details are omitted.

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UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA