

On Veronese manifolds

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From differential geometric point of view, a Veronese surface may be considered as a minimal immersion of a 2-dimensional sphere of curvature $1/3$ into a 4-dimensional unit sphere. S.S. Chern, M. doCarmo and S. Kobayashi [1] gave a local characterization of a Veronese surface. The author [4] also characterized a Veronese surface by non-zero constant normal curvature.

We call an isometric immersion (a submanifold) an *isotropic immersion* (*isotropic*) if all its normal curvature vectors have the same length at each point. Let $P^n(c)$ (resp. $P_n(c)$) be an n -dimensional real (resp. complex) projective space of curvature c and $S^m(c)$ be an m -dimensional sphere of curvature c . B. O'Neill [10] proved the following results:

- (A) *There exists a non-umbilic isotropic minimal imbedding $\phi: P^n(c) \rightarrow S^{n+p}(\tilde{c})$ where $c = \frac{n\tilde{c}}{2(n+1)}$ and $p = \frac{1}{2}n(n+1) - 1$.*
- (B) *There exists a Kaehler imbedding $\phi: P_n(c) \rightarrow P_{n+p}(\tilde{c})$, where $2c = \tilde{c}$ and $p = \frac{1}{2}n(n+1)$.*

Taking account of [8], we may call these submanifolds *the Veronese submanifolds*, in particular, we may call the former *the real Veronese submanifold* and the latter *the complex Veronese submanifold*. M. doCarmo and N. Wallach [2] characterized a real Veronese submanifold. The author and K. Ogiue ([6], [7]) also gave some characterizations of a real Veronese submanifold in terms of isotropic immersion. K. Ogiue [9] gave characterizations of a complex Veronese submanifold.

The purpose of the present paper is to characterize Veronese manifolds by means of geometric invariant functions on submanifolds.

§ 1. Real submanifolds in real space forms.

Let M^n be an n -dimensional submanifold immersed in an $(n+p)$ -dimensional Riemannian manifold \tilde{M}^{n+p} of constant curvature \tilde{c} , (i. e., Riemannian submani-

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fold with induced Riemannian metric). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation on M^n (resp. \tilde{M}^{n+p}), then the second fundamental form (the shape operator) σ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y, \quad \text{where } X \text{ and } Y \text{ are vector fields on } M^n,$$

and it satisfies $\sigma(X, Y) = \sigma(Y, X)$. We choose a local field of orthonormal frames $e_1, e_2, \dots, e_n, e_{n+1}, \dots, e_{n+p}$ in \tilde{M}^{n+p} in such a way that, restricted to M^n , e_1, \dots, e_n are tangent to M^n (and, consequently, the remaining vectors are normal to \tilde{M}^{n+p}). With respect to the frame field of \tilde{M}^{n+p} chosen above, let $\tilde{\omega}_1, \dots, \tilde{\omega}^{n+p}$ be the field of dual frames. Then the structure equations of M^{n+p} are given by^(*)

$$(1.1) \quad d\tilde{\omega}_A = \sum \tilde{\omega}_{AB} \wedge \tilde{\omega}_B, \quad \tilde{\omega}_{AB} + \tilde{\omega}_{BA} = 0,$$

$$(1.2) \quad d\tilde{\omega}_{AB} = \sum \tilde{\omega}_{AC} \wedge \tilde{\omega}_{CB} - \tilde{c}\tilde{\omega}_A \wedge \tilde{\omega}_B.$$

Restricting these forms to M^n , we have the structure equations of the immersion:

$$(1.3) \quad \omega_\alpha = 0,$$

$$(1.4) \quad \omega_{i\alpha} = \sum h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha,$$

$$(1.5) \quad d\omega_i = \sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(1.6) \quad d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l,$$

$$(1.7) \quad R_{ijkl} = \tilde{c}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

$$(1.8) \quad d\omega_{\alpha\beta} = \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum R_{\alpha\beta ij} \omega_i \wedge \omega_j,$$

$$(1.9) \quad R_{\alpha\beta ij} = \sum (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta).$$

Then, the second fundamental form (the shape operator) σ can be written as

$$\sigma(X, Y) = \sum h_{ij}^\alpha \omega_i(X) \omega_j(Y) e_\alpha \quad \text{or} \quad \sigma(e_i, e_j) = \sum h_{ij}^\alpha e_\alpha.$$

If we define h_{ijk}^α by

$$\sum h_{ijk}^\alpha \omega_k := dh_{ij}^\alpha + \sum h_{ik}^\alpha \omega_{kj} + \sum h_{kj}^\alpha \omega_{ki} + \sum h_{ij}^\beta \omega_{\beta\alpha},$$

then, from (1.2), (1.3) and (1.4), we have $h_{ijk}^\alpha = h_{ikj}^\alpha$. The second fundamental form σ is said to be *parallel* if $h_{ijk}^\alpha = 0$ for all α, i, j, k .

Now, we consider the following non-negative functions on M^n :

(*) We use the following convention on the range of indices unless otherwise stated; $A, B, C = 1, 2, \dots, n+p$, $i, j, k, l = 1, 2, \dots, n$, $\alpha, \beta, \gamma = n+1, \dots, n+p$, and we agree that repeated indices under a summation sign without indication are summed over the respective range.

$$\begin{aligned}
 S &:= \|\sigma\|^2 = \sum h_{ij}^\alpha h_{ij}^\alpha, & ([1]), \\
 K_N &:= \sum R_{\alpha\beta ij} R_{\alpha\beta ij} = \sum (\sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta))^2, & ([5]),
 \end{aligned}$$

$$L_N := \sum \langle \sigma(e_i, e_j), \sigma(e_k, e_l) \rangle^2 = \sum (\sum_{ij} h_{ij}^\alpha h_{ij}^\beta)^2 = \sum h_{ij}^\alpha h_{ki}^\alpha h_{ij}^\beta h_{kl}^\beta,$$

where \langle, \rangle is the inner product in the normal space to M^n . We know that S is the square of the length of the second fundamental form and K_N is the square of the length of curvature tensor of the normal bundle, which is called the normal scalar curvature of M^n in \tilde{M} defined in [5]. Set $S_{\alpha\beta} := \sum h_{ij}^\alpha h_{ij}^\beta$, the $(p \times p)$ -matrix $(S_{\alpha\beta})$ is symmetric and can be diagonal for a suitable choice of frames e_{n+1}, \dots, e_{n+p} . Then, setting $S_\alpha := S_{\alpha\alpha} = \sum (h_{ij}^\alpha)^2$, we have $S = \sum S_\alpha$ and $L_N = \sum S_\alpha^2$. Since S_α is the square of the length of the second fundamental form for the direction e_α , we may consider L_N as the sum of the 4-th power of the length of the second fundamental forms for suitable frames.

We know that if M^n is minimal, i.e., $\sum h_{ii}^\alpha = 0$ for any α , then the functions above satisfy a differential equation, that is,

LEMMA 1 ([1]). *Let Δ be the Laplacian. Then we have*

$$\frac{1}{2} \Delta S = n\tilde{c}S - K_N - L_N + \sum (h_{ijk}^\alpha)^2 \quad \text{on } M^n.$$

M^n is said to be λ -isotropic at a point $x \in M^n$ if $\sigma(X, X)$ has the same length λ for any unit tangent vector X to M^n at x . We say M^n is isotropic if it is isotropic at each point of M^n . Then we have the following

LEMMA 2 ([10]). *M^n is λ -isotropic at a point $x \in M^n$ if and only if setting $A_{ij} = \sigma(e_i, e_j)$, for any orthonormal frames of M^n at x , we have*

$$\begin{aligned}
 \|A_{jj}\| &= \lambda, & \langle A_{ii}, A_{ij} \rangle &= 0, & \langle A_{ii}, A_{jj} \rangle + 2\|A_{ij}\|^2 &= \lambda^2, \\
 \langle A_{ii}, A_{jk} \rangle + 2\langle A_{ij}, A_{ik} \rangle &= \langle A_{ij}, A_{kl} \rangle + \langle A_{ik}, A_{jl} \rangle + \langle A_{il}, A_{jk} \rangle &= 0,
 \end{aligned}$$

where \langle, \rangle denotes the inner product in the normal space N_x at x and all indices are different from each other.

§ 2. Real Veronese submanifolds.

We first prove the following

PROPOSITION 1. *Let M^n be an n -dimensional submanifold immersed in an $(n+p)$ -dimensional Riemannian manifold \tilde{M}^{n+p} of constant curvature \tilde{c} . Then we get*

$$K_N \leq nL_N \quad \text{everywhere on } M^n,$$

where the equality holds at a point $x \in M^n$ if and only if M^n is isotropic and minimal at x and the sectional curvature of M^n at x is constant for all tangent

planes to M^n at x .

PROOF. Denoting $A_{ij} := \sigma(e_i, e_j) = \sum h_{ij}^\alpha e_\alpha$, from (1.10) and (1.11) we get

$$\begin{aligned}
 (2.2) \quad K_N &= 2 \sum_{i,j} (\sum_k \langle A_{ik}, A_{kj} \rangle)^2 - 2 \sum \langle A_{ik}, A_{jl} \rangle \langle A_{il}, A_{kj} \rangle \\
 &= 2 \sum_{\neq} \langle A_{ik}, A_{kj} \rangle \langle A_{il}, A_{lj} \rangle + 4 \sum_{\neq} \langle A_{ii}, A_{ij} \rangle \langle A_{jj}, A_{ji} \rangle \\
 &\quad + 8 \sum_{\neq} \langle A_{ii}, A_{ij} \rangle \langle A_{ik}, A_{kj} \rangle - 2 \sum_{\neq} \langle A_{ik}, A_{jl} \rangle \langle A_{il}, A_{kj} \rangle \\
 &\quad - 8 \sum_{\neq} \langle A_{ij}, A_{ik} \rangle \langle A_{ii}, A_{kj} \rangle - 4 \sum_{\neq} \langle A_{ii}, A_{ij} \rangle^2 - 2 \sum_{\neq} \langle A_{ik}, A_{kj} \rangle^2 \\
 &\quad + 2 \sum_{\neq} \|A_{ij}\|^2 \cdot \|A_{ik}\|^2 + 4 \sum_{\neq} \|A_{ii}\|^2 \cdot \|A_{ij}\|^2 - 4 \sum_{\neq} \|A_{ij}\|^2 \cdot \langle A_{ii}, A_{jj} \rangle, \\
 (2.3) \quad L_N &= \sum \langle A_{ij}, A_{kl} \rangle^2 = \sum_{\neq} \langle A_{ij}, A_{kl} \rangle^2 + 4 \sum_{\neq} \langle A_{ij}, A_{ik} \rangle^2 + 4 \sum_{\neq} \langle A_{ii}, A_{ij} \rangle^2 \\
 &\quad + 2 \sum_{\neq} \langle A_{ii}, A_{kl} \rangle^2 + \sum_{\neq} \langle A_{ii}, A_{jj} \rangle^2 + 2 \sum_{\neq} \|A_{ij}\|^4 + \sum \|A_{ii}\|^4,
 \end{aligned}$$

where \sum_{\neq} denotes the summation over different indices.

From these equations we have

$$\begin{aligned}
 (2.4) \quad nL_N - K_N &= \sum_{\neq} \{ \langle A_{ik}, A_{kj} \rangle - \langle A_{il}, A_{lj} \rangle \}^2 + 2 \sum_{\neq} \{ \langle A_{ii}, A_{ij} \rangle - \langle A_{jj}, A_{ji} \rangle \}^2 \\
 &\quad + \sum_{\neq} \{ \langle A_{ij}, A_{kl} \rangle + \langle A_{ik}, A_{jl} \rangle \}^2 + (n-2) \sum_{\neq} \langle A_{ij}, A_{kl} \rangle^2 \\
 &\quad + 4 \sum_{\neq} \{ \langle A_{ij}, A_{ik} \rangle + \langle A_{ii}, A_{kj} \rangle \}^2 + 2n \sum_{\neq} \langle A_{ik}, A_{kj} \rangle^2 \\
 &\quad + 4 \sum_{\neq} \{ \langle A_{ii}, A_{ij} \rangle - \langle A_{ik}, A_{kj} \rangle \}^2 + 2(n-2) \sum_{\neq} \langle A_{ii}, A_{jk} \rangle^2 \\
 &\quad + n \sum_{\neq} \{ \langle A_{ii}, A_{jj} \rangle + 2\|A_{ij}\|^2/n \}^2 + 8 \sum_{\neq} \langle A_{ii}, A_{ij} \rangle^2 \\
 &\quad + \frac{4(n-1)}{n} \sum_{\neq} \{ \|A_{ij}\|^2 - n\|A_{ii}\|^2/2(n-1) \}^2 \\
 &\quad + \sum_{\neq} \{ \|A_{ij}\|^2 - \|A_{ik}\|^2 \}^2 \geq 0,
 \end{aligned}$$

where the equality holds at $x \in M^n$ if and only if for any orthonormal frames e_1, \dots, e_{n+p} in \tilde{M} such that e_1, \dots, e_n are tangent to M^n , we have

$$(2.5) \quad \begin{cases} \langle A_{ii}, A_{ij} \rangle = \langle A_{ij}, A_{ik} \rangle = \langle A_{ij}, A_{kl} \rangle = \langle A_{ii}, A_{jk} \rangle = 0, \\ \|A_{ij}\|^2 = n\|A_{ii}\|^2/2(n-1), \quad n\langle A_{ii}, A_{jj} \rangle + 2\|A_{ij}\|^2 = 0, \\ \|A_{ij}\| = \|A_{ik}\|, \end{cases}$$

where different indices indicate different numbers. From (2.5) we have

$$\|A_{ii}\| = \|A_{jj}\| \quad \text{for all } i, j,$$

and so

$$\| \sum A_{ii} \|^2 = n\|A_{ii}\|^2 + \sum_{\neq} \langle A_{ii}, A_{jj} \rangle = 0,$$

which implies that the immersion is minimal at x .

We assume that the equality of (2.1) holds at a point $x \in M$. Then, using (1.7) and (2.5), we easily have

$$R_{ijkl} = \left(\check{c} - \frac{S}{n(n-1)} \right) (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

which implies that the sectional curvature of M^n at x is constant for all tangent planes to M^n at x . We get (1.13) from (2.5), so the immersion is isotropic at x .

Coverseely, we assume that the immersion is isotropic and minimal at $x \in M^n$ and the sectional curvature of M^n at x is constant c for all tangent planes to M^n at x . It follows from (1.7) that we have

$$\langle A_{ik}, A_{jl} \rangle - \langle A_{il}, A_{jk} \rangle = (\check{c} - c)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

from which we have

$$(2.6) \quad \begin{cases} \langle A_{ij}, A_{jl} \rangle = \langle A_{il}, A_{jk} \rangle & \text{if at least three indices are different} \\ \|A_{ij}\|^2 - \langle A_{ii}, A_{jj} \rangle = \check{c} - c, & (i \neq j), \\ \langle A_{ii}, A_{jk} \rangle = \langle A_{ij}, A_{ik} \rangle, & (j \neq k). \end{cases}$$

Since the immersion is minimal at x , we easily get (2.5) from (1.13) and (2.6), so the equality of (2.1) holds at x . Q. E. D.

Now, using this proposition 1, we can prove the following

THEOREM 1. *Let M^n be an n -dimensional, compact, oriented and connected submanifold minimally immersed in an $(n+p)$ -dimensional sphere $S^{n+p}(\check{c})$ of curvature \check{c} . If the immersion is full and the following inequality holds everywhere on M^n ;*

$$(n+1)L_N \leq n\check{c}S$$

where L_N is the function on M^n defined by (1.11) and S is the square of the length of the second fundamental form, then $p=0$ or $p = \frac{1}{2}n(n+1)-1$ and M^n is of constant curvature $\frac{n\check{c}}{2(n+1)}$, i. e., M^n is a real Veronese submanifold in $S^{n+p}(\check{c})$.

PROOF. It follows from Lemma 1, (2.1) and our assumption that we have

$$(2.7) \quad \frac{1}{2} \Delta S = \sum (h_{ijk}^\alpha)^2 + n\check{c}S - K_N - L_N \geq n\check{c}S - (n+1)L_N \geq 0.$$

Since M^n is compact and oriented, we have $\int_{M^n} \Delta S dV = 0$, where dV is the volume element of M^n . Then from (2.7) we have

$$(2.8) \quad h_{ijk}^\alpha = 0 \quad \text{for all } i, j, k \text{ and } \alpha,$$

$$(2.9) \quad n\tilde{c}S = (n+1)L_n = \frac{n+1}{n}K_n \quad \text{everywhere on } M^n.$$

It follows from Proposition 1 that M^n is isotropic. Next, we shall show that M^n is of constant curvature c and $p=0$ or $p=\frac{1}{2}n(n+1)-1$. Setting $\lambda := \|A_{ii}\|$, from (2.5) we have

$$(2.10) \quad S = \frac{1}{2}n(n+2)\lambda^2.$$

Since S is constant by (2.8), λ is so on M^n . From (1.7) we have

$$(2.11) \quad c = \tilde{c} - \frac{S}{n(n-1)}.$$

It follows from (2.5), (2.9) and (2.10) that we have

$$(2.12) \quad S=0 \quad \text{or} \quad S = \frac{n(n-1)(n+2)\tilde{c}}{2(n+1)} \quad \text{on } M^n.$$

If $S=0$ on M^n , then from (2.11) we have $c=\tilde{c}$ and M^n is totally geodesic (i. e., $p=0$). If $S>0$ on M^n , from (2.11) and (2.12) we get $c = \frac{n\tilde{c}}{2(n+1)}$. In this case, from (2.5) we easily see that the dimension of the vector space generated by $\{\sigma(e_i, e_i)\}_{1 \leq i \leq n}$ and $\{\sigma(e_i, e_j)\}_{1 \leq i < j \leq n}$ is $\frac{1}{2}n(n+1)-1$, that is, the dimension of the first normal space of M^n is $\frac{1}{2}n(n+1)-1$ at each point of M^n . On the other hand, it follows from (2.8) that the second fundamental form σ is parallel everywhere on M^n . Therefore, by Theorem in [3], we see that $p = \frac{1}{2}n(n+1)-1$.

By means of Theorem 1 in [6], we see that M^n is a real Veronese submanifold in $S^{n+p}(\tilde{c})$. Q. E. D.

§ 3. Kaehler submanifolds of complex space forms.

Let $\tilde{M}_{n+p}(\tilde{c})$ be an $(n+p)$ -dimensional complex space form of constant holomorphic sectional curvature \tilde{c} and M_n be an n -dimensional Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$ (i. e., complex submanifold with the induced Kaehler structure). Let J (resp. \hat{J}) be the complex structure of M_n (resp. $\tilde{M}_{n+p}(\tilde{c})$) and let g (resp. \hat{g}) be the Kaehler metric of M_n (resp. $\tilde{M}_{n+p}(\tilde{c})$). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation with respect to g (resp. \hat{g}). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y, \quad \text{for vector fields } X \text{ and } Y \text{ on } M_n,$$

and it satisfies

$$(3.1) \quad \sigma(X, Y) = \sigma(Y, X), \quad \sigma(JX, Y) = \sigma(X, JY) = \hat{J}\sigma(X, Y).$$

We choose a local field of orthonormal frames $e_1, \dots, e_n, e_{1^*} = \check{J}e_1, \dots, e_{n^*} = \check{J}e_n, e_{\tilde{1}}, \dots, e_{\tilde{p}}, e_{\tilde{1}^*} = \check{J}e_{\tilde{1}}, \dots, e_{\tilde{p}^*} = \check{J}e_{\tilde{p}}$ in $\tilde{M}_{n+p}(\check{c})$ in such a way that, restricted to $M_n, e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*}$ are tangent to $M_n^{(*)}$. With respect to the frame field of $\tilde{M}_{n+p}(\check{c})$ chosen above, let $\omega_1, \dots, \omega_n, \omega_{1^*}, \dots, \omega_{n^*}, \omega_{\tilde{1}}, \dots, \omega_{\tilde{p}}, \omega_{\tilde{1}^*}, \dots, \omega_{\tilde{p}^*}$ be the field of dual frames. Then the Kaehler metric can be expressed locally as $g = \sum \omega_j \omega_j$ and $\check{g} = \sum \omega_I \omega_I$. The structure equations of $\tilde{M}_{n+p}(\check{c})$ are given by

$$(3.2) \quad d\omega_I = \sum \omega_{IJ} \wedge \omega_J, \quad \omega_{IJ} + \omega_{JI} = 0,$$

$$(3.3) \quad \begin{aligned} \omega_{ab} &= \omega_{a^*b^*}, & \omega_{\alpha\beta} &= \omega_{\alpha^*\beta^*}, & \omega_{a\beta} &= \omega_{a^*\beta^*}, \\ \omega_{ab^*} &= \omega_{ba^*}, & \omega_{\alpha\beta^*} &= \omega_{\beta^*\alpha^*}, & \omega_{a\beta^*} &= \omega_{\beta^*a^*}, \end{aligned}$$

$$(3.4) \quad \begin{aligned} d\omega_{IJ} &= \sum \omega_{IK} \wedge \omega_{KJ} - \tilde{\Omega}_{IJ}, & \tilde{\Omega}_{IJ} &= \frac{1}{2} \sum \check{R}_{IJKL} \omega_K \wedge \omega_L, \\ \check{R}_{IJKL} &= \frac{\check{c}}{4} (\delta_{IK} \delta_{JL} - \delta_{IL} \delta_{JK} + \check{J}_{IK} \check{J}_{JL} - \check{J}_{IL} \check{J}_{JK} + 2\check{J}_{IJ} \check{J}_{KL}) \end{aligned}$$

where

$$(J_{KL}) = \begin{pmatrix} 0 & -I_n & 0 \\ I_n & 0 & \\ & 0 & -I_p \\ 0 & I_p & 0 \end{pmatrix}, \quad I_s \text{ being the identity matrix of degree } s.$$

Restricting these forms to M_n , we have the structure equations of M_n :

$$(3.5) \quad \begin{aligned} \omega_\mu &= 0, & \omega_{i\mu} &= \sum h_{ij}^\mu \omega_j, & h_{ij}^\mu &= h_{ji}^\mu, \\ d\omega_i &= \sum \omega_{ij} \wedge \omega_j, & \omega_{ij} + \omega_{ji} &= 0, \\ d\omega_{ij} &= \sum \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, & \Omega_{ij} &= \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \\ \Omega_{ij} &= \tilde{\Omega}_{ij} - \sum \omega_{i\mu} \wedge \omega_{\mu j}. \end{aligned}$$

We can easily see that the equation of Gauss is written as

$$(3.6) \quad R_{ijkl} = \frac{\check{c}}{4} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} + J_{ik} J_{jl} - J_{il} J_{jk} + 2J_{ij} J_{kl}) + \sum (h_{ik}^\mu h_{jl}^\mu - h_{il}^\mu h_{jk}^\mu).$$

(*) We use the following convention on the range of indices unless otherwise stated:

$$\begin{aligned} A, B, C, D &= 1, \dots, n, \tilde{1}, \dots, \tilde{p}; & a, b, c, d &= 1, \dots, n; \\ I, J, K, L &= 1, \dots, n, 1^*, \dots, n^*, \tilde{1}, \dots, \tilde{p}, \tilde{1}^*, \dots, \tilde{p}^*; \\ i, j, k, l &= 1, \dots, n, 1^*, \dots, n^*; \\ \mu, \nu, \dots &= 1, \dots, p, \tilde{1}^*, \dots, \tilde{p}^*; & \alpha, \beta, \gamma &= \tilde{1}, \dots, \tilde{p}. \end{aligned}$$

and we agree that repeated indices under a summation sign without indication are summed over the respective range.

Now, we can consider the following non-negative functions on M_n :

$$(3.7) \quad K_N := \sum (\sum_k (h_{ik}^\mu h_{kj}^\nu - h_{jk}^\mu h_{ki}^\nu))^2,$$

$$(3.8) \quad L_N := \sum h_{ij}^\mu h_{ij}^\nu h_{ki}^\mu h_{kl}^\nu, \quad (\text{see Lemma 3.4 in [9]}).$$

Then we know the following useful result.

LEMMA 3 ([9]). *Let Δ denote the Laplacian. Then we have*

$$(3.9) \quad \frac{1}{2} \Delta S = \sum (h_{ijk}^\mu)^2 - K_N - L_N + \frac{n+2}{2} \check{c}S,$$

where S is the square of the length of the second fundamental form.

§ 4. Complex Veronese submanifolds.

We first show the following

PROPOSITION 2. *Let M_n be an n -dimensional Kaehler submanifold immersed in an $(n+p)$ -dimensional complex space form M_{n+p} . Then we have*

$$(4.1) \quad K_N \leq (n+1)L_N \quad \text{everywhere on } M_n$$

where the equality of (4.1) holds at a point $x \in M_n$ if and only if the holomorphic sectional curvature of M_n at x is constant.

PROOF. Denoting $A_{ij} := \sigma(e_i, e_j) = \sum h_{ij}^\mu e_\mu$, from (3.1) we see

$$(4.2) \quad A_{a^*b^*} = -A_{ab}, \quad A_{ab^*} = A_{a^*b} = \check{J}A_{ab}.$$

By means of (2.2), (2.3) and (4.2), we have

$$(4.3) \quad \begin{aligned} K_N = & 2 \sum_{\neq} \langle A_{ik}, A_{kj} \rangle \langle A_{il}, A_{lj} \rangle + 4 \sum_{\neq} \langle A_{ii}, A_{ij} \rangle \langle A_{jj}, A_{ji} \rangle \\ & + 8 \sum_{\neq} \langle A_{ii}, A_{ij} \rangle \langle A_{ik}, A_{kj} \rangle - 8 \sum_{\neq} \langle A_{ij}, A_{ik} \rangle \langle A_{ii}, A_{kj} \rangle \\ & - 4 \sum_{\neq} \langle A_{ii}, A_{ij} \rangle^2 - 2 \sum_{\neq} \langle A_{ik}, A_{kj} \rangle^2 + 16 \sum_{\neq} \|A_{ab}\|^4 \\ & - 2 \sum'_{\neq} \langle A_{ik}, A_{jl} \rangle \langle A_{il}, A_{kj} \rangle + 32 \sum_{\neq} \|A_{aa}\|^2 \cdot \|A_{ab}\|^2 \\ & + 16 \sum_{\neq} \|A_{ab}\|^2 \cdot \|A_{ac}\|^2 + 16 \sum_{\neq} \|A_{aa}\|^4, \end{aligned}$$

$$(4.4) \quad \begin{aligned} L_N = & \sum'_{\neq} \langle A_{ij}, A_{kl} \rangle^2 + 4 \sum_{\neq} \langle A_{ij}, A_{ik} \rangle^2 + 4 \sum_{\neq} \langle A_{ii}, A_{ij} \rangle^2 + 2 \sum_{\neq} \langle A_{ii}, A_{kl} \rangle^2 \\ & + 4 \sum_{\neq} \langle A_{aa}, A_{bb} \rangle^2 + 16 \sum_{\neq} \|A_{ab}\|^4 + 8 \sum_{\neq} \|A_{aa}\|^4, \end{aligned}$$

where \sum_{\neq} denotes the summation over different indices and \sum'_{\neq} denotes the summation over different indices except the case $k=i^*$ and $l=j^*$. From these equations we have

$$\begin{aligned}
 (4.5) \quad & (n+1)L_N - K_N \\
 &= \sum_{\neq} \{ \langle A_{ij}, A_{kl} \rangle + \langle A_{ik}, A_{jl} \rangle \}^2 + (n-1) \sum_{\neq} \langle A_{ij}, A_{kl} \rangle^2 \\
 &+ \sum_{\neq} \{ \langle A_{ik}, A_{kj} \rangle - \langle A_{il}, A_{lj} \rangle \}^2 + 2 \sum_{\neq} \{ \langle A_{ii}, A_{ij} \rangle - \langle A_{jj}, A_{ji} \rangle \}^2 \\
 &+ 4 \sum_{\neq} \{ \langle A_{ij}, A_{ik} \rangle + \langle A_{ii}, A_{jk} \rangle \}^2 + 2(n+1) \sum_{\neq} \langle A_{ik}, A_{jk} \rangle^2 \\
 &+ 4 \sum_{\neq} \{ \langle A_{ii}, A_{ij} \rangle - \langle A_{ik}, A_{kj} \rangle \}^2 + 2(n-1) \sum_{\neq} \langle A_{ii}, A_{jk} \rangle^2 \\
 &+ 4(n+1) \sum_{\neq} \langle A_{aa}, A_{bb} \rangle^2 + 8 \sum_{\neq} \{ \|A_{ab}\|^2 - \|A_{ac}\|^2 \}^2 \\
 &+ 8 \sum_{\neq} \{ \|A_{aa}\|^2 - 2\|A_{ab}\|^2 \}^2 \geq 0,
 \end{aligned}$$

where the equality holds at a point $x \in M_n$ if and only if for any above orthonormal frames of M_n we have

$$\begin{aligned}
 (4.6) \quad & \langle A_{ik}, A_{kj} \rangle = \langle A_{ii}, A_{ij} \rangle = \langle A_{ii}, A_{jk} \rangle = 0, \\
 & \langle A_{ij}, A_{kl} \rangle = 0, \quad (k \neq i^*, l \neq j^*) \\
 & \|A_{ab}\|^2 = \|A_{ac}\|^2 = \frac{1}{2} \|A_{aa}\|^2, \quad \langle A_{aa}, A_{bb} \rangle = 0,
 \end{aligned}$$

where different indices indicate different numbers.

From (4.6) we easily see that the holomorphic sectional curvature of M_n at x is constant.

When the holomorphic sectional curvature of M_n at x is constant c , we easily obtain $K_N = (n+1)L_N = n(n+1)^2(\tilde{c} - c)^2$. Q. E. D.

Using this result, we have the following

THEOREM 2. *Let M_n be an n -dimensional compact Kaehler submanifold immersed in an $(n+p)$ -dimensional complex projective space $P_{n+p}(\tilde{c})$ of curvature \tilde{c} . If the immersion is full and the following inequality holds*

$$2L_N \leq \tilde{c}S \quad \text{everywhere on } M_n,$$

where $S = \|\sigma\|^2$ and L_N is the function on M_n defined by (3.8), then $p=0$ or $p = \frac{1}{2}n(n+1)$ and M_n is of constant holomorphic sectional curvature $\frac{\tilde{c}}{2}$, i. e., M_n is a complex Veronese submanifold in $P_{n+p}(\tilde{c})$.

PROOF. From Lemma 3, (4.1) and our assumption we have

$$(4.7) \quad \frac{1}{2} \Delta S = \sum (h_{ijk}^\mu)^2 + \frac{n+2}{2} \tilde{c}S - K_N - L_N \geq \frac{n+2}{2} (\tilde{c}S - 2L_N) \geq 0.$$

Since M_n is compact, we easily see that all equalities of (4.7) hold everywhere on M_n . Therefore we have

$$(4.8) \quad h_{ijk}^\mu = 0 \quad \text{for all } i, j, k \text{ and } \mu,$$

$$(4.9) \quad 2K_N = 2(n+1)L_N = (n+1)\tilde{c}S \quad \text{everywhere on } M_n.$$

It follows from (3.6) and Proposition 2 that the holomorphic sectional curvature H is given by $H = \tilde{c} - S/\{n(n+1)\}$. Since S is constant by (4.8), H is so, i. e., M_n is of constant holomorphic sectional curvature c . Using (4.6) and (4.8), we easily see that the dimension of the first normal space of M_n is not greater than $\frac{1}{2}n(n+1)$. Then, by Theorem in [3], $p \leq \frac{1}{2}n(n+1)$. Therefore, by means of Theorem 4.4 in [9], we complete the proof. Q. E. D.

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