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ON VIRTUAL CRACK EXTENSION METHODS FOR COMBINED TENSILE AND SHEAR LOADING

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INTRODUCTION

The virtual crack extension methods described by Parks [1] and Hellen [2] have been shown to have advantage over other methods of applying finite element techniques of Linear Elastic Fracture Mechanics [3]. The methods of Parks and Hellen are both designed to compute the energy release rate (G) from (i) the displacement field before crack growth predicted by the finite element or other method, (ii) and the change of stiffness during growth, as described in equation (1),

$$G = -\frac{\partial V}{\partial a} = -\frac{1}{2}\{u\}^T\frac{\partial [K]}{\partial a}\{u\} + \{u\}^T\frac{\partial [f]}{\partial a} \qquad (1)$$

where a is the crack length, u the displacements, [K], the finite element stiffness and f the loads. If the loads remain constant during crack extension then

$$G = -\frac{\partial V}{\partial a} = -\frac{1}{2}\{u\}^T\frac{\partial [K]}{\partial a}\{u\} \; . \qquad (2)$$

Parks [1] computes this expression by summing the contributions to this equation from each element of a contour surrounding the crack. In contrast Hellen [2] bases his calculations on the assembled global stiffness matrix.

In both methods, stiffness derivative terms are approximated by

$$\frac{\partial [K]}{\partial a} \cong \frac{1}{\Delta a}\left([K]_{a+\Delta a} - [K]_a\right) = \frac{\Delta [K]}{\Delta a} \; . \qquad (3)$$

Stiffness terms depend on the nodal coordinates and the difference $\Delta[K]$ is due to the alteration of the coordinates of some nodes by an amount Δa. A complication arises in combined tensile and shear loading because the calculated value of G depends on the direction (angle θ) in which the crack is assumed to extend. Most finite element techniques for the estimation of stress intensities (K_I, K_{II}) do not consider the variation of calculated values with the assumed (instanteous) propagation direction. It has been proposed that a crack will propagate in a direction favouring maximum energy release (G_{max}). If only to provide the most conservative value, the magnitude and direction of the maximum energy release (or equivalently, stress intensity) are desirable from an analysis point of view.

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VARIATION OF K_I, K_{II}, J_1, J_2 WITH θ

Defining W as the strain energy density, u_i the displacement vector, T_i the traction vector and s as the distance and n_K as the normal along a contour, Eshelby [4] has defined

$$J_K = \int \left\{ W n_K - T_i \frac{\partial u_i}{\partial x_K} \right\} ds \tag{4}$$

taken over any open contour starting at the lower crack face, surrounding the crack tip and ending at the upper face. He proves that J_K gives the energy release rate if the crack were to extend in the x_K direction (the crack is initially aligned with the x_K axis). Rice [5] shows further, that, by choosing a convenient contour the energy release rate can be calculated even when local crack-tip yielding is modelled.

Hellen and Blackburn [6] have shown that in two dimensional elasticity problems with combined tensile and shear loading, the stress intensity factors K_I, K_{II} are related to the J_1 and J_2 integrals by

$$J_1 = \frac{(1-\nu)(1+\kappa)}{4E} (K_I^2 + K_{II}^2) \tag{5}$$

$$J_2 = \frac{-(1+\nu)(1+\kappa)}{2E} K_I K_{II} \tag{6}$$

where K_{III} and J_3 are assumed to be zero and ν is Poisson's Ratio, E, Young's Modulus and $\kappa = (3-4\nu)$ for plane strain.

VARIATION OF G WITH J_1, J_2 AND θ

The value of G is simply J_1 for a tensile mode (I) of fracture however for tensile and shear loading (I, II) the calculated value of G depends on the value of θ. The virtual crack extension methods can be used to calculate

$$J_1 = -\left.\frac{\partial V}{\partial a}\right|_{\theta=0} \cong -\frac{1}{2}\{u\}^T \frac{\partial [K]}{\partial a}\{u\}\Big|_{\theta=0}, \tag{7}$$

$$J_2 = -\left.\frac{\partial V}{\partial a}\right|_{\theta=\pi/2} \cong -\frac{1}{2}\{u\}^T \frac{\partial [K]}{\partial a}\{u\}\Big|_{\theta=\frac{\pi}{2}} . \tag{8}$$

It is therefore desirable to have a relationship between J_1, J_2, θ and G. Hellen has predicted the trajectory of a two dimensional crack by calculating G for several value of θ and assuming growth in the direction of maximum G. Upon plotting the results of a test case, Hellen discovered a sinusodial variation of G with θ. Similar results can be obtained independent of the mesh size suggesting the relation between G and θ.

By studying the effect of Δa on the element stiffness matrix it can be shown that G at any angle θ is defined by

$$G(\theta) = J_1\cos\theta + J_2\sin\theta . \tag{9}$$

Furthermore it becomes apparent that this sinusodial behaviour of $G(\theta)$ is a consequence of the assumed linear material behaviour (strain energy is a quadratic form of the nodal displacements). It may be expected that formulations of the problem which include (nonlinear) plastic behaviour or based on plate theories will not exhibit this behaviour.

Considering relation (9) the maximum energy release rate occurs at

$$\theta = \arctan\left(-\frac{2K_I K_{II}}{K_I^2 + K_{II}^2}\right) = \arctan\left(+\frac{J_2}{J_1}\right)$$

and has the value

$$G_{max} = \frac{(1+\nu)(1+x)}{4E}(K_I^4 + 6K_I^2K_{II}^2 + K_{II}^4)^{1/2} = (J_1^2 + J_2^2)^{1/2} .$$

The relationship between G and θ is shown graphically in Figure 1 and illustrates the following:

1) a polar plot of $G(\theta)$ versus θ gives a circle which intersects the origin,
2) if $K_I \neq 0$, $K_{II} = 0$ or if $K_I = 0$, $K_{II} \neq 0$, the circle is centered on the x-axis,
3) if $K_I \neq 0$ and $K_{II} \neq 0$, the centre of the circle will not lie on the x-axis,
4) the minimum and maximum values are the same in absolute value and are oppositely directed,
5) the maximum value of $G(\theta)$ is the same as the vector sum of J_1 and J_2.

In three dimensional applications, similar results are obtained except that, instead of a circle, a sphere is obtained:

$$G = G(\theta, \phi) = J_1\cos\theta\sin\phi + J_2\sin\theta\sin\phi + J_3\cos\phi$$

where θ, ϕ are angles shown in Figure 2 and G varies along the crack front.

If the crack lies on a plane of symmetry and boundary conditions are symmetrical with this plane, the component J_K normal to this plane must be zero. The parts of the contour contained in symmetric parts of the body are non-zero but opposite in sign. Thus, mixed mode crack problems cannot be analyzed using symmetry unless other information is available for the calculation of J_2.

EXAMPLE OF A CIRCULAR CRACK IN A HALF-SPACE

As an example, a circular crack in a half-space was analyzed under the action of 10,000 psi applied perpendicular to the crack face (Figure 3). Although this problem could be analyzed using an appropriate two-dimensional finite element formulation, a 90° segment was analyzed as an example of a crack problem giving rise to the two components J_1, J_2. A modified form of the stiffness derivative technique was used to predict the values of J_1 and J_2 for four points along the crack front. The displacement field was obtained using a variety of constant and linear isoparametric wedge and cube-like elements. Elements bordering the crack front were modified for singular behaviour.

The results of the analysis give the tabulated values of J_1, J_2 shown in Table 1. Because of symmetry the crack remains in its plane when growing and therefore J_3 is zero.

CONCLUSIONS

The method of virtual crack extension can be used to predict J_1, J_2, J_3 the energy release rates for crack growth in three mutually perpendicular directions. It has been shown that this can be related to the energy release rate $G(\theta, \phi)$ for crack growth in the directions described by θ and ϕ at any point along the crack front. $G(\theta, \phi)$ can therefore be considered a vector having both magnitude and direction. Prediction of the vector with maximum amplitude G_{max} is obtained from the vector sum of J_1, J_2, J_3. Since J_1, J_2, J_3 can be related to K_I, K_{II} and K_{III} which may be related to crack growth rate and direction, the value of G or G_{max} should also be indicative of crack growth rate and direction. There remains however the question as to the relation between G and the crack behaviour.

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Table 1 J_1 and J_2 for a Circular Crack in a Half-Space

ψ	J_1	J_2	$G_{max} = (J_1^2+J_2^2)^{1/2}$
11.25°	34.77	6.92	35.45
33.75°	29.48	19.70	35.46
56.25°	19.70	29.48	35.46
78.75°	6.92	34.77	35.35

Derivations by Sneddon [7] show that G_{max} has a constant value along the crack front of 34.76. The combination of the calculated values of J_1 and J_2 give a value of 35.46 within 2% of the theoretical value.

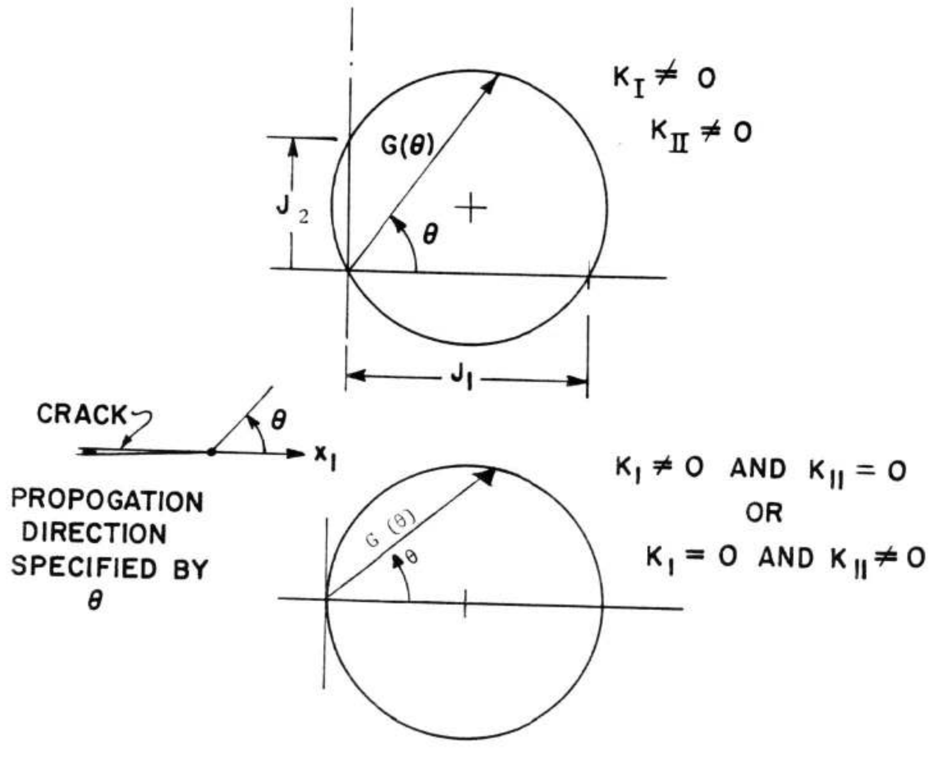

Figure 1

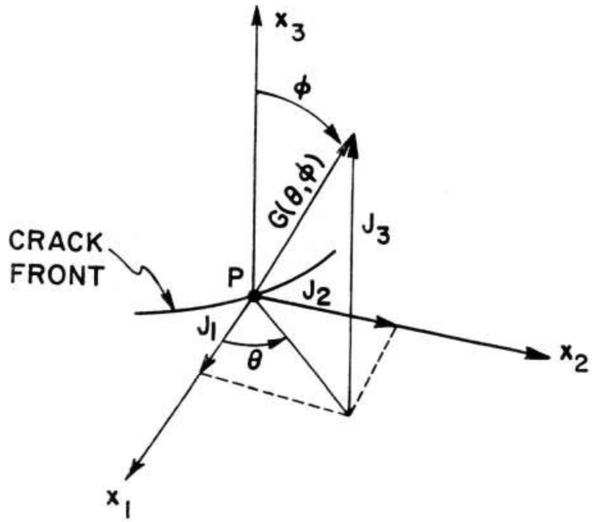

Figure 2

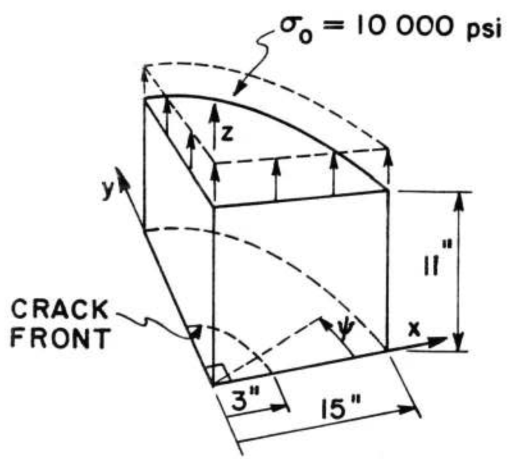

Figure 3