# ON (VON NEUMANN) REGULAR RINGS

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## Introduction

Throughout, $A$ denotes an associative ring with identity and " module " means " left, unitary $A$-module ". In (**3**), it is proved that $A$ is semi-simple, Artinian if $A$ is a semi-prime ring such that every left ideal is a left annihilator. A natural question is whether a similar result holds for a (von Neumann) regular ring. The first proposition of this short note is that if $A$ contains no non-zero nilpotent element, then $A$ is regular iff every principal left ideal is the left annihilator of an element of $A$. It is well-known that a commutative ring is regular iff every simple module is injective (I. Kaplansky, see (**2**, p. 130)). The second proposition here is a partial generalisation of that result.

We write $l(a)$ and $r(a)$ for the left and right annihilators of an element $a$ of $A$ respectively.

**Proposition 1.** *Let $A$ be without non-zero nilpotent elements. The following are then equivalent:*

(a) *$A$ is (von Neumann) regular;*

(b) *Every principal left ideal is the left annihilator of an element of $A$.*

**Proof.** If $A$ is regular, for any $a \in A$, there exists $b \in A$ such that $a = aba$. Since $e = ba$ is an idempotent and $Aa = Ae$, then $Aa$ is the left annihilator of $1-e$. Thus (a) implies (b).

Conversely, assume (b). We first note that since $A$ contains no non-zero nilpotent element, if $ab = 0$ for $a, b \in A$, then $(ba)^2 = baba = 0$ implies $ba = 0$. Thus $l(a) = r(a)$ for every $a \in A$. If $c$ is a non-zero-divisor of $A$, let $s$ be an element of $A$ such that $Ac = l(s)$. Then $cs = 0$ implies $s = 0$ and therefore $Ac = A$ which implies $c$ is left invertible.

Let $0 \neq a \in A$. If $a$ is a non-zero-divisor, then $a = aba$, where $b$ is the left inverse of $a$. If $a$ is a zero-divisor, let $Aa = l(b)$. Then $b$ is non-zero and $ba = ab = 0$. We now show that $c = a+b$ is a non-zero-divisor and then $ca = (a+b)a = a^2$ which will imply $a = da^2$, where $d$ is the left inverse of $c$. Then $(a-ada)^2 = 0$ and by hypothesis, $a = ada$ which will prove that (b) implies (a).

Suppose $cy = (a+b)y = 0$ for some $y \in A$. Then $ay = -by \in r(b) \cap r(a)$. If $w \in r(b) \cap r(a)$, then $w = za$ for some $z \in A$ since $Aa = l(b) = r(b)$ and $aza = aw = 0$ which implies $(za)^2 = zaza = 0$. Since $A$ contains no non-zero nilpotent element, $w = za = 0$. Then $ay = -by = 0$ which implies $y \in r(a) \cap r(b) = 0$. Thus $c = a+b$ is a non-zero-divisor.

**Definition.** A module $M$ is called *p-injective* if, for any principal left ideal $I$ of $A$ and any left $A$-homomorphism $g: I \to M$, there exists $y \in M$ such that $g(b) = by$ for all $b$ in $I$.

It is obvious that injectivity implies $p$-injectivity but the converse is not true as the following lemma shows.

**Lemma 2.** *The following are equivalent:*

(a) *$A$ is regular;*

(b) *Every $A$-module is p-injective;*

(c) *Every cyclic $A$-module is p-injective.*

**Proof.** (a) implies (b). Let $M$ be an $A$-module, $Ab$ a principal left ideal and $g: AB \to M$ a left $A$-homomorphism. If $A$ is regular, $b = bcb$ for some $c \in A$. Let $g(cb) = y \in M$. Then for any $a \in A$,

$$g(ab) = g(abcb) = abg(cb) = aby$$

which implies that $M$ is $p$-injective.

(b) implies (c) evidently.

Assume (c). For any $b \in A$, consider the identity map $i: Ab \to Ab$. Since $Ab$ is $p$-injective, there exists $c \in Ab$ such that $i(ab) = abc$ for all $a$ in $A$. Then $b = i(b) = bc$. Since $c \in Ab$, $c = db$ for some $d$ in $A$ which shows that $b = bdb$. Thus $A$ is regular and (c) implies (a).

The following result partly extends the theorem of Kaplansky.

**Proposition 3.** *The following are equivalent:*

(a) *$A$ is regular without non-zero nilpotent elements;*

(b) *Every simple $A$-module is p-injective and every left ideal of $A$ is two-sided.*

**Proof.** If $A$ is regular without non-zero nilpotent elements, then it is well-known that every left ideal of $A$ is two-sided.

Thus (a) implies (b) by Lemma 2.

Conversely, assume (b). We prove that for any $b \in A$, $Ab + l(b) = A$. Suppose this is not true. Let $J$ be a maximal left ideal containing $Ab + l(b)$. Define $f: Ab \to A/J$ by $f(ab) = a + J$ for all $a$ in $A$. If $a_1 b = a_2 b$, then

$$a_1 - a_2 \in l(b) \subseteq J$$

which implies $f(a_1 b) = a_1 + J = a_2 + J = f(a_2 b)$. Thus $f$ is a well-defined $A$-homomorphism and since, by hypothesis, $A/J$ is $p$-injective, there exists $c \in A$ such that $f(ab) = ab(c + J)$ for all $a$ in $A$. Then

$$1 + J = f(b) = b(c + J) = bc + J$$

and since $bc \in J$ (two-sided), therefore $1 \in J$. This contradiction proves that $A = Ab + l(b)$. Thus $1 = db + s$, for some $d \in A$, $s \in 1(b)$ and therefore $b = db^2 + sb = db^2$ which proves $A$ is regular without nonzero nilpotent elements.

**Corollary 4.** *If A is commutative, then A is regular iff every simple module is p-injective.*

*Remark.* We are grateful to the referee for pointing out that Corollary 4 does not hold when $A$ is non-commutative. J. H. Cozzens (**1**, p. 77) has given an example of a principal ideal domain $A$ such that every simple right $A$-module is injective but which is not a field.

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