Internat. J. Math. & Math. Sci.
Vol. 1 (1978) 133-136

RESEARCH NOTES

ON VON NEUMANN'S INEQUALITY

ARTHUR LUBIN

Department of Mathematics
Illinois Institute of Technology
Chicago, Illinois 60616

(Received October 31, 1977)

Von Neumann's inequality states that for a contraction T acting on a Hilbert space H

$$(v) \qquad ||p(T)|| \leq \sup \{|p(z)| : |z| < 1\}$$

holds for all polynomials p. The analog for a set of commuting contractions $\{T_1,\ldots,T_n\}$,

$$(v_n) \qquad ||p(T_1,\ldots,T_n)|| \leq \sup \{|p(z_1,\ldots,z_n)| : |z_i| < 1\}$$

is known to be false for $n > 2$. In fact, for any $c > o$, there exist $\{T_1,\ldots,T_n\}$, where n is sufficiently large, and a polynomial p such that

$$||(p(T_1,\ldots,T_n)|| > c \sup\{|p(z_1,\ldots,z_n)| : |z_i| < 1\}, \qquad (2)$$

In this note we establish the following weakened version of (v_n):

PROPOSITION 1. Let $\{T_1,\ldots,T_n\}$ be commuting contractions on a Hilbert space H. Then for any polynomial p,

$$||p(T_1,\ldots,T_n)|| \le \sup \{|p(z_1,\ldots,z_n)|:|z_i| < n^{\frac{1}{2}}\},$$

i.e., $D_n = \{(z_1,\ldots,z_n): |z_i| < n^{\frac{1}{2}}\}$ is a spectral set for $(T_1,\ldots,T_n)$.

Our proof is an easy consequence of the following proposition.

<u>PROPOSITION 2 (3, I.9.2)</u>. Let $\{S_1,\ldots,S_n\}$ be commuting contractions with $\sum_{i=1}^{n} ||S_i||^2 \le 1$. Then $\{S_1,\ldots,S_n\}$ has a commuting unitary dilation (in fact a regular one) and it therefore follwos immediately that $\{S_1,\ldots,S_n\}$ satisfies (v_n).

<u>PROOF OF PROPOSITION 1</u>. Given $\{T_1,\ldots,T_n\}$, let $S_i = n^{-\frac{1}{2}} T_i$, $i = 1,\ldots,n$. Then $\sum_{i=1}^{n} ||S_i||^2 = n^{-1} \sum_{i=1}^{n} ||T_i||^2 \le 1$ so (v_n) holds for $\{S_1,\ldots,S_n\}$.

Given any polynomial $p(z_1,\ldots,z_n)$, let $q(z_1,\ldots,z_n) = p(n^{\frac{1}{2}}z_1,\ldots,n^{\frac{1}{2}}z_n)$. Then

$$\begin{aligned}
||p(T_1,\ldots,T_n)|| &= ||p(n^{\frac{1}{2}}S_1,\ldots,n^{\frac{1}{2}}S_n)|| \\
&= ||q(S_1,\ldots,S_n)|| \\
&\le \sup \{|q(w_1,\ldots,w_n)|:|w_i| < 1\} \\
&= \sup \{|p(n^{\frac{1}{2}}w_1,\ldots,n^{\frac{1}{2}}w_n)|:|w_i| < 1\} \\
&= \sup \{|p(z_1,\ldots,z_n)|:|z_i| < n^{\frac{1}{2}}\}
\end{aligned}$$

<u>COROLLARY 3</u>. (see (1) p. 279). Any set $\{T_1,\ldots,T_n\}$ of commuting contractions on H has the polydisc $D_n = \{(z_1,\ldots,z_n):|z_i| < n^{\frac{1}{2}}\}$ as a complete spectral set.

<u>PROOF</u>. By proposition 2, there exist commuting unitary operators $U_1,\ldots,U_n$ on a Hilbert space K containing H such that $q(S_1,\ldots,S_n) = P\, q(U_1,\ldots,U_n)$ for all polynomials q, where $S_i = n^{-\frac{1}{2}} T_i$ and P projects K onto H. Setting $N_i = n^{\frac{1}{2}} U_i$, we have that $\{N_1,\ldots,N_n\}$ is a normal dilation of $\{T_1,\ldots,T_n\}$ with joint spectrum sp(N) contained in the boundary of D_n and the corollary follows as in (1).

Similarly, it follows that $D_a = \{(z_1,\ldots,z_n):|z_i| < a_i\}$ is a complete spectral set for all commuting contractions $\{T_1,\ldots,T_n\}$ if $\Sigma\, a_i^{-2} < 1$.

Since the common intersection of such D_a is the unit polydisc D, which is not in general a complete spectral set since (v_n) can fail if $n \geq 3$, we have

COROLLARY 4. If $\{T_1,\ldots,T_n\}$ is a set of commuting contractions such that the intersection of any two complete spectral sets is also a complete spectral set, then the unit polydisc D is also a complete spectral set.

We note that von Neumann's original paper (4) showed that for a single contraction the intersection of two spectral sets need not be a spectral set.

Since (v_n) holds for $n = 2$, we see that proposition 1 is not the best possible result. This prompts the following

PROBLEM. Find

$$V(n) = \inf\{r: \|p(T_1,\ldots,T_n)\| \leq \sup\{|p(z_1,\ldots,z_n)|:|z_i| < r\}\}$$

We note that Theorem 1.2(b) of (2) yields information concerning the growth of V(n) as n increases. Since

$\sup\{|p(z_1,\ldots,z_n)|:|z_i| < r\} = r^s \sup\{|p(z)|:|z_i| < 1\}$ for homogeneous polynomials of degree s, we have for any $\varepsilon > o$, $V(n) \geq n^{(\frac{1}{4}-\varepsilon)}$ for n sufficiently large.

ACKNOWLEDGMENT. This research was partially supported by NSF Grant MCS 76-06516 A01.

REFERENCES

1. Arveson, W. Subalgebras of C*-algebras II, Acta. Math. 128 (1972) 271-308.

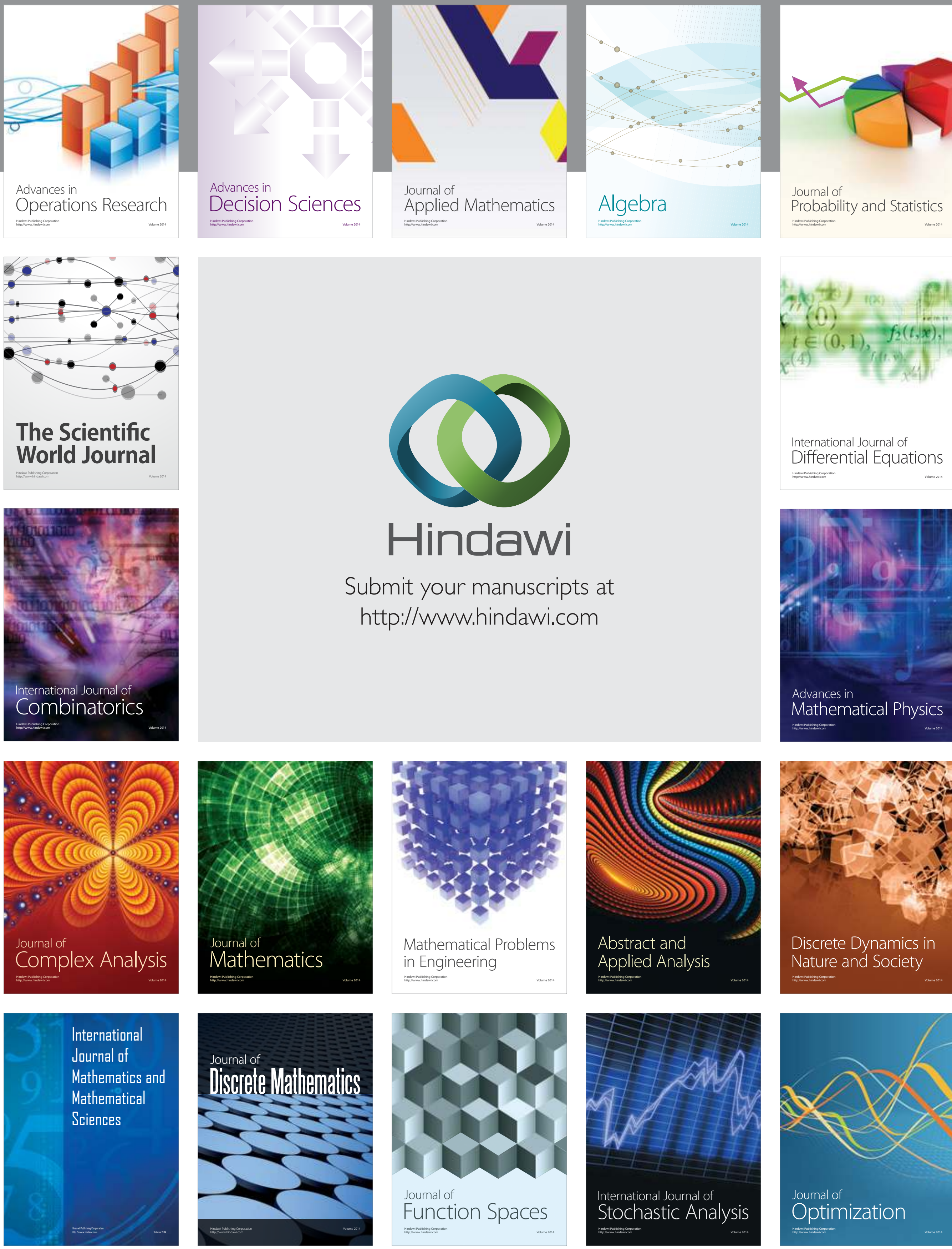
2. Dixon, P. G. The von Neumann Inequality for Polynomials of Degree Greater Than 2, J. London Math. Soc. 14 (1976) 369-375.

3. Sz.-Nagy, B. and C. Foias. Harmonic Analysis of Operators on Hilbert Space, North-Holland, 1970.

4. von Neumann, J. Eine Spektraltheorie fur Allgemeine Operatoren Eines Unitaren Raumes, Math. Nachr. 4 (1951) 258-281.

KEY WORDS AND PHRASES. Commuting contraction on a Hilbert Space, spectral set, Inequality.

AMS(MOS) SUBJECT CLASSIFICATIONS(1970). 47A25.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization