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ON VORONTSOV'S THEOREM ON K3 SURFACES WITH NON-SYMPLECTIC GROUP ACTIONS

KEIJI OGUISO AND DE-QI ZHANG

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ABSTRACT. We shall give a proof for Vorontsov's Theorem and apply this to classify log Enriques surfaces with large prime canonical index.

INTRODUCTION

A K3 surface is, by definition, a simply connected smooth projective surface over the complex numbers $\mathbf{C}$ with a nowhere vanishing holomorphic 2-form. For a K3 surface X, we denote by S_X, T_X and ω_X the Néron-Severi lattice, the transcendental lattice and a nowhere vanishing holomorphic 2-form of X. We write $t(X) = \mathrm{rank} T_X$.

Nukulin [Ni1] considered the kernel H_X of the natural representation $\mathrm{Aut}(X) \longrightarrow O(S_X)$ and proved that H_X is a finite cyclic group with $\varphi(\mathrm{ord}(H_X))|t(X)$ and acts faithfully on the space $H^{2,0}(X) = \mathbf{C}\omega_X$, where φ is the Euler function. We set $h(X) = \mathrm{ord}(H_X)$. The interesting case here is when $\varphi(h(X)) = t(X)$.

Kondo [Ko, Main Theorem] has studied the case where T_X is unimodular and shown the following complete classification:

Theorem 1. *Set* $\Sigma := \{66, 44, 42, 36, 28, 12\}$.

(1) *Let* X *be a* $K3$ *surface with* $\varphi(h(X)) = t(X)$ *whose transcendental lattice* T_X *is unimodular. Then* $h(X) \in \Sigma$.

(2) *Conversely, for each* $N \in \Sigma$*, there exists, modulo isomorphisms, a unique* $K3$ *surface* X *such that* $h(X) = N, \varphi(h(X)) = t(X)$*. Moreover,* T_X *is unimodular for this* X.

In the case where T_X is not unimodular, about 15 years ago, Vorontsov [Vo] announced the following complete classification:

Theorem 2. *Set* $\Omega := \{3^k(1 \le k \le 3), 5^l(l = 1, 2), 7, 11, 13, 17, 19\}$.

(1) *Assume that* X *is a* $K3$ *surface satisfying* $\varphi(h(X)) = t(X)$ *and that* T_X *is non-unimodular. Then* $h(X) \in \Omega$.

(2) *Conversely, for each* $N \in \Omega$*, there exists, modulo isomorphisms, a unique* $K3$ *surface* X *such that* $h(X) = N, \varphi(h(X)) = t(X)$*. Moreover,* T_X *is non-unimodular for this* X.

However, till now, he gave neither proof of this theorem nor construction of such K3 surfaces. In fact the original statement of (1) in [Vo] was weaker than here.

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Later, in [Ko, Sections 6 and 7], Kondo has sharpened the statement of (1) as in the present form and also given a complete proof of the statement (1). He has also shown the existence part of the statement (2) by constructing such K3 surfaces explicitly as follows:

Kondo's example. For each $h \in \Omega$, the following pair $(X_h, \langle g_h \rangle)$ of a K3 surface X_h defined by the indicated Weierstrass equation (or a weighted homogeneous equation in a weighted projective space) and its cyclic automorphism group $\langle g_h \rangle$ satisfies $H_{X_h} = \langle g_h \rangle$ and $\varphi(h(X_h)) = t(X_h)$ and T_X is not unimodular:
$X_{19} : y^2 = x^3 + t^7 x + t,\quad g_{19}^*(x, y, t) = (\zeta_{19}^7 x, \zeta_{19} y, \zeta_{19}^2 t);$
$X_{17} : y^2 = x^3 + t^7 x + t^2,\quad g_{17}^*(x, y, t) = (\zeta_{17}^7 x, \zeta_{17}^2 y, \zeta_{17}^2 t);$
$X_{13} : y^2 = x^3 + t^5 x + t^4,\quad g_{13}^*(x, y, t) = (\zeta_{13}^5 x, \zeta_{13} y, \zeta_{13}^2 t);$
$g_{11}^*(x, y, t) = (\zeta_{11}^5 x, \zeta_{11}^2 y, \zeta_{11}^2 t);$
$X_7 : y^2 = x^3 + t^3 x + t^8,\quad g_7^*(x, y, t) = (\zeta_7^3 x, \zeta_7 y, \zeta_7^2 t);$
$X_{25} : \{y^2 + x_0^6 + x_0 x_1^5 + x_1 x_2^5 = 0\} \subset \mathbf{P}(1, 1, 1, 3);$
$g_{25}^*([x_0 : x_1 : x_2 : y]) = [x_0 : \zeta_{25}^{20} x_1 : \zeta_{25} x_2 : y];$
$X_5 : y^2 = x^3 + t^3 x + t^7,\quad g_5^*(x, y, t) = (\zeta_5^3 x, \zeta_5^2 y, \zeta_5^2 t);$
$X_{27} : y^2 = x^3 + t(t^9 - 1),\quad g_{27}^*(x, y, t) = (\zeta_{27}^2 x, \zeta_{27}^3 y, \zeta_{27}^6 t);$
$X_9 : y^2 = x^3 + t^5(t^3 - 1),\quad g_9^*(x, y, t) = (\zeta_9^2 x, \zeta_9^3 y, \zeta_9^3 t);$
$X_3 : y^2 = x^3 + t^2(t^{10} - 1),\quad g_3^*(x, y, t) = (\zeta_3 x, y, t).$

However, Kondo did not touch the uniqueness part of (2), either. Only the uniqueness in the case where $h(X) = 5^2$ has been just settled by [MO, Theorem 3].

The main purpose of this short article is to give a complete proof for the uniqueness part of (2) to guarantee Vorontsov's Theorem. This together with Kondo's Theorem completes the classification of K3 surfaces X with $\varphi(h(X)) = t(X)$.

We shall also show the following strong uniqueness result as an application of Theorem 2:

Corollary 3. *Let X be a K3 surface with an automorphism g of order $I \in \{19, 17, 13\}$, the three largest possible prime orders. Then we have:*
$(X, \langle g \rangle) \simeq (X_{19}, \langle g_{19} \rangle)$ *when* $I = 19$*;*
$(X, \langle g \rangle) \simeq (X_{17}, \langle g_{17} \rangle)$ *when* $I = 17$*; and*
$(X, \langle g \rangle) \simeq (X_{13}, \langle g_{13} \rangle)$ *when* $I = 13$*,*
where $(X_I, \langle g_I \rangle)$ are pairs defined in Kondo's example.

Besides its own interest, our motivation for this project lies also in its applicability to the study of log Enriques surfaces initiated by the second author ([Z1]). We should also mention here that log Enriques surfaces are regarded as a log version of K3 surfaces and play an increasingly important role in higher dimensional algebraic geometry. For instance, base spaces of elliptically fibered Calabi-Yau threefolds $\Phi_D : X \to S$ with $D.c_2(X) = 0$ are necessarily log Enriques surfaces ([Og]).

A log Enriques surface Z is, by definition, a projective rational surface with at worst quotient singularities, or in other words, at worst klt singularities and with numerically trivial canonical Weil divisor. Passing to the maximal crepant partial resolution, we may also assume in the definition the following maximality for Z:

$(*)$ *any birational morphism $Z' \to Z$ from another log Enriques surface Z' must be an isomorphism.*

For a log Enriques surface Z, we define the canonical index $I(Z)$ or index for short, by

$$I(Z) := \min\{n \in \mathbf{Z}_{>0} | \mathcal{O}_Z(nK_Z) \simeq \mathcal{O}_Z\}.$$

A log Enriques surface of index I is closely related to a K3 surface admitting a non-symplectic group ($\simeq \mathbf{Z}/I\mathbf{Z}$) action via the canonical cover and its minimal resolution:

$$X \xrightarrow{\nu} \overline{X} := Spec(\bigoplus_{n=0}^{I-1} \mathcal{O}_Z(-iK_Z)) \xrightarrow{\pi} Z.$$

In fact, it is well known that $\overline{X}$ is either an abelain surface or a normal K3 surface with at worst Du Val singularities and that $\pi : \overline{X} \to Z$ is a cyclic Galois cover of order I which acts faithfully on the space $H^0(\overline{X}, \mathcal{O}_{\overline{X}}(K_{\overline{X}})) = \mathbf{C}\omega_{\overline{X}}$ and is ramified only over Sing(Z) ([Ka], [Z1]).

In the case where $\overline{X}$ is an abelain surface, Blache [Bl] shows that there are exactly two such log Enriques surfaces up to isomorphisms.

Let us consider the case where X is a K3 surface. In [OZ1], [OZ2], [OZ3], we regard the rank of the sublattice of S_X generated by the exceptional curves of π as an invariant to measure how bad Sing(Z) is and to classify the worst case, namely, the "extremal" case where the rank is 19. As a result, we found that there exist exactly 7 such surfaces up to isomorphisms. However one of them is of index 2 and the others are all of index 3. Note that these indices are rather small.

Now, as a counterpart, it is also interesting to consider the canonical index $I(Z)$ as an invariant measuring how bad Sing(Z) is. It is known that $2 \le I(Z) \le 21$ and $I(Z) \in \{2, 3, 5, 7, 9, 11, 13, 17, 19\}$ if $I(Z)$ is prime ([Z1], [Bl]).

As an application of Corollary 3, we show the following uniqueness result for log Enriques surfaces Z with the three largest prime indices:

Corollary 4. *Let Z be a log Enriques surface with $I(Z) = 19, 17$ or 13 satisfying the maximality $(*)$. Then we have:*
$Z \simeq Z_{19} := X_{19}/\langle g_{19}\rangle$ when $I(Z) = 19$;
$Z \simeq Z_{17} := X_{17}/\langle g_{17}\rangle$ when $I(Z) = 17$; and
$Z \simeq Z_{13} := \overline{X_{13}}/\langle g_{13}\rangle$ when $I(Z) = 13$,
where (X_I, g_I) are pairs defined in Kondo's example and $\overline{X_{13}}$ is the surface obtained from X_{13} by contracting the unique rational curve in the fixed locus $X_{13}^{g_{13}}$.

The second author constructed log Enriques surfaces of indices $19, 17, 13$ in a completely different way ([Z1]). However, it looks very hard to show directly that they are isomorphic to Z_{19}, Z_{17} and Z_{13}.

§1. Existence of Jacobian fiber space structures

Throughout this section we assume that X is a K3 surface with $\varphi(h(X)) = t(X)$ and with $p^r = N = h(X) \in \Omega$, where p is prime and fix a generator g of H_X with $g^*\omega_X = \zeta_N\omega_X$. In what follows, set $S_X^* = \mathrm{Hom}(S_X, \mathbf{Z})$, $T_X^* = \mathrm{Hom}(T_X, \mathbf{Z})$ and regard $S_X \subset S^* \subset S_X \otimes \mathbf{Q}$, $T_X \subset T^* \subset T_X \otimes \mathbf{Q}$ via the bilinear form of S_X and T_X induced by the cup product on $H^2(X, \mathbf{Z})$. We denote by $l(S_X)$ the minimal number of generators of the finite abelian group S_X^*/S_X. We call S_X p-elementary if there exists a non-negative integer a such that S_X^*/S_X is isomorphic to $(\mathbf{Z}/p)^{\oplus a}$. In this case we denote this a by $l(S_X)$.

Recall that S_X (resp. T_X) is an even lattice of signature $(1, \mathrm{rank} S_X - 1)$ (resp. of signature $(2, \mathrm{rank} T_X - 2)$) and $\mathrm{rank} S_X + \mathrm{rank} T_X = 22$.

The goal of this section is to show the following:

Proposition (1.1). *X admits a Jacobian fibration $\Phi : X \to \mathbf{P}^1$ if $N \neq 25$.*

First we notice the following:

Lemma (1.2) ([MO], [Ni1])**.** (1) *Each eigenvalue of $g^*|T_X$ is a primitive N-th root of* 1.

(2) *$Ann(T_X) = \langle \Phi_N(g^*) \rangle$, and T_X is then naturally a torsion free $\mathbf{Z}[\langle g^* \rangle]/\langle \Phi_N(g^*) \rangle$-module, where $\Phi_N(x)$ denotes the minimal polynomial over $\mathbf{Q}$ of a primitive N-th root of* 1.

(3) *Under the identification $\mathbf{Z}[\langle g^* \rangle]/\langle \Phi_N(g^*) \rangle = \mathbf{Z}[\zeta_N]$ through the correspondence $g^*(mod \langle \Phi_N(g^*) \rangle) \leftrightarrow \zeta_N$, $T_X \simeq \mathbf{Z}[\zeta_N]$ as $\mathbf{Z}[\zeta_N]$-modules.*

Proof. This is proved in [MO, Lemma(1.1)]. But it is so easy that we reproduce the verfication here from [MO]. The statement (1) is shown by Nukulin ([Ni1, Theorem 3.1, Corollary 3.3]). The statement (2) is a simple reinterpretation of (1) in terms of group algebra. Recall that torsion free modules are in fact free if the coefficient ring is PID. Now, combining (2) with the fact that $\mathbf{Z}[\zeta_N]$ is PID for $N \in \Omega$ [MM, Main Theorem], we get the assertion (3). □

Lemma (1.3). *S_X is a p-elementary lattice with $l(S_X) = 1$.*

Proof. Since there exists a natural isomorphism $T_X^*/T_X \simeq S_X^*/S_X$ which commutes with the action of $\mathrm{Aut}(X)$, it is enough to show that $T_X^*/T_X \simeq \mathbf{Z}/p$. Since $g^*|S_X = id$ by the definition of H_X, $g^*|(S_X^*/S_X) = id$, whence $g^*|(T_X^*/T_X) = id$. This means $g^*(x) \equiv x \pmod{T_X}$ for each $x \in T_X^*$. Set $n = N/p$ and $h = g^n$. Then h is of order p. Using (1.2)(1), we get $px \equiv x + h^*(x) + ... + (h^*)^{p-1}(x) = (1 + h^* + ... + (h^*)^{p-1})(x) = 0 \pmod{T_X}$. Thus, T_X^*/T_X is p-elementary. We determine $l(T_X)$. □

We shall treat the case where $N = p$. The verification for the case where $N = 3^2, 3^3, 5^2$ is quite similar and left to the reader as an exercise (cf. [MO, Claim(3.4)] for the case where $N = 5^2$). Let e_i $(i = 1, ..., p-1)$ be a $\mathbf{Z}$-basis of T_X corresponding to the $\mathbf{Z}$-basis $1, \zeta_p, ..., \zeta_p^{p-2}$ of $\mathbf{Z}[\zeta_p]$ via the isomorphism in (1.2). Then $g^*(e_i) = e_{i+1}$ for $i = 1, ..., p-1$ and $g^*(e_{p-1}) = -(e_1 + e_2 + ... + e_{p-1})$ (corresponding to the equality $\Phi_p(\zeta_p) = 0$ in $\mathbf{Z}[\zeta_p]$).

Choose $y \in T_X^* (\subset T_X \otimes \mathbf{Q})$ arbitrary. Since T_X^*/T_X is p-elementary, we can write $y = 1/p(\sum_{i=1}^{p-1} a_i e_i)$, where $a_i \in \mathbf{Z}$. Then

$$g^*(y) - y = 1/p(-(a_1 + a_{p-1})e_1 + \sum_{i=1}^{p-3}(a_i - a_{p-1} - a_{i+1})e_{i+1} + (a_{p-2} - 2a_{p-1})e_{p-1}).$$

Since $g^*|(T_X^*/T_X) = id$, we have $g^*(y) - y \in T_X$, whence $a_1 + a_{p-1} \equiv 0$, $a_i - a_{p-1} - a_{i+1} \equiv 0$ and $a_{p-2} - 2a_{p-1} \equiv 0 \pmod{p}$. This implies $a_i \equiv ia_1 \pmod{p}$ and then $y = a_1 \times (1/p)(e_1 + 2e_2 + ... + (p-1)e_{p-1})$ in T_X^*/T_X. Thus,

$$T_X^*/T_X = \langle (1/p)(e_1 + 2e_2 + ... + (p-1)e_{p-1}) \rangle \simeq \mathbf{Z}/p$$

because $l(T_X) \neq 0$ if $N \in \Omega$ (cf. [Ko]). This implies the result.

Proof of Proposition (1.1). Let U be the even unimodular hyperbolic lattice of rank 2. If $N \in \Omega - \{5^2\}$, then $\mathrm{rank}(S_X) \geq 4 = 3 + l(S_X)$. We can then apply the so-called splitting theorem due to Nikulin [Ni3, Corollary 1.13.5] for S_X to split U out from S_X, namely, $S_X \simeq U \oplus S'$. Now the result follows from [Ko, Lemma 2.1]. □

§2. Uniqueness theorem when $h(X) = 3^3, 3^2, 3$

In this section we show the uniqueness of K3 surfaces X with $\varphi(h(X)) = t(X)$ and with $N := h(X) = 3^3$ (resp.3^2, resp.3). Let us set $H_X = \langle g \rangle$. Then rank$S_X = 22 - t(X) = 4$ (resp. 16, resp. 20). Since S_X is an even hyperbolic 3-elementary lattice with $\ell(S_X) = 1$ by (1.3), applying [RS, Section 1], we find that $S_X \simeq U \oplus A_2$, $U \oplus E_8 \oplus E_6$, and $U \oplus E_8 \oplus E_8 \oplus A_2$. Thus X has a Jacobian fibration $\Phi : X \to \mathbf{P}^1$ whose reducible fibers are exactly I_3 or IV (resp. $II^* + IV^*$, resp. $II^* + II^* + I_3$ or $II^* + II^* + IV$).

Since $g^*|S_X = id$, there exists $\overline{g} \in \mathrm{Aut}(\mathbf{P}^1)$ such that $\Phi \circ g = \overline{g} \circ \Phi$. Note also that each smooth rational curve on X must be g-stable whence each reducible fiber of Φ is also g-stable.

First consider the case where $N = 3$. Since there exist three reducible fibers, $\overline{g} = id$. Thus each smooth fiber E is g-stable and $(g|E)^*\omega_E = \zeta_3\omega_E$. Thus, the J-invariant map $J : \mathbf{P}^1 \to \mathbf{P}^1$ is $j(\mathbf{C}/\mathbf{Z} + \mathbf{Z}\zeta_3) = 0$. In particular, each singular fiber is either of Type II, II^*, IV or IV^* by the classification of singular fibers ([Kd]). Thus, the reducible fibers of Φ are $II^* + II^* + IV$. We may adjust an inhomogeneous coordinate t of the base so that X_{-1} and X_1 are of type II^* and X_0 is of type IV. Since $\chi_{top}(X) = 24 = \chi_{top}(X_1) + \chi_{top}(X_{-1}) + \chi_{top}(X_0)$, there are no other singular fibers.

Let us determine the minimal Weierstrass equation $y^2 = x^3 + a(t)x + b(t)$ of Φ. We use the notation in [Ne, Table on the last page]. Since

$$J(t) = 4a(t)^3/(4a(t)^3 + 27b(t)^2) = 0,$$

we have $a(t) = 0$ as polynomials. Thus, $\Delta(t) = 27b(t)^2$. This has exactly two zeros of order 10 (mod12) at $t = 1, -1$ and one zero of order 4 (mod12) at $t = 0$. Note that deg$\Delta(t) \le 24$, because X is a K3 surface. Thus, $\Delta(t) = C(t^{10} - 1)^2 t^4$ for some constant $C \neq 0$, whence $b(t) = c(t^{10} - 1)t^2$ for some constant $c \neq 0$. This means the equation is written as $y^2 = x^3 + c(t^{10} - 1)t^2$. Then changing the coordinates x, y to $c^{1/3}x, c^{1/2}y$, we normalise this equation as $y^2 = x^3 + (t^{10} - 1)t^2$. This shows that X is isomorphic to the Jacobian K3 surface $y^2 = x^3 + (t^{10} - 1)t^2$.

Next consider the case where $N = 9$. We may take an inhomogeneous coordinate t so that X_0 is of type II^* and X_∞ is of type IV^*. First determine ord$(\overline{g})$. A priori ord$(\overline{g}) = 1, 3$ or 9. If ord$(\overline{g}) = 1$, a smooth fiber E is g-stable and $(g|E)^*\omega_E = \zeta_9\omega_E$. However there exists no such elliptic curve. If ord$(\overline{g}) = 9$, then $\overline{g}$ permutes nine fibers $\{X_{\zeta_9^i t}\}_{i=0}^{p-1}$, and there exists an integer m with $24 = \chi_{top}(X) = \chi_{top}(X_0) + \chi_{top}(X_\infty) + 9m = 18 + 9m$, a contradiction. Thus, ord$(\overline{g}) = 3$. Then g^3 acts on each fiber $(g|E)^*\omega_E = \zeta_3\omega_E$. Thus, the J-invariant map $J : \mathbf{P}^1 \to \mathbf{P}^1$ is $j(\mathbf{C}/\mathbf{Z} + \mathbf{Z}\zeta_3) = 0$. In particular, each singular fiber is either of Type II, II^*, IV or IV^*. Then by counting the Euler number of $\chi_{top}(X)$, we see that there exist three other singular fibers of Φ of type II permuted by g. Thus, we may adjust an inhomogeneous coordinate t so that singular fibers of Φ are X_0, X_∞ and $X_{\zeta_3^i}$ $(i = 0, 1, 2)$. Now by the same argument as before, we can readily see that X is isomorphic to the Jacobian K3 surface $y^2 = x^3 + t^5(t^3 - 1)$.

Finally consider the case where $N = 27$. As in the previous case, we readily see that ord$(\overline{g}) = 9$, the J-invariant map is the constant map $J(t) = j(\mathbf{C}/\mathbf{Z} + \mathbf{Z}\zeta_3) = 0$, the reducible singular fiber is of Type IV and the remaining singular fibers consist of one singular fiber of Type II stable under g and nine singular fibers of Type II permuted by g. Then, we may normalise inhomogeneous coordinate t of the base so that X_0 and $X_{\zeta_9^i}$ $(0 \le i \le 8)$ are of Type II and X_∞ is of Type IV. Now, writing the Weierstrass equation and adjusting coordinates of fibers suitably just as before, we can readily see that X is isomorphic to the Jacobian K3 surface $y^2 = x^3 + t(t^9 - 1)$.

This complete the uniqueness for the case where $N = 3, 3^2$, or 3^3.

§3. Determination of singular fibers when $h(X)$ equals a prime p (≥ 5) and satisfies $\varphi(h(X)) = t(X)$

Let $p \ge 5$ be a prime number in Ω and X a K3 surface with $\varphi(h(X)) = t(X)$ and with $h(X) = p$. Let us fix a solution of $4t^p + 27 = 0$ and denote it by α_p.

The goal of this section is to show the following:

Proposition (3.1). *For each p, X admits a Jacobian fibration $\Phi_p : X \to \mathbb{P}^1$ whose singular fibers are as follows:*

X_0 is of Type II, X_∞ is of Type III, and $X_{\alpha_{19}\zeta_{19}^i}$ $(1 \le i \le 19)$ is of Type I_1 when $p = 19$;

X_0 is of Type IV, X_∞ is of Type III, and $X_{\alpha_{17}\zeta_{17}^i}$ $(1 \le i \le 17)$ is of Type I_1 in the case where $p = 17$;

X_0 is of Type II, X_∞ is of Type III^*, and $X_{\alpha_{13}\zeta_{13}^i}$ $(1 \le i \le 13)$ is of Type I_1 in the case where $p = 13$;

X_0 is of Type II^*, X_∞ is of Type III, and $X_{\alpha_{11}\zeta_{11}^i}$ $(1 \le i \le 11)$ is of Type I_1 in the case where $p = 11$;

X_0 is of Type IV^*, X_∞ is of Type III^*, and $X_{\alpha_7\zeta_7^i}$ $(1 \le i \le 7)$ is of Type I_1 in the case where $p = 7$;

X_0 is of Type II^*, X_∞ is of Type III^*, and $X_{\alpha_5\zeta_5^i}$ $(1 \le i \le 5)$ is of Type I_1 in the case where $p = 5$.

Proof. By (1.1), there is a Jacobian fibration $\Phi : X \to \mathbf{P}^1$. For a generator g of H_X, there is an element $\overline{g} \in \mathrm{Aut}(\mathbf{P}^1)$ such that $\overline{g} \circ \Phi = \Phi \circ g$ because $g^*|S_X = id$. Note also that each smooth rational curve on X is g-stable. □

Claim (3.2). $\overline{g}$ is of order p.

Proof. Suppose to the contrary that the assertion is false. Then $\overline{g} = id$. Let E be a smooth fiber of Φ. Then $g(E) = E$. Since $\omega_E \wedge \Phi^*(dt)$ gives a nowhere vanishing 2-form around E, $g^*\omega = \zeta_p\omega$ implies that $(g|E)^*\omega_E = \zeta_p\omega_E$. But there is no such elliptic curve with such action.

We adjust an inhomogeneous coordinate t of $\mathbf{P}^1$ such that $(\mathbf{P}^1)^{\overline{g}} = \{0, \infty\}$. Then only X_0 and X_∞ are the g-stable fibers. Note that singular fibers X_a where $a \ne 0, \infty$ (and hence X_a is not g-stable) are of Kodaira type I_1 or II, for otherwise X_a contains a smooth rational curve which is g-stable for $g^*|S_X = id$. Since $\overline{g}$ permutes $\{X_a, X_{\zeta_p a}, \ldots, X_{\zeta_p^{p-1} a}\}$, we have

$$24 = \chi_{top}(X_0) + \chi_{top}(X_\infty) + pc_1 + 2pc_2, \tag{3.0.1}$$

where pc_1, pc_2 denote the numbers of singular fibers of types I_1, II, respectively. Moreover, $X^g = (X_0)^g \coprod (X_\infty)^g$, whence

$$\chi_{top}(X^g) = \chi_{top}((X_0)^g) + \chi_{top}((X_\infty)^g). \tag{3.0.2}$$

□

Lemma (3.3). *When X_t is smooth (i.e., of type I_0), we set $n_t = 0$, and when X_t is singular, we let n_t denote the number of irreducible components of X_t. Then each of X_0 and X_∞ is either of type $I_{pm}, I_{pm}^*, II, III, IV, II^*, III^*, IV^*$. For both $t = 0, \infty$, $\chi_{top}(X_t) = \chi_{top}((X_t)^g) = n_t$ (resp. $n_t + 1$) if X_t is of type I_{pm} (resp. otherwise).*

Proof. We only consider X_0, for X_∞ is exactly the same. □

By the classification of elliptic fibers, $\chi_{top}(X_0) = n_0$ (resp. $n_0 + 1$) if X_0 is of type I_{n_0} (resp. otherwise). We now show that $\chi_{top}(X_0^g) = \chi_{top}(X_0)$.

If X_0 is a smooth fiber, then either $X_0 \subseteq X^g$ or $X_0 \cap X^g = \emptyset$ because there is no elliptic curve with an automorphism g of prime order p (≥ 5) fixing at least one point. It follows that $\chi_{top}(X_0^g) = \chi_{top}(X_0) = 0 = n_0$ in this case.

Now assume that X_0 is singular. Notice the following facts (cf. 3-Go lemma in [OZ1, §2]):

(1) If $Q \in X_0^g$, then there exist local coordinates (x_Q, y_Q) around Q and an integer a such that $g^*(x_Q, y_Q) = (\zeta_p^a x_Q, \zeta_p^{-a+1} y_Q)$ (as $g^*\omega_X = \zeta_p \omega_X$);

(2) If $g|C \neq id$ for a smooth rational curve C, then C^g consists of two points, say, Q_1, Q_2. If $(g|C)^*(t_{Q_1}) = \zeta_p^b t_{Q_1}$ around Q_1, then $(g|C)^*(t_{Q_2}) = \zeta_p^{-b} t_{Q_2}$ around Q_2.

Now, using these facts and passing to the normalisation of X_0 in the case of Types I_1 and II, we can identify X_0^g for each possible type of X_0 and hence deduce easily the result.

Claim (3.4). We have $\chi_{top}(X_0) + \chi_{top}(X_\infty) = 24 - p$. In particular, all singular fibers other than X_0, X_∞ are of type I_1. Moreover these are permuted by g.

Proof. By (3.0.2) and (3.2),

$$\chi_{top}(X_0) + \chi_{top}(X_\infty) = \chi_{top}(X_0^g) + \chi_{top}(X_\infty^g) = \chi(X^g)$$

$$= \sum_{i=0}^{4} tr(g^*|H^i(X, \mathbb{Z})) = 2 + tr(g^*|S_X) + tr(g^*|T_X)$$

$$= 2 + (22 - (p-1)) + (-1) = 24 - p.$$

Now (3.0.1) implies that $24 = \chi_{top}(X) = (24 - p) + pc_1 + 2pc_2$, and $c_1 + 2c_2 = 1$. Hence $c_1 = 1, c_2 = 0$. This proves Claim (3.4). □

Lemma (3.5). *The pair of g-stable fibers (X_0, X_∞) of the elliptic fibration $\Phi : X \to \mathbf{P}^1$ is one of the following types, after switching the indices $0, \infty$ if necessary:*
(II, III) if $p = 19$;
(IV, III) if $p = 17$;
(II, III^*), or (IV^*, III) if $p = 13$;
(II^*, III), or (IV, III^*), or (I_{11}, II) if $p = 11$;
(IV^*, III^*), or (IV, I_7^*), or (I_7, II^*), or (III, I_{14}) if $p = 7$;
(II^*, III^*), or (IV^*, I_5^*), (III, I_{10}^*), (III^*, I_{10}), or (IV, I_{15}) if $p = 5$.

Proof. This readily follows from (3.3) and (3.4). □

In order to complete (3.1), it is enough to show the following:

Lemma (3.6). *In Lemma* (3.5)*, replacing* Φ *by a new one, we may assume that* (X_0, X_∞) *has the following type:* (II, III)*, or* (IV, III)*, or* (II, III^*)*, or* (II^*, III)*, or* (IV^*, III^*)*, or* (II^*, III^*) *if* $p = 19$*, or* 17*, or* 13*, or* 11*, or* 7 *or* 5*.*

Proof. In the case $p = 5$ (resp. $p = 7$ or $p = 11$), (X_0, X_∞) has one of 5 (resp. 4, 3) types in (3.5). Suppose that (X_0, X_∞) is not of the first type in (3.5). Let F be a section of Φ. Clearly, $X_0 + F + X_\infty$ contains a weighted rational tree X_0'' of Kodaira type II^* (resp. III^*, or II^*). Then X_0'' is nef. Now the Riemann-Roch theorem implies that there is an elliptic fibration Ψ on X with X_0'' as a (g-stable) fiber.

It is easy to see that X_0 or X_∞ contains a cross-section of Ψ. Applying (3.4) to Ψ, we see that the only two g-stable fibers of Ψ are of the first type in (3.4). Now (3.6) follows by replacing Φ by Ψ.

Next consider the case $p = 13$. Suppose that the pair of the only two g-stable fibers (X_0, X_∞) is of the second type (IV^*, III) in (3.5). □

Claim. There are two cross-sections F_1, F_2 of Φ such that $F_1 \cap F_2 = \emptyset$ and such that F_1 and F_2 meet different (multiplicity one) components in X_t for both $t = 0, \infty$.

Once this Claim is proved to be true, (3.6) follows by replacing Φ by the elliptic fibration one of whose singular fibers is of type III^* and contained in $X_0 + F_1 + F_2 + X_\infty$.

Now we prove the Claim. We fully use the notation and results in [Sh, Theorems 8.4, 8.6 and 8.7]. Fix one section F_1 as the zero in the Mordell-Weil lattice $E(K)$ of Φ. First, $E(K)$ is torsion free. Indeed, if F ($\neq F_1$) is a torsion in $E(K)$, then the height pairing $0 = \langle F_1, F_1\rangle = 2\chi(\mathcal{O}_X) + 2F.F_1 - \sum_{v\in R} contr_v(F) = 4 + 2F.F_1 - (4/3 \text{ or } 0) - (1/2 \text{ or } 0) \geq 2F.F_1 + 13/6 \geq 13/6 > 0$, a contradiction. So $E(K)$ is a torsion free lattice of rank 1 [Sh, Corollary 5.3]. Write $E(K) = \mathbb{Z}F_2$.

Denoting by n the index of the sublattice $E(K)^0$ in $E(K)$, we have $n^2\langle F_2, F_2\rangle = \det(E(K)^0) = (\det S_X)n^2/(3 \times 2)$, and $\langle F_2, F_2\rangle = 13/6$. Now the equality $13/6 = \langle F_2, F_2\rangle = 2\chi(\mathcal{O}_X) + 2F_2.F_1 - \sum_{v\in R} contr_v(F_2)$ and the description of $contr_v(F_2)$ in [Sh, (8.16)] imply the Claim. This also completes the proof of (3.6).

§4. Weierstrass equations of K3 surfaces when $h(X)$ equals a prime p (≥ 5) and satisfies $\varphi(h(X)) = t(X)$

Let $y^2 = x^3 + a_p(t)x + b_p(t)$ be the minimal Weierstrass equation of $\Phi_p : X \to \mathbb{P}^1$ in (3.1). In this section, we determine this equation for each p by applying the Néron-Tate algorithm ([Ne, Table on the last page]). This will imply the uniqueness of a K3 surface X with $\varphi(h(X)) = t(X)$ and with $h(X) = p \geq 5$ for each p.

Since g acts on the base as $\overline{g}^*(t) = \zeta_p^k t$ (for some k with $(k, p) = 1$), the J-invariant function $J_p(t) := 4a_p(t)^3/\Delta_p(t)$ is $\langle\zeta_p\rangle$-invariant, and $\Delta_p(t) := 4a_p(t)^3 + 27b_p(t)^2$, which defines the discriminant divisor of Φ_p, is semi $\langle\zeta_p\rangle$-invariant. Thus, $a_p(t)$ is semi $\langle\zeta_p\rangle$-invariant. Since $J_p(t) \neq 0$, we have $a_p(t) \neq 0$. This together with the invariance of $J_p(t)$ also implies the semi-invariance of $b_p(t)$.

On the other hand, by the description of singular fibers and by the fact that $\deg\Delta_p(t) \leq 24$, we have $\Delta_{19}(t) = C_{19}t^2(4t^{19} + 27)$; $\Delta_{17}(t) = C_{17}t^4(4t^{17} + 27)$; $\Delta_{13}(t) = C_{13}t^2(4t^{13} + 27)$; $\Delta_{11}(t) = C_{11}t^{10}(4t^{11} + 27)$; $\Delta_7(t) = C_7t^8(4t^7 + 27)$;

$\Delta_5(t) = C_5 t^{10}(4t^5 + 27)$. Here $C_p \neq 0$ are some constants. Moreover, in each case, the singular fiber X_∞ is the form of the finite quotient of $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta_4)$. Then we have $1 = J_p(\infty) = \lim_{t\to\infty} J_p(t)$. This implies: $a_p(t) = A_p t^7$ if $p = 19, 17, 11$, $a_p(t) = A_p t^5$ if $p = 13, 7$ and (using also the semi-invariance) $a_5(t) = A_5 t^5 + C$ if $p = 5$, where A_p are constants with $A_p^3 = C_p$. In the case $p = 5$, using $\Delta_5(t) = 4a_5(t)^3 + 27b_5(t)^2$ and the semi-invariance of $b_5(t)$, we readily see that $C = 0$. Thus, $a_5(t) = A_5 t^5$.

Now, substituting these into $\Delta_p(t) = 4a_p(t)^3 + 27b_p(t)^2$, we obtain $b_{19}(t) = B_{19}t$; $b_{17}(t) = B_{17}t^2$; $b_{13}(t) = B_{13}t$; $b_{11}(t) = B_{11}t^5$; $b_7(t) = B_7 t^4$; $b_5(t) = B_5 t^5$, where B_p are constants with $B_p^2 = C_p$. Then, there exists a constant $D_p \neq 0$ such that $A_p = D_p^4$ and $B_p = D_p^6$. Thus, the Weierstrass equation of X_p is:

$y^2 = x^3 + D_p^4 t^7 x + D_p^6 t$ if $p = 19$; $y^2 = x^3 + D_p^4 t^7 x + D_p^6 t^2$ if $p = 17$; $y^2 = x^3 + D_p^4 t^5 x + D_p^6 t$ if $p = 13$; $y^2 = x^3 + D_p^4 t^7 x + D_p^6 t^5$ if $p = 11$; $y^2 = x^3 + D_p^4 t^5 x + D_p^6 t^4$ if $p = 7$; $y^2 = x^3 + D_p^4 t^5 x + D_p^6 t^5$ if $p = 5$.

Now changing the coordinates of fibers (x, y) by $(D_p^2 x, D_p^3 y)$, we can normalise the equation as:

$y^2 = x^3 + t^7 x + t$ if $p = 19$; $y^2 = x^3 + t^7 x + t^2$ if $p = 17$; $y^2 = x^3 + t^5 x + t$ if $p = 13$; $y^2 = x^3 + t^7 x + t^5$ if $p = 11$; $y^2 = x^3 + t^5 x + t^4$ if $p = 7$; $y^2 = x^3 + t^5 x + t^5$ if $p = 5$.

This shows the uniqueness of a K3 surface X with $\varphi(h(X)) = t(X)$ and with $h(X) = p \geq 5$ for each p.

§5. Conclusion

In this section, we complete the proof of the uniqueness part of Theorem 2(2) and Corollaries 3 and 4.

The uniqueness part of Theorem 2(2) follows from Section 2 (the case where $h(X) = 3, 3^2, 3^3$), Section 4 (the case where $h(X) = p \geq 5$ is prime) and [MO, Theorem 3] (the case where $h(X) = 5^2$). Q.E.D.

Next we show Corollary 3. Set $p = 19$ (resp. 17 or 13). Since $g^*\omega_X \neq \omega_X$ by [Ni1, §5], $g^*|T_X$ is of order p. Then $t(X) = p - 1$ by [Ni1, Theorem 3.1 and Corollary 3.3] whence $\mathrm{rank} S_X = 22 - (p - 1) = 4$ (resp. 6 or 10). In each case, $\mathrm{rank} S_X < \varphi(p) = p - 1$. This implies $g^*|S_X = id$, whence $\langle g\rangle \subset H_X$. Combining this with Theorems 1(1) and 2(1), we get $H_X = \langle g\rangle$. Now we may apply Theorem 2(2) to conclude the result. Q.E.D.

Finally, we show Corollary 4. Let $\overline{X}$ be the canonical cover of Z, $\langle g\rangle$ the Galois group of this covering and X the minimal resolution of $\overline{X}$. Then X is a K3 surface and g induces an automorphism of X of order $I(Z)$. Now we can apply Corollary 3 to get $(X, \langle g\rangle) \simeq (X_I, \langle g_I\rangle)$. Since $\overline{X} \to Z$ has no ramification curves, every g-fixed curve on X must be contracted under $X \to \overline{X}$. Now the result follows from the maximality assumption (*) on Z. Q.E.D.

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Department of Mathematical Sciences, University of Tokyo, Komaba, Meguro, Tokyo, Japan

E-mail address: oguiso@ms.u-tokyo.ac.jp

Department of Mathematics, National University of Singapore, Lower Kent Ridge Road, Singapore 119260

E-mail address: matzdq@math.nus.edu.sg