Title
ON VORTEX METHODS

Authors
Anderson, C.
Greengard, C.

Publication Date
1984

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Physics Division

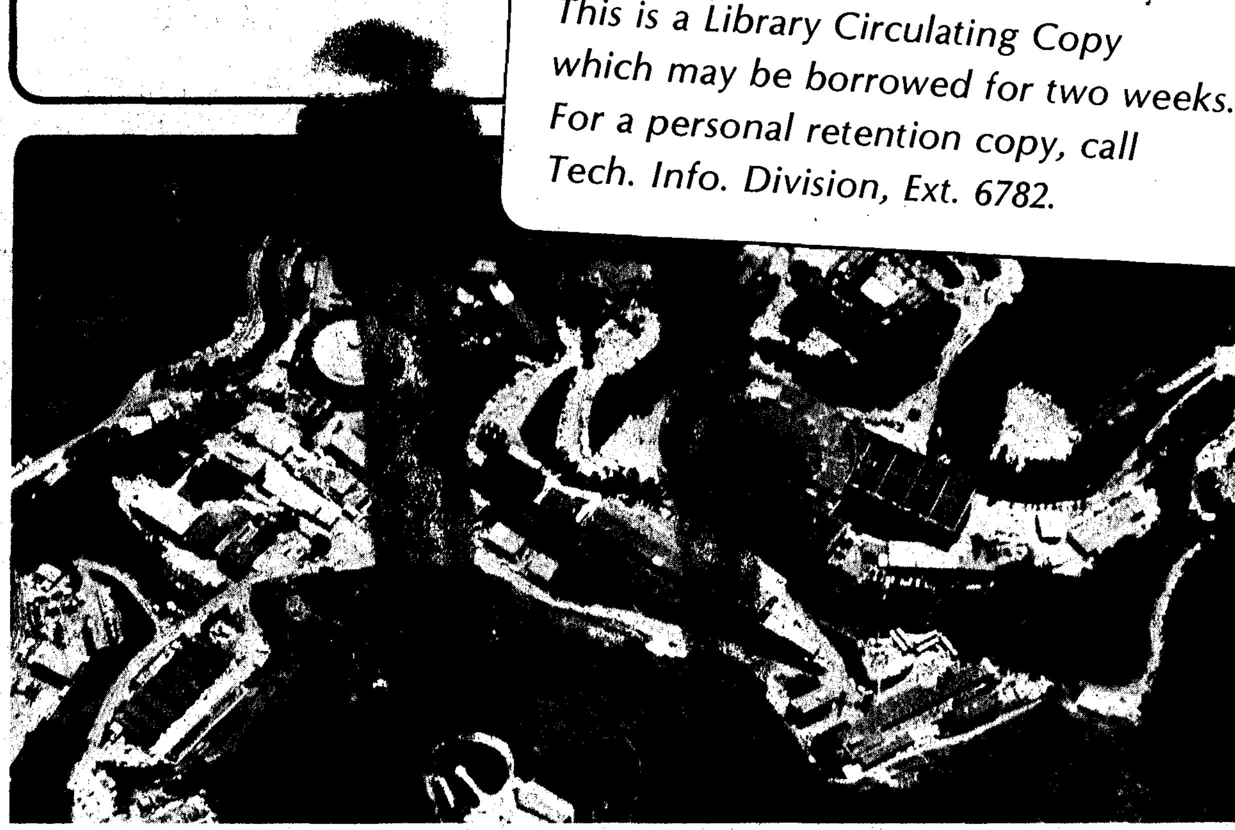
Mathematics Department

To be submitted for publication

ON VORTEX METHODS

C. Anderson and C. Greengard

January 1984



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

LBL-16376

ON VORTEX METHODS[1]

Christopher Anderson† and Claude Greengard‡

† Department of Mathematics
Stanford University
Stanford, California 94305

‡ Lawrence Berkeley Laboratory
and
Department of Mathematics
University of California
Berkeley, California 94720

January 1984

[1]This work was supported in part by the Director, Office of Energy Research, Office of Basic Energy Sciences, Engineering, Mathematical, and Geosciences Division of the U.S. Department of Energy under contract DE-AC03-76SF00098.

ABSTRACT

We present a collection of results on two and three dimensional vortex methods. We discuss the convergence proofs of Beale and Majda, and present a simple proof of their consistency result. We give convergence results which take into account time discretization in the vortex method. We describe how to obtain accurate vortex methods for problems in which the initial computational points are distributed on the nodes of nonrectangular grids. We introduce a new three dimensional vortex method and contrast it with previous three dimensional vortex methods.

INTRODUCTION

We present a collection of proofs and observations on two and three-dimensional vortex methods for Euler's equations, including convergence results for the time-discretized vortex method, and we introduce a new three-dimensional vortex algorithm.

In the vortex method Euler's equations are replaced by a system of ordinary differential equations whose solution gives approximate trajectories of a finite number of fluid particles, called vortices. The first proof of convergence of vortex methods was given by Hald ([16]), who showed, in two dimensions, that these approximate trajectories converge to the exact particle trajectories of the fluid as the number of vortices (and hence the order of the system of ordinary differential equations) increases. Subsequently, Beale & Majda found a different convergence proof which enabled them to establish higher order convergence in two dimensions, and convergence, also of high order, for a new three-dimensional algorithm which they propose ([2],[3]). They show that systems of vortex blobs are stable in the sense that small perturbations in the positions of vortex blobs lead to small perturbations in the corresponding velocity fields. They also prove a consistency result, which says that the approximation of continuous distributions of vorticity by finite sums of vortex blobs causes a small change in the induced velocity. Cottet ([10]), following work of Cottet and Raviart ([11]), obtained a somewhat stronger consistency result in two dimensions, relying on the Bramble-Hilbert Lemma, and avoiding the difficult analysis using pseudodifferential operators of the Beale and Majda paper. In chapter 2, we cite the stablity result of Beale and Majda, with a minor improvement, and give a simple proof of Cottet's consistency result, together with its three-dimensional analogue. We show, following Beale and Majda, how the stability and consistency results lead directly to their convegence theorems.

The implementation of the vortex method requires the numerical solution of the system of ordinary differential equations. All of the convergence results for vortex methods which have been given neglect this source of error. In chapter 3, we give convergence results for numerical approximations of fluid flows by the vortex method. Our proof relies on the stability and consistency results described above.

We propose a new three-dimensional vortex method in chapter 1, in which the computed velocity field is differentiated in order to calculate the stretching of vorticity. Beale and Majda have been able to modify their proof and have established the convergence of our algorithm ([5]). In chapter 4, we contrast our algorithm with those of Chorin and of Beale and Majda, and explain how the method of Beale and Majda is very similar to other vortex methods which have been used in practice.

Vortex methods are based upon following finite numbers of particles and evaluating velocities by discretizing the singular integral equation which relates velocity fields to the corresponding vorticity distributions. In all of the three-dimensional algorithms which we discuss here, the cutoff functions, which are used to smooth the singular kernel, are invariant in time. This differs from some other algorithms which have been proposed, in which the detailed structure of the filaments is followed in order to determine more accurately the interaction of nearby particles, either by using the self-induction approximation ([19]) or by allowing the core functions to change in time ([21]).

When dealing with problems which have some kind of symmetry (such as radial symmetry), one often wants the initial configuration of computational points to reflect this symmetry. In chapter 5, we describe how to obtain vortex methods of high order accuracy in which the computational points are initially distributed on the nodes of nonrectangular grids.

CHAPTER 1

In this chapter we describe some vortex methods for approximating solutions to Euler's equations. Euler's equations govern the evolution of incompressible, inviscid fluids of constant density. The equations involve the velocity u and the pressure p, and are given by

$$u_t+(u\cdot\nabla)u=-\nabla p \tag{1}$$

$$\nabla\cdot u=0, \tag{2}$$

where the number of space dimensions, and hence the number of components of u, is either two or three. We denote by N the number of space dimensions.

We shall assume that a solution of Euler's equations exists on some space-time interval $R^N\times[0,T]$. Smooth solutions are known to exist for all time in the two-dimensional case, and for sufficiently small T when there are three space dimensions, provided the initial conditions are sufficiently smooth (see [22] and [20], respectively).

Vortex methods involve the tracking of particle trajectories. Consequently, an important role in the theory is played by the flow map $x:R^N\times[0,T]\to R^N$, defined so that $x(\alpha,t)$ is the trajectory of the fluid particle which at time $t=0$ is at the point α. The trajectory $x(\alpha,t)$, for fixed α, is obtained from the velocity field u as the solution of the ordinary differential equation

$$\frac{dx}{dt}(\alpha,t) = u(x(\alpha,t),t) \qquad x(\alpha,0)=\alpha\,. \tag{3}$$

We begin by discussing vortex methods in two space dimensions.

Two dimensions

Before presenting the vortex method, we need to introduce the vorticity stream formulation of Euler's equations. The vorticity ω is defined by

$$\omega=\partial_1 u_2-\partial_2 u_1,$$

where $u=(u_1,u_2)$ and ∂_i denotes partial differentiation with respect to the i^{th} space variable. By taking the curl of equation (1), and using equation (2), we get

$$\frac{D\omega}{Dt}=0\ , \tag{4}$$

where D/Dt is defined to be $\partial_t+(u\cdot\nabla)$ and is called the material derivative. Equation (4) says that the vorticity is transported passively by the velocity field u, that is, $\omega(x(\alpha,t),t)=\omega(\alpha,0)$, for $\alpha\varepsilon R^2$ and $t\,\varepsilon[0,T]$ (see [9]). We shall assume that the vorticity is of compact support, a hypothesis which is not necessary for the convergence results (it is sufficient to assume that the vorticity decays rapidly at infinity), but one which makes the proofs simpler.

We now show how the vorticity determines the velocity. From the incompressibility condition (2), it follows that there is a stream function ψ such that

$$\partial_2\psi=u_1 \qquad -\partial_1\psi=u_2. \tag{5}$$

Then

$$\omega=\partial_1 u_2-\partial_2 u_1=-\partial_1^2\psi-\partial_2^2\psi=-\Delta\psi. \tag{6}$$

We denote by G the fundamental solution of the Laplace operator

$$G(x)=\frac{-1}{2\pi}\log(|x|)\ . \tag{7}$$

It follows from (6) that $\psi=G*\omega$. Thus, setting $K_1=\partial_2 G$, $K_2=-\partial_1 G$, and $K=(K_1,K_2)$, we have

$$u_1=\partial_2\psi=\partial_2(G*\omega)=K_1*\omega$$
$$u_2=-\partial_1\psi=-\partial_1(G*\omega)=K_2*\omega$$

or,

$$u(x)=K*\omega(x)=\int K(x-x')\omega(x')dx' . \tag{8}$$

We note that

$$K(x)=\frac{1}{2\pi}\frac{1}{|x|^2}(-x_2,x_1). \tag{9}$$

Equations (4) and (8) are the vorticity stream formulation of Euler's equations.

We can now present the equations of motion in Lagrangian form. We have, by (3) and (8),

$$x(\alpha,0)=\alpha \tag{10}$$

and

$$\begin{aligned}\frac{dx}{dt}(\alpha,t) &= \int K(x(\alpha,t)-x')\omega(x',t)dx' \\ &=\int K(x(\alpha,t)-x(\alpha',t))\omega(x(\alpha',t),t)d\alpha' \\ &=\int K(x(\alpha,t)-x(\alpha',t))\omega_0(\alpha')d\alpha' ,\end{aligned} \tag{11}$$

where we have set $\omega_0(\alpha')=\omega(\alpha',0)$. To get the second equality in (11), we changed variables in the integral, using the transformation $x(\alpha,t)$. Since the flow is incompressible, the Jacobian of this transformation is 1, and hence the new integral does not contain derivatives of the transformation $x(\alpha,t)$. The system of equations (10)-(11) is equivalent to Euler's equations. In fact, existence theorems to Euler's equations have been proved on the basis of this formulation ([22]).

The vortex method is a discretization of the equations (10)-(11). The solution to these equations is approximated by solving (10)-(11) for a finite number of particles, with the integrals on the right hand side of (11) approximated by finite summations. The particles we track are those that are initially located on the nodes of a grid. We denote by Λ^h the set of nodes of the rectangular grid centered at the origin and of mesh width h, and which are contained in the support of the initial vorticity distribution. Thus Λ^h consists of the points $\alpha_i = h \cdot i = (h \cdot i_1, h \cdot i_2)$ such that $\omega_0(\alpha_i) \neq 0$.

Denoting by $\tilde{x}_i(t)$ the trajectory starting at α_i, an obvious approximation consists in solving the following system of ordinary differential equations:

$$\tilde{x}_i(0) = \alpha_i \tag{12}$$

$$\frac{d\tilde{x}_i}{dt}(t) = \sum_{\substack{jh \in \Lambda^h \\ j \neq i}} K(\tilde{x}_i(t) - \tilde{x}_j(t))\omega_j h^2, \tag{13}$$

where $\omega_j = \omega_0(\alpha_j, 0)$, and where i is excluded from the summation because the integral in (11) is improper. The numerical integration of equations (12)-(13) is known as the point vortex method.

It is clear from (9) that $K(|x|)$ has a singularity at the origin of order $1/|x|$, so that whenever two vortices approach one another, the velocity that each induces on the other goes to infinity. Chorin ([6]) introduced the idea of replacing (12)-(13) by the system of equations

$$\tilde{x}_i(0) = \alpha_i \tag{14}$$

$$\frac{d\tilde{x}_i}{dt}(t) = \sum_{jh \in \Lambda^h} K_\delta(\tilde{x}_i(t) - \tilde{x}_j(t))\omega_j h^2, \tag{15}$$

with a new kernel K_δ close to K except at the origin, where K_δ is bounded. Hald ([16]) has shown that for the theory of vortex methods, it is convenient to obtain K_δ by convolving K with a smoothing function f_δ obtained from a fixed function f of integral one by the relation

$$f_\delta(x)=\frac{1}{\delta^2}f\left(\frac{1}{\delta}x\right)$$

for $x \varepsilon R^2$. Then we define

$$K_\delta=K*f_\delta .$$

The numerical integration of equations (14)-(15) constitutes the vortex blob algorithm. We remark that in some formulations of the vortex method, such as in [16], the constants ω_j of equations (13) and (15) are replaced by average values of the initial vorticity distribution in neighborhoods of the points α_i.

We would like now to derive the equations (15) from a different point of view, one which will clarify the name "vortex blob". We wish to approximate solutions to Euler's equations by solving the system of ordinary differential equations

$$\tilde{x}_i(0)=\alpha_i$$
$$\frac{d\tilde{x}_i}{dt}(t)=\tilde{u}(x,t) ,$$

where $\tilde{u}$ is some velocity field determined by the information at our disposal, namely, the positions $\tilde{x}_i$ and the vorticity values ω_i. We can obtain $\tilde{u}$ by first creating an approximate vorticity distribution $\tilde{\omega}$, and then using this approximation in the right hand side of equation (8). Suppose we take $\tilde{\omega}$ to be the sum of blobs of strengths $\omega_i h^2$, centered at $x_i(t)$, and of common shape f_δ, so that

$$\tilde{\omega}(x,t)=\sum_{jh\varepsilon\Lambda^h} f_\delta(x-\tilde{x}_j(t))\omega_j h^2 .$$

Then the velocity field $\tilde{u}$ corresponding to this vorticity distribution is given by

$$\tilde{u}(x)=K*\tilde{\omega}(x)$$

$$= \sum_{jh \varepsilon \Lambda^h} (K * f_\delta)(x - \tilde{x}_j(t)) \omega_j h^2$$

$$= \sum_{jh \varepsilon \Lambda^h} K_\delta(x - \tilde{x}_j(t)) \omega_j h^2.$$

This is precisely the velocity field used to move the particles in the vortex blob algorithm (15).

Three dimensions

In three dimensions, the behavior of solutions to Euler's equations is inordinately more complicated than it is in two dimensions. The reason for this becomes evident as soon as one looks at the vorticity stream formulation of Euler's equations. Whereas equations (1)-(2) look identical in dimensions two and three, the equation for the evolution of the vorticity ω, defined by

$$\omega = \nabla \times u \ ,$$

becomes

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) u \qquad (16)$$

in three dimensions. Hence, vorticity is no longer preserved along particle trajectories.

As we did in two dimensions, we now seek a formula that relates the vorticity to the velocity. Instead of the scalar stream function ψ we had in two dimensions, we now have a vector stream function Ψ such that

$$\Delta \Psi = -\omega,$$

and u is determined from Ψ by the relation

$$u = \nabla \times \Psi \ .$$

We denote again by G the fundamental solution to the Laplace operator. Then $\Psi = G * \omega$, and we have

$$u = \nabla\times\Psi = \nabla\times(G * \omega) = K * \omega,$$

so that for all $x \varepsilon R^3$,

$$u(x,t) = \int K(x-x')\omega(x',t)dx' . \tag{17}$$

We note that $G(x) = -1/(4\pi|x|)$, and that K is the matrix

$$K(x) = \frac{1}{4\pi}\begin{pmatrix} 0 & \frac{x_3}{|x|^3} & \frac{-x_2}{|x|^3} \\ \frac{-x_3}{|x|^3} & 0 & \frac{x_1}{|x|^3} \\ \frac{x_2}{|x|^3} & \frac{-x_1}{|x|^3} & 0 \end{pmatrix}$$

Just as in the two-dimensional case, particle trajectories satisfy the equations

$$x(\alpha,0) = \alpha \tag{18}$$

$$\frac{dx}{dt}(\alpha,t) = \int K(x(\alpha,t)-x')\omega(x',t)dx'$$

$$= \int K(x(\alpha,t)-x(\alpha',t)\omega(x(\alpha',t),t)d\alpha'. \tag{19}$$

This formulation suggests one way of obtaining vortex methods in three dimensions. We shall again assume that the vorticity is of compact support. We denote by Λ^h the set of points $\alpha_i = hi = h(i_1,i_2,i_3)$ such that $\omega(\alpha_i,0)\neq 0$, and we track those particles which are initially located at the points in Λ^h. We use the following set of differential equations to calculate approximate particle positions $\tilde{x}_i(t)$:

$$\tilde{x}_i(0) = \alpha_i \tag{20}$$

$$\frac{d\tilde{x}_i}{dt}(t) = \sum_{jh\epsilon\Lambda^h} K_\delta(\tilde{x}_i(t)-\tilde{x}_j(t))\tilde{\omega}_j(t)h^3 . \tag{21}$$

Here $\tilde{\omega}_j(t)$ is an approximation to $\omega_j(t) = \omega(x_j(t),t)$. Of course, equations (20)-(21) must be supplemented by a system of equations which determines the $\tilde{\omega}_j(t)$. We describe now two different procedures.

Algorithm (A): By (16), the evolution of vorticity along particle trajectories is described by the equations

$$\frac{d\omega}{dt}(x(\alpha,t),t)=(\omega(x(\alpha,t),t)\cdot\nabla_x)u(x(\alpha,t),t)\ . \tag{22}$$

Since, according to equation (21), the particles are moved by the velocity field $\tilde{u}$ defined by

$$\tilde{u}(x,t)=\sum_{jh\varepsilon\Lambda^h} K_\delta(x-\tilde{x}_j(t))\tilde{\omega}_j(t)h^3,$$

and since we can evaluate

$$\nabla_x\tilde{u}(x,t)=\sum_{jh\varepsilon\Lambda^h}\nabla_x K_\delta(x-\tilde{x}_j(t))\tilde{\omega}_j(t)h^3$$

for cutoff functions whose corresponding kernels K_δ can be given explicitly, equation (22) suggests the differential equations

$$\frac{d\tilde{\omega}_i}{dt}(t)=(\tilde{\omega}_i(t)\cdot\nabla_x)\tilde{u}^h(\tilde{x}_i(t),t)\ .$$

Algorithm (B): It can be shown ([9]) that (16) implies

$$\omega_i(t)=\left[\nabla_\alpha x(\alpha,t)\right]\cdot\omega_0(\alpha_i)\ , \tag{23}$$

where $\omega_0(\alpha_i)=\omega(\alpha_i,0)$. In algorithm (B), the vorticity is computed by approximating the gradient in (23) by a finite difference operator ∇_α^h. Thus, we set

$$\tilde{\omega}_i(t)=\left[\nabla_\alpha^h\tilde{x}(\alpha,t)\right]\cdot\omega_0(\alpha_i). \tag{24}$$

We shall define ∇_α^h precisely later in this chapter.

We remark that Beale and Majda ([2]) propose, as a three-dimensional algorithm, coupling equations (20)-(21) to the set of equations

$$\tilde{\omega}_i(0)=\omega_0(\alpha_i) \tag{25}$$

$$\frac{d\tilde{\omega}_i}{dt}(t)=\left[\nabla_\alpha^h\tilde{u}(\tilde{x}(\alpha_i,t),t)\right]\omega_0(\alpha_i). \tag{26}$$

The systems of equations {(20)-(21),(24)} and {(20)-(21),(25)-(26)} are equivalent. For, differentiating (24) with respect to time yields equation (26). Thus, algorithm (B) is in fact the Beale and Majda method, presented here in simpler form.

Actually, the grid Λ^h as defined above is not big enough for this algorithm, for when applying the finite difference operator ∇_α^h to $\tilde{x}$ at points near the edge of the grid, the positions of neighbors of these points are needed. Thus, in algorithm (B) we need to keep track of the positions of a number of particles which carry no vorticity. We shall assume, whenever referring to algorithm (B), that Λ^h contains sufficiently many neighbors of those grid points at which the vorticity is initially nonzero.

We summarize by displaying the three vortex methods which we have described. We have the two-dimensional

Algorithm (T)

$$\tilde{x}_i(0)=\alpha_i$$

$$\frac{d\tilde{x}_i}{dt}(t)=\sum_{jh\epsilon\Lambda^h} K_\delta(\tilde{x}_i(t)-\tilde{x}_j(t))\omega_j h^2,$$

and the three-dimensional algorithms

Algorithm (A)

$$\tilde{x}_i(0)=\alpha_i \qquad \tilde{\omega}_i(0)=\omega_0(\alpha_i)$$

$$\frac{d\tilde{x}_i}{dt}(t)=\sum_{jh\epsilon\Lambda^h} K_\delta(\tilde{x}_i(t)-\tilde{x}_j(t))\tilde{\omega}_j(t)h^3$$

$$\frac{d\tilde{\omega}_i}{dt}(t)=\tilde{\omega}_i(t)\cdot\sum_{jh\epsilon\Lambda^h} \nabla_x K_\delta(\tilde{x}_i(t)-\tilde{x}_j(t))\tilde{\omega}_j(t)h^3,$$

and

Algorithm (B)

$$\tilde{x}_i(0)=\alpha_i$$

$$\tilde{\omega}_j(t)=\left[\nabla_\alpha^h\tilde{x}(\alpha_j,t)\right]\omega_0(\alpha_j)$$

$$\frac{d\tilde{x}_i}{dt}(t)=\sum_{jh\in\Lambda^h} K_\delta(\tilde{x}_i(t)-\tilde{x}_j(t))\tilde{\omega}_j(t)h^3.$$

For these methods, the parameters that must be chosen are the initial mesh width h and the smoothing parameter δ; the cut-off function f must also be chosen, as well as the finite difference approximation ∇_α^h for algorithm (B).

The cutoff function

It turns out that by choosing the function f carefully, one can obtain vortex methods of high order accuracy ([2]). The cutoff functions f that we consider belong to the class $M^{L,p}$, broader than the class of functions $F\varepsilon S^{L,p}$ considered by Beale and Majda, and similar to Cottet's class $\mathcal{L}'_p$.

Definition. The class $M^{L,p}$ is the collection of functions $f:R^N\to R$ which satisfy all of the following conditions:

(i) $\int_{R^N} f(x)dx=1$

(ii) $\int_{R^N} x^\alpha f(x)dx=0,$ for all multi-indices α such that $1\le |\alpha|\le p-1$

$\int_{R^N} |x|^p |f(x)|dx < \infty$

(iii) $f\varepsilon C^L(R^N)$

(iv) $|x|^{N+|\beta|}|D^\beta f(x)|\le C$ for some C, and all β s.t. $|\beta|\le L$

(v) $|x|^{p+N+2}|f(x)|\le C$ for some constant C

The conditions (i) and (ii), as will be seen in the proof of the Moment Lemma, ensure that smoothing the vorticity of a fluid by convolving with the function f_δ causes a change in the velocity of the fluid of order $O(\delta^p)$. Conditions (iii)-(v) are needed for the Stability and Discretization Lemmas.

Beale and Majda ([4]) describe cutoff functions f for which the corresponding kernels K_δ can be calculated explicitly. We remark that the cutoff functions f_δ are often called smoothing functions, blob functions, or core functions in the literature.

The discrete norms

We introduce some notation now, somewhat different from that in Beale and Majda. Let g be either a function of the Lagrangian variables α, or a function defined only on the grid Λ^h. We denote by $\| g \|_{0,h}$ the discrete L^2-norm of the function g, that is,

$$\| g \|_{0,h} = \Big(\sum_{i h \in \Lambda^h} | g(\alpha_i) |^2 h^N \Big)^{1/2}.$$

When g is a function of α and of t, we denote by $\| g(t) \|_{0,h}$ the discrete L^2-norm of the function $g(t)$ of α obtained by fixing t, so that, for example,

$$\| x(t) \|_{0,h} = \Big(\sum_{i h \in \Lambda^h} | x_i(t) |^2 h^N \Big)^{1/2}.$$

We shall occasionally use the notations $g_i = g(\alpha_i)$ and $g_i(t) = g(\alpha_i, t)$.

For later reference, we remark that for N=2, N=3, and all functions g,

$$\begin{aligned} \| g \|_{0,h} &\le \Big(\sum_{i h \in \Lambda^h} h^N \Big)^{1/2} \max_{i h \in \Lambda^h} | g_i | \\ &\le \Big(\frac{\pi}{(6-N)} (D + \sqrt{N} h)^N \Big)^{1/2} \max_{i h \in \Lambda^h} | g_i | \, . \end{aligned} \tag{27}$$

where D is the diameter of Λ^h.

In the discussion of the proof of the 3-d Stability Lemma, we refer to the discrete analogue of the Sobolev H^{-1}-norm. Set

$$\| g \|_{-1,h} = \sup \frac{| h^3 \sum_{i h \in \Lambda^h} g(\alpha_i) \cdot g'(\alpha_i) |}{\| g' \|_{0,h}^2 + \sum_{k=1}^{3} \| D_k^+ g' \|_{0,h}^2},$$

where D_k^+ is the forward divided difference operator in the k^{th}-direction, and the suprĕmum is over all functions g' defined on Λ^h and on each of its neighboring grid points.

The Lagrangian finite difference operator

In the convergence proof of algorithm (B), we will need to assume certain stability and accuracy properties of the approximate gradient ∇_α^h. We give two definitions.

Definition. We say that ∇_α^h is r^{th}-order accurate if for any C^{r+1} function φ of compact support, there is a constant C such that

$$\| \nabla_\alpha^h \varphi - \nabla_\alpha \varphi \|_{-1,h} \le C h^r. \tag{28}$$

Definition. We say that ∇_α^h is stable if if there exist a constant C and an h_0 such that for all $h \le h_0$,

$$\| \nabla_\alpha^h f \|_{-1,h} \le C \| f \|_{0,h}. \tag{29}$$

The requirement that ∇_α^h be r^{th}-order accurate and stable, which shall be imposed in the next chapter, is not a very restrictive one. For, denote by T^l grid translation in the direction of the multi-index l. Then any operator of the form

$$\nabla_\alpha^h = \frac{1}{h} \sum_{|l| \le l_0} a_l(h) T^l,$$

where the a_l are bounded functions of h, and which is r^{th}-order accurate in the usual pointwise sense, satisfies the two definitions given above (see [2]).

CHAPTER 2

In this chapter, we give the stability and consistency results for algorithms (T) and (B) which are needed for the convergence proofs in the next chapter. The stability results are proved in [2] and [3]; we do not give the proofs here. The consistency error is estimated by considering separately the moment error and the discretization error. We give a proof here of the Moment Lemma, reproducing the argument of Beale and Majda. The Discretization Lemma we present is a strengthening of the result of Beale and Majda, and was first obtained by Cottet ([10]) in the two-dimensional case. We give an elementary proof of Cottet's result, valid in both two and three dimensions. Finally, for the sake of completeness, we show, as Beale and Majda do, how it follows from the consistency and stability results that the trajectories which are the solutions to (T) and (B) converge to the correct particle trajectories as the number of particles goes to infinity.

The proofs rely on *a priori* knowledge of the smoothness of the solutions to Euler's equations. We state now all of our assumptions about the flow and the cutoff function; we will assume hereafter that these hold.

We recall that we denote by N the number of space dimensions, which can be either two or three. We shall take the cutoff function f to belong to the class $M^{L,p}$, defined in the last chapter, with $L \geq N+1$ and $p \geq 4$.

We suppose that we have a solution u to Euler's equations defined on $R^N \times [0,T]$ with vorticity $\omega = \nabla \times u$ of compact support and belonging to C^{p+N}. We assume that for all multi-indices β, γ such that $|\beta| \leq L$ and $|\gamma| \leq L+1$, we have

$$\| D_\alpha^\beta x \|_{L^\infty(R^N \times [0,T])} < \infty \qquad\qquad \| D_x^\gamma u \|_{L^\infty(R^N \times [0,T])} < \infty . \tag{1}$$

Before presenting the stability and consistency results, we need to introduce some notation. Each transformation $X:\Lambda^h \to R^N$ induces the velocity function $V[X]:\Lambda^h \to R^N$ defined, for $N=2$, by setting

$$V[X]_i = \sum_{jh\varepsilon\Lambda^h} K_\delta(X(\alpha_i)-X(\alpha_j))\omega_j h^2 ,$$

and for $N=3$, by setting

$$V[X]_i = \sum_{jh\varepsilon\Lambda^h} K_\delta(X(\alpha_i)-X(\alpha_j))\Omega[X]_j h^3 ,$$

where

$$\Omega[X]_j = \left(\nabla_\alpha^h X(\alpha_j)\right)\omega_0(\alpha_j) .$$

Observe that algorithms (T) and (B) may be expressed in the form

$$\frac{d\tilde{x}_i}{dt}(t) = V[\tilde{x}(t)]_i .$$

Define $\dot{x}(\alpha,t)=u(x(\alpha,t),t)$, and consider the quantity

$$\| \dot{x}(t) - V[\tilde{x}(t)] \|_{0,h} .$$

It is a measure of the difference between the fluid velocity at the positions $x_i(t)$ and the approximate velocity, determined by the ordinary differential equations of the algorithm, at the positions $\tilde{x}_i(t)$. Using the triangle inequality, we have

$$\| \dot{x}(t) - V[\tilde{x}(t)] \|_{0,h} \le \| \dot{x}(t) - V[x(t)] \|_{0,h} + \| V[x(t)] - V[\tilde{x}(t)] \|_{0,h} .$$

The first term $\| \dot{x}(t) - V[x(t)] \|_{0,h}$ is called the consistency error. It is the error in velocity due to the replacement of the continuous distribution of vorticity ω by a finite number of vortex blobs centered at the exact particle positions $x_i(t)$ and with strengths $\omega_i(t)h^N$.

The next term $\| V[x(t)] - V[\tilde{x}(t)] \|_{0,h}$ is called the stability error and measures the error in velocity due to summing over approximate particle positions rather than the exact ones.

We begin by considering the consistency error. Define

$$e_c(x,t)=\left|\sum_{i h \varepsilon \Lambda^h} K_\delta(x-x_i(t))\omega_i(t)h^N-u(x,t)\right| .$$

As we shall see later, pointwise estimates for e_c yield the discrete L^2-norm estimate of the Consistency Lemma. Applying the triangle inequality and recalling (1.8), we have

$$\begin{aligned} e_c(x,t)&=\left|\sum_{i h \varepsilon \Lambda^h} K_\delta(x-x_i(t))\omega_i(t)h^N-\int K(x-x')\omega(x',t)dx'\right| \\ &\leq\left|\sum_{i h \varepsilon \Lambda^h} K_\delta(x-x_i(t))\omega_i(t)h^N-\int K_\delta(x-x')\omega(x',t)dx'\right| \\ &\quad+\left|\int K_\delta(x-x')\omega(x',t)dx'-\int K(x-x')\omega(x',t)dx'\right| \\ &=\left|\sum_{i h \varepsilon \Lambda^h} K_\delta(x-x_i(t))\omega_i(t)h^N-\int K_\delta(x-x')\omega(x',t)dx'\right| \\ &\quad+\left|(K_\delta * \omega(t))(x)-(K * \omega(t))(x)\right| \\ &=e_d(x,t)+e_m(x,t) \end{aligned}$$

So we have bounded the error function e_c by the sum of the functions e_d and e_m, which we call the discretization error and the moment error, respectively.

The error e_m is estimated by the following result, which may be found within the discussion of the Consistency Lemma given by Beale and Majda ([3]).

Moment Lemma : If f is in $M^{L,p}$, then for some constant C_m,

$$\max_{0 \leq t \leq T} \| e_m(t) \|_{L^\infty(R^N)} \leq C_m \delta^p \tag{2}$$

Proof: Define

$$g(x,t)=(K_\delta * \omega(t))(x)-(K * \omega(t))(x) .$$

Note that $e_m(x,t)=|g(x,t)|$. We have, taking Fourier transforms with respect to the space variables only,

$$\begin{aligned}\hat{g}(\zeta,t)&=(K_\delta * \omega(t))\hat{}(\zeta)-(K * \omega(t))\hat{}(\zeta)\\&=(\hat{K}_\delta(\zeta)-\hat{K}(\zeta))\hat{\omega}(\zeta,t)\\&=\hat{K}(\zeta)(\hat{f}_\delta(\zeta)-1)\hat{\omega}(\zeta,t)\\&=\hat{K}(\zeta)\hat{\omega}(\zeta,t)(\hat{f}(\delta\zeta)-\hat{f}(0)),\end{aligned}$$

since by condition (i) on $M^{L,p}$, $\hat{f}(0)=1$. From condition (ii) on $M^{L,p}$ we have that $D^\alpha\hat{f}(0)=0$, when $1\le|\alpha|\le p-1$, and that $|D^\alpha\hat{f}|$ is bounded, for $|\alpha|=p$. Hence, it follows from Taylor's theorem that $|\hat{f}(\zeta)-\hat{f}(0)|/|\zeta|^p$ is a bounded function of ζ. Since $\omega\varepsilon C^{p+N}$, $(1+|\zeta|^{p+N})\ |\hat{\omega}(\zeta,t)|$ is also bounded. Noting that

$$|\hat{K}(\zeta)|=\frac{C}{|\zeta|}, \tag{3}$$

for some C which depends only on the dimension N, we have for some constant C',

$$\begin{aligned}|\hat{g}_m(\zeta,t)|&\le\frac{C}{|\zeta|}C'\frac{|\delta\zeta|^p}{1+|\zeta|^{p+N}}\\&=CC'\delta^p\frac{|\zeta|^{p-1}}{1+|\zeta|^{p+N}}.\end{aligned}$$

Since $\int\frac{|\zeta|^{p-1}}{1+|\zeta|^{p+N}}d\zeta<\infty$, we have

$$\|e_m(t)\|_{L^\infty(R^N)}\le\|\hat{g}_m(t)\|_{L^1(R^N)}\le C_m\delta^p$$

for some constant C_m which does not depend on t. Thus, the lemma is proved.

In order to estimate e_d, we first show that the approximation of the integral of a compactly supported function in $L^1(R^N)$ by the trapezoidal rule is a procedure whose order of accuracy is restricted only by the degree of differentiability of the function. We need the following result:

Lemma 1. For $N=2$ and $N=3$ set $i=(i_1,\ldots,i_N)$ and $\|i\|=\max_{j=1,\ldots,N}|i_j|$. Then whenever either $L\geq 3$ and $N=2$, or $L\geq 4$ and $N=3$, we have

$$\sum_{\substack{i\varepsilon Z^2\\ i\neq 0}}\frac{1}{\|i\|^L}=8\sum_{k=1}^{\infty}\frac{1}{k^{L-1}}<16$$

and

$$\sum_{\substack{i\varepsilon Z^3\\ i\neq 0}}\frac{1}{\|i\|^L}=24\sum_{k=1}^{\infty}\frac{1}{k^{L-2}}+2\sum_{k=1}^{\infty}\frac{1}{k^L}<52$$

Proof: For each integer k, $\{i\varepsilon Z^N : \|i\|=k\}$ is the set of i which lie on the surface of the box centered at the origin and with edges of length $2k$. The number of such points is given by $(2k+1)^N-(2k-1)^N$. Since

$$(2k+1)^2-(2k-1)^2=8k$$

and

$$(2k+1)^3-(2k-1)^3=24k^2+2,$$

we have

$$\sum_{\substack{i\varepsilon Z^2\\ i\neq 0}}\frac{1}{\|i\|^L}=\sum_{k=1}^{\infty}\frac{8k}{k^L}=8\sum_{k=1}^{\infty}\frac{1}{k^{L-1}}$$

and

$$\sum_{\substack{i\varepsilon Z^3\\ i\neq 0}}\frac{1}{\|i\|^L}=\sum_{k=1}^{\infty}\frac{24k^2+2}{k^L}=24\sum_{k=1}^{\infty}\frac{1}{k^{L-2}}+2\sum_{k=1}^{\infty}\frac{1}{k^L}$$

The lemma now follows from the fact that $\sum_{j=1}^{\infty}\frac{1}{j^{L-1}}<2$ for $L\geq 3$ and that $\sum_{j=1}^{\infty}\frac{1}{j^L}<\sum_{j=1}^{\infty}\frac{1}{j^{L-2}}<2$ for $L\geq 4$.

The next lemma gives us the trapezoidal rule error estimate that we use in the Discretization Lemma.

Lemma 2. Suppose that $g \varepsilon C^r(R^N)$ and is of compact support. Set $\| g \|_r = \max_{l=1,\dots,N} \| \partial_l^r g \|_{L^1(R^N)}$. Then if $r \geq N+1$,

$$| \sum_{i \varepsilon Z^N} g(ih)\, h^N - \int_{R^N} g(x)dx | \leq \frac{52}{(2\pi)^r} \| g \|_r h^r. \tag{4}$$

Proof: The Poisson summation formula ([13],p.139) tells us that

$$h^N \sum_{i \varepsilon Z^N} g(ih) = \sum_{i \varepsilon Z^N} \hat{g}(i/h),$$

so that

$$| h^N \sum_{i \varepsilon Z^N} g(ih) - \int_{R^N} g(x)dx | = | \sum_{i \varepsilon Z^N} \hat{g}(i/h) - \hat{g}(0) |$$

$$= | \sum_{\substack{i \varepsilon Z^N \\ i \neq 0}} \hat{g}(i/h) | .$$

Since $| \hat{g}(\zeta) | \leq \int_{R^N} | g(x) | \, dx$ and $(D^\alpha g)\hat{}\,(\zeta) = (2\pi\sqrt{-1})^{|\alpha|} \zeta^\alpha \hat{g}(\zeta)$, we have

$$| (2\pi\sqrt{-1})^r \zeta_l^r \hat{g}(\zeta) | \leq \int_{R^N} | \partial_l^r g(x) | \, dx , \tag{5}$$

for all $\zeta \varepsilon R^N$ and $l=1,\dots,N$. Define

$$\| \zeta \| = \max_{l=1,\dots,N} | \zeta_l | .$$

It follows from (5) that for all $\zeta \neq 0$,

$$| \hat{g}(\zeta) | \leq \frac{\| g \|_r}{(2\pi)^r} \frac{1}{\| \zeta \|^r} .$$

Thus,

$$| \sum_{\substack{i \varepsilon Z^N \\ i \neq 0}} \hat{g}(\tfrac{1}{h} i) | \leq \frac{\| g \|_r}{(2\pi)^r} \sum_{\substack{i \varepsilon Z^N \\ i \neq 0}} \frac{1}{\| i/h \|^r}$$

$$= \frac{\| g \|_r}{(2\pi)^r} h^r \sum_{\substack{i \varepsilon Z^N \\ i \neq 0}} \frac{1}{\| i \|^r}$$

$$\leq \frac{52}{(2\pi)^r} \| g \|_r \, h^r$$

The last inequality follows from Lemma 1.

In order to apply Lemma 2 to the estimate of e_d, we need the bounds on derivatives of K_δ given by the following lemma.

Lemma 3 . Let A be any compact set. There is a constant C_A such that for all $\delta \leq \frac{1}{2}$ and all multi-indices β such that $|\beta| \leq L$,

$$\int_A |D^\beta K_\delta(x)|\, dx \leq C_A\, \delta^{1-L} . \tag{6}$$

Proof : Pointwise estimates on derivatives of K_δ of the kind given by Beale and Majda can be obtained under our hypotheses on the cutoff function. Suppose $|\beta| \leq L$. Then there is a constant C_β such that for all $x \varepsilon R^N$,

$$|D^\beta K_\delta(x)| \leq C_\beta \delta^{1-N-|\beta|} \tag{7}$$

and

$$|D^\beta K_\delta(x)| \leq C_\beta |x|^{1-N-|\beta|} . \tag{8}$$

We denote by $B_\delta(x)$ the ball

$$B_\delta(x) = \{x' \varepsilon R^N : |x - x'| \leq \delta\}.$$

We have, for any x and any β, that

$$\int_A |D^\beta K_\delta(x - x')|\, dx' = \int_{B_\delta(x) \cap A} |D^\beta K_\delta(x - x')|\, dx' + \int_{A - B_\delta(x)} |D^\beta K_\delta(x - x')|\, dx' . \tag{9}$$

Using (7) to estimate the first term in (9), we find that

$$\int_{B_\delta(x) \cap A} |D^\beta K_\delta(x - x')|\, dx' \leq \frac{4}{3}\pi\, \delta^N\, C_\beta\, \delta^{1-N-|\beta|} = \frac{4}{3}\pi\, C_\beta \delta^{1-|\beta|} . \tag{10}$$

We estimate the second term in (9) using (8), and obtain

$$\int_{A-B_\delta(x)} |D^\beta K_\delta(x-x')|\,dx' \le 4\pi \int_\delta^{\mathrm{diam}(A)} C_\beta\, r^{1-N-|\beta|}\, r^{N-1}\,dr$$

$$\le 4\pi C_\beta \int_\delta^{\mathrm{diam}(A)} r^{-|\beta|}\,dr \tag{11}$$

$$= \begin{cases} 4\pi C_\beta(\mathrm{diam}(A)-\delta) & \beta = 0 \\ 4\pi C_\beta(\log(\mathrm{diam}(A))-\log(\delta)) & |\beta| = 1 \\ \dfrac{4\pi C_\beta}{1-|\beta|}((\mathrm{diam}(A))^{1-|\beta|}-\delta^{1-|\beta|}) & |\beta| > 1 \end{cases}.$$

Combining (10) and (11) gives the desired result.

With Lemma 2 and 3 at our disposal, we are ready to estimate e_d.

Discretization Lemma. For some constant C_d and all $\delta \le \frac{1}{2}$,

$$\max_{0\le t\le T} \| e_d(t) \|_{L^\infty(R^N)} \le C_d \left(\frac{h}{\delta}\right)^L \delta\,. \tag{12}$$

Proof: Fix x and t. From the change of variables theorem and the fact that volumes are preserved by the flow, it follows that

$$e_d(x,t) = \left| \int K_\delta(x-x')\omega(x',t)dx' - \sum K_\delta(x-x_i(t))\omega_i(t)h^N \right|$$

$$= \left| \int K_\delta(x-x(\alpha,t))\omega(x(\alpha,t),t)d\alpha - \sum K_\delta(x-x(\alpha_i,t))\omega(x(\alpha_i,t),t)h^N \right|.$$

Defining

$$F(\alpha) = K_\delta(x-x(\alpha,t))\omega(x(\alpha,t),t),$$

we observe that

$$e_d(x,t) = \left| \int F(\alpha)d\alpha - \sum_{ih\in\Lambda^h} F(ih)h^N \right|,$$

which is simply the error due to numerical integration by the trapezoidal rule.

We use Lemma 2 to bound e_d, and so we need bounds on integrals of derivatives of F. By repeated application of the chain rule and the product rule, we can see that derivatives of F up to order L are sums of derivatives of K_δ up to order L multiplied by derivatives with respect to α of $x(\alpha,t)$ and derivatives with respect to x of $\omega(x,t)$. By hypothesis (1) on the flow, these latter are all bounded. Moreover ω, and hence F, have compact support. Denote by $\bar{\Omega}$ the support of the vorticity at time t. Then there are constants C and C', independent of x and t, such that for any $|\beta|=L$,

$$\int_{R^N} |D_\alpha^\beta F(\alpha)|\,d\alpha = \int_{\bar{\Omega}} |D^\beta F(\alpha)|\,d\alpha$$

$$\leq C' \sum_{0\leq|\gamma|\leq L} \int_{\bar{\Omega}} |D_x^\gamma K_\delta(x)|\,dx$$

$$\leq C\delta^{1-L}$$

It follows now from Lemma 2 that

$$|e_d(x,t)| \leq \frac{52}{(2\pi)^L} C\,\delta^{1-L}\cdot h^L = \frac{52}{(2\pi)^L} C\left(\frac{h}{\delta}\right)^L \delta$$

Since the constant C depends neither on x nor on t, the lemma is proved.

The Consistency Lemma now follows immediately.

Consistency Lemma. If $f\,\varepsilon M^{L,p}$, then for some constant C_c and all $t\,\varepsilon[0,T]$, $h\leq 1$, and $\delta\leq\frac{1}{2}$,

$$\|\dot{x}(t)-V[x(t)]\|_{0,h} \leq C_c\left(\delta^p+\left(\frac{h}{\delta}\right)^L\delta\right). \tag{13}$$

Proof: It follows from the Discretization and Moment Lemmas that

$$e_c(x_i(t),t) \leq (C_d+C_m)\left(\delta^p+\left(\frac{h}{\delta}\right)^L\delta\right)$$

for all i such that $ih\,\varepsilon\Lambda^h$ and all $t\,\varepsilon[0,T]$. Since

$$|\dot{x}_i(t)-V[x(t)]_i| = e_c(x_i(t),t),$$

the lemma follows from (1.27).

We come now to the Stability Lemmas, which estimate the errors in velocity due to summing over the contributions from approximate particle positions rather than the exact ones.

In the case of two space dimensions, we have the

2-d Stability Lemma. There is a constant C_s such that for all $t \varepsilon [0,T]$ and all $X:\Lambda^h \to R^2$ such that $\| X-x(t) \|_{0,h} \leq h\delta$, we have

$$\| V[X]-V[x(t)] \|_{0,h} \leq C_s (\| X-x(t) \|_{0,h}) .$$

In three dimensions, we have the

3-d Stability Lemma. Let ∇_α^h be stable and of r^{th}-order accuracy, and assume $x \varepsilon C^{r+1}(R^3 \times [0,T])$. Then there is a constant C_s such that for all $t \varepsilon [0,T]$ and all $X:\Lambda^h \to R^3$ such that $\| X-x(t) \|_{0,h} \leq h^3$, we have

$$\| V[X]-V[x(t)] \|_{0,h} \leq C_s (\| X-x(t) \|_{0,h} + h^r) . \tag{14}$$

We do not present a proof of these lemmas. The proof of the 2-d Stability Lemma is given by Beale and Majda ([3]), although they state a weaker version of the lemma (in which $\tilde{x}(t)$ takes the place of our X above). We shall need the lemma as formulated here for the proofs in chapter 3.

In their proof of stability of the three-dimensional algorithm ([2]), Beale and Majda show that there is a constant C such that for arbitrary $X:\Lambda^h \to R^3$ satisfying $\| X-x(t) \|_{0,h} \leq h^3$, and for arbitrary $\xi:\Lambda^h \to R^3$,

$$\| v - V[x(t)] \|_{0,h} \leq C(\| X-x(t) \|_{0,h} + \| \xi - \omega(t) \|_{-1,h}) , \tag{15}$$

where $v(\alpha_i) = \sum_j K_\delta(X(\alpha_i)-X(\alpha_j))\xi_j h^3$. By the stability and accuracy conditions on ∇_α^h, we have

$$\| \Omega[X]-\omega(t) \|_{-1,h} = \| \left[\nabla_\alpha^h X\right]\omega_0 - \left[\nabla_\alpha x(t)\right]\omega_0 \|_{-1,h}$$

$$\le \| \left[\nabla_\alpha^h (X-x(t))\right]\omega_0 \|_{-1,h} + \| \left[(\nabla_\alpha^h x - \nabla_\alpha x)(t)\right]\omega_0 \|_{-1,h}$$

$$\le C'(\| X-x(t) \|_{0,h} + h^r) . \qquad (16)$$

The constant C' depends only on ω_0 and on the flow map x. (14) follows from (15) and (16), setting $\xi=\Omega[X]$.

We are now ready to present the convergence theorems of Beale and Majda. The proof of Theorem 1 is a slight improvement over that given by Beale and Majda, since we are using a stability result in which the vorticity does not appear explicitly. Otherwise, the argument here is the same as that which they give.

Theorem 1. Let $N=3$. Assume the hypothesis (1) on the smoothness of the flow and that $f \varepsilon M^{L,p}$. Assume further that ∇_α^h is a stable difference operator of r^{th}-order accuracy, and that $x \varepsilon C^{r+1}$. Set $C_1=C_s+C_c$, where C_s and C_c are the constants from the Stability and Consistency Lemmas, respectively, and set $\overline{C}=\exp(C_1T)-1$. Then for all sufficiently small h and δ,

$$\| \tilde{x}(t)-x(t) \|_{0,h} \le \overline{C}(\delta^p + (\frac{h}{\delta})^L \delta + h^r)$$

for $0 \le t \le T$, provided δ and h are chosen so that $\overline{C}(\delta^p + (\frac{h}{\delta})^L\delta + h^r) \le \frac{h^3}{2}$.

Proof: We denote by $e(\alpha,t)$ the error in position of the particle $\tilde{x}(\alpha,t)$, so that

$$e(\alpha,t)=\tilde{x}(\alpha,t)-x(\alpha,t),$$

and set

$$e_i(t)=e(\alpha_i,t)=\tilde{x}_i(t)-x_i(t).$$

Differentiating in time, we have for each i,

$$\dot{e}_i(t)=V[\tilde{x}(t)]_i-\dot{x}_i(t)$$
$$=(V[\tilde{x}(t)]_i-V[x(t)]_i)+(V[x(t)]_i-\dot{x}_i(t))$$

The Consistency Lemma gives us

$$\| V[x(t)]-\dot{x}(t)\|_{0,h}\le C_c(\delta^p+(\frac{h}{\delta})^L\delta).$$

Set

$$T^*=\min\{T,\ \inf\{t:\| e(t)\|_{0,h}\ge h^3\}\}.$$

Then we have from the Stability Lemma that in the time interval $[0,T^*]$,

$$\| V[\tilde{x}(t)]-V[x(t)]\|_{0,h}\le C_s(\| e(t)\|_{0,h}+h^r)$$

Hence, for $0\le t\le T^*$,

$$\| \dot{e}(t)\|_{0,h}\le C_1(\| e(t)\|_{0,h}+h^r+\delta^p+(\frac{h}{\delta})^L\delta). \tag{17}$$

Define $g:R\to R$ by setting

$$g(\alpha)=C_1\,\alpha+\ C_1\ (h^r+\delta^p+(\frac{h}{\delta})^L\delta).$$

for all real α. We have, rewriting (17), that in the time interval $[0,T^*]$,

$$\| \dot{e}(t)\|_{0,h}\le g(\| e(t)\|_{0,h}).$$

We now cite the following lemma from [18].

> **Lemma.** Let $g:R\to R$ be a smooth function, let $\|\ \|$ be a norm on R^n and let e be a continuously differentiable n-vector function on $[0,T^*]$ such that $e(0)=0$ and $\| \dot{e}(t)\|\le g(\| e(t)\|)$. Let y be the real-valued function such that $y(0)=0$ and $\dot{y}(t)=g(y(t))$. Then for $t\,\varepsilon[0,T^*]$, $\| e(t)\|\le y(t)$.

It follows from this lemma that for $0\le t\le T^*$,

$$\| e(t) \|_{0,h} \leq \bar{C} \left(\delta^p + \left(\frac{h}{\delta}\right)^L \delta + h^r\right). \tag{18}$$

Thus, by hypothesis, we have

$$\| e(t) \|_{0,h} \leq \frac{h^3}{2}$$

for $0 \leq t \leq T^*$. Hence $T^*=T$ and (18) holds on the entire interval $[0,T]$. Thus the theorem has been proved.

The two-dimensional version of the theorem is proved similarly. We merely state

Theorem 2. Suppose $N=2$. Assume the hypothesis (1) on the smoothness of the flow and that f belongs to $M^{L,p}$. Set $\bar{C}=\exp((C_s+C_c)T)-1$. Then for all sufficiently small h and δ,

$$\| \tilde{x}(t) - x(t) \|_{0,h} \leq \bar{C}\left(\delta^p + \left(\frac{h}{\delta}\right)^L \delta\right)$$

for $0 \leq t \leq T$, provided δ and h are chosen so that $\bar{C}\left(\delta^p + \left(\frac{h}{\delta}\right)^L \delta\right) \leq \frac{h\delta}{2}$.

Parameter Choices and Rates of Convergence

We restrict our discussion to two-dimensional vortex methods.

In their proof of convergence of vortex methods ([3]), Beale and Majda take δ to be a function of h in order to determine a rate of convergence in terms of h alone. We see from Theorem 2 that, by setting $\delta=h^q$ for some $q \in (0,1)$, as Beale and Majda do, we have

$$\| e(t) \|_{0,h} \leq \bar{C}(h^{pq} + h^{L(1-q)+q}).$$

Since one can find cutoff functions which are infinitely differentiable, one has $f \in M^{L,p}$ for arbitrarily large L. If the vorticity is $L+1$-times differentiable, with

$$L > \frac{(p-1)q}{(1-q)},$$

then choosing h_0 so that $\overline{C}(h_0^{pq}+h_0^{L(1-q)+q})<h_0^{1+q}/2$, we have

$$\| e(t) \|_{0,h} \leq \overline{C}h^{pq},$$

for all $h<h_0$. For infinitely differentiable flows, we see that one can let q approach 1, and thus have an order of convergence in h as close to p as one likes. Kernels $K_\delta=K * f_\delta$, with $f \varepsilon M^{L,p}$, are called p^{th}-order kernels.

However, as q approaches 1, and we need correspondingly larger values of L, the constant C_d of the Discretization Lemma also grows. Perlman ([25]) carries out numerical tests on problems in which the vorticity is confined to a disc and is radially symmetric. She separately examines the discretization and the moment errors and finds that once there is substantial stretching in the flow map, the order of accuracy predicted by the Discretization Lemma in equation (12) is not seen when q is close to 1 and when reasonable (the evaluation of the velocities requires $O(h^{-2})$ operations) values are chosen for h.

Nevertheless, numerical studies by Beale and Majda ([4]) and by Perlman ([25]) show that using p^{th}-order kernels, with p large, can significantly improve the accuracy of calculations with the vortex method.

Second-Order Kernels

It turns out to be convenient, in practice, to use second-order kernels. If one wishes to use a cutoff function which belongs only to the class $M^{L,2}$, and to take $\delta=h^q$, for some q, then the analysis we have presented does not apply. For, the hypotheses $\bar{C}(\delta^p+(\frac{h}{\delta})^L\delta+h^r)\le\frac{h^3}{2}$, and $\bar{C}(\delta^p+(\frac{h}{\delta})^L\delta)\le\frac{h\delta}{2}$, of Theorems 1 and 2, respectively, can then never be satisfied for $p=2$.

However, as Beale and Majda have explained, convergence results can be obtained for second-order kernels by carrying out the analysis using discrete L^μ-norms, instead of L^2-norms, with μ sufficiently greater than two. Define, for functions $g:\Lambda^h\to R$,

$$\| g \|_{L_h^\mu}=(\sum g_i^\mu h^N)^{1/\mu} .$$

It can be shown that the Stability Lemmas still hold, with the hypotheses

$$\| X-x(t) \|_{0,h} \le h^3 \qquad\qquad \| X-x(t) \|_{0,h} \le h\delta$$

replaced by the hypotheses

$$\| X-x(t) \|_{L_h^\mu} \le h^{1+(3/\mu)} \qquad\qquad \| X-x(t) \|_{L_h^\mu} \le h^{2/\mu}\delta .$$

The two-dimensional result is given in [3]; the three-dimensional result is unpublished ([5]). It is easy to see that the Consistency Lemma can be replaced by a more general assertion with arbitrary L_h^μ-norms replacing L_h^2-norms. Convergence results may thus be obtained with L_h^μ-norms rather than L_h^2-norms. Theorems 1 and 2 still hold with the new norm and with the conditions

$$\bar{C}(\delta^p+(\frac{h}{\delta})^L\delta+h^3)\le\frac{h^{1+(3/\mu)}}{2} \qquad\qquad \bar{C}(\delta^p+(\frac{h}{\delta})^L\delta)\le\frac{h^{2/\mu}\delta}{2}.$$

For $p=2$ and $\delta=h^q$, with $q\,\varepsilon(0,1)$ arbitrary, one can find a suitably large μ such that these conditions hold, and hence prove convergence in L_h^μ. But, just as L^μ-norms are bounded by $L^{\mu'}$-norms, for compact measure spaces and $\mu' > \mu$, so is the norm $\| \ \|_{0,h} = \| \ \|_{L_h^2}$ bounded by $\| \ \|_{L_h^\mu}$ for all $2 \le \mu < \infty$. Thus, convergence in L_h^μ implies convergence in L_h^2, so that convergence can be obtained in the discrete L^2-norm even for second-order kernels.

CHAPTER 3

With the results of the last chapter on the stability and consistency of the vortex method at our disposal, we are ready to prove convergence of the time-discretized vortex method. We give proofs for the three-dimensional algorithm (B). The two-dimensional results are very similar.

We analyze how well solutions to the system of equations

$$\tilde{x}_i(0)=\alpha_i \tag{1}$$

$$\frac{d\tilde{x}_i}{dt}(t)=\sum_{jh\in\Lambda^h} K_\delta(\tilde{x}_i(t)-\tilde{x}_j(t))\tilde{\omega}_j(t)h^2 \tag{2}$$

obtained from approximate difference schemes approach the solution to Euler's equations. Here $\tilde{\omega}_j(t)=\left[\nabla_\alpha^h\tilde{x}(\alpha_j,t)\right]\omega_j(0)$. We denote by $\tilde{x}_i^n$ the approximation to $\tilde{x}_i(n\Delta t)$ given by some discretized version of (1)-(2), with time steps of size $\Delta t=T/n_1$. Set $x^n=x(n\Delta t)$. The error we estimate in this chapter is

$$e_i^n=\tilde{x}_i^n-x_i^n. \tag{3}$$

We can obtain crude estimates immediately. For,

$$\| e^n \|_{0,h}\le \| \tilde{x}^n-\tilde{x}(n\Delta t)\|_{0,h}+\|\tilde{x}(n\Delta t)-x^n\|_{0,h}.$$

The second term on the right-hand-side above is the error estimated by Theorem 1 of chapter 2. The first term is the error due to the time discretization of the system of ordinary differential equations (1)-(2). If we use an m^{th}-order integration scheme, we have

$$\|\tilde{x}^n-\tilde{x}(n\Delta t)\|_{0,h}\le C_1(\Delta t)^m.$$

Unfortunately, the constant C_1 depends on the equations (1)-(2), and hence on δ and on h. In fact, C_1 typically involves derivatives of K_δ of order $m+1$. Thus, we have

$$\| e^n \|_{0,h} \le \bar{C}(\delta^p + (\frac{h}{\delta})^L \delta + h^r) + C_1(\delta,h)(\Delta t)^m, \tag{4}$$

where $C_1(\delta,h)$ has as one of its factors inverse powers of δ (see equation 2.7).

But one need not consider the errors made in the numerical approximation to (1)-(2) and in the approximation by (1)-(2) of Euler's equations separately. In fact, by estimating the *total* error at each time step, it turns out that one can obtain estimates of the form (4) with constants C_1 independent of δ and h.

We shall give such results for second-order Runge-Kutta methods and for a broad class of multistep methods.

Multistep Methods

We consider explicit multistep methods of order $m \ge 1$ of the form

$$\tilde{x}_i^{n+1} = \sum_{j=0}^{q} a_j \tilde{x}_i^{n-j} + \Delta t \sum_{j=0}^{q} b_j V[\tilde{x}^{n-j}]_i \qquad n = q, q+1, \ldots, n_1 - 1 \tag{6}$$

where $a_j \ge 0$, all j, and

$$\sum_{j=0}^{q} a_j = 1 \tag{7}$$

$$-\sum_{j=0}^{q} j a_j + \sum_{j=0}^{q} b_j = 1 \tag{8}$$

$$\sum_{j=0}^{q} (-j)^i a_j + i \sum_{j=0}^{q} (-j)^{i-1} b_j = 1 \qquad i = 2, \ldots, m \tag{9}$$

Atkinson ([1]) discusses this class of methods and gives a convergence proof (the lines of which we have followed) for the approximation of ordinary differential equations by these methods. We remark that Euler's method and the Adams-Bashforth methods are of this form.

We need the following lemma in the proof of our convergence result.

Lemma 1. Suppose $u \varepsilon C^{m+1}(R^2 \times [0,T])$ and that the coefficients $\{a_j\}$ and $\{b_j\}$ satisfy conditions (7)-(9). Then there is a constant C_2 such that for all h and all $n \geq q$ satisfying $n \Delta t \leq T$,

$$\| x^{n+1} - \sum_{j=0}^{q} a_j x^{n-j} - \Delta t \sum_{j=0}^{q} b_j \dot{x}((n-j)\Delta t) \|_{0,h} \leq C_2 (\Delta t)^{m+1} \qquad (10)$$

Proof: Since the multistep method (6)-(9) has a local truncation error of order $m+1$, it follows that for each i there is a constant $C_{i,h}$ such that

$$| x_i^{n+1} - \sum_{j=0}^{q} a_j x_i^{n-j} - \Delta t \sum_{j=0}^{q} b_j \dot{x}_i((n-j)\Delta t) | \leq C_{i,h} (\Delta t)^{m+1}$$

The $C_{i,h}$ are all bounded, being derivatives of u of order $m+1$. The lemma now follows.

We have the following convergence result.

Theorem 1. Assume $f \varepsilon M^{L,p}$, and, in addition to the smoothness hypothesis (2.1) on the flow, that $u \varepsilon C^{m+1}(R^2 \times [0,T])$. Suppose that one uses an m^{th}-order multistep method of the form (6)-(9), with time steps of size $\Delta t = T/n_1$. Suppose the initial errors satisfy

$$\max_{0 \leq k \leq q} \| e^k \|_{0,h} \leq C_3 (\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^m).$$

Set $C_4 = \max(C_c \sum_{j=0}^{q} | b_j |, C_s \sum_{j=0}^{q} | b_j |, C_2)$, where C_c and C_s are given by the Consistency and Stability Lemmas, respectively, and set $C_5 = (1+C_3)\exp(C_4 T)$. Then if $\Delta t \leq 1/C_4$,

$$\max_{0 \leq n \leq n_1} \| e^n \|_{0,h} \leq C_5 (\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^m),$$

provided that h and δ are sufficiently small and satisfy the condition $C_6(\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^m) \leq h^3$.

Proof: The proof is by induction on n. Define, for $0 \le n \le n_1$,

$$E^n = \max_{0 \le k \le n} \| e^k \|_{0,h} .$$

Suppose that n satisfies $q \le n < n_1$ and that

$$E^n \le (\Delta t\, C_4 \sum_{j=0}^{n-q-1} (1+\Delta t\, C_4)^j + C_3 (1+\Delta t\, C_4)^{n-q})(\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^m). \quad (11)$$

We note that, by hypothesis, (11) is satisfied for $n=q$. It follows from (11) that

$$E^n \le C_5 (\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^m), \quad (12)$$

which permits the Stability Lemma to be applied in the argument that follows. Set $t = n\Delta t$. It follows from (6),(7), and (10) that

$$\begin{aligned} \| e^{n+1} \|_{0,h} &= \| \tilde{x}^{n+1} - x(t+\Delta t) \|_{0,h} \\ &\le \sum_{j=0}^{q} a_j \| e^{n-j} \|_{0,h} \quad (13) \\ &\quad + \Delta t \sum_{j=0}^{q} | b_j | \, \| V[\tilde{x}^{n-j}] - \dot{x}((n-j)t) \|_{0,h} + C_2 (\Delta t)^{m+1}. \end{aligned}$$

For each j, we have

$$\begin{aligned} \| \dot{x}((n-j)t) - V[\tilde{x}^{n-j}] \|_{0,h} \le& \| \dot{x}((n-j)t) - V[x((n-j)t)] \|_{0,h} \quad (14) \\ &+ \| V[x((n-j)t)] - V[\tilde{x}^{n-j}] \|_{0,h}. \end{aligned}$$

The Consistency Lemma gives

$$\| \dot{x}((n-j)t) - V[x((n-j)t)] \|_{0,h} \le C_c (\delta^q + (\frac{h}{\delta})^L \delta) .$$

By (12), we can apply the Stability Lemma to the second term in (14) and obtain

$$\| V[x((n-j)t)] - V[\tilde{x}^{n-j}] \|_{0,h} \le C_s (\| e^{n-j} \|_{0,h} + h^r) .$$

Thus, we have

$$\| \dot{x}((n-j)t) - V[\tilde{x}^{n-j}] \|_{0,h} \le C_c(\delta^p + (\frac{h}{\delta})^L \delta) + C_s(\| e^{n-j} \|_{0,h} + h^r) . \quad (15)$$

so that , by (13) and (15),

$$\| e^{n+1} \|_{0,h} \le \sum_{j=0}^{q} a_j \| e^{n-j} \|_{0,h} + \Delta t\, C_s \sum_{j=0}^{q} | b_j | (\| e^{n-j} \|_{0,h} + h^r)$$

$$+ \Delta t \sum_{j=0}^{q} | b_j | \; C_c(\delta^p + (\frac{h}{\delta})^L \delta) + C_2 (\Delta t)^{m+1}$$

$$\le \sum_{j=0}^{q} a_j E^n + \Delta t\, C_s \sum_{j=0}^{q} | b_j | E^n + \Delta t\, C_s \sum_{j=0}^{q} | b_j | h^r$$

$$+ \Delta t\, C_c \sum_{j=0}^{q} | b_j | (\delta^p + (\frac{h}{\delta})^L \delta) + C_2 (\Delta t)^{m+1} .$$

Thus,

$$E^{n+1} \le E^n + \Delta t\, C_4 E^n + \Delta t\, C_4(\delta^p + (\frac{h}{\delta})^L \delta + h^r) + C_4 (\Delta t)^{m+1} \quad (16)$$

$$\le (\Delta t\, C_4 \sum_{j=0}^{n-q} (1+\Delta t\, C_4)^j \; + C_3 (1+\Delta t\, C_4)^{n+1-q} \;)(\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^m).$$

The first inequality here follows from (7) and the definition of C_4; the second inequality follows from (11). Thus, the induction argument is complete and equation (11), and hence (12), holds for all n such that $q \le n < n_1$.

Runge-Kutta Methods

We now consider Runge-Kutta methods. We have been able to derive estimates of the kind given in Theorem 1 for second-order Runge-Kutta methods; we present the proof of our result below. This proof can be generalized in a straightforward way to higher order Runge-Kutta methods. Unfortunately, a term such as the last term in (22) below occurs, which restricts the convergence result to be of second order accuracy in time. However, Hald has used a different technique, which requires him to establish a stronger version of the Stability Lemma, and has obtained convergence results for the classical fourth-order Runge-Kutta method ([17]).

In order to minimize the number of constants occurring in the proof of the next theorem, we derive the estimate only for the modified Euler method (see [14]). The other second-order Runge-Kutta methods can be shown to converge in the same way.

Set

$$\tilde{x}_i^{n+\frac{1}{2}} = \tilde{x}_i^n + \tfrac{1}{2}\Delta t\, V[\tilde{x}^n]_i \,.$$

In the modified Euler method, the particle positions at time step $n+1$ are given by

$$\tilde{x}_i^{n+1} = \tilde{x}_i^n + \Delta t\, V[\tilde{x}^{n+\frac{1}{2}}]_i \,. \tag{17}$$

We merely state

Lemma 2. Suppose $u \varepsilon C^8(R^2 \times [0,T])$. Then there are constants C_2 and C_3 such that for all $t \varepsilon [0, T-\Delta t]$,

$$\| x(t+\Delta t) - x(t) - \Delta t \dot{x}(t+\tfrac{1}{2}\Delta t) \|_{0,h} \le C_2 (\Delta t)^3 \tag{18}$$

and

$$\| x(t+\tfrac{1}{2}\Delta t) - x(t) - \tfrac{1}{2}\Delta t \dot{x}(t) \|_{0,h} \le C_3 (\Delta t)^2 \tag{19}$$

Theorem 2. Assume $f \varepsilon M^{L,p}$ and that the flow satisfies the smoothness conditions (2.1). Suppose that particle positions are updated according to equation (17), with time steps of size $\Delta t = T/n_1$. Set $C_4 = \max(C_2, C_s C_3, \tfrac{1}{2} C_s C_c, (C_s + C_c)(1+\tfrac{1}{2} C_s \Delta t))$ and $C_5 = \exp(C_4 T)$. Then if $0 \le n \le n_1$, we have

$$\| \tilde{x}^n - x(n\Delta t) \|_{0,h} \le C_5 (\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^2) \,,$$

for all h, δ which are sufficiently small and which satisfy the relation $C_5(1+\tfrac{1}{2}\Delta t (C_s + C_c) + C_3)(\delta^p + (\frac{h}{\delta})^L + h^r + (\Delta t)^2) \le h^3$. Here, the constants C_c and C_s are given by the Consistency and Stability Lemmas, respectively.

Proof: The proof is again by induction on n. By hypothesis, $\| e^0 \|_{0,h}=0$. Suppose that $1 \le n < n_1$ and that

$$\| e^n \|_{0,h} \le C_4 \Delta t \left(\sum_{j=0}^{n-1} (1+C_4\Delta t)^j\right) \left(\delta^p + \left(\frac{h}{\delta}\right)^L \delta + h^r + (\Delta t)^2\right). \tag{20}$$

It follows that

$$\| e^n \|_{0,h} \le C_5 \left(\delta^p + \left(\frac{h}{\delta}\right)^L \delta + h^r + (\Delta t)^2\right) \le h^3 . \tag{21}$$

Set $t = n\Delta t$. From (17) and (18), we have

$$\begin{aligned}
\| e^{n+1} \|_{0,h} &= \| \tilde{x}^{n+1} - x(t+\Delta t) \|_{0,h} \\
&\le \| \tilde{x}^n + \Delta t \; V[\tilde{x}^{n+1/2}] - x(t+\Delta t) \|_{0,h} \\
&\le \| \tilde{x}^n - x(t) \|_{0,h} + C_2 (\Delta t)^3 + \Delta t \; \| V[\tilde{x}^{n+1/2}] - \dot{x}(t+\tfrac{1}{2}\Delta t) \|_{0,h} \\
&\le \| e^n \|_{0,h} + C_2 (\Delta t)^3 + \Delta t \; \| V[\tilde{x}^{n+1/2}] - V[x(t+\tfrac{1}{2}\Delta t)] \|_{0,h} \qquad (22) \\
&\qquad + \Delta t \; \| V[x(t+\tfrac{1}{2}\Delta t)] - \dot{x}(t+\tfrac{1}{2}\Delta t) \|_{0,h}
\end{aligned}$$

An estimate for the last term above is given by the Consistency Lemma. We have

$$\| V[x(t+\tfrac{1}{2}\Delta t)] - \dot{x}(t+\tfrac{1}{2}\Delta t) \|_{0,h} \le C_c \left(\delta^p + \left(\frac{h}{\delta}\right)^L \delta\right) .$$

There remains the second to last term to estimate. We would like to apply the Stability Lemma to this term, but we need first to verify that its hypotheses are satisfied. We have

$$\begin{aligned}
\| \tilde{x}^{n+1/2} - x(t+\tfrac{1}{2}\Delta t) \|_{0,h} &\le \| \tilde{x}^{n+1/2} - (x^n + \tfrac{1}{2}\Delta t \, V[x^n]) \|_{0,h} \\
&\quad + \| x^n + \tfrac{1}{2}\Delta t \, V[x^n] - (x^n + \tfrac{1}{2}\Delta t \, \dot{x}(t)) \|_{0,h} \qquad (23) \\
&\quad + \| x^n + \tfrac{1}{2}\Delta t \, \dot{x}(t) - x(t+\tfrac{1}{2}\Delta t) \|_{0,h}
\end{aligned}$$

We estimate these three terms in turn. As a consequence of (21) and the hypothesis of the theorem, $\| e^n \|_{0,h} \le h^3$. This fact allows us to apply the Stability Lemma to the first term on the right hand side in (23), and we have

$$\| \tilde{x}^{n+\frac{1}{2}} - (x^n + \tfrac{1}{2}\Delta t\, V[x^n]) \|_{0,h} = \| \tilde{x}^n + \tfrac{1}{2}\Delta t\, V[\tilde{x}^n] - (x^n + \tfrac{1}{2}\Delta t\, V[x^n]) \|_{0,h}$$

$$\le \| \tilde{x}^n - x^n \|_{0,h} + \tfrac{1}{2}\Delta t \, \| V[\tilde{x}^n] - V[x^n] \|_{0,h}$$

$$\le \| e^n \|_{0,h} + \tfrac{1}{2}\Delta t\, C_s (\| \tilde{x}^n - x^n \|_{0,h} + h^r) .$$

$$\le (1 + \tfrac{1}{2}\Delta t\, C_s) \| e^n \|_{0,h} + \tfrac{1}{2}\Delta t\, C_s h^r .$$

An estimate for the second term in (23) follows from the Consistency Lemma, for

$$\| x^n + \tfrac{1}{2}\Delta t\, V[x^n] - (x^n + \tfrac{1}{2}\Delta t \dot{x}(t)) \|_{0,h} = \tfrac{1}{2}\Delta t \, \| V[x^n] - \dot{x}(t) \|_{0,h}$$

$$\le \tfrac{1}{2}\Delta t \, C_c (\delta^p + (\frac{h}{\delta})^L \delta) .$$

The final term is bounded, from Lemma 2, by $C_3(\Delta t)^2$. Thus, we have

$$\| \tilde{x}^{n+\frac{1}{2}} - x(t + \tfrac{1}{2}\Delta t) \|_{0,h} \le (1 + \tfrac{1}{2}\Delta t\, C_s) \| e^n \|_{0,h} + \tfrac{1}{2}\Delta t\, C_s h^r$$

$$+ \tfrac{1}{2}\Delta t\, C_c (\delta^p + (\frac{h}{\delta})^L \delta) + C_3(\Delta t)^2$$

$$\le (1 + \tfrac{1}{2}\Delta t (C_s + C_c) + C_3)\, C_5\, (\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^2)$$

$$\le h^3 .$$

Thus, we can apply the Stability Lemma to the next to last term of (22), and we have

$$\| V[\tilde{x}^{n+\frac{1}{2}}] - V[x(t + \tfrac{1}{2}\Delta t)] \|_{0,h} \le C_s (\| \tilde{x}^{n+\frac{1}{2}} - x(t + \tfrac{1}{2}\Delta t) \|_{0,h} + h^r)$$

$$\le C_s ((1 + \tfrac{1}{2}\Delta t\, C_s) \| e^n \|_{0,h}$$

$$+ \tfrac{1}{2}\Delta t (C_c + C_s)(\delta^p + (\frac{h}{\delta})^L \delta + h^r) + C_3(\Delta t)^2 + h^r) .$$

Combining the above estimates, we have

$$\| e^{n+1} \|_{0,h} \le \| e^n \|_{0,h} + C_2(\Delta t)^3 + C_s C_3(\Delta t)^3 + \Delta t (C_c + C_s)(\delta^p + (\frac{h}{\delta})^L \delta + h^r)$$

$$+ C_s (C_c + C_s) \tfrac{1}{2} (\Delta t)^2 (\delta^p + (\frac{h}{\delta})^L \delta + h^r) + C_s (1 + \tfrac{1}{2} \Delta t\, C_s) \Delta t \, \| e^n \|_{0,h}$$

$$\le (1 + C_4 \Delta t) \| e^n \|_{0,h} + C_4 \Delta t \ (\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^2),$$

so that, by (20),

$$\| e^{n+1} \|_{0,h} \le (\sum_{j=0}^{n} (1 + C_4 \Delta t)^j) \ (\delta^p + (\frac{h}{\delta})^L \delta + h^r + (\Delta t)^2).$$

Thus, (20), and hence (21), holds for all $n \le n_1$, and the proof is complete.

CHAPTER 4

We now turn to a discussion of three-dimensional vortex methods. We begin by describing briefly the method of Chorin.

Chorin takes vorticity to be concentrated at the midpoints of line segments between particles $X^{(1)}$ and their successors $X^{(2)}$, and to be proportional to the vector $X^{(1)}-X^{(2)}$. Thus, the velocity field used to move the particles is

$$U(x)=\sum K_\delta(x-m)\ dX\,\Gamma,$$

where the summation is over all of the "segments". Here

$$m=\frac{1}{2}(X^{(1)}+X^{(2)})\,,$$

$$dX=X^{(2)}-X^{(1)}\,,$$

and Γ is a constant multiple of the initial intensity of the segment.

This idea is used by Chorin in two different ways. In a simulation of viscous flow, segments are generated at the boundary through the no-slip condition ([7]). In another calculation ([8]), Chorin follows a filament, and the segments of the algorithm are the pieces of the filament; in this calculation, each particle is both the upper end of one segment and the lower end of the following segment. Vorticity updating does not need to be carried out explicitly in either version of the algorithm, since the stretching is taken into account intrinsically by the change in $X^{(1)}-X^{(2)}$.

Algorithm (B) has been thought of as being much different, in its Lagrangian updating of vorticity, from previous three-dimensional vortex methods, such as those of Nakamura et al ([23]) and of Chorin. But the vortex stretching in these "filament" methods, just as that in (B), is carried out by applying a finite difference operator to the initial configuration of vortices. The stretching in (B) differs only in that higher order finite difference operators, rather than forward difference operators, are used. In particular, it seems likely that Chorin's algorithm converges, though without the high order of accuracy, of course.

To each set of vortex trajectories $\{(\tilde{x}_i(t),\tilde{\omega}_i(t))\}$, of any of the algorithms, there is associated a natural approximate flow map $\tilde{x}(t):R^3\times[0,T]\to R^3$, namely, define $\tilde{x}(\alpha,t)$ to be the solution of the ordinary differential equations

$$\frac{d\tilde{x}}{dt}(\alpha,t)=\sum_{i\in\Lambda^h} K_\delta(\tilde{x}(\alpha,t)-\tilde{x}_i(t))\tilde{\omega}_i(t)h^3 \qquad \tilde{x}(\alpha,0)=\alpha\ .$$

The following is an interesting fact about algorithm (A).

Proposition. Let $\tilde{x}$ denote the approximate flow map corresponding to algorithm (A), with vortex trajectories $\tilde{\omega}_i(t)$. Then for each i, we have

$$\tilde{\omega}_i(t)=\left[\nabla_\alpha\tilde{x}(\alpha_i,t)\right]\cdot\omega_i(0)\ .$$

Proof. Define

$$\xi_i(t)=\left[\nabla_\alpha\tilde{x}(\alpha_i,t)\right]\cdot\omega_i(0)\ .$$

Then we have

$$\frac{d}{dt}\xi_i(t)=\left[\nabla_\alpha\frac{d\tilde{x}}{dt}(\alpha_i,t)\right]\cdot\omega_i(0)$$

$$=\left[\nabla_x\tilde{u}(\tilde{x}(\alpha_i,t),t)\right]\cdot\left[\nabla_\alpha\tilde{x}(\alpha_i,t)\right]\cdot\omega_i(0)$$

$$=\left[\nabla_x\tilde{u}(\tilde{x}_i(t),t)\right]\cdot\xi_i(t)\ ,$$

where $\tilde{u}(x,t)=\sum_{j\in\Lambda^h} K_\delta(x-\tilde{x}_j(t))\tilde{\omega}_j(t)h^3$. Moreover, since

$$\xi_i(0)=\omega_i(0)\ ,$$

and since

$$\frac{d}{dt}\tilde{\omega}_i(t)=\left[\nabla_x\tilde{u}(\tilde{x}_i(t),t)\right]\cdot\tilde{\omega}_i(t)\,,$$

ξ_i and $\tilde{\omega}_i$ satisfy the same ordinary differential equation and hence are identical.

We display now the Lagrangian vortex-stretching equations for the Euler equations and for algorithms (A) and (B):

Euler: $$\omega_i(t)=\left[\nabla_\alpha x(\alpha,t)\right]\cdot\omega_i(0)$$

A: $$\tilde{\omega}_i(t)=\left[\nabla_\alpha\tilde{x}(\alpha,t)\right]\cdot\omega_i(0)$$

B: $$\tilde{\omega}_i(t)=\left[\nabla_\alpha^h\tilde{x}(\alpha,t)\right]\cdot\omega_i(0)$$

Thus, we see that the transformation in time of the approximate vorticity under algorithm (A) is uniquely determined by the approximate flow map, and that the approximate flow map transforms the approximate vorticity just as the vorticity is transformed by the flow map in solutions of Euler's equations. Also, we see that although the systems of ordinary differential equations that constitute methods (A) and (B) appear very different, the transformation of vorticity by the approximate flow map in (B) differs from that in (A) only in that a discretized version of the spatial derivative of $\tilde{x}(t)$, rather than the real derivative, is applied to the initial vorticity.

But this difference is important. Whereas in (A), the evolution of vorticity is determined by derivatives of the velocity field $\tilde{u}$, and hence does not depend on which particles were originally close together, algorithm (B) remembers neighbors. As long as the fluid has not undergone substantial deformation, all of the algorithms will produce similar results (though (A) has one less source of error, of course). Once serious stretching has occurred, however, the difference becomes consequential.

As a matter of fact, it is not reasonable to use either of the algorithms in the form in which they have been described past times when considerable stretching has occurred. A procedure is needed to add new vortices in places where the original ones have become too widely separated; the alternative is to use an enormous number of computational elements from the beginning, with pointless over-resolution of regions of the fluid which are not very interesting.

In algorithm (B), as in the filament algorithms, there is a natural way to add vortices. Namely, between two vortices which were neighbors on the original mesh, one can interpolate a new vortex, with the vorticity contained in the two original vortices now shared among the three. It is easy to modify the finite difference operator which governs the stretching of vorticity, then, though the high order of accuracy is sacrificed by such a procedure. This loss of accuracy may not be important, however, for the improvement in accuracy in using a high order method over a lower order one becomes extremely small after the initial configuration of the particles has been sufficiently distorted. We mention that Chorin uses a linear interpolation procedure in [8].

In an algorithm such as (A), where one does not keep track of neighbors, vortices can also be added, though in a less natural way. We have experimented with the following procedure. Once a value of vorticity $\tilde{\omega}_i(t)$ has become sufficiently large, which suggests that the vortex filament to which $\tilde{\omega}_i(t)$ is tangent has becomes stretched, we replace the vortex $(\tilde{x}_i(t), \tilde{\omega}_i(t))$ by the two vortices $(\tilde{x}_i(t) \pm \gamma, \frac{1}{2}\tilde{\omega}_i(t))$, where γ is some parameter,

a natural choice of which is one third of the original inter-particle spacing. Numerical experiments with such an algorithm will be reported in [15].

Vortex methods can be used for the simulation both of inviscid and of slightly viscous flows. In inviscid flows, vortex filaments remain filaments. Between two fluid particles in the same filament, the vorticity always lies along the material curve joining the particles, and changes in magnitude in proportion to the stretching of the filament. Thus, the filament methods in [8] and [23], as well as method (B), are good candidates for doing inviscid flow calculations.

The dynamics of viscous flow involves, in addition to the stretching of vorticity, the diffusion of vorticity. Viscosity is of course of particular importance in boundary layers, but even away from boundaries, and for small values of viscosity, diffusion can play an important role. In regions of space where nearby pairs of vortex concentrations of opposite sign exist, viscosity can cause the bulk of the vorticity in those regions to cancel itself. Changes in topology can result. This process has been seen in the dissipation of trailing vortices ([12]), and in the experiments of Oshima and Asaka([24]), for example.

Thus, if one wants a vortex method to simulate such flows, one needs to be able to model viscous diffusion. Chorin proposed diffusing vorticity by random walks for two-dimensional flows in [6], and in [7] for three-dimensional flows. Thus, in addition to being convected by the other vortices, each of the particles undergoes a random displacement at every time step, of mean 0 and variance $2\nu\Delta t$, in each direction. The diffusion of vorticity by random walks can be used in conjunction with algorithm (A) as well. In this viscous version of algorithm (A), the competing processes of stretching and diffusion may be studied. Vortex structures are modeled by several parallel filaments. As the fluid is stretched in the direction of the filaments, the vorticity grows and the vortices are split so that resolution is not lost in the physically interesting part of the fluid. The vortex structure itself spreads through the random dispersion of the pieces of the filaments. One of the authors ([15]) shall report elsewhere numerical studies of the fusing of two initially parallel vortex rings into one larger ring (as in [24]). Leonard has also studied this problem, using a variable-core filament method ([21]).

CHAPTER 5

In this chapter, we explain how the vortex method described earlier may be modified so that one can use different distributions of the initial points and still retain high order accuracy. For simplicity, we discuss only the two-dimensional problem.

Let $\{\alpha_j\}$ denote, as before, some set of initial vortex positions, all of which are assumed to lie within the support of the initial vorticity distribution. In place of formula (1.15), we consider, more generally, calculating the motion of the particles using velocity fields of the form

$$\tilde{u}(x,t)=\sum_j K_\delta(x-\tilde{x}_j(t))\omega_j p_j \ ,$$

where, as before, $\omega_j=\omega_0(\alpha_j)$ and $\tilde{x}_j(t)$ is the computed trajectory of the particle of fluid such that $\tilde{x}_j(0)=\alpha_j$. Whereas, in chapter 1, we had $p_j=h^2$ for all j, we will now allow the p_j to vary.

In order to see how the p_j should be chosen, we consider the error in approximating the actual fluid velocity u by the velocity field v, defined by

$$v(x,t)=\sum_j K_\delta(x-x_j(t))\omega_j p_j \ .$$

We have

$$\begin{aligned} |u(x,t)-v(x,t)| \le \; & |(K*\omega)(x,t)-(K_\delta*\omega)(x,t)| \\ & + |(K_\delta*\omega)(x,t)-\sum_j K_\delta(x-x_j(t))\omega_j p_j| \\ = \; & e_m+e_d, \end{aligned}$$

so that the consistency error is again a sum of moment and discretization errors.

The error e_m is the moment error we estimated in chapter 2; it does not depend on the choice of computational points. We will have a high order accurate vortex method provided the constants p_j are chosen so that e_d is suitably bounded. We note that, fixing x and t, and defining

$$F(\alpha)=K_\delta(x-x(\alpha,t))\omega_0(\alpha),$$

we have

$$\begin{aligned} e_d(x,t) &= \int_{R^2} K_\delta(x-x')\omega(x')dx' - \sum_j K_\delta(x-x_j(t))\omega_j p_j \\ &= \int_{R^2} K_\delta(x-x(\alpha,t))\omega_0(\alpha)d\alpha - \sum_j K_\delta(x-x(\alpha_j,t))\omega_0(\alpha_j)p_j \\ &= \int_{R^2} F(\alpha)d\alpha - \sum_j F(\alpha_j)p_j . \end{aligned} \tag{1}$$

Thus we see that α_j and p_j should be chosen to be the nodes and weights of an accurate integration formula.

Given a choice of points and weights, a convergence proof for the resulting vortex method follows closely the proof given in chapter 2. Among the changes which must be made in the proof, the discrete L^2-norm is redefined by setting

$$\| f \|_{0,h} = \left(\sum_j f(\alpha_j)^2 p_j\right)^{\frac{1}{2}}, \tag{2}$$

and an estimate for the error (1) replaces Lemma 2.2. A new Stability Lemma is also needed, since some of the details of the proof of stability depend upon the choice of the grid.

We consider now a particular example. Suppose we have a problem with radial symmetry. We seek radially symmetric integration points and weights such that the error e_d is small. Let $\bar{R}$ be sufficiently small so that $F(\alpha)=0$ whenever $|\alpha| \geq \bar{R}$. We will form an integration scheme by using one-dimensional formulas in the radial and in the azimuthal directions.

In the radial direction we divide $[0, \bar{R}]$ into intervals of length δr and use k-point Gaussian quadrature formulas over each interval. If we denote by r_j the endpoints of the intervals of length δr, then we have

$$\int_{R^2} F(\alpha)d\alpha = \int_0^{\bar{R}}\int_0^{2\pi} F(r,\vartheta)\, r\, d\vartheta dr$$

$$\cong \sum_{j=1}^{P}\sum_{i=1}^{k}\left[\int_0^{2\pi} F(r_j+g_i,\vartheta)d\vartheta\right](r_j+g_i)\, w_i \delta r$$

where P is the number of divisions of $[0, \bar{R}]$, and w_i and g_i are the Gaussian weights and nodes, respectively, for an interval of length δr. We approximate the integrals $\int_0^{2\pi} F(r_j+g_i,\vartheta)d\vartheta$ using the trapezoidal rule since, F being periodic in the ϑ direction, the accuracy of this integration procedure is of arbitrarily high order (see the Euler-MacLaurin formula in [1]). We shall take the corresponding partition of $[0,2\pi]$ to depend only on r_j. Then we may express our integration formula in the form

$$\sum_{j=1}^{P}\sum_{i=1}^{K}\sum_{l=1}^{T_j} F(r_j+g_i,\frac{2\pi l}{\delta\vartheta_j})\, \delta\vartheta_j\, (r_j+g_i)\, w_i\, \delta r \quad . \tag{3}$$

where $T_j \cdot \delta\vartheta_j = 2\pi$. We choose the $\delta\vartheta_j$ so that

$$\delta\vartheta_j \cong \delta r \qquad \text{if } r_j < 1 \tag{4}$$

and

$$r_j\, \delta\vartheta_j \cong \delta r \qquad \text{if } r_j > 1 \tag{5}$$

If α_i and p_i denote the computational points and weights of the above scheme, and if δr and the $\delta\vartheta_j$ satisfy (4)-(5), then by combining error estimates for the one-dimensional integration formulas used to make up (3), we obtain

$$\left| \int F(\alpha)d\alpha - \sum_j F(\alpha_j)p_j \right| \le Ch^{2k} \max_{\alpha \in R^2} \left| D^{2k} F(\alpha) \right| ,$$

where k is the number of points in Gaussinan integration formula and C depends only on k. Proceeding now as in the Discretization Lemma, we find that there is a constant C_k such that

$$\| e_d \|_{0,h} \leq C_k \left(\frac{h}{\delta}\right)^{2k} \delta^{-2} ,$$

and the consistency error is bounded by $C'(\delta^p + (\frac{h}{\delta})^{2k} \delta^{-2}))$.

With minor changes in the proof, a version of the Stability Lemma, can be shown to hold. From such a lemma and the above consistency estimate, one can prove convergence for the above scheme. Since the proof is nearly identical to the convergence proof given earlier, we do not present it here.

Acknowledgements

We would like to thank J.T. Beale, A. Chorin, O. Hald, and A. Majda for the very useful conversations that we have had with them.

REFERENCES

[1] K.E. Atkinson, An Introduction to Numerical Analysis, John Wiley & Sons, New York, 1978.

[2] J.T. Beale and A. Majda, Vortex Methods I : Convergence In Three Dimensions, Math. Comp., 39 (1982), pp. 1-27.

[3] J.T. Beale and A. Majda, Vortex Methods II : Higher Order Accuracy in Two and Three Dimensions, Math. Comp., 39 (1982), pp. 29-52.

[4] J.T. Beale and A. Majda, Explicit Smooth Velocity Kernels and Vortex Method Accuracy, Submitted to J. Comp. Phys..

[5] J.T. Beale and A. Majda, *private communication.*

[6] A.J. Chorin, Numerical Study of Slightly Viscous Flow, J. Fluid Mech., 57 (1973), pp. 785-796.

[7] A.J. Chorin, Vortex Models and Boundary Layer Instability, SIAM J. Sci. Stat. Comp., 1 (1980), pp. 1-21.

[8] A.J. Chorin, The Evolution of a Turbulent Vortex, Comm. Math. Phys., 83 (1982), pp. 517-533.

[9] A.J. Chorin, and Marsden, J., A Mathematical Introduction to Fluid Mechanics, Springer-Verlag, New York, 1979.

[10] G.H. Cottet, PhD. Thesis, l' Universite' Pierre et Marie Curie, -Paris 6-, France, (1982).

[11] G.H. Cottet, and P.A. Raviart, Particle Methods for one- dimensional Vlasov Poisson equations, *preprint.*

[12] S.C. Crow, Stability Theory for a Pair of Trailing Vortices, AIAA Journal, 8 (1970), pp. 2172-2179.

[13] H. Dym and H.P. McKean, Fourier Series and Integrals, Academic Press, New York, 1972.

[14] G.W. Gear, Numerical Initial Value Problems in Ordinary Differential Equations, Prentice-Hall, Englewood Cliffs, 1971.

[15] C. Greengard, Ph.D. Thesis, University of California at Berkeley, (1984)

[16] O. Hald, The Convergence of Vortex Methods, II., SIAM J. Num. Anal., 16 (1979), pp. 726-755.

[17] O. Hald, private communication

[18] J.K.Hale, Ordinary Differential Equations, Wiley Interscience, New York, 1969.

[19] F.R. Hama, Progressive Deformation of a Perturbed Line Vortex Filament, Phys. Fluids, 6 (1963), pp.526-534

[20] T. Kato, Nonstationary Flows of Viscous and Ideal Fluids in R^3, J. Funct. Anal., 9 (1972), pp.296-305.

[21] A. Leonard, Numerical Simulation of Interacting Three Dimensional Vortex Filaments, Proc. 4th Int. Conf. on Num. Methods in Fluid Dynamics., R.D. Richtmeyer, ed., Springer-Verlag, New York, 1975, p.245-250.

[22] F.J. McGrath, Nonstationary Plane Flow of Viscous and Ideal Fluids, Arch. Rational Mech. Anal., (1968), pp. 329-348.

[23] Y. Nakamura, A. Leonard, and P.R. Spalart, Numerical Simulation of Vortex Breakdown by the Vortex-Filament Method, AGARD Proceedings #342, Rotterdam, April 1983.

[24] Y. Oshima and S. Asaka, Interaction of two Vortex Rings Along Parallel Axes in Air", Jour. Phys. Soc. Japan, 42 (1977), pp. 708-713.

[25] M. Perlman, Ph.D. Thesis, University of California at Berkeley, (1983).

TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720