On wave propagation in a random micropolar generalized thermoelastic medium

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Abstract   This paper endeavours to study aspects of wave propagation in a random generalized-thermal micropolar elastic medium. The smooth perturbation technique conformable to stochastic differential equations has been employed. Six different types of waves propagate in the random medium. The dispersion equations have been derived. The effects due to random variations of micropolar elastic and generalized thermal parameters have been computed. Randomness causes change of phase speed and attenuation of waves. Attenuation coefficients for high frequency waves have been computed. Second moment properties have been briefly discussed with application to wave propagation in the random micropolar elastic medium. Integrals involving correlation functions have been transformed to radial forms. A special type of generalized thermo-mechanical auto-correlation functions has been used to approximately compute effects of random variations of parameters. Uncoupled problem has been briefly outlined.

Keywords: Waves; Random; Micropolar; Thermoelastic; Propagation

1   Introduction

A large number of research papers covering various branches of theoretical and computational micropolar elasticity are being reported in the literature

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every year in recent time. Eringen has published many pioneering papers and treatises on various aspects of micropolar elasticity. The coupled stress theory developed by Eringen comprises granular materials as also composite fibrous materials [1]. For this reason authors from different research fields have been taking interest in micropolar elasticity. The associated microrotational motions, spin, couple stress inertia, couple stress and distributed body couples were defined. Micropolar thermoelasticity also has become an important field of research these days. Eringen derived equations of motion, constitutive equations and boundary conditions for a class of micromorphic elastic solids whose microelements can undergo expansions or contractions defined as stretch [2]. He also defined and proposed a theory of thermomicrostretch fluids and bubbly liquids [3]. Microstretch and micropolar continua and all other aspects of micropolar studies can be found in Eringen's foundational treatise entitled *Microcontinuum Field Theories* [4]. Micropolar thermoelastcity and stretch have attracted the attention of many authors. Marin formulated some theorems on elastostatics of micropolar material with voids [5]. He investigated the behaviour of porous solids in which the matrix material is elastic and the interstices are voids of materials. In a more recent paper Marin [6] considered the concept of domain of influence in the context of displacement and microrotation fields along with the microstretch function. Marin and Lupu [7] on the other hand discussed the problem of harmonic vibrations of micropolar elastic materials under thermoelasticity. Marin and Marinescu [8] studied thermoelasticity of initially stressed bodies. Kumar [9] examined the problem of wave propagation in a micropolar viscoelastic medium under generalized thermoelasticity. Singh [10] studied plane wave propagation in a homogeneous transversely isotropic thermally conducting elastic solid with two relaxation times. Singh and Kumar [11] investigated the problem of reflection and refraction of plane waves at an interface between micropolar elastic solid and viscoelastic solid. Kumar and Deswal [12] discussed the problem of surface wave propagation in a micropolar thermoelastic medium. Kumar and Singh [13] studied effects of stretch in wave propagation in a micropolar material again under generalized thermoelasticity. Kumar and Tomar [14] studied aspects of reflected and refracted micropolar waves. Aouadi [15] considered a medium with a microstructure and derived general equations of motion and constitutive equations. On the other hand, Suiker, Borst and Chang [16] derived a second-gradient micro-polar constitutive theory on micro-mechanical modelling of granular medium. Majewski [17] dealt

with seismic rotation waves in a micropolar elastic earth. In fact varieties of papers on wave propagation in micropolar elastic, micropolar thermoelastic and coupled micropolar elastic media are being frequently reported in the literature. It is not found necessary to cite more of these recent publications simply because they fall exclusively to the non-random domain.

The present paper instead proposes to focus on the procedure to evaluate effects of random variation of parameters of the inhomogeneous micropolar medium on propagation of waves and associated phenomena. To date none other than a single paper by Mitra and Bhattacharyya [18] has appeared in the literature dwelling on randomness in relation to waves in micropolar medium.

The present paper therefore aims at investigating wave propagation phenomenon in a random generalized thermal micropolar elastic medium, and associated statistical properties such as second moment and its application. The parameters representing inhomogeneities of the coupled medium are assumed to vary slightly from their mean values. The originality of the paper lies in attempting to measure effects of random variations of parameters on wave propagation in the micropolar generalized thermoelastic medium following procedures of the authors' earlier paper [18].

The heat conduction equation and the coupling model have been chosen under the generalized thermoelasticity proposed by Lord and Shulman [19] and Green-Lindsay [20]. The details of L-S and G-L theories of generalized thermoelasticity can be found in Ignaczak and Starzewski [21] and need not to be repeated here. Generalized thermoelasticity attempts at nullifying the physical anomaly of classical thermoelasticity which insists that thermal speed assumes infinite speed at infinity ([21], p. xii). Consequently two relaxation parameters t_0, t_1, appear in the generalized model heat conduction equation.

The methodology adopted is the smooth perturbation technique enunciated by Keller [22]. Karal and Keller [23], Keller and Karal [24], Chow [25], Chen and Tien [26] and many others used it in studying elastic, electromagnetic, thermal and other waves. Bhattacharyya [27,28] and Bera [29] pursued the method in the study of wave propagation phenomena in the coupled media.

In fact the study of problems of wave propagation in random media flourished since the time of Chernov [30]. He first suggested adoption of an exponentially decaying form for a two-point dielectric correlation function. Beran and McCoy [31] proposed the technique of iterative perturbation in

studying mean field variations in the dielectric and other media. Beran, Frankenthal, Deshmukh and Whitman studied propagation of radiation in time-dependent three-dimensional random media [32]. Sobczyk studied elastic wave propagation in a discrete random medium [33]. He developed a general formalism of the analysis of coherent elastic waves in terms of scatters. In fact, Wenzel [34], Sobczyk, Wedrychowicz and Spencer [35], Frankenthal and Beran [36], Uscinski [37] and Frisch [38] have made extensive studies on various aspects of randomness in physical sciences, wave propagation phenomena and random characteristics of media. In particular, Chen and Soong [39] worked on covariance properties of waves propagating in a medium. A detailed discussion on properties and applications of random differential equations can be found, among others, in the treatise of Soong [40]. Akira Ishimaru discussed the development of the theory of wave propagation and scattering in random media in his landmark treatise on the subject [41]. Choudhury, Basu and Bhattacharyya discussed the phenomenon of wave propagation in a rotating random granular medium under generalized thermoelasticity [42]. On the basis of these citations one cannot but admit the importance of the study of randomness in applied mathematics and physical sciences. The present paper however proposes to discuss randomness in respect of waves in a micropolar medium impressed by a generalized thermoelastic field.

2   The problem

Using the smooth perturbation technique [22] the field equations have been put in the form

$$L(\vec{x}, t)V(\vec{x}, t) = F(\vec{x}, t) \ , \tag{1}$$

$$L = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 \ , \tag{2}$$

where: L – random linear operator, V – field vector, F – non-random source, L_0 – unperturbed part of L, L_1, L_2 – first and second order perturbation of L respectively, and ε – small parameter measuring the scale of random fluctuation of generalized thermo-micropolar elastic inhomogeneities of the medium.

Then employing iterative operations [23] it can be shown that the mean field $\langle V \rangle$ satisfies the equation

$$[L_0 + \varepsilon\langle L_1 \rangle + \varepsilon^2\{\langle L_2 \rangle + \langle L_1 \rangle L_0^{-1}\langle L_1' \rangle - \langle L_1 L_0^{-1} L_1' \rangle\}]\langle V \rangle = F \tag{3}$$

with

$$L_0 G_{ij}(\vec{x}, \vec{x}', t, t') = \delta(\vec{x}, \vec{x}')\delta(t, t')\delta_{ij} \ . \tag{4}$$

Here$\langle V(\vec{x}, t)\rangle$ is defined as the mean field quantity being the statistical average of the field vectors, an ensemble average. Also G_{ij} is the appropriate Green's matrix (computed in the infinite domain) for L_0, $\delta(\vec{x}, \vec{x'})$ is the Dirac delta function and δ_{ij} is the Kronecker delta. Equation (3) is evidently an integro-differential equation, as in terms of G_0, L_0^{-1} becomes

$$L_0^{-1} f = \int G(\vec{x}, \vec{x'})f(\vec{x'})d\vec{x'}.$$

The present analysis aims at investigating the effects of random inhomogeneities on the propagation of waves in an interacting generalized thermo-micropolar elastic medium. The equations of motion, constitutive equations and boundary conditions for the micropolar elastic field have been derived by Eringen [2]. The generalized thermal coupling is assumed to vary randomly only in the perturbed field. It may be recorded here that this assumption was necessitated to facilitate computation of the associated Green's matrix [18,25,42]. Therefore L_0 effectively represents the linear partial differential operator involving parameters of the unperturbed micropolar elastic medium only. Operator L acts on the displacement vector $\vec{u}$, the microrotation vector $\vec{\phi}$ and θ, the temperature. Six different types of body waves, which propagate in the non-random medium, depend upon the random inhomogeneities of the medium. The dispersion equations for these six longitudinal and transverse types of waves have been obtained. They involve terms up to the order of ε^2, since $\langle L_1 \rangle \neq 0$ in this case. However, finally the effect of randomness comes to alter the wave number to the order of ε^2 only. Theoretically therefore the effect of randomness comes to be small to the order of ε^2. Deviations in the propagation constants from their unperturbed values have been calculated in terms of $\delta_l, \delta_n, \delta_c, \delta_s$. for different types of waves defined in the text in Sec. 4. These quantities involve auto- and cross-correlation functions between micropolar elastic and generalized thermal field parameters. Assuming simple auto-correlation functions only for density and two thermo-mechanical coupling parameters in the forms [25,29]

$$R_{\rho\rho}(r) = \langle \rho_1^2 \rangle e^{-\frac{r}{a}} \ \text{ and } \ R_{mm}(r) = \langle \rho_1^2 \rangle e^{-\frac{r}{b}}, R_{mm^*}(r)e^{-\frac{r}{b_1}} \ ,$$

where $\langle \rho_1^2 \rangle$ is the mean square value of the perturbation density ρ_1, and a, b, b_1 are correlation lengths of inhomogeneities, and all other parameters

viz., λ, μ, α, etc., to be nonrandom, it has been shown that body waves for which the deviations are measured in terms of δ_c, δ_s, attenuate for high frequencies. It has been proved earlier [18] that second moments of the form $\langle \bar{\bar{v}}^T \bar{v} \rangle - \langle \bar{\bar{v}}^T \rangle \langle \bar{v} \rangle$ is a small quantity of the order of ε only but certainly larger than the order of ε^2, (v being the solution of the equation $Lv(\vec{x}) = 0$ referred to in Sec. 9). The same result applies in this case too. Components of associated Green's matrix which were computed earlier have been quoted in Appendix I as a ready reference. The various integrals *viz.*, a_{11}, b_{63}, etc. involving correlation functions between different micropolar, elastic and generalized thermal parameters have been reduced to radial forms; these are presented in Sec. 6. In all these cases, lengthy and cumbersome computations become unavoidable. Radial transformations ensure symmetrical propagation of radiations in all directions in the infinite medium.

Field equations for a micropolar elastic medium under the influence of a generalized thermal field are written explicitly. The displacement equation of motion in the randomly varying inhomogeneous micropolar elastic medium is written following Eringen [1] and Chow [25] as

$$
\begin{aligned}
&(\lambda + \mu)\vec{\nabla}(\vec{\nabla} \cdot \vec{u}) + (\mu + \kappa)\nabla^2 \vec{u} + \vec{\nabla}\lambda(\vec{\nabla} \cdot \vec{u}) + \vec{\nabla}\mu \times (\vec{\nabla} \times \vec{u}) \\
&+ (\vec{\nabla}(2\mu + \kappa) \cdot \vec{\nabla})\vec{u} + \vec{\nabla} \times (\kappa\vec{\phi}) - \vec{\nabla}[m\{\theta + t_1\dot{\theta}\}] + \vec{f} = \rho\ddot{\vec{u}} \,.
\end{aligned}
\tag{5}
$$

The microrotation equation of motion is represented by [1]

$$
\begin{aligned}
&\kappa(\vec{\nabla} \times \vec{u}) + (\alpha + \beta)\vec{\nabla}(\vec{\nabla} \cdot \vec{\phi}) + (\vec{\nabla}\alpha(\vec{\nabla} \cdot \vec{\phi})) + \gamma\nabla^2\vec{\phi} \\
&+ \vec{\nabla}\beta \times (\vec{\nabla} \times \vec{\phi}) + \{\vec{\nabla}(\beta + \gamma) \cdot \vec{\nabla}\}\vec{\phi} - 2\kappa\vec{\phi} + \vec{l} = \rho j\ddot{\vec{\phi}} \,.
\end{aligned}
\tag{6}
$$

Finally the generalized thermal equation is written following Ignaczak and Ostoja-Starzewski [21], Bera [29], and Singh [10] as

$$
\eta[\dot{\theta} + t_0\ddot{\theta}] = \vec{\nabla} \cdot [\nu\vec{\nabla}\theta] - \theta_0 m\vec{\nabla} \cdot [\dot{\vec{u}} + \delta_{lk}t_0\ddot{\vec{u}}] + q \,,
\tag{7}
$$

where $\dot{\theta} = \frac{\theta(\vec{x},t)}{\partial t}$, $\ddot{\theta} = \frac{\partial^2 \theta(\vec{x},t)}{\partial t^2}$, etc. Here

$\vec{u}(\vec{x}, t)$	–	displacement vector,
$\vec{\phi}(\vec{x}, t)$	–	microrotation vector,
$\theta(\vec{x}, t)$	–	temperature,
$m(\vec{x})$	–	thermomechanical coupling parameter,
$= (\lambda + \mu)\bar{\alpha}, \ \bar{\alpha}$	–	thermal expansion coefficient,
$(\alpha, \beta, \gamma, \kappa)(\vec{x})$	–	micropolar elastic moduli,
$(\lambda, \mu)(\vec{x}) = Lame'$	–	elastic parameters,
$\rho(\vec{x})$	–	mass density,
j	–	a nonrandom micropolar constant such that $j \geq 0$,
$\vec{f}(\vec{x}, t), .\vec{l}(\vec{x}, t), q(\vec{x}, t)$	–	body force, body couple per unit mass and heat source respectively,
$\eta(\vec{x}) = \rho c, \ c$	–	specific heat,
$\nu(\vec{x})$	–	thermal diffusivity,
$v =$	–	solution of the differential equation $Lv(\vec{x}) = 0$,
v_0	–	solution of the differential equation $Lv_0(\vec{x}) = 0$,
θ_0	–	constant reference temperature,
t_0	–	initial thermal relaxation time,
$t_1(\vec{x})$	–	thermal relaxation time, $t_1 \geq t_0 \geq 0$,
$\delta_{lk} = 1, t_1 = 0, k = 1$	–	for Lindsay-Shulman (L-S) model of generalized thermoelasticity,
$\delta_{lk} = 0, \ t_1 > 0, \ k = 2$	–	for Green-Lindsay (G-L) model of generalized thermoelasticity.

Let the differential operators $(M, N, R), (P, Q), (K, S)$ act respectively on $\vec{u}, \vec{\phi}$, and θ. Then field Eqs. (5)–(7) can be put in matrix form:

$$LV(\vec{x}, t) = \begin{bmatrix} M & P & K \\ N & Q & 0 \\ R & 0 & S \end{bmatrix} (\vec{x}, t) \begin{bmatrix} \vec{u}(\vec{x}, t) \\ \vec{\phi}(\vec{x}, t) \\ T(\vec{x}, t) \end{bmatrix} = F(\vec{x}, t) = \begin{bmatrix} \vec{f}(\vec{x}, t) \\ \vec{l}(\vec{x}, t) \\ q(\vec{x}, t) \end{bmatrix}. \quad (8)$$

Differential operators $(M, P, K, N, Q, R, S)(\vec{x}, t)$ are explicitly defined as

$$M\vec{u} = \rho \frac{\partial^2 \vec{u}}{\partial t^2} - (\lambda + \mu)\vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - (\mu + \kappa)\nabla^2 \vec{u} - \vec{\nabla}\lambda(\vec{\nabla} \cdot \vec{u})$$
$$-\vec{\nabla}\mu \times (\vec{\nabla} \times \vec{u}) - (\vec{\nabla}(2\mu + \kappa) \cdot \vec{\nabla})\vec{u},$$

$$P\vec{\phi} = -\vec{\nabla} \times (\kappa\vec{\phi}),$$

$$K\theta = \vec{\nabla}\left[m(\theta + t_1 \frac{\partial \theta}{\partial t})\right] = \vec{\nabla}(m\theta) + \vec{\nabla}(mt_1 \frac{\partial \theta}{\partial t}) = \left[\vec{\nabla}(m\times) + \vec{\nabla}(mt_1 \frac{\partial}{\partial t})\right]\theta,$$

$$N\vec{u} = -\kappa(\vec{\nabla} \times \vec{u}),$$

$$Q\vec{\phi} = \rho j \frac{\partial^2 \vec{\phi}}{\partial t^2} - (\alpha + \beta)\vec{\nabla}(\vec{\nabla} \cdot \vec{\phi}) - \vec{\nabla}\alpha(\vec{\nabla} \cdot \vec{\phi}) - \gamma\nabla^2\vec{\phi} - \vec{\nabla}\beta \times (\vec{\nabla} \times \vec{\phi})$$
$$- (\vec{\nabla}(\beta + \gamma) \cdot \vec{\nabla})\vec{\phi} + 2\kappa\vec{\phi} \,,$$

$$R\vec{u} = \theta_0 m \vec{\nabla} \cdot \left[\frac{\partial \vec{u}}{\partial t} + \delta_{lk} t_0 \frac{\partial^2 \vec{u}}{\partial t^2}\right] = \left[\theta_0 m \vec{\nabla} \cdot \left(\frac{\partial}{\partial t} + \delta_{lk} t_0 \frac{\partial^2}{\partial t^2}\right)\right]\vec{u} \,,$$

$$S\theta = \eta\left(\frac{\partial \theta}{\partial t} + t_0 \frac{\partial^2 \theta}{\partial t^2}\right) - \vec{\nabla} \cdot (\nu\nabla\theta) = \left[\eta\left(\frac{\partial}{\partial t} + t_0 \frac{\partial^2}{\partial t^2}\right) - \vec{\nabla} \cdot (\nu\vec{\nabla})\right]\theta \,.$$
$$(9)$$

Next let us assume

$$V(\vec{x}, t) = \begin{bmatrix} \vec{u}(\vec{x}) \\ \vec{\phi}(\vec{x}) \\ \theta(\vec{x}) \end{bmatrix} e^{-i\omega t} = V_0(\vec{x})e^{-i\omega t}, (say) \qquad (10)$$

and

$$F(\vec{x}, t) = \begin{bmatrix} \vec{f}_0(\vec{x}) \\ \vec{l}_0(\vec{x}) \\ q_0(\vec{x}) \end{bmatrix} e^{-i\omega t} = F_0(\vec{x})e^{-i\omega t} \,. \qquad (11)$$

Here ω is the frequency of propagating wave and $i = \sqrt{-1}$. The mean field Eq. (3), involving the Green's matrix, can be put in the form
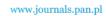
$$\left[L_0(\vec{x}) + \varepsilon\langle L_1(\vec{x})\rangle + \varepsilon^2 \left\{\langle L_2(\vec{x})\rangle + \langle L_1(\vec{x})\rangle L_0^{-1}(|\vec{x} - \vec{x}'|)\langle L_1(\vec{x}')\rangle \right.\right.$$
$$\left.\left. - \langle L_1(\vec{x}) L_0^{-1}(|\vec{x} - \vec{x}'|) L_1(\vec{x}')\rangle\right\}\right]\langle V_0(\vec{x}')\rangle = F_0 \,, \qquad (12)$$

where the associated Green's tensor is the kernel of the nonrandom operator equation represented by

$$L_0 G_{ij}(\vec{x}, \vec{x}') = \delta(\vec{x}, \vec{x}')\delta_{ij} \,. \qquad (13)$$

Then Eqs. (10) and (11) redefine operators $(M, P, K, N, Q, R, S)(\vec{x}, t)$ as

$M = -\rho\omega^2 - (\lambda + \mu)\vec{\nabla}(\vec{\nabla}\cdot) - (\mu + \kappa)\nabla^2 - \vec{\nabla}\lambda(\vec{\nabla}\cdot) - \vec{\nabla}\mu \times (\vec{\nabla}\times) - \left(\vec{\nabla}(2\mu + \kappa) \cdot \vec{\nabla}\right)$,

$P = -\vec{\nabla} \times (\kappa)$,

$K = \vec{\nabla}(m\times) - i\omega\vec{\nabla}(mt_1) = \vec{\nabla}(m\times) - i\omega\vec{\nabla}(m^\bullet), m^\bullet = mt_1(say)$,

$N = -\kappa(\vec{\nabla}\times)$

$$Q = -\rho j\omega^2 - (\alpha+\beta)\vec{\nabla}(\vec{\nabla}\cdot) - \vec{\nabla}\alpha(\vec{\nabla}\cdot) - \gamma\nabla^2 - \vec{\nabla}\beta\times(\vec{\nabla}\times) - \{\vec{\nabla}(\beta+\gamma)\cdot\vec{\nabla}\} + 2\kappa$$

$$R = -\omega\theta_0 m\vec{\nabla}\cdot\left[i + \delta_{lk}t_0\omega)\right] = -i\omega\theta_0\left[1 - i\delta_{lk}t_0\omega)\right]m(\vec{\nabla}\cdot),$$

$$S = -i\omega(1-it_0\omega)\eta - (\vec{\nabla}\cdot\left(\nu\nabla\right)) .$$

$$(14)$$

3 Solution

Let us assume plane wave propagation in the medium and set

$$\begin{bmatrix} \vec{u}(\vec{x}) \\ \vec{\phi}(\vec{x}) \\ \theta(\vec{x}) \end{bmatrix} = \begin{bmatrix} \vec{A} \\ \vec{B} \\ C \end{bmatrix} e^{i\vec{k}\cdot\vec{x}} \tag{15}$$

Here $\vec{k}$ indicates direction of the propagating wave and $\left|\vec{k}\right| = k$ is the wave number. Then the physical parameters $\lambda, \mu, \rho, \alpha, \beta, \gamma, \kappa, \eta, \nu, m$, and $m^\bullet = t_1 m$, are random functions of $\vec{x}(x, y, z)$, and since random deviations are small, we write, following, Karal and Keller [22] and Chow [25]:

$$(\lambda, \mu, \rho, \alpha, \beta, \gamma, \kappa, \eta, \nu, m, m^\bullet) = (\lambda_0, \mu_0, \rho_0, \alpha_0, \beta_0, \gamma_0, \kappa_0, \eta_0, \nu_0,$$
$$m_0 = 0, m_0^\bullet = 0) + \varepsilon(\lambda_1, \mu_1, \rho_1, \alpha_1, \beta_1, \gamma_1, \kappa_1, \eta_1, \nu_1, m_1, m_1^\bullet)(\vec{x}) , \tag{16}$$

where $(\lambda_0, ..., \nu_0)$ are constants and $(\lambda_1, ..., \nu_1, m_1, m^\bullet)$ are functions of $\vec{x}$ representing random fluctuations of the corresponding quantities such that

$$\langle\lambda_1, ..., \nu_1\rangle = 0 . \tag{17}$$

Conditions (16) and (17) ensure that fluctuations from nonrandom values of parameters remain small. It is further assumed that

$$m = \varepsilon m_1(\vec{x}), \text{ such that } \langle m_1(\vec{x})\rangle = m_2, (say), m_2 \neq 0 ,$$
$$\text{and } m^\bullet = \varepsilon m_1^\bullet(\vec{x}), \text{ such that } \langle m_1^\bullet(\vec{x})\rangle = m_3, (say), m_3 \neq 0 . \tag{18}$$

The last two assumptions confirm that the generalized thermal field is taken to be weakly random. A new parameter $m^\bullet(\vec{x}) = (t_1 m)(\vec{x})$, is introduced characterizing the random thermal coupling with the micropolar elastic medium. By the help of Eqs. (16)–(18) one gets

$$L_0 = \begin{bmatrix} M_0 & P_0 & 0 \\ N_0 & Q_0 & 0 \\ 0 & 0 & S_0 \end{bmatrix} , \tag{19}$$

$$L_1(\vec{x}) = \begin{bmatrix} M_1 & P_1 & K_1 \\ N_1 & Q_1 & 0 \\ R_1 & 0 & S_1 \end{bmatrix}(\vec{x}) \tag{20}$$

and

$$M_0 = -\rho_0\omega^2 - (\lambda_0 + \mu_0)\vec{\nabla}(\vec{\nabla}\cdot) - (\mu_0 + \kappa_0)\nabla^2 \,,$$
$$P_0 = -\kappa_0(\vec{\nabla}\times) = N_0 \,,$$
$$Q_0 = (2\kappa_0 - \rho_0 j\omega^2) - (\alpha_0 + \beta_0)\vec{\nabla}(\vec{\nabla}\cdot) - \gamma_0\nabla^2 \,,$$
$$S_0 = -i\omega(1 - it_0\omega)\eta_0 - \nu_0\nabla^2 \,. \tag{21}$$

The deterministic operators save S_0 are independent of generalized thermal relaxation parameters and the thermo-mechanical parameters. However the perturbed operators K_1, R_1, S_1, are characterized by $m_1, m_1^\bullet, t_0, t_1$, such that

$M_1 = -\rho_1\omega^2 - (\lambda_1 + \mu_1)\vec{\nabla}(\vec{\nabla}\cdot) - (\mu_1 + \kappa_1)\nabla^2 - \vec{\nabla}\lambda_1(\vec{\nabla}\cdot) - \vec{\nabla}\mu_1 \times (\vec{\nabla}\times) - \{\vec{\nabla}(2\mu_1 + \kappa_1)\cdot\vec{\nabla}\}$

$P_1 = -\vec{\nabla}\times(\kappa_1),$

$K_1 = \vec{\nabla}(m_1) - i\omega\vec{\nabla}m_1^\bullet \,,$

$N_1 = -\kappa_1(\vec{\nabla}\times) \,,$

$Q_1 = -\rho_1 j\omega^2 - (\alpha_1 + \beta_1)\vec{\nabla}(\vec{\nabla}\cdot) - \vec{\nabla}\alpha_1(\vec{\nabla}\cdot) - \gamma_1\nabla^2 - \vec{\nabla}\beta_1 \times (\vec{\nabla}\times) - (\vec{\nabla}(\beta_1 + \gamma_1)\cdot\vec{\nabla}) + 2\kappa_1 \,,$

$R_1 = -i\omega\theta_0(1 - i\delta_{lk}t_0\omega)m_1(\vec{\nabla}\cdot) \,,$

$S_1 = -i\omega(1 - it_0\omega)\eta_1 - \{\vec{\nabla}\cdot(\nu_1\nabla)\} \,. \tag{22}$

A computational note is given below:

$$S = -i\omega(1 - it_0\omega)\eta - (\vec{\nabla}\cdot(\nu\nabla)$$
$$= -i\omega(1 - it_0\omega)(\eta_0 + \varepsilon\eta_1) - \vec{\nabla}\cdot[(\nu_0 + \varepsilon\nu_1)\vec{\nabla}]$$
$$= -i\omega(1 - it_0\omega)\eta_0 - \vec{\nabla}\cdot(\nu_0\vec{\nabla}) - \varepsilon i\omega(1 - it_0\omega)\eta_1 - \varepsilon\vec{\nabla}\cdot(\nu_1\vec{\nabla})$$
$$= \{-i\omega(1 - it_0\omega)\eta_0 - \nu_0\nabla^2] + \{-\varepsilon i\omega(1 - it_0\omega)\eta_1\varepsilon\vec{\nabla}\cdot(\nu_1\vec{\nabla})\}$$
$$= S_0 + \varepsilon S_1] \,.$$

By virtue of Eqs. (17)–(18) one obtains

$$\langle L_1\rangle = \begin{bmatrix} 0 & 0 & (m_2 - i\omega m_3)\vec{\nabla} \\ 0 & 0 & 0 \\ -i\omega\theta_0(1 - i\delta_{lk}t_0\omega)m_2(\vec{\nabla}\cdot) & 0 & 0 \end{bmatrix} \neq 0, \tag{23}$$

where

$$m_2 = \langle m_1(\vec{x}) \rangle = \text{const.} \neq 0, m_3 = \langle m_1^{\bullet}(\vec{x}) \rangle = \text{const.} \neq 0 \;\; \text{and} \;\; \langle L_2 \rangle = 0 \,. \tag{24}$$

Thus Eq. (3) involves first and second perturbation terms; with vanishing body force, body couple and the heat source, Eq. (12) reduces to

$$\left[L_0(\vec{x}) + \varepsilon \langle L_1(\vec{x}) \rangle + \varepsilon^2 \Big\{ \langle L_1(\vec{x}) \rangle L_0^{-1}(|\vec{x} - \vec{x}'|) \langle L_1(\vec{x}') \rangle \right.$$
$$\left. - \langle L_1(\vec{x}) L_0^{-1}(|\vec{x} - \vec{x}'|) L_1(\vec{x}') \rangle \Big\} \right] \langle V_0(\vec{x}') \rangle = 0 \,. \tag{25}$$

The components of Green's tensor corresponding to L_0^{-1} are given by (say),

$$G_{ij} = \begin{bmatrix} G_0 & G_1 & 0 \\ G_2 & G_3 & 0 \\ 0 & 0 & G_4 \end{bmatrix}. \tag{26}$$

The components $G_0(r), G_1(r), G_2(r)$, and $G_3(r)$ have already been evaluated [18] and have been given in the Appendix I. The component $G_4(r)$ is defined by

$$\nabla^2 G_4(r) + \frac{i\omega(1 - i\omega t_0)\eta_0}{\nu_0} G_4(r) = \delta(\vec{x} - \vec{x}')\delta_{ij} \,.$$

Hence

$$G_4(r) = -\frac{e^{i\beta r}}{4\pi r} I_3 \,, \tag{27}$$

where

$$\beta = \left[\frac{i\omega(1 - i\omega t_0)\eta_0}{\nu_0} \right]^{\frac{1}{2}} \,,$$

and I_3 is the identity matrix.
Hence

$$\beta = \sqrt{\frac{\omega\eta_0}{2\nu_0}} \left\{ \left[\sqrt{\omega^2 t_0^2 - 1} + \omega t_0 \right]^{\frac{1}{2}} + i \left[\sqrt{\omega^2 t_0^2 - 1} - \omega t_0 \right]^{\frac{1}{2}} \right\}$$
$$= \beta_1 + i\beta_2 (say). \tag{28}$$

It is observed that $G_i(r), i = 0, 1, 2, 3$ are independent of thermo-mechanical coupling parameter $m(\vec{x})$ and generalized thermoelastic relaxation times t_0, t_1, and $m^{\bullet}(\vec{x})$. On the other hand $G_4(r)$ is independent of $m(\vec{x})$ but depends upon the generalized thermoelastic relaxation time t_0. Substituting (15) in the mean field Eq. (25) one gets the following three vector equations:

$$[\rho_0\omega^2\vec{A} - (\lambda_0 + \mu_0)(\vec{k}\cdot\vec{A})\vec{k} - (\mu_0 + \kappa_0)k^2\vec{A}] + i\kappa_0(\vec{k}\times\vec{B}) - \varepsilon(im_2 + \omega m_3)C\vec{k}$$

$$+\varepsilon^2\left[\omega\theta_0(1 - \omega t_0\delta_{lk})m_2(m_2 - i\omega m_3)(\vec{k}\cdot\vec{A})\right]\int\left[(\vec{\nabla}G_4)\right]e^{-i\vec{k}\cdot\vec{r}}d\vec{r}$$

$$-\varepsilon^2\int\left[\begin{array}{c}\langle[M_1\{G_0(M_1'\vec{A} + P_1'\vec{B} + K_1'C) + G_1(N_1'\vec{A} + Q_1'\vec{B})\}\\ +P_1\{G_2(M_1'\vec{A} + P_1'\vec{B} + K_1'C) + G_3(N_1'\vec{A} + Q_1'\vec{B})\}\\ +K_1G_4(R_1'\vec{A} + S_1'C)]\rangle\end{array}\right]e^{-i\vec{k}\cdot\vec{r}}d\vec{r} = 0,$$

$$(29)$$

$$i\kappa_0(\vec{k}\times\vec{A}) - [(2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2)\vec{B} + (\alpha_0 + \beta_0)(\vec{k}\cdot\vec{B})\vec{k}]$$

$$-\varepsilon^2\int\left[\begin{array}{c}\langle[N_1\{G_0(M_1'\vec{A} + P_1'\vec{B} + K_1'C) + G_1(N_1'\vec{A} + Q_1'\vec{B})\}\\ +Q_1\{G_2(M_1'\vec{A} + P_1'\vec{B} + K_1'C) + G_3(N_1'\vec{A} + Q_1'\vec{B})\}]\rangle\end{array}\right]e^{-i\vec{k}\cdot\vec{r}}d\vec{r} = 0,$$

$$(30)$$

and
$$[i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2]C - \varepsilon\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(\vec{k}\cdot\vec{A})$$

$$+\varepsilon^2[\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(m_2 - i\omega m_3)]C\int[(\vec{\nabla}\cdot G_0\vec{k})]e^{-i\vec{k}\cdot\vec{r}}d\vec{r}$$

$$-\varepsilon^2\int\left[\begin{array}{c}\langle[R_1\{G_0(M_1'\vec{A} + P_1'\vec{B} + K_1'C) + G_1(N_1'\vec{A} + Q_1'\vec{B})\}\\ +S_1G_4(R_1'\vec{A} + S_1'C)]\rangle\end{array}\right]e^{-i\vec{k}\cdot\vec{r}}d\vec{r} = 0.$$

$$(31)$$

Eliminating C from (29) and (30) with the help of (31) one gets the following two equations:

$$[\rho_0\omega^2\vec{A} - (\lambda_0 + \mu_0)(\vec{k}\cdot\vec{A})\vec{k} - (\mu_0 + \kappa_0)k^2\vec{A}] + i\kappa_0(\vec{k}\times\vec{B})$$

$$-\varepsilon^2\frac{\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)(\vec{k}\cdot\vec{A})}{i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2}\vec{k}$$

$$+\varepsilon^2\left[\omega\theta_0(1 - \omega t_0\delta_{lk})m_2(m_2 - i\omega m_3)(\vec{k}\cdot\vec{A})\right]\int\left[(\vec{\nabla}G_4)\right]e^{-i\vec{k}\cdot\vec{r}}d\vec{r}$$

$$-\varepsilon^2\int\left[\begin{array}{c}[\langle M_1G_0M_1'\rangle\vec{A} + \langle M_1G_0P_1'\rangle\vec{B} + \langle M_1G_1N_1'\rangle\vec{A}\\ +\langle M_1G_1Q_1'\rangle\vec{B} + \langle P_1G_2M_1'\rangle\vec{A} + \langle P_1G_2P_1'\rangle\vec{B}\\ +\langle P_1G_3N_1'\rangle\vec{A} + \langle P_1G_3Q_1'\rangle\vec{B} + \langle K_1G_4R_1'\rangle\vec{A}]\end{array}\right]e^{-i\vec{k}\cdot\vec{r}}d\vec{r} = 0, \quad (32)$$

and

$$i\kappa_0(\vec{k} \times \vec{A}) - [(2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2)\vec{B} + (\alpha_0 + \beta_0)(\vec{k} \cdot \vec{B})\vec{k}]$$

$$-\varepsilon^2 \int \left[\begin{array}{l} \langle N_1 G_0 M_1' \rangle \vec{A} + \langle N_1 G_0 P_1' \rangle \vec{B} + \langle N_1 G_1 N_1' \rangle \vec{A} \\ + \langle N_1 G_1 Q_1' \rangle \vec{B} + \langle Q_1 G_2 M_1' \rangle \vec{A} + \langle Q_1 G_2 P_1' \rangle \vec{B} \\ + \langle Q_1 G_3 N_1' \rangle \vec{A} + \langle Q_1 G_3 Q_1' \rangle \vec{B} \end{array} \right] e^{-i\vec{k}\cdot\vec{r}} d\vec{r} = 0 \,. \tag{33}$$

Some computations of the form

$$\langle M_1 G_o M_1' \rangle \vec{A} = e^{-i\vec{k}\cdot\vec{x}'} \langle M_1 G_o M_1' \rangle (\vec{A} e^{i\vec{k}\cdot\vec{x}'}) \,,$$

are shown in the Appendix II. The integral in the third term of (32)

$$= \int [\vec{\nabla}(G_4(r)]e^{-i\vec{k}\cdot\vec{r}} d\vec{r}$$

$$= -i\hat{k} \int_0^{\infty} e^{i\beta r} \sin kr dr, \beta = \beta_1 + i\beta_2, \beta_1 = Re\beta, \beta_2 = Im\beta. \tag{34}$$

$$= \frac{-i\hat{k}}{2} \left[\frac{\beta_1 + k}{\beta_2^2 + (\beta_1 + k)^2} - \frac{\beta_1 - k}{\beta_2^2 + (\beta_1 - k)^2} \right.$$

$$\left. + i\beta_2 \left(\frac{1}{\beta_2^2 + (\beta_1 - k)^2} - \frac{1}{\beta_2^2 + (\beta_1 + k)^2} \right) \right]. \tag{35}$$

Hence (32) can be rewritten as

$$[\rho_0\omega^2 \vec{A} - (\lambda_0 + \mu_0)(\vec{k} \cdot \vec{A})\vec{k} - (\mu_0 + \kappa_0)k^2\vec{A}] + i\kappa_0(\vec{k} \times \vec{B})$$
$$-\varepsilon^2 \frac{\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)(\vec{k} \cdot \vec{A})}{i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2}\vec{k}$$

$$-\varepsilon^2 \left[\omega\theta_0(1 - \omega t_0\delta_{lk})m_2(im_2 + \omega m_3)(\vec{k} \cdot \vec{A}) \; \widehat{\vec{k}} \right] \int_0^{\infty} e^{i\beta r} \sin kr dr$$

$$-\varepsilon^2 \int \left[\begin{array}{l} [\langle M_1 G_0 M_1' \rangle \vec{A} + \langle M_1 G_0 P_1' \rangle \vec{B} + \langle M_1 G_1 N_1' \rangle \vec{A} \\ + \langle M_1 G_1 Q_1' \rangle \vec{B} + \langle P_1 G_2 M_1' \rangle \vec{A} + \langle P_1 G_2 P_1' \rangle \vec{B} \\ + \langle P_1 G_3 N_1' \rangle \vec{A} + \langle P_1 G_3 Q_1' \rangle \vec{B} + \langle K_1 G_4 R_1' \rangle \vec{A}] \end{array} \right] e^{-i\vec{k}\cdot\vec{r}} d\vec{r} = 0. \tag{36}$$

Thus we are left with two vector Eqs. (36) and (33), determining the propagation of waves in the interacting random medium. It is to be noted that none of these equations include any term of the order of ε even though $\langle L_1 \rangle \langle V_0(\vec{x}) \rangle \neq 0$. Effects of randomness therefore is small to the order of ε^2.

4   Analysis of first perturbation dispersion equations

We get back to considering Eqs. (29)–(31) to ε-order terms only, assuming without loss of generality:

$$\vec{k} = (k, 0, 0), \quad \vec{A} = (A_1, A_2, 0), \quad \vec{B} = (B_1, B_2, B_3). \tag{37}$$

Hence (29)–(31) reduce to

$$[\rho_0\omega^2\vec{A} - (\lambda_0+\mu_0)(\vec{k}\cdot\vec{A})\vec{k} - (\mu_0+\kappa_0)k^2\vec{A}] + i\kappa_0(\vec{k}\times\vec{B}) - \varepsilon(im_2+\omega m_3)C\vec{k} = 0, \tag{38}$$

$$i\kappa_0(\vec{k}\times\vec{A}) - [(2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2)\vec{B} + (\alpha_0+\beta_0)(\vec{k}\cdot\vec{B})\vec{k}] = 0, \tag{39}$$

and

$$[i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2]C - \varepsilon\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(\vec{k}\,\vec{A}) = 0. \tag{40}$$

Eliminating C as before from Eq. (38) one gets

$$\begin{aligned}
&[\rho_0\omega^2\vec{A} - (\lambda_0+\mu_0)(\vec{k}\cdot\vec{A})\vec{k} - (\mu_0+\kappa_0)k^2\vec{A}] + i\kappa_0(\vec{k}\times\vec{B}) \\
&-\varepsilon^2(im_2+\omega m_3)\frac{\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(\vec{k}\cdot\vec{A})}{i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2}\vec{k} = 0.
\end{aligned} \tag{41}$$

Thus (39) and (41) represent two dispersion equations. Terms to the order ε however disappears from these equations. The equations almost reduce to two deterministic equations except for the presence of a second perturbation term in (41) inducing a weak thermoelastic influence on propagation of waves. These equations lead to the following six equations:

$$[\rho_0\omega^2 - (\lambda_0+2\mu_0+\kappa_0)k^2]A_1 - \varepsilon^2(im_2+\omega m_3)\frac{\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2 k^2}{i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2}A_1 = 0, \tag{42}$$

$$[\rho_0\omega^2 - (\mu_0+\kappa_0)k^2]A_2 + i\kappa_0 k B_3 = 0, \tag{43}$$

$$i\kappa_0 k B_2 = 0. $$

This last equation indicates

$$B_2 = 0. \tag{44}$$

This indicates that microrotation waves do not propagate along the direction of B_2. Also Eq. (39) gives three Eqs. (45), (46), and (47):

$$(2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2) + (\alpha_0+\beta_0)k^2]B_1 = 0 \tag{45}$$

and

$$(2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2)B_2 = 0.$$

This indicates that $B_2 = 0$, but

$$2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2 \neq 0 \,. \tag{46}$$

This confirms that there is no propagation along the direction of B_2. The third equation is

$$i\kappa_0 k A_2 - (2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2)B_3 = 0 \,. \tag{47}$$

The dispersion equation for longitudinal waves propagating along the direction of $\vec{k}$ is represented by (42), that is, by

$$\left[\rho_0\omega^2 - (\lambda_0 + 2\mu_0 + \kappa_0)k^2\right] - \varepsilon^2(im_2 + \omega m_3)\frac{\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2 k^2}{i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2} = 0 \,. \tag{48}$$

This is a fourth degree equation in the wave number k. From Eq. (48) effects of randomness of the generalized thermal field determined by the presence of m_2, and m_3 is discernible to terms to the order of ε^2.

Equation (45) determines the propagation of microrotation waves along the direction of B_1; the wave number is determined by

$$k^2 = \frac{\rho_0 j\omega^2 - 2\kappa_0}{\alpha_0 + \beta_0 + \gamma_0} \,. \tag{49}$$

Waves having amplitudes A_2 and B_3 are coupled by Eqs. (43) and (47). Eliminating A_2 and B_3 one gets the dispersion equation for the propagation of coupled waves in the form

$$\frac{A_2}{B_3} = -\frac{i\kappa_0 k}{\rho_0\omega^2 - (\mu_0 + \kappa_0)k^2} = \frac{2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2}{i\kappa_0 k} \,,$$

or

$$\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k^2\right]\left[2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2\right] = \kappa_0^2 k^2 \,. \tag{50}$$

This equation which is again a fourth degree equation in the wave number k however determines the dispersion equation of waves propagating in the non-thermal micropolar elastic medium. Equation (48) only includes one ε^2-order term which represents small generalized thermal effects. Equations (48), (49), and (50) can easily be solved numerically.

5   Analysis of dispersion equations

We proceed to analyze dispersion Eqs. (33) and (36). Each term of the form $\int \langle M_1 G_0 M_1' \rangle \vec{A} e^{i\vec{k}\cdot\vec{x}'}$ contains linear functions of the displacement amplitudes, A_1, A_2, A_3, and the microrotation amplitudes B_1, B_2, B_3. Each of the integrals involves correlation functions between different micropolar elastic and generalized thermal parameters including the generalized thermomechanical coupling parameters m and $m^\bullet$. It is observed that none of the integrands of the forms $\langle M_1 G_0 M_1' \rangle \vec{A} e^{i\vec{k}\cdot\vec{x}'}$ contains the perturbed operator S_1. Thus wave propagation in the coupled random medium is independent of the perturbation in the classical thermal parameters $\eta(\vec{x})$ and $\nu(\vec{x})$. Moreover the only integrand representing effects of random variation of generalized thermal field is $\langle K_1 G_4 R_1' \rangle \vec{A} e^{i\vec{k}\cdot\vec{x}'}$. The integrand involves the random coupling variable $m^\bullet$. In a latter section we will analyze (33) and (36) retaining the integral $\int \langle K_1 G_4 R_1' \rangle \vec{A} e^{i\vec{k}\cdot\vec{x}'} d\vec{x}'$ and making all other correlation functions vanish.

Now equations (36) and (33) give rise to six equations. Then terms of the order ε^2 in each of these equations are arranged as linear combinations of five amplitudes A_1, A_2, B_1, B_2, B_3, the coefficients of each of which being integrals involving different correlation functions. Rewriting one gets:

$$
\begin{aligned}
&\left[\rho_0 \omega^2 - (\lambda_0 + 2\mu_0 + \kappa_0)k^2\right]A_1 - \varepsilon^2 \frac{\omega\theta_0(1-i\omega t_0 \delta_{lk})m_2(im_2+\omega m_3)k^2}{i\omega(1-it_0\omega)\eta_0 - \nu_0 k^2}A_1 \\
&-\varepsilon^2 \omega\theta_0(1-i\omega t_0 \delta_{lk})m_2(im_2+\omega m_3)kA_1 \int_0^\infty e^{i\beta r}\sin kr \, dr \\
&-\varepsilon^2[a_{11}A_1 + a_{12}A_2 + b_{11}B_1 + b_{12}B_2 + b_{13}B_3] = 0,
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
&[\rho_0 \omega^2 - (\mu_0 + \kappa_0)k^2]A_2 - i\kappa_0 k B_3 \\
&\qquad -\varepsilon^2[a_{21}A_1 + a_{22}A_2 + b_{21}B_1 + b_{22}B_2 + b_{23}B_3] = 0,
\end{aligned}
\tag{52}
$$

$$
i\kappa_0 k B_2 - \varepsilon^2[a_{31}A_1 + a_{32}A_2 + b_{31}B_1 + b_{32}B_2 + b_{33}B_3] = 0,
\tag{53}
$$

$$
\begin{aligned}
&[\rho_0 j\omega^2 - 2\kappa_0 - (\alpha_0 + \beta_0 + \gamma_0)k^2]B_1 \\
&\qquad -\varepsilon^2[a_{41}A_1 + a_{42}A_2 + b_{41}B_1 + b_{42}B_2 + b_{43}B_3] = 0,
\end{aligned}
\tag{54}
$$

$$
(\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k^2)B_2 - \varepsilon^2[a_{51}A_1 + a_{52}A_2 + b_{51}B_1 + b_{52}B_2 + b_{53}B_3] = 0,
\tag{55}
$$

$$
\begin{aligned}
&i\kappa_0 k A_2 + (\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k^2)B_3 \\
&\qquad -\varepsilon^2[a_{61}A_1 + a_{62}A_2 + b_{61}B_1 + b_{62}B_2 + b_{63}B_3] = 0.
\end{aligned}
\tag{56}
$$

The integrands in the coefficients, a_{lj} and $b_{lk}(l = 1, 2, ..., 6; j = 1, 2; k = 1, 2, 3)$, are long expressions involving: (i) components of Green's tensor and their first and second order derivatives with respect to Cartesian coordinates, and (ii) various correlation functions (there are 49+2 in all, the last two being the auto-correlation function of the non-generalized thermo-mechanical coupling parameter and the cross-correlation function between the two thermo-mechanical coupling parameters) and their first and second order derivatives with respect to Cartesian coordinates.

6 Transformation to radial forms

The transformation of these integrals from Cartesian to radial forms are therefore carried out under the substitutions

$$\vec{r} = \vec{x} - \vec{x}' = (\xi, \eta, \zeta) = (r\cos\theta, \ r\sin\theta\cos\phi, \ r\sin\theta\sin\phi)$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} = -\frac{\partial}{\partial x'}, \quad \text{etc.} \tag{57}$$

The Green's tensor and the correlation functions are all functions of $r = |\vec{r}|$ alone. Thus 30 coefficients a_{lj} and b_{lk} are all functions of the wave number k alone. The expressions for $a_{11}, a_{22}, a_{62}, b_{23}, b_{51}, b_{63}$, which enter into the dispersion relations governing wave propagation in the medium are transformed into radial forms; the other coefficients which involve similar lengthy and cumbersome expressions are omitted. The expression for a_{11} dependent on thermal parameters is computed as

$$
\begin{aligned}
a_{11} = 4\pi k \int_0^\infty r^2 & \Big[\{kR_1 - \frac{1}{k}\omega^2(R_{\lambda\rho}+R_{\mu\rho})\}\{f'(G_1)_{11}'' + \frac{(G_1)_{11}'}{r}(f+f'')\} - R_1'\{f'''(G_1)_{11}'' \\
& - \frac{1}{r}(G_1)_{11}'(f'+f''')\} + \frac{1}{2}(R_{\lambda\lambda}'+R_{\mu\lambda}')\{G_1)_{11}'' + (G_1)_{33}'' - \frac{1}{r}((G_1)_{11}'+(G_1)_{33}')\}(f'+f''') \\
& + ((G_1)_{11}''+2(G_1)_{11}')[\{\frac{1}{k}\omega^2(R_{\mu\rho}+R_{\kappa\rho})-kR_2\}f+\frac{1}{k}R_2'f'] + (G_1)_{11}'[k\{(R_1'+R_2')f'' + \\
& R_2'(f+f'')\}-R_{\lambda\lambda}''f'''-\frac{1}{r}R_{\lambda\lambda}'(f'+f''')] + \frac{1}{2}[(R_{\lambda\lambda}''-\frac{1}{2}R_{\lambda\lambda}')((G_1)_{11}'+(G_1)_{33}')+(R_{\mu\lambda}''-\frac{1}{r}R_{\mu\lambda}') \\
& \{(G_1)_{33}'f'+(f'+f''')(G_1)_{11}\}] + \frac{1}{r}R_{\mu\lambda}'f'((G_1)_{11}'+(G_1)_{33}') + (R_2''-\frac{1}{r}R_2')(G_1)_{11}'(f'+f''') \\
& - \omega^2[\frac{1}{k}(G_1)_{11}'\{f''(R_{\lambda\rho}'+2R_{\mu\rho}'+R_{\kappa\rho}')+(R_{\mu\rho}+R_{k\rho})(f+f'')\}-(G_1)_{11}\{\frac{1}{k}f\omega^2R_{\rho\rho} \\
& - kf(R_{\rho\lambda}+2R_{\rho\mu}+R_{\rho\kappa}) + f'(R_{\rho\lambda}'+2R_{\rho\mu}'+R_{\rho\kappa}')\}]\Big] \, dr \\
& - 4i\pi k\omega\theta_0(1-i\delta_{lk}t_0\omega k)\int_0^\infty[(R_{mm}-i\omega R_{mm\bullet})G_4'+(R_{mm}'-i\omega R_{mm\bullet}')G_4]r^2f'dr.
\end{aligned}
\tag{58}
$$

where

$$
\begin{aligned}
R_1(r) &= \langle (\lambda_1+\mu_1)(\lambda_1'+2\mu_1'+\kappa_1') \rangle, \\
R_2(r) &= \langle (\mu_1+\kappa_1)(\lambda_1'+2\mu_1'+\kappa_1') \rangle,
\end{aligned}
$$

$$
R_{\lambda\rho}(r) = \langle \lambda_1\rho_1'\rangle, R_{mm}(r) = \langle mm'\rangle, R_{mm\bullet}(r) = \langle mm^\bullet{}'\rangle, \text{etc.}
$$

$$
\begin{aligned}
\lambda_1 &= \lambda_1(\vec{x}), \lambda_1' = \lambda_1'(\vec{x}'), \\
m_1 &= m_1(\vec{x}), \ m_1^\bullet = m_1 t = m_1^\bullet(\vec{x}), \\
m_1' &= m_1(\vec{x}'), \ m_1^{\bullet}{}' = m_1 t = m_1^\bullet(\vec{x}'), \text{etc.}
\end{aligned}
\tag{59}
$$

and

$$
f \ = \ \frac{\sin kr}{kr}, \ f' \ = \ \frac{df}{d(kr)}
$$

$$
(G_1)_{11}' \ = \ \frac{d(G_1)_{11}}{dr} \ , \quad R_1' \ = \ \frac{dR_1}{dr}, \ etc.
\tag{60}
$$

$$
R_7(r) \ = \ \left\langle (\alpha_1+\beta_1)(2\kappa_1'-j\omega^2\rho_1') \right\rangle.
$$

The radial expressions for coefficients $a_{22}, a_{62}, b_{23}, b_{51}, b_{63}$, which are independent of thermal field can be found in [18]. However the expression for b_{63} is reproduced illustratively as also as a ready reference for the analysis given below.

$$b_{63} = 4\pi k \int_0^\infty r^2 [\{R_{\kappa\kappa} f' + \frac{1}{k} R'_{\kappa\kappa} f'' + \frac{1}{2k} R'_{\kappa\kappa}(f + f')\}(G_1)'_{11} - \frac{1}{2k}\{k^2(R_{\alpha\gamma} + R_{\beta\gamma}) + R_7\}$$

$$\{\frac{2}{r} f(G_4)'_{33} + (f + f'')(\frac{1}{r}(G_4)'_{33} - (G_4)''_{33})\} + \frac{1}{2}(R'_{\alpha\beta} + R'_{\beta\beta})((G_4)''_{11} - (G_4)'_{11})$$

$$(f' + f'') + (R'_{\alpha\gamma} + R'_{\beta\gamma})\{\frac{1}{r} f'(G_4)'_{33} + \frac{1}{2}(f' + f''')((G_4)''_{33} - \frac{1}{r}(G_4)'_{33})\} + \frac{1}{2k}(k^2 R'_{\alpha\gamma}$$

$$+ 2R'_{\alpha\kappa} - j\omega^2 R_{\alpha\rho})(f + f'')(G_4)'_{33} + \{\frac{1}{2}(R''_{\alpha\beta} - R'_{\alpha\beta})(f' + f''') + R'_{\alpha\beta} f'\}(G_4)'_{11}$$

$$+ \frac{1}{2}(R''_8 - R'_8)(f' + f''')(G_4)'_{33} + \{R'_{\gamma\gamma} f' - \frac{1}{k}(R_{\gamma\gamma} k^2 + 2R_{\gamma\kappa} - j\omega^2 R_{\gamma\rho})f\}\{(G_4)''_{33} + \frac{2}{r}(G_4)'_{33}\}$$

$$+ \frac{1}{2k}(G_4)'_{33}[f''\{R_9 + k^2(R'_{\beta\gamma} + 4R'_{\gamma\gamma})\} + \{k^2(R'_{\beta\gamma} + 2R'_{\gamma\gamma}) + R_{10}\}f] + \frac{1}{2}(R''_{\beta\beta} - \frac{1}{r} R'_{\beta\beta})$$

$$(G_4)'_{11}(f' + f''') + (G_4)'_{33}[R''_{\gamma\gamma} f + \frac{1}{2}(f' + f''')(R''_{\beta\gamma} - \frac{1}{r} R'_{\beta\gamma})] + [k(2R_{\kappa\gamma} - j\omega^2 R_{\beta\gamma}) + \frac{1}{k} R_{11}]$$

$$\times f(G_4)_{33} - (G_4)'_{33} f'(2R'_{\kappa\beta} - j\omega^2 R'_{\rho\beta})]dr \tag{61}$$

where

$$R_7(r) = \langle(\alpha_1 + \beta_1)(2\kappa'_1 - j\omega^2\rho'_1)\rangle,$$
$$R_8(r) = \langle(\alpha_1 + \beta_1 + \gamma_1)\gamma'_1\rangle,$$
$$R_9(r) = \langle(\beta_1 + 4\gamma_1)(2\kappa'_1 - j\omega^2\rho'_1)\rangle,$$
$$R_{10}(r) = \langle(\beta_1 + 2\gamma_1)(2\kappa'_1 - j\omega^2\rho'_1)\rangle,$$
$$R_{11}(r) = \langle(2\kappa_1 - j\omega^2\rho_1)(2\kappa'_1 - j\omega^2\rho'_1)\rangle =$$
$$= 4\langle\kappa_1\kappa'_1\rangle - 2j\omega^2[\langle\rho_1\kappa'_1\rangle + \langle\rho'_1\kappa'_1\rangle] + j^2\omega^4\langle\rho_1\rho'_1\rangle. \tag{62}$$

7   Analysis of equations (51)–(56)

(i) Let

$$k^2 = k_l^2 + \varepsilon^2\delta_l , \tag{63}$$

where

$$k_l^2 = \frac{\rho_0\omega^2}{\alpha_0 + \beta_0 + \gamma_0} \tag{64}$$

be taken as a solution of Eqs. (51)–(56). Substituting in (54) one gets the equation

$$\{-[2\kappa_0 - \rho_0 j\omega^2 + (\alpha_0 + \beta_0 + \gamma_0)]k_l^2\}B_1$$
$$-\varepsilon^2\{[a_{41}A_1 + a_{42}A_2 + [b_{41} + (\alpha_0 + \beta_0 + \gamma_0)\delta_l]B_1 +$$
$$b_{42}B_2 + b_{43}B_3]_{k=k_l}\} = 0.$$

Hence

$$B_1 \approx O(\varepsilon^2) \ . \tag{65}$$

Similarly it can be shown from Eq. (57) that

$$B_2 \approx O(\varepsilon^2) \ . \tag{66}$$

From (52) one gets

$$\begin{aligned}
&\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_l^2\right]A_2 - i\kappa_0 k_l B_3 \\
&-\varepsilon^2\big[a_{21}A_1 + \{a_{22} + (\mu_0 + \kappa_0)\delta_l\}A_2 + b_{21}B_1 + b_{22}B_2 + \\
&\{b_{23} - \tfrac{i\kappa_0}{2k_l}\}B_3\big]_{k=k_l} = 0 \ .
\end{aligned}$$

Similarly from (56) one gets

$$\begin{aligned}
&i\kappa_0 k_l A_2 + (\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_l^2)B_3 \\
&-\varepsilon^2\big[a_{61}A_1 + (a_{62} + \tfrac{i\kappa_0}{2k_l})A_2 + b_{61}B_1 + b_{62}B_2 + (\gamma_0\delta_l + b_{63})B_3\big]_{k=k_l} = 0.
\end{aligned}$$

Thus from these two equations it can be concluded that for finite A_1

$$A_2 \approx O(\varepsilon^2) \ \text{ and } \ B_3 \approx O(\varepsilon^2) \ . \tag{67}$$

Lastly the first Eq. (51) becomes, on neglecting A_2, B_1, B_2, and B_3, as these amplitudes are small to the order ε^2 only:

$$\begin{aligned}
&\left[\rho_0\omega^2 - (\lambda_0 + 2\mu_0 + \kappa_0)(k_l^2 + \varepsilon^2\delta_l)\right]A_1 \\
&-\varepsilon^2 \frac{\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)k_l^2}{i\omega(1 - it_0\omega)\eta_0 - \nu_0 k_l^2}A_1 \\
&-\varepsilon^2\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)k_l A_1 \int_0^\infty e^{i\beta r}\sin k_l r \, dr \\
&-\varepsilon^2[a_{11}]_{k=k_l}A_1 = 0.
\end{aligned}$$

Rewriting, one obtains, assuming A_1 to be finite,

$$-(\lambda_0 + 2\mu_0 + \kappa_0)\varepsilon^2\delta_l A_1 + \varepsilon^2 D_3 A_1 + \varepsilon^2 D_4 A_1 - \varepsilon^2 a_{11}(k_l)A_1 = 0 \ ,$$

where

$$D_3(k_l) = -\frac{\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)k^2}{i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2} \ ,$$

$$D_4(k_l) = -\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)k_l \int_0^\infty e^{i\beta r}\sin k_l r \, dr \ .$$

Hence

$$\delta_l = \frac{D_3(k_l) + D_4(k_l) - a_{11}(k_l)}{\lambda_0 + 2\mu_0 + \kappa_0} \ . \tag{68}$$

where

$$D_3(k_l) = -\frac{\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)k^2}{i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2}$$

$$= \frac{-\omega\theta_0 m_2^2 k^2(1 - it_0\omega)\big[(\omega\eta_0 - \nu_0 k^2) + it_0\omega\eta_0\big]}{(\omega\eta_0 - \nu_0 k^2)^2 + (t_0\omega\eta_0)^2}, t_1 = 0, \delta_{lk} = 1, (L - S),$$

$$= \frac{-\omega\theta_0 m_2 k^2\big[(m_2 - i\omega m_3)(\omega\eta_0 - \nu_0 k^2) + it_0\omega\eta_0\big]}{(\omega\eta_0 - \nu_0 k^2)^2 + (t_0\omega\eta_0)^2}, \delta_{lk} = 0, t_1 > 1, (G - L). \tag{69}$$

and

$$D_4(k_l) = -\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)k_l \int_0^\infty e^{i\beta r} \sin k_l r \, dr$$
$$= -\omega\theta_0 m_2^2 k_l[t_0\omega + i] \int_0^\infty e^{i\beta r} \sin k_l r \, dr, \ t_1 = 0, \delta_{lk} = 1, (L - S),$$
$$= -\omega\theta_0 m_2 k_l[\omega m_3 + im_2] \int_0^\infty e^{i\beta r} \sin k_l r \, dr, \ t_1 > 0, \delta_{lk} = 0, (G - L). \tag{70}$$

Thus Eq. (68) determines δ_l given by (63). The integral in D_4 can be easily evaluated. The change in wave number δ_l which is small therefore depends upon the random variation of the generalized thermal field and measured in terms of statistical means, m_2, m_3, for both L-S and G-L theories. For L-S, the change depends upon m_2 only, and for G-L upon m_2, m_3, (for, L-S, $t_1 = 0, \delta_{lk} = 1, k = 1$, and for G-L, $t_1 > 0, \delta_{lk} = 0, k = 2$. Generalized thermal relaxation times t_0 and t_1 satisfy ([10,21,29,42]) the inequalities $t_1 \geq t_0 \geq 0$).

Thus it is observed that for waves for which the wave number k satisfies Eqs. (63)–(64), the amplitude A_1 is important in comparison to other amplitudes A_2, B_1, B_2, and B_3. Therefore Eq. (63), which represents longitudinal type of displacement waves, is modified due to random fluctuation of inhomogeneities of the medium such that the wave number is increased by $\varepsilon^2 \frac{\delta_l}{2k_l}$.

(ii) Next let us consider waves for which

$$k^2 = k_n^2 + \varepsilon^2\delta_n, \quad \text{where} \ k_n^2 = \frac{\rho_0 j\omega^2 - 2\kappa_0}{\alpha_0 + \beta_0 + \gamma_0} \ . \tag{71}$$

From (51) one gets

$$[\rho_0\omega^2 - (\lambda_0 + 2\mu_0 + \kappa_0)k_n^2]A_1 - \varepsilon^2(\lambda_0 + 2\mu_0 + \kappa_0)\delta_n A_1$$

$$-\varepsilon^2\frac{\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)k_n^2}{i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2}A_1$$
$$-\varepsilon^2\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(im_2 + \omega m_3)k_n A_1 \int_0^\infty e^{i\beta r}\sin k_n r dr$$
$$-\varepsilon^2[a_{11}A_1 + a_{12}A_2 + b_{11}B_1 + b_{12}B_2 + b_{13}B_3]_{k=k_n} = 0 .$$

This shows that

$$A_1 \approx O(\varepsilon^2) . \tag{72}$$

From (53) it follows that

$$\left[i\kappa_0 + \frac{\varepsilon^2\delta_n}{2k_n^2}i\kappa_0\right]k_n B_2 - \varepsilon^2[a_{31}A_1 + a_{32}A_2 + b_{31}B_1 + b_{32}B_2 + b_{33}B_3] = 0.$$

This shows that

$$B_2 \approx O(\varepsilon^2) . \tag{73}$$

Also from (52) and (56) it can be shown that for finite B_1,

$$[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_n^2]A_2 - i\kappa_0 k_n B_3 - \varepsilon^2\delta_n(\mu_0 + \kappa_0)A_2 - \frac{\varepsilon^2\delta_n}{2k_n^2}i\kappa_0 k_n B_3$$
$$-\varepsilon^2[a_{21}A_1 + a_{22}A_2 + b_{21}B_1 + b_{22}B_2 + b_{23}B_3] = 0 \tag{73a}$$

and

$$i\kappa_0 k_n A_2 + (\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_n^2)B_3 + \frac{\varepsilon^2\delta_n}{2k_n^2}i\kappa_0 k_n A_2 - \varepsilon^2\delta_n\gamma_0 B_3$$
$$-\varepsilon^2 \int [a_{61}A_1 + a_{62}A_2 + b_{61}B_1 + b_{62}B_2 + b_{63}B_3] = 0 , \tag{73b}$$

respectively. From these two Eqs. (73a) and (73b) it is clear that

$$A_2 \approx O(\varepsilon^2) \ \text{ and } \ B_3 \approx O(\varepsilon^2) . \tag{74}$$

Equation (54) however, determines δ_n, since

$$-\varepsilon^2\delta_n(\alpha_0 + \beta_0 + \gamma_0)B_1$$

$$-\varepsilon^2 \int [a_{41}A_1 + a_{42}A_2 + b_{41}B_1 + b_{42}B_2 + b_{43}B_3] = 0$$

and $(A_1, A_2, B_2, B_3) \approx O(\varepsilon^2)$. Hence, if B_1 is finite, then

$$\delta_n = \frac{-b_{41}}{\alpha_0 + \beta_0 + \gamma_0} . \tag{75}$$

Therefore, $\delta_n = 0$ if $b_{41} = 0$ and amplitude B_1 is finite.

It is observed that for the waves represented by (71), the amplitude B_1 is important in comparison to the amplitudes A_1, A_2, B_2, and B_3. These are amplitudes of longitudinal type microrotation waves propagating in the random generalized thermoelastic micropolar field but are scarcely affected by the random inhomogeneities of the medium.

(iii) Next let us examine the propagation of elastic waves for which

$$k^2 = k_c^2 + \varepsilon^2 \delta_c, \quad k_c^2 = \frac{\rho_0 \omega^2}{\lambda_0 + 2\mu_0}, \tag{76}$$

and

$$k^2 = k_s^2 + \varepsilon^2 \delta_s, k_s^2 = \frac{\rho_0 \omega^2}{\mu_0} . \tag{77}$$

Substituting (76) in (51) one observes

$$[-\kappa_0 \rho_0 \omega^2 - \varepsilon^2 (\lambda_0 + 2\mu_0 + \kappa_0)(\lambda_0 + 2\mu_0)\delta_c]A_1$$

$$-\varepsilon^2 \frac{\omega \theta_0 (1 - i\omega t_0 \delta_{lk}) m_2 (im_2 + \omega m_3) k_c^2}{i\omega(1 - it_0 \omega)\eta_0 - \nu_o k_c^2}(\lambda_0 + 2\mu_0)A_1$$
$$-\varepsilon^2 \omega \theta_0 (1 - i\omega t_0 \delta_{lk}) m_2 (im_2 + \omega m_3)(\lambda_0 + 2\mu_0) k_c A_1 \int_0^\infty e^{i\beta r} \sin kr dr$$

$$-\varepsilon^2 (a_{11} A_1 + a_{12} A_2 + b_{11} B_1 + b_{12} B_2 + b_{13} B_3)(\lambda_0 + 2\mu_0) = 0 .$$

Hence

$$A_1 \approx O(\varepsilon^2) . \tag{78a}$$

Substituting (76) in (55) it can be similarly shown that

$$B_2 \approx O(\varepsilon^2) . \tag{78b}$$

From (54) one can similarly conclude that

$$B_1 \approx O(\varepsilon^2) . \tag{79}$$

Two equations (52) and (56) are coupled with the transverse amplitudes A_2 and B_3 such that

$$[\rho_0 \omega^2 - (\mu_0 + \kappa_0)k^2]A_2 - i\kappa_0 k B_3 - \varepsilon^2 (b_{22} B_2 + b_{23} B_3) = 0$$

and

$$i\kappa_0 k A_2 + (\rho_0 j \omega^2 - 2\kappa_0 - \gamma_0 k^2)B_3 - \varepsilon^2 (b_{62} B_2 + b_{63} B_3) = 0 .$$

Eliminating A_2 and B_3 one obtains the dispersion equation for two sets of transverse type of displacement and microrotation waves propagating in the uncoupled micropolar elastic medium and independent of thermal field, as no thermal parameter is involved in any of the integrals b_{22}, b_{23}, b_{62}, and b_{63}, given by

$$\begin{aligned}
&\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k^2\right](\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k^2) - \kappa_0^2 k^2 \\
&-\varepsilon^2\bigg[a_{22}(\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k^2) + b_{63}\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k^2\right] \\
&+i\kappa_0 k(a_{62} - b_{23})\bigg] = 0.
\end{aligned} \tag{80}$$

This equation involves integrals a_{22}, a_{62}, b_{23}, and b_{63} each involving correlation functions between non-thermal medium-parameters only. Substituting (76) in (80) and neglecting terms to the order ε^2, the deviation δ_c can be computed from:

$$\begin{aligned}
&\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_c^2\right](\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_c^2) - \kappa_0^2 k_c^2 \\
&-\varepsilon^2\gamma_0\delta_c\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_c^2\right] - \varepsilon^2(\mu_0 + \kappa_0)\delta_c(\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_c^2) - \varepsilon^2\kappa_0^2\delta_c \\
&-\varepsilon^2\bigg[a_{22}(\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_c^2) + b_{63}\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_c^2\right] \\
&+i\kappa_0(a_{62} - b_{23})k_c\bigg] = 0 \ .
\end{aligned}$$

If $\varepsilon^2 = 0$, then we know

$$\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_c^2\right](\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_c^2) - \kappa_0^2 k_c^2 = 0 \ .$$

Hence δ_c and similarly δ_s are obtained in the forms:

$$\delta_c \approx -\frac{b_{63}\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_c^2\right] + a_{22}(\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_c^2) + (a_{62} - b_{23})i\kappa_0 k_c}{\gamma_0\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_c^2\right] + (\mu_0 + \kappa_0)(\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_c^2) + \kappa_0^2}, \tag{81}$$

and

$$\delta_s \approx -\frac{b_{63}\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_s^2\right] + a_{22}(\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_s^2) + (a_{62} - b_{23})i\kappa_0 k_s}{\gamma_0\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k_s^2\right] + (\mu_0 + \kappa_0)(\rho_0 j\omega^2 - 2\kappa_0 - \gamma_0 k_s^2) + \kappa_0^2} \ . \tag{82}$$

Both δ_c and δ_s are complex quantities. Computing real and imaginary parts one can write:

$$k = k_c + \frac{\varepsilon^2\delta_c}{2k_c} = k_c + \varepsilon^2\frac{\text{Re}\delta_c}{2k_c} + i\varepsilon^2\frac{\text{Im}\delta_c}{2k_c} \ , \tag{83}$$

and

$$k = k_s + \frac{\varepsilon^2 \delta_s}{2k_s} = k_c + \varepsilon^2 \frac{\mathrm{Re}\delta_s}{2k_s} + i\varepsilon^2 \frac{\mathrm{Im}\delta_s}{2k_s} \; . \tag{84}$$

Hence for longitudinal waves

$$i\vec{k}\,\vec{x} = ikx = ix\left[k_c + \frac{\varepsilon^2 \delta_c}{2k_c}\right] = ix\left[k_c + \varepsilon^2 \frac{\mathrm{Re}\delta_c}{2k_c} + i\varepsilon^2 \frac{\mathrm{Im}\delta_c}{2k_c}\right] \; ,$$

and for transverse waves

$$i\vec{k}\,\vec{x} = ikx = ix\left[k_s + \frac{\varepsilon^2 \delta_s}{2k_s}\right] = ix\left[k_s + \varepsilon^2 \frac{\mathrm{Re}\delta_s}{2k_s} + i\varepsilon^2 \frac{\mathrm{Im}\delta_s}{2k_s}\right] \; .$$

Thus change in the wave number and attenuation occurs in both cases, having attenuation factors of the forms

$$\exp\left[-\frac{\varepsilon^2 x}{2}\mathrm{Im}\left(\frac{\delta_c}{k_c}, \frac{\delta_s}{k_s}\right)\right]. \tag{85}$$

It may be recalled here that the only term in Eq. (36) that might contribute to some effect of thermal parameters on $\delta_{c,s}$ is

$$\int \langle K_1 G_4 R_1'\rangle(\vec{A}e^{i\vec{k}\cdot\vec{x}'})d\vec{x}'$$

$$= -\omega\theta_0(1 - i\delta_{lk}t_0\omega)(\vec{k}\,\vec{A})e^{i\vec{k}\,\vec{x}}\int\{\vec{\nabla}[G_4(r)R_{mm}(r)]$$

$$-i\omega\vec{\nabla}[G_4(r)R_{mm\bullet}]\}e^{-i\vec{k}'\cdot\vec{r}}d\vec{r} \; .$$

This term clearly does not add anything to a_{22} since in our case $\vec{k}\,\vec{A} = (kA_1, 0, 0)$. Therefore the integrals a_{22}, a_{62}, b_{23} and b_{63} are all independent of the thermal field. However the integral $\int\langle K_1 G_4 R_1'\rangle(\vec{A}e^{i\vec{k}\cdot\vec{x}'})d\vec{x}'$ contributes an additional term to a_{11} which is

$$-4i\pi\omega\theta_0(1-i\delta_{lk}t_0\omega)k\int_0^\infty [(R_{mm}-i\omega R_{mm\bullet})G_4'+(R_{mm}'-i\omega R_{mm\bullet}')G_4]r^2 f'dr \; .$$

This term has already been included in the radial expression for a_{11} in (58).

8   Attenuation of high frequency nonthermal waves in a particular case

This topic has already been discussed in [18]. However for the sake of continuity of discussion, the salient features are briefly enumerated below.

It may be noted that the expressions derived for δ_c and δ_s do not depend on generalized thermal parameters. Hence the phenomenon of propagation of transverse type of displacement and microrotation waves in the random micropolar elastic medium is considered taking

$$R_{\rho\rho}(r) = \langle \rho_1^2 \rangle e^{\frac{-r}{a}} \neq 0, a\rangle 0 \ , \tag{86}$$

and making all other auto- and cross-correlation functions vanish. (Auto-correlation functions $\langle \kappa_1 \kappa_1' \rangle, \langle \beta_1 \beta_1' \rangle, \langle \gamma_1 \gamma_1' \rangle$, signifying effects of random variation of micropolar elastic properties of the medium appear several times in b_{63} making calculations unmanageably lengthy and cumbersome and hence are omitted). Then

$$b_{63}(k_c) = \frac{4j\pi\omega^2 \langle \rho_1^2 \rangle}{k_c} \int_0^{\infty.} re^{-\frac{r}{a}} \left[G_4(r)\right]_{33} \sin(k_c r) dr \tag{87}$$

and

$$a_{22}(k_c) = a_{62}(k_c) = b_{23}(k_c) = 0 \ . \tag{88}$$

where $[G_4(r)]_{33}$ has already been evaluated [18]. Components including $[G_4(r)]_{33}$ of the associated Green's tensor have been reproduced in the App. III for ready-reference.

For large ω the approximate values of k_c^2, k_s^2, k_m^2, and k_n^2 were computed to show that transverse type of displacement waves attenuate. In a similar way it was shown that high frequency transverse type microrotation waves attenuate if $j > 1$. Results indicate that body waves of these types attenuate due to randomness as they propagate through the medium, only in the non-thermal environment.

9 Computation of the dispersion of the field from the mean field

In this section we propose to give an outline of the procedure to evaluate an expression for dispersion of the field from the mean field. It is assumed that $v_0(\vec{x})$ and $v(\vec{x})$ represent solutions of operator equations $L_0 v_0(\vec{x}) = 0$ and $Lv(\vec{x}) = 0$ respectively, where $L = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + O(\varepsilon^3)$.

Following Karal and Keller ([23], Eq. (5)) it can be easily shown that

$$v = v_0 + \varepsilon L_0^{-1} L_1 v_0 + \varepsilon^2 (L_0^{-1} L_1 L_0^1 l_1 + L_0^{-1} l_2) v_0 + O(\varepsilon^2) \ . \tag{89a}$$

Hence

$$\bar{v}^T = \bar{v}_0^T + \varepsilon \overline{(L_0^{-1}L_1v_0)}^T + \varepsilon^2 \left[\overline{(L_0^{-1}L_1L_0^{-1}L_1v_0)}^T + \overline{(L_0^{-1}L_2v_0)}^T \right] + O(\varepsilon^3) ,$$

(89b)

where v^T is the transpose of v and $\bar{v}^T$ is the complex conjugate of v^T.
Therefore

$$\bar{v}^T v = \bar{v}_0^T v_0 + \varepsilon \left[\overline{(L_0^{-1}L_1v_0)}^T v_0 + \bar{v}_0^T (L_0^{-1}L_1v_0) \right]$$

$$+\varepsilon^2 \left[\overline{(L_0^{-1}L_1v_0)}^T (L_0^{-1}L_1v_0) + \overline{(L_0^{-1}L_1L_0^{-1}L_1v_0)}^T v_0 \right.$$

$$+ \overline{(L_0^{-1}L_2v_0)}^T v_0 + \bar{v}_0^T (L_0^{-1}L_1L_0^{-1}L_1v_0) + \bar{v}_0^T (L_0^{-1}L_2v_0) \left. \right] + O(\varepsilon^3)$$

We take expectation of this equation taking that $\langle L_1 \rangle = 0$, so that

$$\left\langle \bar{v}_0^T (L_0^{-1}L_1v_0) \right\rangle = \bar{v}_0^T L_0^{-1} \langle L_1 \rangle v_0 = 0$$

and

$$\left\langle (L_0^{-1}L_1v_0)^T v_0 \right\rangle = \left(L_0^{-1} \langle L_1 \rangle v_0 \right)^T v_0 = 0 .$$

The condition $\langle L_1 \rangle = 0$ with the help of Eq. (23), leads to

$$m_2 = i\omega m_3 \quad \text{and} \quad 1 = it_0\omega,$$

taking $\delta_{lk} = 1$ and assuming ω to be complex. Combining these two relations one gets

$$m_2 = im_3 \times \frac{1}{it_0} .$$

Hence

$$m_3 = t_0 m_2 .$$

(90)

Therefore

$$\bar{v}^T v = \bar{v}_0^T v_0 + \varepsilon^2 \left[\left\langle \overline{(L_0^{-1}L_1v_0)}^T (L_0^{-1}L_1v_0) \right\rangle + \left\langle \overline{(L_0^{-1}L_1L_0^{-1}L_1v_0)}^T v_0 \right\rangle \right.$$

$$+ \left\langle \overline{(L_0^{-1}L_2v_0)}^T v_0 \right\rangle + \left\langle \bar{v}_0^T (L_0^{-1}L_1L_0^{-1}L_1v_0) \right\rangle$$

$$+ \left\langle \bar{v}_0^T (L_0^{-1}L_2v_0) \right\rangle \left. \right] + O(\varepsilon^3) .$$

(91)

From (89a) we obtain on taking the expectation

$$\langle v \rangle \; = \; v_0 \; + \; \varepsilon^2 \Big[\left\langle L_0^{-1} L_1 L_0^{-1} L_1 v_0 \right\rangle \; + \; \left\langle L_0^{-1} L_2 v_0 \right\rangle \Big] \; + \; O(\varepsilon^3) \; .$$

Hence

$$v_0 \; = \; \langle v \rangle \; - \; \varepsilon^2 \Big[\left\langle L_0^{-1} L_1 L_0^{-1} L_1 v_0 \right\rangle \; + \; \left\langle L_0^{-1} L_2 v_0 \right\rangle \Big] \; + \; O(\varepsilon^3) \; .$$

and by iteration it follows that

$$v_0 \; = \; \langle v \rangle \; - \; \varepsilon^2 \Big[\left\langle L_0^{-1} L_1 L_0^{-1} L_1 \right\rangle \langle v \rangle \; + \; \left\langle L_0^{-1} L_2 \right\rangle \langle v \rangle \Big] \; + \; O(\varepsilon^3) \tag{92}$$

and

$$\bar{v}_0^T \; = \; \left\langle \bar{v}^T \right\rangle \; - \; \varepsilon^2 \left[\left\langle \overline{\left(L_0^{-1} L_1 L_0^{-1} L_1 \langle v \rangle \right)}^T \right\rangle \; + \; \left\langle \overline{\left(L_0^{-1} L_2 \langle v \rangle \right)}^T \right\rangle \right] \; + \; O(\varepsilon^3) \tag{93}$$

Thus

$$\bar{v}_0^T v_0 = \left\langle \bar{v}^T \right\rangle \langle v \rangle \; - \; \varepsilon^2 \left[\left\langle \overline{\left(L_0^{-1} L_1 L_0^{-1} L_1 \langle v \rangle \right)}^T \right\rangle \langle v \rangle \; + \; \left\langle \bar{v}^T \right\rangle \right.$$
$$\times \left\langle \left(L_0^{-1} L_1 L_0^{-1} L_1 \langle v \rangle \right) \right\rangle + \left\langle \overline{\left(L_0^{-1} L_2 \langle v \rangle \right)}^T \right\rangle \langle v \rangle$$
$$\left. + \left\langle \bar{v}^T \right\rangle \left\langle \left(L_0^{-1} L_2 \langle v \rangle \right) \right\rangle \right] \; + \; O(\varepsilon^3) \tag{94}$$

Substituting Eqs. (92)–(94) into (91) we obtain the expression

$$\left\langle \bar{v}^T v \right\rangle \; - \; \left\langle \bar{v}^T \right\rangle \langle v \rangle \; = \; \varepsilon^2 \left[\left\langle \overline{\left(L_0^{-1} L_1 \langle v \rangle \right)}^T \left(L_0^{-1} L_1 \langle v \rangle \right) \right\rangle \right] \; + \; O(\varepsilon^3) \; , \tag{95}$$

which is equal to sum of variances of real and imaginary parts of all the components of v. This relation holds only under the condition that $m_3 = t_0 m_2$. Taking in general

$$L = (M_{lj}), \quad (lj) = 1, 2, 3 \tag{96}$$

$$L_0^{-1} f \; = \; \int (G_{lj}) e^{i \vec{k} \cdot \vec{x'}} \, d\vec{x'} \; , \tag{97}$$

$$\langle v(\vec{x}) \rangle \; = \; [A_j] e^{i \vec{k} \cdot \vec{x}} \; , \tag{98}$$

$$\vec{k} = \vec{k}_1 + i\vec{k}_2, \quad k_1, k_2 \equiv \text{real}, \tag{99}$$

$$\langle L_1 \rangle = 0,$$

it has been shown that

$$
\begin{aligned}
\left\langle \bar{v}^T v \right\rangle - \left\langle \bar{v}^T \right\rangle \langle v \rangle &= \varepsilon^2 e^{-2Im\vec{k}\cdot\vec{x}} \left(\sum |C_l|^2 \right) \\
&= \varepsilon^2 e^{-2Im\vec{k}\,\vec{x}} \ell(A_l).
\end{aligned}
\tag{100}
$$

where $\ell(A_l) \geq 0$ is seen from (92) to be a bilinear function of the amplitudes A_l. Then Eq. (100) shows that the dispersion ($\sqrt{variance}$) of the field from the mean field represented by the plane wave (98), is a small quantity of order ε and that it decays as the wave propagates in those cases where $Im\,k > 0$. It follows that [18] the field may differ from the plane wave mean field by a quantity of order ε with finite probability.

10 Variance for micropolar generalized thermoelastic wave propagation in the random medium

This analysis is next applied to the waves in the random micropolar generalized thermoelastic medium. In this case

$$
(L_0^{-1}L_1 <v.>)^T = [e^{i\vec{k}\cdot\vec{x}} \int
\begin{bmatrix} G_0 & G_1 & 0 \\ G_2 & G_3 & 0 \\ 0 & 0 & G_4 \end{bmatrix}
\begin{bmatrix} M_1 & P_1 & K_1 \\ N_1 & Q_1 & 0 \\ R_1 & 0 & S_1 \end{bmatrix}
\begin{bmatrix} \vec{A} \\ \vec{B} \\ C \end{bmatrix}
e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}]^T \, d\vec{x}'
$$

$$
= [e^{i\vec{k}\cdot\vec{x}} \int
\begin{bmatrix} G_0 & G_1 & 0 \\ G_2 & G_3 & 0 \\ 0 & 0 & G_4 \end{bmatrix}
\begin{bmatrix} M_1\vec{A} + P_1\vec{B} + K_1 C \\ N_1\vec{A} + Q_1\vec{B} \\ R_1\vec{A} + S_1 C \end{bmatrix}
e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}]^T \, d\vec{x}'
$$

$$
= [e^{i\vec{k}\cdot\vec{x}} \int
\begin{bmatrix} G_0(M_1\vec{A} + P_1\vec{B} + K_1 C) + G_1(N_1\vec{A} + Q_1\vec{B}) \\ G_2(M_1\vec{A} + P_1\vec{B} + K_1 C) + G_3(N_1\vec{A} + Q_1\vec{B}) \\ G_4(R_1\vec{A} + S_1 C) \end{bmatrix}
e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}]^T \, d\vec{x}'
\tag{101}
$$

where

$$M\vec{u} = \rho\,\frac{\partial^2\vec{u}}{\partial t^2} - (\lambda + \mu)\vec{\nabla}(\vec{\nabla}\,\vec{u}) - (\mu + \kappa)\nabla^2\vec{u} - \vec{\nabla}\lambda(\vec{\nabla}\,\vec{u}) - \vec{\nabla}\mu \times (\vec{\nabla}\times\vec{u}) - (\vec{\nabla}(2\mu + \kappa)\vec{\nabla})\vec{u}$$

$$M_1(\vec{A}e^{i\vec{k}\cdot\vec{x}}) = [-\rho_1\omega^2\vec{A} + (\lambda_1 + \mu_1)(\vec{k}\,\vec{A})\vec{k} + (\mu_1 + \kappa_1)k^2\vec{A}$$

$$- i(\vec{k}\,\vec{A})\vec{\nabla}\lambda_1 - i\vec{\nabla}\mu_1 \times (\vec{k}\times\vec{A}) - i(\vec{\nabla}(2\mu_1 + \kappa_1)\vec{k})\vec{A}]e^{i\vec{k}\cdot\vec{x}}$$

$$P_1(\vec{B}e^{i\vec{k}\cdot\vec{x}}) = -\vec{\nabla}\times(\kappa_1\vec{B}e^{i\vec{k}\cdot\vec{x}}) = -[\vec{\nabla}k_1\times\vec{B} + i\kappa_1\vec{k}\times\vec{B}]e^{i\vec{k}\cdot\vec{x}},$$

$$K_1(Ce^{i\vec{k}\cdot\vec{x}}) = [\vec{\nabla}(m_1) - i\omega\vec{\nabla}m_1^\bullet]Ce^{i\vec{k}\cdot\vec{x}} = [\vec{\nabla}m_1 + im_1\vec{k} - i\omega(\vec{\nabla}m_1^\bullet + im_1^\bullet)]Ce^{i\vec{k}\cdot\vec{x}},$$

$$N_1(\vec{A}e^{i\vec{k}\cdot\vec{x}}) = -\kappa_1(\vec{\nabla}\times)(\vec{A}e^{i\vec{k}\cdot\vec{x}}) = -i\kappa_1(\vec{k}\times\vec{A})e^{i\vec{k}\cdot\vec{x}}$$

$$Q_1(\vec{B}e^{i\vec{k}\cdot\vec{x}}) = [(-\rho_1 j\omega^2 + 2\kappa_1)\vec{B} + (\alpha_1 + \beta_1)(\vec{k}\cdot\vec{B})\vec{k} - i(\vec{k}\,\vec{B})(\vec{\nabla}\alpha_1)$$

$$+\gamma_1 k^2\vec{B} - i(\vec{\nabla}\beta_1)\times(\vec{k}\times\vec{B}) - i\{\vec{\nabla}(\beta_1 + \gamma_1)\vec{k}\}\vec{B}]e^{i\vec{k}\cdot\vec{x}}$$

$$R_1(\vec{A}e^{i\vec{k}\cdot\vec{x}}) = -i\omega\theta_0(1 - i\delta_{lh}t_0\omega)m_1(\vec{k}\,\vec{A})e^{i\vec{k}\cdot\vec{x}},$$

$$S_1(Ce^{i\vec{k}\cdot\vec{x}}) = [-i\omega(1 - it_0\omega)\eta_1 Ce^{i\vec{k}\cdot\vec{x}}] - iC\vec{\nabla}\{\nu_1\vec{\nabla}(e^{i\vec{k}\cdot\vec{x}})\}$$

$$= -i[\omega(1 - it_0\omega)\eta_1 + (\vec{\nabla}\nu_1 + i\nu_1\vec{k})\vec{k}]Ce^{i\vec{k}\cdot\vec{x}}$$

and $\langle L_1\rangle = 0.$

Let us write

$$\left(L_0^{-1}L_1\langle v\rangle\right)^T \;=\; e^{i\vec{k}_2\cdot\vec{x}}\,e^{i\vec{k}_1\cdot\vec{x}}(c_1 + id_1\,,\,c_2 + id_2)\,, \qquad (102)$$

where c_i, d_i, are linear functions of $A_1, A_2, B_1, B_2,$ and B_3. Then it can be shown as before that

$$\overline{\left(L_0^{-1}L_1\langle v\rangle\right)}^T\left(L_0^{-1}L_1\langle v\rangle\right) \;=\; e^{-2(\mathrm{Im}k)x}\,\ell\,(A_l\,,B_l)$$

where $\ell(A_l\,,\,B_l)$ is clearly a non-negative, bilinear function of A_l and B_l. Hence the amount of dispersion of waves from the mean field comes out to be $\varepsilon e^{-(\mathrm{Im}k)x}\,\sqrt{\ell}.$

It may be concluded that the dispersion of the wave field from the mean field may be obtained following this procedure in many other cases examined by employing Keller's perturbation theory.

11   The uncoupled problem

This section proposes to outline, approximately following Chow [25] the method of examining the uncoupled random micropolar generalized-thermal problem. With this end in view the $\vec{u}-$term is dropped from the temperature Eq. (7), so that the reduced temperature equation becomes

$$\eta(\vec{x})\big[\dot{\theta}(\vec{x},t) + t_0\ddot{\theta}(\vec{x},t)\big] = \vec{\nabla}\left\{\nu(\vec{x})\vec{\nabla}[\theta(\vec{x},t)]\right\} + q\,.$$

The objective is to study the random elasticity-displacement Eq. (5) independently. The three dispersion equations (29)–(31) now become:

$$\left[\rho_0\omega^2\vec{A} - (\lambda_0 + \mu_0)(\vec{k}\,\vec{A})\vec{k} - (\mu_0 + \kappa_0)k^2\vec{A}\right] + i\kappa_0(\vec{k}\times\vec{B}) - \varepsilon(im_2 + \omega m_3)C\vec{k}$$

$$+\varepsilon^2\left[\omega\theta_0(1 - \omega t_0\delta_{lk})m_2(m_2 - i\omega m_3)(\vec{k}\,\vec{A})\right]\int\left[(\vec{\nabla}G_4)\right]e^{-i\vec{k}\,\vec{r}}d\vec{r}$$

$$-\varepsilon^2\int\left[\begin{array}{l}\langle[M_1\{G_0(M_1'\vec{A} + P_1'\vec{B} + K_1'C) + G_1(N_1'\vec{A} + Q_1'\vec{B})\}\\ +P_1\{G_2(M_1'\vec{A} + P_1'\vec{B} + K_1'C) + G_3(N_1'\vec{A} + Q_1'\vec{B})\}\\ +K_1G_4(R_1'\vec{A} + S_1'C)]\rangle\end{array}\right]e^{-i\vec{k}\cdot\vec{r}}d\vec{r} = 0\,,$$

$$\tag{103}$$

$$i\kappa_0(\vec{k}\times\vec{A}) - \left[(2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2)\vec{B} + (\alpha_0 + \beta_0)(\vec{k}\,\vec{B})\vec{k}\right]$$

$$-\varepsilon^2\int\left[\begin{array}{l}\langle[N_1\{G_0(M_1'\vec{A} + P_1'\vec{B} + K_1'C) + G_1(N_1'\vec{A} + Q_1'\vec{B})\}\\ +Q_1\{G_2(M_1'\vec{A} + P_1'\vec{B} + K_1'C) + G_3(N_1'\vec{A} + Q_1'\vec{B})\}]\rangle\end{array}\right]e^{-i\vec{k}\cdot\vec{r}}d\vec{r} = 0$$

$$\tag{104}$$

and

$$\left[i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2\right]C = 0\,.\tag{105}$$

Simplified Case I. Equation (105) is easily solvable since $C \neq 0$. The first two equations can be analyzed as before or otherwise. Considering equations up to ε-order terms one gets

$$\left[\rho_0\omega^2\vec{A} - (\lambda_0 + \mu_0)(\vec{k}\,\vec{A})\vec{k} - (\mu_0 + \kappa_0)k^2\vec{A}\right] + i\kappa_0(\vec{k}\times\vec{B}) - \varepsilon(im_2 + \omega m_3)C\vec{k} = 0$$

and

$$i\kappa_0(\vec{k}\times\vec{A}) - \left[(2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2)\vec{B} + (\alpha_0 + \beta_0)(\vec{k}\,\vec{B})\vec{k}\right] = 0\,.$$

The six equations are

$$\left[\rho_0\omega^2 - (\lambda_0 + 2\mu_0 + \kappa_0)k^2\right]A_1 - \varepsilon(im_2 + \omega m_3)Ck = 0\,,$$

$$\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k^2\right]A_2 + i\kappa_0 kB_3 = 0\,,$$

$$i\kappa_0 kB_2 = 0$$

and

$$\left[(2\kappa_0 - \rho_0 j\omega^2) + (\alpha_0 + \beta_0 + \gamma_0)k^2\right]B_1 = 0\,,$$

$$(2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2)B_2 = 0\,,$$

$$i\kappa_0 k A_2 - (2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2)B_3 = 0 \ .$$

It is clear that the amplitude ratio

$$\frac{A_1}{C} = \frac{\varepsilon(im_2 + \omega m_3)k}{\rho_0\omega^2 - (\lambda_0 + 2\mu_0 + \kappa_0)k^2} = O(\varepsilon)$$

and is dependent upon expectation values of generalized thermal parameters m and $m^\bullet = t_1 m$. Moreover A_1 is finite but small such that $A_1 = O(\varepsilon)$. Also $B_2 = 0$ indicates that there is no propagation along B_2. Further $B_1 \neq 0$ and $B_1 \equiv$ finite, indicates that waves with wave number given by

$$k^2 = \frac{\rho_0 j\omega^2 - 2\kappa_0}{\alpha_0 + \beta_0 + \gamma_0}$$

propagate in the medium. Finally the coupled dispersion equation for waves propagating along A_2 and B_3 directions is represented by

$$\frac{A_2}{B_3} = -\frac{i\kappa_0 k}{\rho_0\omega^2 - (\mu_0 + \kappa_0)k^2} = \frac{2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2}{i\kappa_0 k} \ ,$$

$$\left[\rho_0\omega^2 - (\mu_0 + \kappa_0)k^2\right]\left[2\kappa_0 - \rho_0 j\omega^2 + \gamma_0 k^2\right] = \kappa_0^2 k^2 \ .$$

Case II. Unlike Chow ([25], cf. Eq. (4.3)), we retain the ε-order term in (31) to get

$$\left[i\omega(1 - it_0\omega)\eta_0 - \nu_0 k^2\right]C - \varepsilon\omega\theta_0(1 - i\omega t_0\delta_{lk})m_2(\vec{k}\,\vec{A}) = 0 \ . \tag{106}$$

Then Eqs. (103), (104), and (105) can be analyzed as in Sec. 4 or otherwise.

12   Summary and conclusions

1. The integrand of a_{11} only is dependent on generalized thermal parameters including thermomechanical auto-correlation function and the cross-correlation function between thermomechanical and generalized thermomechanical coupling parameters. This dependence holds good both for L-S and G-L thermoelastic fields.

2. Numerical solutions can be easily obtained by setting

$$R_{mm}(r) = \langle\rho_1^2\rangle e^{-\frac{r}{b}}, \ \ R_{mm}(r) = \langle\rho_1^2\rangle e^{-\frac{r}{b}}.$$

3. On the other hand integrands of a_{22}, a_{62}, b_{23}, and b_{63} are clearly independent of auto- and cross-correlation functions $R_{mm}(r)$ and $R_{mm\bullet}(r)$.

4. The three vector wave equations considered up to first order perturbation only leads to the dispersion equation for the wave propagation in the deterministic micropolar elastic medium. However effects of generalized thermal field are discernible only if the second order perturbation term is retained in this dispersion equation. This will be evident from the Eq. (48) of Sec. 4.

5. Equation (64) shows that longitudinal type displacement waves get modified due to random fluctuation of inhomogeneities of the micropolar medium under generalized thermoelasticity such that the wave number is increased by $\varepsilon^2 \frac{\delta_l}{2k_l}$, where δ_l is a function of $R_{mm}(r)$ and $R_{mm\bullet}(r)$. These results are valid for both L-S and G-L generalized thermoelasticity.

6. Equation (68) shows that the change in wave number δ_l depends upon the random variation of the generalized thermal field and measured in terms of statistical means, m_2 and m_3, for both L-S and G-L theories. For L-S, the change depends upon m_2 only, and for G-L upon m_2 and m_3.

7. Equation (100) shows that the dispersion ($\sqrt{\text{variance}}$) of the field from the mean field is a small quantity of order ε and that it decays as the wave propagates in those cases where $\operatorname{Im} k > 0$. This result holds good only if $m_3 = t_0 m_2$.

8. It is quite clear from Eqs. (70) and (75) that longitudinal type microrotation waves propagate in the random generalized thermoelastic micropolar medium but are scarcely affected by the random inhomogeneities of the medium, at least up to the domain of second order perturbation. In this case the amplitude B_1 is important in comparison to the amplitudes A_1, A_2, B_2, and B_3.

   The study of micropolar materials and micropolar generalized-thermoelasticity in particular has been enormously drawing the interest of applied mathematicians and engineers in recent times. Micropolar materials include fibrous and granular or composite substances. The domain of micropolarity is therefore expanding in time.

The study of micropolar properties in relation to wave propagation in coupled media therefore has been gaining importance from the perspective of probable applications in industry, engineering and to a great extent in earthquake prediction and seismology. Research studies are being carried out through various methodologies. The present paper has introduced the smooth perturbation technique [21] in examining the phenomenon of wave propagation in an infinite random generalized thermoelastic micropolar medium. In future certain other methods, *viz.*, iterative perturbation method [31] or the method of scatters [37] may be employed in measuring effects of random variation of parameters. The domain decomposition method developed by Adomian [43] may also be employed to study micropolar elasticity in coupled dynamic problems. There is possibility of solving micropolar coupled elastic problems with the help of fractional calculus theories [44,45].

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Appendix I

Components of the associated Green's tensor

The 36+9 components of the Green's tensor

$$
G = \left|
\begin{array}{ccc}
G_1 & G_2 & 0 \\
G_3 & G_4 & 0 \\
0 & 0 & G_5
\end{array}
\right| ,
$$

where G_i, $i = 1, 2, 3, 4$, are 3×3 matrices, are then obtained and given by (for $\omega^2 >> \kappa_0(\kappa_0 + 2\mu_0)$):

$$
(G_j)_{kl} = (G_j)_{kk}\delta_{kl} \tag{1}
$$

with $(G_1)_{11} = (G_1)_{22}$ for $j = 1, 4$;

$$
(G_j)_{kl} = 0 \quad \text{if } (k, l) \neq (1, 2), (2, 1); \tag{2}
$$

with $(G_1)_{12} = -(G_1)_{21}$, $j = 2, 3$;
and $(G_3)_{kl} = -(G_2)_{kl}$.
Components of G_5 have been defined in (27) and (28).

The independent components $(G_1)_{11}, (G_1)_{33}, (G_2)_{12}, (G_4)_{11}$ and $(G_4)_{33}$ are given by:

$$
(G_1)_{11} = \frac{i\kappa_0^2}{2\pi\gamma_0(\mu_0 + \kappa_0)^2 r^3} \left[\frac{r^2 - (i + k_m r)}{(k_m^2 - k_c^2)(k_m^2 - k_s^2)} e^{ik_m r} \right.
$$

$$
\left. + \frac{r^2 - (i + k_c r)}{(k_c^2 - k_m^2)(k_c^2 - k_s^2)} e^{ik_c r} + \frac{r^2 - (i + k_s r)}{(k_s^2 - k_m^2)(k_s^2 - k_c^2)} e^{ik_s r} \right]
$$

$$
- \frac{1}{4\pi r^3} \left[\frac{r^2 k_{s'}^2 + ir k_{s'} - 1}{\mu_0 k_{s'}^2} e^{ik_{s'} r} - \frac{ik_{c'} r - 1}{(\lambda_0 + 2\mu_0)k_{c'}^2} e^{ik_{c'} r} \right]
$$

$$
(G_1)_{33} = \frac{-i\kappa_0}{\pi\gamma_0(\mu_0 + \kappa_0)^2 2 r^3} \left[\frac{i + k_m r}{(k_m^2 - k_c^2)(k_m^2 - k_s^2)} e^{ik_m r} + \frac{i + k_c r}{(k_c^2 - k_m^2)(k_c^2 - k_s^2)} e^{ik_c r} \right.
$$

$$
\left. + \frac{i + k_s r}{(k_s^2 - k_m^2)(k_s^2 - k_c^2)} e^{ik_s r} \right] + \frac{1}{4\pi r^3} \left[\frac{2(ik_{s'} r - 1)}{\mu_0 k_{s'}^2} e^{ik_{s'} r} - \frac{k_{c'}^2 r^2 + 2ik_{c'} r - 2}{(\lambda_0 + 2\mu_0)k_{c'}^2} e^{ik_{c'} r} \right]
$$

$$
(G_2)_{12} = \frac{-k_0}{2\pi\gamma_0(k_c^2 - k_s^2)(\mu_0 + k_0)r^2} \left[(i + k_c r)e^{iK_c r} - (i + k_s r)e^{iK_s r} \right]
$$

The other 9 components have been computed in the text.

$$(G_4)_{11} = \frac{-1}{2\pi\gamma_0(\mu_0+\kappa_0)(k_c^2-k_s^2)r}\Big[\big\{(\mu_0+\kappa_0)k_c^2-\rho_0\omega^2\big\}e^{ik_c r} - \big\{(\mu_0+\kappa_0)k_s^2-\rho_0\omega^2\big\}e^{ik_s r}\Big]$$

$$+ \frac{1}{2\pi\gamma_0(\mu_0+\kappa_0)(\alpha_0+\beta_0+\gamma_0)r^3}\Bigg[\frac{[(\alpha_0+\beta_0)\{(\mu_0+\kappa_0)k_c^2-\rho_0\omega^2\}+\kappa_0^2](1-ik_c r)}{(k_c^2-k_s^2)(k_c^2-k_n^2)}e^{ik_c r}$$

$$+ \frac{[(\alpha_0+\beta_0)\{(\mu_0+\kappa_0)k_s^2-\rho_0\omega^2\}+\kappa_0^2](1-ik_s r)}{(k_s^2-k_c^2)(k_s^2-k_n^2)}e^{ik_s r}$$

$$+ \frac{[(\alpha_0+\beta_0)\{(\mu_0+\kappa_0)k_n^2-\rho_0\omega^2\}+\kappa_0^2](1-ik_n r)}{(k_n^2-k_c^2)(k_n^2-k_s^2)}e^{ik_n r}\Bigg]$$

$$(G_4)_{33} = \frac{-1}{2\pi\gamma_0(\mu_0+\kappa_0)(k_c^2-k_s^2)r}\Big[\big\{(\mu_0+\kappa_0)k_c^2-\rho_0\omega^2\big\}e^{ik_c r} - \big\{(\mu_0+\kappa_0)k_s^2-\rho_0\omega^2\big\}e^{ik_s r}\Big]$$

$$+ \frac{1}{2\pi\gamma_0(\mu_0+\kappa_0)(\alpha_0+\beta_0+\gamma_0)r^3}\Bigg[\frac{[(\alpha_0+\beta_0)\{(\mu_0+\kappa_0)k_c^2-\rho_0\omega^2\}+\kappa_0^2](k_c^2 r^2+2ik_c r-2)}{(k_c^2-k_s^2)(k_c^2-k_n^2)}e^{ik_c r}$$

$$+ \frac{[(\alpha_0+\beta_0)\{(\mu_0+\kappa_0)k_s^2-\rho_0\omega^2\}+\kappa_0^2](k_s^2 r^2+2ik_s r-2)}{(k_s^2-k_c^2)(k_s^2-k_n^2)}e^{ik_s r}$$

$$+ \frac{[(\alpha_0+\beta_0)\{(\mu_0+\kappa_0)k_n^2-\rho_0\omega^2\}+\kappa_0^2](k_n^2 r^2+2ik_n r-2)}{(k_n^2-k_c^2)(k_n^2-k_s^2)}e^{ik_n r}\Bigg]$$

Appendix II

Method of forming products and taking expectation values $\langle M_1 G_0 M_1'\rangle(\vec{A}e^{i\vec{k}\cdot\vec{x}'})$, **etc.**

$$\langle N_1 G_0 M_1'\rangle(\vec{A}e^{i\vec{k}\,\vec{x}'})$$
$$= \kappa_1\vec{\nabla}\times G_0[-(\lambda_1'+\mu_1')(\vec{k}\,\vec{A})\vec{k}-(\mu_1'+\kappa_1')k^2\vec{A}+$$
$$i\vec{\nabla}'\lambda_1'(\vec{k}\,\vec{A})+i\vec{\nabla}'\mu_1'\times(\vec{k}\times\vec{A})+i[\vec{\nabla}'(2\vec{\mu}'+\kappa_1')\,\vec{k}]\vec{A}+\rho_1'\omega^2\vec{A}]e^{i\vec{k}\,\vec{x}'}$$

$$= [-(R_{\kappa\lambda}+R_{\kappa\mu})A_1 k^2(0,\frac{\partial(G_0)_{11}}{\partial z},-\frac{\partial(G_0)_{11}}{\partial y})-k^2(R_{\kappa\mu}+R_{\kappa\kappa})(-A_2\frac{\partial(G_0)_{22}}{\partial z},A_1\frac{\partial(G_0)_{11}}{\partial z},$$

$$A_2\frac{\partial(G_0)_{22}}{\partial x}-A_1\frac{\partial(G_0)_{11}}{\partial y})+ikA_1(\frac{\partial R_{\kappa\lambda}}{\partial z'}\cdot\frac{\partial(G_0)_{33}}{\partial y}-\frac{\partial R_{\kappa\lambda}}{\partial y'}\cdot\frac{\partial(G_0)_{22}}{\partial z},$$

$$\frac{\partial R_{\kappa\lambda}}{\partial x'}\cdot\frac{\partial(G_0)_{11}}{\partial z}-\frac{\partial R_{\kappa\lambda}}{\partial z'}\cdot\frac{\partial(G_0)_{22}}{\partial x},\frac{\partial R_{\kappa\lambda}}{\partial y'}\cdot\frac{\partial(G_0)_{22}}{\partial x}-\frac{\partial R_{\kappa\lambda}}{\partial x'}\cdot\frac{\partial(G_0)_{11}}{\partial y})+ikA_2(\frac{\partial R_{\kappa\mu}}{\partial x'}\cdot\frac{\partial(G_0)_{22}}{\partial z},$$

$$A_2\frac{\partial(G_0)_{22}}{\partial x}-A_1\frac{\partial(G_0)_{11}}{\partial y})\frac{\partial(2R_{\kappa\mu}+R_{\kappa\kappa})}{\partial x'}.$$

$$\frac{\partial R_{\kappa\mu}}{\partial y'} \cdot \frac{\partial (G_0)_{11}}{\partial z}, -\frac{\partial R_{\kappa\mu}}{\partial x'} \cdot \frac{\partial (G_0)_{22}}{\partial x} - \frac{\partial R_{\kappa\mu}}{\partial y'} \cdot \frac{\partial (G_0)_{11}}{\partial y}) + ik(-A_2 \frac{\partial (G_0)_{22}}{\partial z}, A_1 \frac{\partial (G_0)_{11}}{\partial z},$$

$$+\omega^2 R_{\kappa\rho}(-A_2 \frac{\partial (G_0)_{22}}{\partial z}, A_1 \frac{\partial (G_0)_{11}}{\partial z}, A_2 \frac{\partial (G_0)_{22}}{\partial x} - A_1 \frac{\partial (G_0)_{11}}{\partial y})]e^{i\vec{k}\cdot\vec{x'}}.$$

Appendix III

A note on transforming to radial forms

$$\vec{r} = \vec{x} - \vec{x'} = (\xi, \eta, \varsigma) = (r\cos\theta, r\sin\theta\cos\phi, r\sin\theta\sin\phi);$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}, \frac{\partial}{\partial z} = \frac{\partial}{\partial \varsigma}; \frac{\partial}{\partial x'} = -\frac{\partial}{\partial \xi}, \frac{\partial}{\partial y'} = -\frac{\partial}{\partial \eta}, \frac{\partial}{\partial z'} = -\frac{\partial}{\partial \varsigma};$$

$$r^2 = \xi^2 + \eta^2 + \varsigma^2; \ r\frac{\partial r}{\partial \xi} = \xi; \therefore \frac{\partial r}{\partial \xi} = \frac{\xi}{r}, \frac{\partial r}{\partial \eta} = \frac{\eta}{r}, \frac{\partial r}{\partial \varsigma} = \frac{\varsigma}{r};$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} = (\frac{\partial}{\partial r})\frac{\partial r}{\partial \xi} = \frac{\xi}{r}\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial r};$$

$$\frac{\partial}{\partial y} = \frac{\eta}{r}\frac{\partial}{\partial r} = \sin\theta\cos\phi\frac{\partial}{\partial r}; \frac{\partial}{\partial z} = \frac{\varsigma}{r}\frac{\partial}{\partial r} = \sin\theta\sin\phi\frac{\partial}{\partial r};$$

$$\frac{\partial^2}{\partial x^2} = \frac{\xi}{r}\frac{\partial}{\partial r}(\frac{\xi}{r}\frac{\partial}{\partial r}) = \cos^2\theta\frac{\partial^2}{\partial r^2}; \frac{\partial^2}{\partial y^2} = \frac{\eta}{r}\frac{\partial}{\partial r}(\frac{\eta}{r}\frac{\partial}{\partial r}) = \sin^2\theta\cos^2\phi\frac{\partial^2}{\partial r^2};$$

$$\frac{\partial^2}{\partial z^2} = \frac{\varsigma}{r}\frac{\partial}{\partial r}(\frac{\varsigma}{r}\frac{\partial}{\partial r}) = \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial r^2}.$$

1. $\int_0^{2\pi} \cos^2\phi d\phi = . \int_0^{2\pi} \sin^2\phi d\phi = \pi;$

2. $f(r) = \frac{\sin kr}{kr}$

3. $f'(r) = \frac{d}{d(kr)}\frac{\sin kr}{kr}; f''(r) = \frac{d^2}{d(kr)^2}\frac{\sin kr}{kr}$

4. $\int_0^\pi e^{-ikr\cos\theta}\sin\theta d\theta = 2\frac{\sin kr}{kr} = 2f(r) = \int_{-1}^1 e^{-ikrx}dx$

5. $\int_0^\pi e^{-ikr\cos\theta}\sin\theta\cos\theta d\theta = 2if'(r) = \int_{-1}^1 xe^{-ikrx}dx$

6. $\int_0^\pi e^{-ikr\cos\theta}\sin\theta\cos^3\theta d\theta = -2if'''(r)$

7. $\int_0^\pi e^{-ikr\cos\theta} \sin^3\theta \cos\theta d\theta = 2i[f'(r) + f'''(r)]$

8. $\int_0^\pi e^{-ikr\cos\theta} \sin\theta \cos^2\theta d\theta = -2f''(r)$

9. $\int_0^\pi e^{-ikr\cos\theta} \sin^3\theta d\theta = -2[f(r) + f''(r)]$

10. $\int_0^{2\pi} \cos^3\phi d\phi = \left|\sin\phi - \frac{1}{3}\sin^3\phi\right|_0^{2\pi} = 0.$

Appendix IV

Finite parts of some integrals [46]

(i) $Pf \int_0^\infty \frac{e^{-pt}}{t} dt = -\log p - \gamma = -\log Cp, \ \ \mathrm{Re}p > 0,$
where $\gamma = \log C = .577$ (Euler's constant) .

(ii) $Pf \int_0^\infty \frac{e^{-pt}}{t^2} dt = p(\log p + \gamma - 1)$.

(iii) $\int_0^\infty e^{-pt}\frac{\sin bt}{t} dt = \tan^{-1}\frac{b}{p}, |\arg p| < \pi/2$.

(iv) $Pf \int_0^\infty e^{-pt}\frac{\cos bt}{t} dt = -\frac{1}{2}\log(p^2 + b^2) - \gamma, \ |\arg p| < \pi/2$.

(v) $Pf \int_0^\infty e^{-pt}\frac{\sin bt}{t^2} dt = \frac{-r}{1}[\theta\cos\theta + \{\log r - \Psi(2)\}\sin\theta, \ |\arg p| < \pi/2$,

$\theta = \tan^{-1}\frac{b}{p}, r = \sqrt{p^2 + b^2}.$

(vi) $Pf \int_0^\infty e^{-pt}\frac{\cos bt}{t^2} dt = \frac{-r}{1}\left\{\left[\log r - \Psi(2)\right]\cos\theta - \theta\sin\theta\right\}, \ |\arg p| < \pi/2,$

$\Psi(2) = -\gamma + 1 + \frac{1}{2}$.