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ON WAVE PROPAGATION PROBLEMS IN WHICH $c_f = c_s = c_2$ OCCURS*

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Abstract. The combined longitudinal and torsional plastic waves in a thin-walled tube of rate-independent isotropic work-hardening material are used to illustrate the problems involved when the situation $c_f = c_s = c_2$ occurs. Two examples are presented. In the first example, the stress paths in the $\sigma \sim \tau$ plane for the fast and slow simple waves are examined in the region near the singular point $(\sigma^*, 0)$ where $c_f = c_s = c_2$. For $\eta \geq \frac{1}{2}$, where η is a nondimensional material constant defined in the paper, there is *no* stress path passing through the singular point $(\sigma^*, 0)$ other than the σ-axis itself. For $0 < \eta < \frac{1}{2}$, there is a family of stress paths emanated from $(\sigma^*, 0)$ which span an angle of $\tan^{-1} (1 - 2\eta)^{1/2}$ with the σ-axis. In any case, the stress paths for the fast and slow simple waves are *not* orthogonal to each other at the singular point. In the second example, a study is made of the propagation of the plastic wave front into the tube which is initially prestressed at the stress state $(\sigma^*, 0)$. It is shown that the solution in the region next to a region of constant state is *not* necessarily a simple wave solution. In fact, an unloading can occur at the plastic wave front which changes its speed from c_2 to c_0 at the onset of the unloading.

1. Introduction. The elastic-plastic wave propagation of combined stress has been studied by various authors in recent years [1–6]. The governing equations can be written as a system of first-order partial differential equations in which the characteristic wave speeds are the fast wave speed c_f and the slow wave speed c_s. In some problems, the shear wave speed c_2 is also a characteristic wave speed in the plastic region. In any case, $c_s \leq c_2 \leq c_f$ holds and the system is *totally* hyperbolic provided c_s and c_f are different from c_2.

The problems studied in [1–6] do not pay special attention to the situation when $c_f = c_s = c_2$ occurs because this particular situation does not happen very often. When this particular situation happens, one may obtain the result from the general solution by letting c_f and c_s approach to c_2. Unfortunately, this limiting process may not work, and if it does work it sometimes leads to erroneous conclusions. For instance, the fact that the stress paths for the fast and slow simple waves are orthogonal to each other for all values of c_f and c_s may not be true for the special case when $c_f = c_s = c_2$. This fact does not seem to have been mentioned in the literature. Also not mentioned is the fact that there may be no stress paths passing through the point in the stress space where $c_f = c_s = c_2$ other than the trivial one, namely the stress axis itself. These new facts, and the consequences of these facts, are presented in Secs. 2 and 3 by considering

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the combined longitudinal and torsional waves in a thin-walled tube as an example. As it turns out, the material constant η introduced in Sec. 3 plays an important role in the analysis. A means for determining this material constant is presented in Sec. 4.

When $c_f = c_s = c_2$, the system of the partial differential equations is not totally hyperbolic. Theorems which apply to a totally hyperbolic system are therefore not necessarily applicable to this case. For instance, a generalized Lax's theorem on simple waves [7, 8] states that: *if a region is bounded by a characteristic, and the dependent variables are constants on this characteristic, the solution in the region is a simple wave solution.* If the boundary characteristic in the theorem is the one with $c_f = c_s = c_2$, the solution next to this characteristic is not necessarily a simple wave solution. Therefore, a different approach is needed to obtain the solution in the neighborhood of this characteristic. This is illustrated in Sec. 5 by considering the plastic wave propagation in the tube which is initially pre-stressed at the stress state for which $c_f = c_s = c_2$. Indeed, it is shown that the solution in the region next to the plastic wave front is in general not a simple wave solution. Moreover, it is shown that an unloading can occur at the plastic wave front, a phenomenon which would never occur if the system at the wave front were totally hyperbolic.

2. The basic equations. The governing equations for the combined longitudinal and torsional waves in a thin-walled tube of rate-independent isotropic work-hardening materials are derived by Clifton [1] and can be written in matrix notation as (see also [9])

$$\mathbf{A}\mathbf{w}_t + \mathbf{B}\mathbf{w}_x = \mathbf{0} \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & (1/E) + H\sigma^2/\theta^2 & H\sigma\tau \\ 0 & 0 & H\sigma\tau & (1/\mu) + H\theta^2\tau^2 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} u \\ v \\ \sigma \\ \tau \end{bmatrix}. \tag{2}$$

The subscripts t and x denote partial differentiation with respect to t and x. u and v are the longitudinal and the circumferential particle velocities, E is Young's modulus, μ is the shear modulus, and ρ is the mass density of the tube. In the elastic region $H = 0$, while in the plastic region H is positive, and is an increasing function of yield stress k:

$$k^2 = (\sigma/\theta)^2 + \tau^2, \tag{3}$$

where $\theta = \sqrt{3}$ for the von Mises yield condition and $\theta = 2$ for the Tresca yield condition.

The characteristics c of Eq. (1) are the roots of $||c\mathbf{A} - \mathbf{B}|| = 0$, or

$$\left(\frac{c^2}{c_0^2} - 1\right)\left(\frac{c^2}{c_2^2} - 1\right) + \rho c^2 H\left\{\left(\frac{c^2}{c_2^2} - 1\right)\frac{\sigma^2}{\theta^2} + \left(\frac{c^2}{c_0^2} - 1\right)\theta^2\tau^2\right\} = 0 \tag{4}$$

where

$$c_0^2 = E/\rho, \qquad c_2^2 = \mu/\rho. \tag{5}$$

Since $H = 0$ in the elastic region, $\pm c_0$ and $\pm c_2$ are the characteristic wave speeds in the elastic region. In the plastic region $H > 0$, and the left-hand side of Eq. (4) is positive when $c = 0$, negative when $c = c_2$, and positive again when $c = c_0$. Therefore, if $\pm c_f$ and $\pm c_s$ are respectively the fast and slow wave speeds in the plastic region, we must have [1]

$$0 \leq c_s \leq c_2 \leq c_f \leq c_0 . \tag{6}$$

Since $c_2 \neq c_0$, the system is totally hyperbolic in the elastic region. In the plastic region, the system is totally hyperbolic unless $c_f = c_s$. If $c_f = c_s$, then in view of Eq. (6) c_f and c_s both must have the value c_2. Therefore $c_f = c_s$ when c_2 is the double root of Eq. (4). Let (σ^*, τ^*) be the stress state for which $c = c_2$ is the double root of Eq. (4). It is easily seen from Eq. (4) that

$$\tau^* = 0, \qquad 1/c_2^2 = 1/c_0^2 + \rho H^*(\sigma^*/\theta)^2, \tag{7a, b}$$

where

$$H^* = H(\sigma^*/\theta). \tag{8}$$

Here H^* is the value of $H(k)$ evaluated at $k = \sigma^*/\theta$ because $\tau^* = 0$. Since Hk^2 is an increasing function of k (see Eq. (26)), there is only one value of σ^* satisfying Eq. (7b) if σ^* exists. (Here we restrict ourselves to positive σ^* only.) When σ^* exists, σ^* corresponds to the stress magnitude on the stress-strain curve for a simple tension test where the slope of the curve is equal to μ (see [1] or Sec. 4).

Since $\mathbf{A}$ depends on σ and τ only, Eq. (1) admits simple wave solutions. According to the theory of simple waves [10, 11], $\mathbf{w}$ of Eq. (1) is a function of c only for a simple wave solution and $d\mathbf{w}/dc$ is proportional to the right eigenvector $\mathbf{r}$ defined by

$$(c\mathbf{A} - \mathbf{B})\mathbf{r} = \mathbf{0}. \tag{9}$$

The fact that $d\mathbf{w}/dc$ is proportional to $\mathbf{r}$ reduces to the following result [4]:

$$\frac{d\sigma}{d\tau} = \frac{\sigma}{\theta^2 \tau}\left(\frac{1}{c_2^2} - \frac{1}{c^2}\right) \Big/ \left(\frac{1}{c_0^2} - \frac{1}{c^2}\right). \tag{10}$$

Since c as given by Eq. (4) depends on σ and τ only, Eq. (10) can be integrated. For $c = c_f$ one obtains the stress paths for the fast simple waves and for $c = c_s$ one obtains other stress paths for the slow simple waves. It can be shown that the stress paths for the fast and slow simple waves are orthogonal to each other.

At the singular point $\sigma = \sigma^*$ and $\tau = 0$, $c = c_2$ and the right-hand side of Eq. (10) is indeterminate. In the next section, we will expand the right-hand side of Eq. (10) about the point $(\sigma^*, 0)$ and integrate the resulting equation which is valid in the neighborhood of $(\sigma^*, 0)$.

3. The stress paths for simple waves near $(\sigma^*, 0)$. Before we expand the right-hand side of Eq. (10) about the singular point $(\sigma^*, 0)$, we have to find c of Eq. (4) for (σ, τ) in the neighborhood of $(\sigma^*, 0)$. To this end, let

$$s = \sigma - \sigma^*, \qquad r^2 = s^2 + \tau^2, \tag{11}$$

and consider the values of σ and τ such that $r \ll 1$. We have

$$\sigma = \sigma^* + s, \qquad k = \sigma^*/\theta + s/\theta + 0(r^2), \qquad\qquad (12)$$
$$H(k) = H^* + H'^*s/\theta + 0(r^2),$$

where

$$H'^* = \left.\frac{dH(k)}{dk}\right|_{k=\sigma^*/\theta}. \qquad\qquad (13)$$

For the present purpose, it is more convenient to write the roots of Eq. (4) in the following form:

$$\frac{1}{c^2} = \frac{1}{2}\left(\frac{1}{c_2^2} + \frac{1}{c_0^2} + \rho H\frac{\sigma^2}{\theta^2} + \rho H\theta^2\tau^2\right)$$
$$\pm \frac{1}{2}\left\{\left(-\frac{1}{c_2^2} + \frac{1}{c_0^2} + \rho H\frac{\sigma^2}{\theta^2} - \rho H\theta^2\tau^2\right)^2 + (2\rho H\sigma\tau)^2\right\}^{1/2} \qquad (14)$$

where the plus sign is for c_s and the minus sign is for c_f. Substitution into the above equation from Eqs. (12) yields, after making use of Eq. (7b),

$$\frac{1}{c^2} = \frac{1}{c_2^2} + \rho\frac{\sigma^*}{\theta^2}h^*s \pm \left\{\left(\rho\frac{\sigma^*}{\theta^2}h^*s\right)^2 + \left(\frac{1}{c_2^2} - \frac{1}{c_0^2}\right)^2\frac{\theta^4\tau^2}{\sigma^{*2}}\right\}^{1/2} + 0(r^2) \qquad (15)$$

where

$$h^* = H^* + \tfrac{1}{2}\sigma^*H'^*/\theta. \qquad\qquad (16)$$

If we ignore the higher-order terms, it is seen from Eq. (15) that the contour lines for constant values of c are parabolas in the $s \sim \tau$ plane with the s-axis as the axis of the parabolas.

With c given by Eq. (15), Eq. (10) can now be written, after ignoring the terms of $0(r)$, as:

$$ds/d\tau = \eta(s/\tau) \pm \{(\eta(s/\tau))^2 + 1\}^{1/2} \quad \text{for} \quad s^2 + \tau^2 \ll 1, \qquad (17)$$

where η is the non-dimensional material constant defined by

$$\eta = h^*/(\theta^2H^*). \qquad\qquad (18)$$

In the next section we will show that $\eta > 0$ if the stress-strain curve for a simple tension test is concave to the strain axis.

Eq. (17) can be integrated analytically. Before we present the solution, the following observations should be noted:

(1) The plus sign in Eq. (17) is for the slow simple wave paths and the minus sign is for the fast simple wave paths. It is readily shown that

$$(ds/d\tau)_+ \cdot (ds/d\tau)_- = -1 \qquad\qquad (19)$$

provided s and τ are not both zero. Therefore the stress paths for the fast and slow simple waves are orthogonal to each other in the neighborhood of the origin $s = 0$, $\tau = 0$ not including the origin.

(2) If we replace s by $-s$, $(ds/d\tau)_+$ becomes $-(ds/d\tau)_-$. The stress paths for the

fast simple waves can therefore be obtained from the stress paths for the slow simple waves by a simple reflection on the τ-axis in the $s \sim \tau$ plane (see Fig. 1). In the following, it suffices to discuss Eq. (17) with the plus sign in the right-hand side.

(3) If we replace s and τ by αs and $\alpha\tau$ respectively where α is an arbitrary constant, Eq. (17) remains unchanged. Hence all curves are *similar*. If we know one curve which is a solution of Eq. (17), the other curves can be obtained by a "homothetic transformation", i.e. by a uniform enlargement or contraction about the origin.

(4) From Eq. (17) it is seen that $ds/d\tau$ is constant along the radial line s/τ = constant. Thus all curves intersect the radial line at the same angle. In particular, they intersect the line $s = 0$ at 45° and intersect the negative s axis at 90°. For $0 < \eta < \frac{1}{2}$, $ds/d\tau$ becomes identical to s/τ at

$$\tau = s \tan \phi^* \tag{20}$$

where

$$\tan \phi^* = (1 - 2\eta)^{1/2}, \qquad 0 < \eta < \tfrac{1}{2}. \tag{21}$$

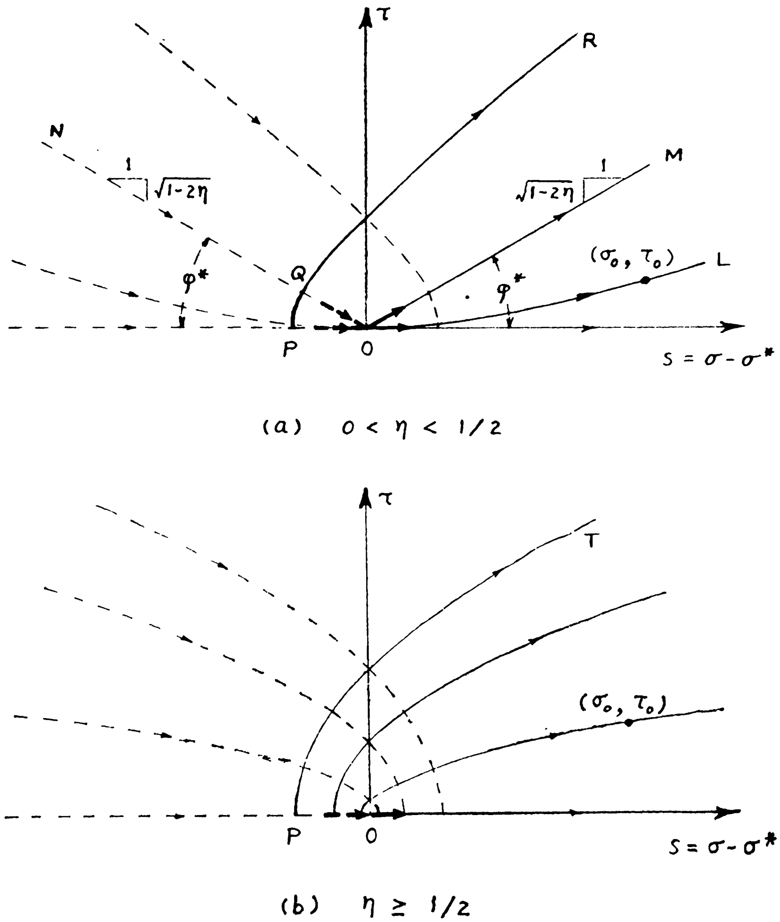

Fig. 1. Stress paths for fast (dotted lines) and slow (solid lines) simple waves.

Therefore, the radial line given by Eq. (20) is also a solution of Eq. (17) when $0 < \eta < \frac{1}{2}$ (see Fig. 1a).

To integrate Eq. (17) analytically, all we have to do is to use (s/τ) as the new dependent variable instead of s. The resulting equations can then be integrated. If we use the polar coordinate system

$$s = r \cos \phi, \qquad \tau = r \sin \phi, \tag{22}$$

the solution can be expressed as

$$\left(\frac{r}{r_0}\right)^{1-2\eta} = \frac{|((\eta \cos \phi)^2 + \sin^2 \phi)^{1/2} - \eta \cos \phi|^\eta}{(2\eta)^\eta\, |((\eta \cos \phi)^2 + \sin^2 \phi)^{1/2} - (1 - \eta) \cos \phi|^{1-\eta}} \tag{23a}$$

for $\eta \neq \frac{1}{2}$, and

$$\frac{r}{r_0} = \frac{2}{(4 - 3 \cos^2 \phi)^{1/2} - \cos^2 \phi} \exp \left\{\frac{1}{2} + \frac{\cos \phi}{(4 - 3 \cos^3 \phi)^{1/2} - \cos \phi}\right\} \tag{23b}$$

for $\eta = \frac{1}{2}$. r_0 is an integration constant. In Eq. (23a), the absolute values should be taken in the numerator and the denominator before applying the exponential powers η and $(1 - \eta)$.

For $0 < \eta < \frac{1}{2}$, the curves in the $s - \tau$ plane can be divided into two groups. In Fig. 1(a), the line OM is itself a stress path for slow simple waves and is given by Eq. (20). The curve PQR is a typical stress path in the region above the line OM. Other stress paths in this region can be obtained by a uniform enlargement or contraction of the curve PQR. It can be shown that the curve PQR is nearest to the origin not at point P but at point Q, where PQR intersects the line ON, even though the curve is everywhere concave to the s-axis. In the region below OM the curve OL is a typical stress path for slow simple waves. All curves in this region are tangent to the s-axis at the origin. It is seen from Fig. 1(a) that at the origin the stress paths for the slow simple waves (solid arrowheads) and the stress paths for the fast simple waves (dotted arrowheads) are not orthogonal to each other.

For $\eta \geq \frac{1}{2}$, the stress paths are all similar to the typical curve PT shown in Fig. 1(b). The positive s-axis is itself a stress path for slow simple waves. Again, the stress paths for the fast and slow simple waves are not orthogonal to each other at the origin. In fact, there is no stress path passing through the origin other than the s-axis itself. It is interesting to point out that, when $\eta = 1$, Eq. (23a) reduces to an equation for a parabola. Hence the curves in Fig. 1(b) are parabolas when $\eta = 1$.

The value of η plays a significant role in the wave propagation. To illustrate this, let us consider a tube, initially pre-stressed at the stress state $(\sigma^*, 0)$, which is subjected to a constant longitudinal stress σ_0 and the torsional stress τ_0 at the end $x = 0$ of the tube. Suppose that $\sigma_0 > \sigma^*$ and τ_0 is small (see Fig. 1). The resulting wave propagations are shown in Figs. 2(a) and 2(b). The wave patterns are quite different depending on whether $0 < \eta < \frac{1}{2}$ or $\eta \geq \frac{1}{2}$.

4. The material constant η. From Eqs. (16) and (18), η can be written as

$$\eta = (1/\theta^2) + (\sigma^*/2\theta^3)(H'^*/H^*). \tag{24}$$

The function H is related to the slope of the stress-strain curve for the simple tension test. If σ and ϵ are respectively the tensile stress and the strain, and

$$g(\sigma) = d\sigma/d\epsilon \tag{25}$$

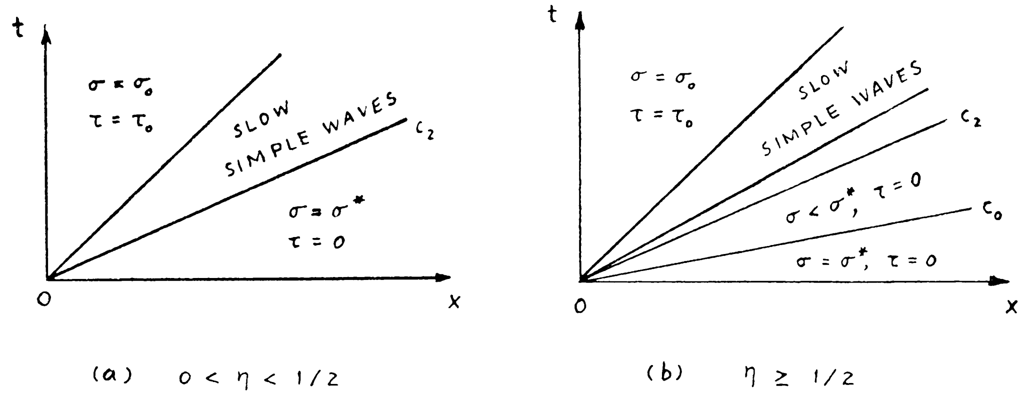

FIG. 2. Simple wave solutions.

is the slope which can be expressed as a function of the tensile stress, it is shown in [1] that

$$H(k) = \frac{1}{k^2}\left\{\frac{1}{g(\theta k)} - \frac{1}{E}\right\}. \tag{26}$$

Substitution of Eq. (26) into Eq. (24) yields the result

$$2\theta^2\eta\left\{\frac{E}{g(\sigma^*)} - 1\right\} = \sigma^* \frac{d}{d\sigma^*}\left\{\frac{E}{g(\sigma^*)} - 1\right\} \tag{27}$$

or

$$2\theta^2\eta\left\{\frac{1}{g(\sigma^*)} - \frac{1}{E}\right\} = -\sigma^* g'(\sigma^*)/g^2(\sigma^*) \tag{28}$$

where $g'(\sigma) = dg/d\sigma$ which is negative if the stress-strain curve for the simple tension test is concave to the strain axis. Hence η as obtained from Eq. (28) is always positive.

Eq. (27) can be rewritten as

$$2\theta^2\eta = \left.\frac{d[\log(E/g(\sigma) - 1)]}{d[\log(\sigma/\sigma_Y)]}\right|_{\sigma=\sigma^*} \tag{29}$$

where σ_Y is the initial yield stress. From Eqs. (7b) and (26), $g(\sigma^*) = \mu = E/2(1 + \nu)$ and

$$E/g(\sigma^*) - 1 = 1 + 2\nu, \tag{30}$$

where ν is Poisson's ratio. Eqs. (29) and (30) suggest how one may determine the value η graphically if the relation between $(E/g(\sigma) - 1)$ and (σ/σ_Y) is drawn on a logarithmic scale. This is depicted in Fig. 3.

5. The unloading of a plastic wave front. Let us consider the solution of Eq. (1) subject to the following initial and boundary conditions:

$$\mathbf{w}(x, 0) = \mathbf{w}^*, \qquad \sigma(0, t) = \sigma^* + at, \qquad \tau(0, t) = bt, \tag{31}$$

where $\mathbf{w}^*$ has the elements $(0, 0, \sigma^*, 0)$ and a and b are positive constants. Clearly the tube is initially at rest and is pre-stressed at the stress state $(\sigma^*, 0)$ such that $c_f = c_s = c_2$.

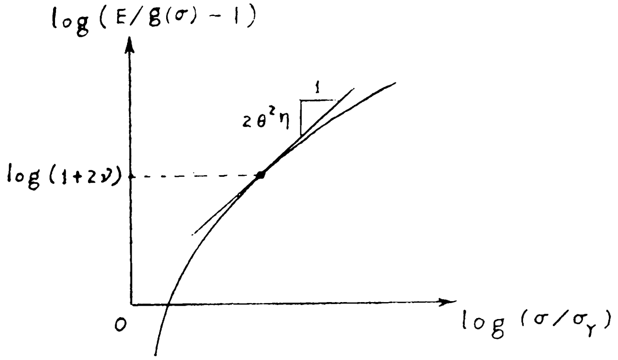

FIG. 3. The material constant η.

Therefore $w(x, t) = w^*$ for $x \geq c_2 t$, and the initial disturbance is propagated at the speed c_2 unless an unloading occurs first. Although $\mathbf{w}(x, t) = \mathbf{w}^* =$ constant along $x = c_2 t$ which is a characteristic, the solution in the region $x \leq c_2 t$ is not necessarily a simple wave solution. This is so because the system along the line $x = c_2 t$ is not totally hyperbolic and Lax's theorem [7, 8] does not apply here.

We will try to find the solution in the region $x \leq c_2 t$ and in the neighborhood of $x = c_2 t$. To this end, let

$$\mathbf{w}(x, t) = \mathbf{w}^* + \mathbf{w}_1(x)(t - x/c_2) + \tfrac{1}{2}\mathbf{w}_2(x)(t - x/c_2)^2 + \cdots \tag{32}$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are functions of x only. In particular, we have

$$\begin{aligned} \sigma(x, t) &= \sigma^* + \sigma_1(x)(t - x/c_2) + O(t - x/c_2)^2, \\ \tau(x, t) &= 0 + \tau_1(x)(t - x/c_2) + O(t - x/c_2)^2, \end{aligned} \tag{33}$$

and use of this expression for the matrix $\mathbf{A}$ of Eq. (2) yields

$$\mathbf{A} = \mathbf{A}^* + \mathbf{A}_1(t - x/c_2) + O(t - x/c_2)^2 \tag{34}$$

where

$$\mathbf{A}^* = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & 1/\rho c_2^2 & 0 \\ 0 & 0 & 0 & 1/\rho c_2^2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sigma^* h^* \sigma_1/\theta^2 & H^*\sigma^*\tau_1 \\ 0 & 0 & H^*\sigma^*\tau_1 & 0 \end{bmatrix}.$$

If we substitute Eqs. (32) and (34) into (1), rearrange the equation in a power series of $(t - x/c_2)$ and put the coefficients of each term equal to zero, we obtain:

$$(c_2\mathbf{A}^* - \mathbf{B})\mathbf{w}_1(x) = 0 \tag{35}$$

$$(c_2\mathbf{A}^* - \mathbf{B})\mathbf{w}_2(x) = -c_2(\mathbf{A}_1\mathbf{w}_1 + \mathbf{B}\mathbf{w}_1') \tag{36}$$

where a prime stands for differentiation with respect to x. From Eq. (35) we obtain

$$\mathbf{w}_1 = \begin{bmatrix} -\sigma_1(x)/\rho c_2 \\ -\tau_1(x)/\rho c_2 \\ \sigma_1(x) \\ \tau_1(x) \end{bmatrix}. \tag{37}$$

With $\mathbf{w}_1$ given by Eq. (37), the right-hand side of Eq. (36) can be written as

$$-c_2(\mathbf{A}_1\mathbf{w}_1 + \mathbf{B}\mathbf{w}_1') = -c_2 \begin{bmatrix} -\sigma_1' \\ -\tau_1' \\ 2\sigma^* h^* \sigma_1^2/\theta^2 + H^*\sigma^*\tau_1^2 + \sigma_1'/\rho c_2 \\ H^*\sigma^*\sigma_1\tau_1 + \tau_1'/\rho c_2 \end{bmatrix}. \tag{38}$$

Now $\mathbf{w}_2$ of Eq. (36) has a solution only if the vector on the right hand side, namely Eq. (38), is orthogonal to every $\mathbf{1}$ of the following equation [12]:

$$\mathbf{1}^T(c_2\mathbf{A}^* - \mathbf{B}) = 0. \tag{39}$$

Since $\mathbf{1}^T$ has two independent solutions,

$$(-1/\rho c_2\,,\, 0,\, 1,\, 0), \qquad (0,\, -1/\rho c_2\,,\, 0,\, 1), \tag{40}$$

the orthogonality of the vector given by Eq. (38) to the vectors given by (40) furnishes the following two differential equations for $\sigma_1(x)$ and $\tau_1(x)$:

$$\sigma_1' = -\beta(2\eta\sigma_1^2 + \tau_1^2), \qquad \tau_1' = -\beta\sigma_1\tau_1, \tag{41a,b}$$

where η is defined in Eq. (18) and

$$\beta = \tfrac{1}{2}\rho c_2 H^*\sigma^* = \tfrac{1}{2}c_2\left(\frac{1}{c_2^2} - \frac{1}{c_0^2}\right)\frac{\theta^2}{\sigma^*} \tag{42}$$

The initial conditions for Eqs. (41) are obtained from Eqs. (31) and (33):

$$\sigma_1(0) = a, \qquad \tau_1(0) = b. \tag{43}$$

Since both η and β are positive constants, we conclude from Eq. (41a) that $\sigma_1(x)$ is a monotonically decreasing function of x. In view of the fact that $\sigma_1 = a > 0$ at $x = 0$, σ_1 may vanish at certain x, say at $x = \xi$. Thus

$$\begin{aligned} \sigma_1(x) &> 0, \qquad 0 \le x < \xi, \\ \sigma_1(x) &< 0, \qquad \xi < x. \end{aligned} \tag{44}$$

It should be noticed that if ξ exists, an unloading will occur at $x = \xi$. This is so because if we substitute Eq. (33) into (3), we obtain

$$k^2(x, t) = (\sigma^*/\theta)^2 + 2(\sigma^*/\theta^2)\sigma_1(x)(t - x/c_2) + O(t - x/c_2)^2. \tag{45}$$

Therefore $\partial k/\partial t < 0$ at the wave front $x = c_2 t$ if $\sigma_1(x) < 0$.

Now Eqs. (41) can be integrated as follows. Elimination of σ_1 between the two equations gives

$$(2\eta + 1)(\tau_1')^2 + \beta^2\tau_1^4 - \tau_1\tau_1'' = 0. \tag{46}$$

Let

$$\tau_1' = p, \qquad \tau'' = dp/dx = (dp/d\tau_1)p. \tag{47}$$

Eq. (46) is then reduced to a first-order differential equation which can be integrated. The result is, after taking into consideration Eq. (43),

$$p = -\beta b^2(\tau_1/b)^{(1+2\eta)}\{([\tau_1/b]^{2(1-2\eta)} - [1 - (1 - 2\eta)a^2/b^2])/(1 - 2\eta)\}^{1/2} \tag{48a}$$

for $\eta \neq \frac{1}{2}$ and

$$p = -\beta\tau_1^2\{(a/b)^2 - 2\log(\tau_1/b)\}^{1/2} \tag{48b}$$

for $\eta = \frac{1}{2}$. Since $p = d\tau_1/dx$, integration of Eqs. (48) yields

$$x = \frac{1}{\beta b}\int_{\tau_1/b}^{1} \frac{d\lambda}{\lambda^{1+2\eta}\{(\lambda^{2(1-2\eta)} - [1 - (1 - 2\eta)a^2/b^2])/(1 - 2\eta)\}^{1/2}} \tag{49a}$$

for $\eta \neq \frac{1}{2}$ and

$$x = \frac{1}{\beta b}\int_{\tau_1/b}^{1} \frac{d\lambda}{\lambda^2(a^2/b^2 + 2\log\lambda)^{1/2}} \tag{49b}$$

for $\eta = \frac{1}{2}$. Eqs. (49) are the solution for $\tau_1(x)$.

The solution for $\sigma_1(x)$ can be obtained by several means. We will use the following approach which is useful in locating the value ξ. From Eqs. (41a) and (41b),

$$d\sigma_1/d\tau_1 = (2\eta\sigma_1^2 + \tau_1^2)/\sigma_1\tau_1 .$$

Upon integration, taking into account the initial conditions Eq. (43), we obtain

$$\tau_1^{2(1-2\eta)}[1 - (1 - 2\eta)\sigma_1^2/\tau_1^2] = b^{2(1-2\eta)}[1 - (1 - 2\eta)a^2/b^2] \tag{50a}$$

for $\eta \neq \frac{1}{2}$ and

$$\sigma_1^2/\tau_1^2 = a^2/b^2 - \log(b/\tau_1)^2 \tag{50b}$$

for $\eta = \frac{1}{2}$. With $\tau_1(x)$ given by Eqs. (49), Eqs. (50) furnish the solution for $\sigma_1(x)$.

From Eqs. (50) we notice that τ_1 cannot become zero before σ_1 does. Otherwise the left-hand sides of the equations would not be equal to the right-hand sides of the equations. Hence

$$0 < \tau_1(x) < b \quad \text{for} \quad 0 < x < \xi.$$

Also from Eqs. (50) we notice that $\sigma_1 = 0$ when

$$[\tau_1(\xi)/b]^{2(1-2\eta)} = 1 - (1 - 2\eta)a^2/b^2 \tag{51a}$$

for $\eta \neq \frac{1}{2}$ and

$$[\tau_1(\xi)/b]^2 = \exp(-a^2/b^2) \tag{51b}$$

for $\eta = \frac{1}{2}$. Since $\tau_1(\xi)/b < 1$, Eqs. (51a) and (51b) have a solution for $\tau_1(\xi)$ except possibly when $0 < \eta < \frac{1}{2}$. When $0 < \eta < \frac{1}{2}$, Eq. (51a) can be written as

$$[\tau_1(\xi)/b]^{2(1-2\eta)} = 1 - (\tan\phi^*/\tan\psi)^2 \tag{52}$$

where ϕ^* is defined in Eq. (21) and

$$\tan\psi = b/a. \tag{53}$$

In view of the fact that $\tau_1(\xi)/b < 1$, Eq. (52) has a solution for $\tau_1(\xi)$ only when $\phi^* < \psi$. We conclude therefore that an unloading occurs at $x = \xi$ for all cases except when $0 < \eta < \frac{1}{2}$ and $\phi^* \geq \psi$. When ξ exists, ξ is determined by substituting τ_1/b of Eqs. (51) into Eqs. (49) with x in the latter replaced by ξ. A typical wave pattern when ξ exists is shown in Fig. 4, where N, E and P stand for neutral, elastic and plastic regions respectively. The solution in the region next to $x = c_2 t$ is clearly not a simple wave solution.

When $0 < \eta < \frac{1}{2}$ and $\phi^* = \psi$, $\sigma(0, t)$ and $\tau(0, t)$ are following the path OM (Fig. 1(a)) and the solution should be a simple wave solution. Indeed, when $0 < \eta < \frac{1}{2}$ and $\phi^* = \psi$, (i.e. $(1 - 2\eta)^{1/2} = b/a$), Eq. (49a) can be integrated to give

$$\tau_1/b = (1 + \beta a x)^{-1} \tag{54a}$$

and Eq. (50a) yields

$$\sigma_1/a = \tau_1/b. \tag{54b}$$

Another case when Eq. (49a) can be integrated is the case $\eta = 0$. Notice from Eq. (28) that $\eta = 0$ implies $g'(\sigma^*) = 0$. In other words, $\eta = 0$ corresponds to the case when the curvature of the stress-strain curve for a simple tension test vanishes at $\sigma = \sigma^*$. An example of this case is when the material is linearly work-hardening and $g(\sigma) = \mu$ for $\sigma > \sigma_Y$. A detailed study of wave propagation of combined stress in linearly work-hardening materials can be found in [13]. When $\eta = 0$, Eqs. (49a) and (51a) furnish $\tau_1(x)$ and $\sigma_1(x)$ in the following forms:

$\eta = 0,\ a > b$:

$$\tau_1 = (a^2 - b^2)^{1/2} \operatorname{csch} [\cosh^{-1} (a/b) + (a^2 - b^2)^{1/2} \beta x], \tag{55a}$$

$$\sigma_1 = (a^2 - b^2)^{1/2} \coth [\cosh^{-1} (a/b) + (a^2 - b^2)^{1/2} \beta x]. \tag{55b}$$

$\eta = 0,\ a = b$:

$$\tau_1 = \sigma_1 = b/(1 + \beta b x). \tag{56}$$

$\eta = 0,\ a < b$:

$$\tau_1 = (b^2 - a^2)^{1/2} \sec [\sin^{-1} (a/b) - (b^2 - a^2)^{1/2} \beta x], \tag{57a}$$

$$\sigma_1 = (b^2 - a^2)^{1/2} \tan [\sin^{-1} (a/b) - (b^2 - a^2)^{1/2} \beta x], \tag{57b}$$

$$0 \leq x \leq \xi, \qquad (b^2 - a^2)^{1/2} \beta \xi = \sin^{-1} (a/b). \tag{57c}$$

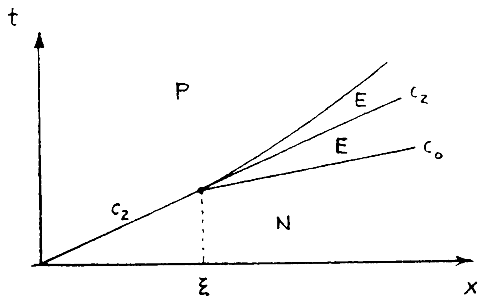

Fig. 4. Unloading at a plastic wave front.

Notice that when $a < b$, an unloading occurs at $x = \xi$ and ξ is given by Eq. (57c). Notice also that since

$$\cosh^{-1}(a/b) = \operatorname{csch}^{-1}(b/(a^2 - b^2)^{1/2}) = \coth^{-1}(a/(a^2 - b^2)^{1/2}),$$
$$\sin^{-1}(a/b) = \sec^{-1}(b/(b^2 - a^2)^{1/2}) \mp \tan^{-1}(a/(b^2 - a^2)^{1/2}),$$

the solutions given by Eqs. (55) and (57) satisfy the initial conditions (Eq. (43)).

6. Concluding remarks. In this paper unexpected phenomena in elastic-plastic wave propagation are presented for the particular situation in which $c_f = c_s = c_2$ occurs. The problem of combined longitudinal and torsional waves in a thin-walled tube is used as an example because this is the simplest problem in combined stress waves which possesses most features of more complicated combined stress waves. It is expected that phenomena similar to the ones presented here can be found in other types of combined stress waves [3, 6] and other types of materials [5].

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