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On Weak and Monotone σ-Closures of C^*-Algebras

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Abstract. It is proved that the monotone σ-closure of the self-adjoint part of any C^*-algebra A is the self-adjoint part of a C^*-algebra $\mathscr{B}$. If A is of type I it is proved that $\mathscr{B}$ is weakly σ-closed, i.e. $\mathscr{B}$ is a Σ^*-algebra. The physical importance of Σ^*-algebras was explained in [1] and [7].

We recall that the class of bounded real Baire functions $\mathscr{B}^R(X)$ on a locally compact Hausdorff space X is defined as the monotone σ-closure of $C_0^R(X)$. It is immediately verified that $\mathscr{B}^R(X)$ is closed under pointwise limits of sequences hence $\mathscr{B}^R(X)$ is also the weak (pointwise) σ-closure of $C_0^R(X)$.

Regarding a C^*-algebra A as the non-commutative analogue of some $C_0(X)$ we may for a convenient representation of A as operators on a Hilbert space H form the monotone σ-closure $\mathscr{B}_A^R$ of A^R in $B(H)$. This class of Baire operators was introduced in [5] by R. V. KADISON in order to give measure-theoretic conditions on a representation between two concrete C^*-algebras to have a normal extension. His result together with those of [6] seem to indicate that $\mathscr{B}_A^R$ is able to take over the rôle played by the Baire functions in commutative theory.

Recently E. B. DAVIES in [1], [2] and [3] has considered instead the weak σ-closure of A and has outlined an interesting theory of Σ^*-algebras i.e. C^*-algebras which are weakly σ-closed. Since for non-commutative C^*-algebras one cannot use lattice arguments it is no more an easy matter to determine whether the weak and monotone σ-closure of A^R coincide. We prove in this paper that such is indeed the case if A is of type I. Unfortunately the proof will not be applicable for other types but since we are able to show in general that $\mathscr{B}_A^R$ is the self-adjoint part of a C^*-algebra we feel rather optimistic that the result is true in general i.e. that $\mathscr{B}_A^R + i\mathscr{B}_A^R$ is a Σ^*-algebra.

We shall use [4] as a standard reference on notations and terminology. In particular for a C^*-algebra A we shall write A'' for the enveloping von Neumann algebra of A in its universal representation. When no confusion may arise we shall drop the subscript and write $\mathscr{B}^R$ for the monotone σ-closure of A^R in A''.

Theorem 1. *$\mathscr{B}^R$ is the self-adjoint part of a C^*-algebra.*

Proof. By [5, p. 317] $\mathscr{B}^R$ is a uniformly closed Jordan algebra. Therefore $\mathscr{B} = \mathscr{B}^R + i\mathscr{B}^R$ is a uniformly closed subspace clearly self-adjoint and since by polarization any product of elements from $\mathscr{B}$ can be expressed as linear combination of elements of the form $(x+iy)^*(x+iy)$, $x, y \in \mathscr{B}^R$ we see that all we have to prove is that the commutator $[x, y] = i(xy - yx)$ is in $\mathscr{B}^R$ since then

$$(x + iy)^*(x + iy) = x^2 + y^2 + [x, y] \in \mathscr{B}^R .$$

We have

$$[x, y] = (1 + iy)^* x(1 + iy) - x - yxy .$$

Since $x, y \in A^R$ implies $[x, y] \in A^R$ and since the above formula shows that the operators $x \in \mathscr{B}^R$ such that $[x, y] \in \mathscr{B}^R$ for any $y \in A^R$ is a monotone σ-class we have $[x, y] \in \mathscr{B}^R$ for any $x \in \mathscr{B}^R$, $y \in A^R$. But since $-[x, y] = [y, x]$ we can also use the formula to show that the operators $y \in \mathscr{B}^R$ such that $[x, y] \in \mathscr{B}^R$ for any $x \in \mathscr{B}^R$ is a monotone σ-class. Since this class contains A^R by the first statement we have $[x, y] \in \mathscr{B}^R$ for any $x, y \in \mathscr{B}^R$.

Henceforth we shall refer to $\mathscr{B} = \mathscr{B}^R + i\mathscr{B}^R$ as the Baire operators of A.

The next lemma is a somewhat technical result which allow us to identify the Baire operators of a C^*-subalgebra B of A with the monotone σ-closure of B in $\mathscr{B}_A$.

Lemma 2. *If Φ is a *-isomorphism of the C^*-algebra B into the C^*-algebra A then the extension of Φ from B'' into A'' is a normal isomorphism.*

Proof. The extension (again denoted Φ) defined in [4, 12.1.5. Proposition] maps B'' onto the weak closure of $\Phi(B)$. However, if H is the universal Hilbert space of B then Φ^{-1} is a representation of $\Phi(B)$ on H hence by [4, 2.10.2. Proposition] there is a representation ψ of A on a Hilbert space K containing H as a subspace such that if p is the projection of K onto H then $p\psi(x) = \Phi^{-1}(x)$ for all $x \in \Phi(B)$. Since ψ has a normal extension as well, we see that the map $x \to p\psi(\Phi(x))$ is a normal automorphism of B'' which is the identity on B. It follows that $\Phi^{-1}(x) = p\psi(x)$ for all $x \in B''$.

Lemma 3. *If a subset L of $\mathscr{B}^R$ consists of commuting elements then the weak σ-closure of L is contained in $\mathscr{B}^R$.*

Proof. Since $\mathscr{B}^R$ is a uniformly closed Jordan algebra we may assume that L is a uniformly closed algebra over the reals. Then $L + iL$ is a commutative C^*-algebra hence $L = C_0^R(X)$ for some locally compact space X. It is then known that the weak σ-closure of L coincides with the monotone σ-closure.

By [1, Theorem 3.2.] the map from A'' to the weak closure of A in its reduced atomic representation is isomorphic on the weak σ-closure of A. In particular the elements of $\mathscr{B}$ are determined completely by their values under the irreducible representations of A, hence the following definition makes sense: For any two operators x, y in $\mathscr{B}$ we call y *the pointwise normalized of* x if for all $\pi \in \hat{A}$

$$\pi(y) = \|\pi(x)\|^{-1} \pi(x) \quad \text{for} \quad \pi(x) \neq 0\,, \qquad \pi(y) = 0 \quad \text{for} \quad \pi(x) = 0\,.$$

Lemma 4. *If A is a separable C^*-algebra with continuous trace then for any $x \in \mathscr{B}^R$ the pointwise normalized of x is also in $\mathscr{B}^R$.*

Proof. By [6, Proposition 5.3] the functions f_n on $\hat{A}$ defined by $f_n(\pi) = \left(\|\pi(x)\| + \frac{1}{n}\right)^{-1}$ are bounded Baire functions hence by [6, Proposition 4.6] the elements $f_n \cdot x$ belong to $\mathscr{B}^R$. $\{f_n \cdot x\}$ is a commuting sequence converging weakly to the pointwise normalized of x hence this element is in $\mathscr{B}^R$ by Lemma 3.

For the sake of completeness we insert a proof of the following result from [5, p. 323]:

Lemma 5. *If $y \in A''^+$, $x \in \mathscr{B}^+$ and $xyx \in \mathscr{B}^+$ then $[x]\, y\, [x] \in \mathscr{B}^+$.*

Proof. We define the real Baire functions f_n by

$$f_n(t) = 0 \quad \text{for} \quad t \leqq \frac{1}{n}\,, \qquad f_n(t) = \frac{1}{t} \quad \text{for} \quad t > \frac{1}{n}$$

and have projections $p_n = f_n(x)\, x \in \mathscr{B}^+$ with $p_n \nearrow p = [x]$.
For $m \geqq n$ this gives

$$\begin{aligned}
&p_n y p_m + p_m y p_n = p_n (p_m y p_m) + (p_m y p_m)\, p_n \in \mathscr{B}^R \\
&\quad\Rightarrow (p_n y p + p)\, p_m (p + p y p_m) = p_n y p_m y p_n + p_n y p_m + p_m y p_n \\
&\qquad + p_m \in \mathscr{B}^R \\
&\quad\Rightarrow (p_n y p + p)\,(p + p y p_n) \in \mathscr{B}^R \wedge p_n y p y p_n \in \mathscr{B}^R \\
&\quad\Rightarrow p_n y p + p y p_n \in \mathscr{B}^R \Rightarrow (p_n y p + p y p_n)^2 \in \mathscr{B}^R \Rightarrow p y p_n y p \in \mathscr{B}^R \\
&\quad\Rightarrow p y p y p \in \mathscr{B} \\
&\quad\Rightarrow p y p \in \mathscr{B}
\end{aligned}$$

Lemma 6. *Let A be a separable C^*-algebra with continuous trace and homogeneous of degree $d \leqq \aleph_0$. There exist a set of pairwise orthogonal projections $\{p_n\} \subset \mathscr{B}$, $\operatorname{card}\{p_n\} = d$, $\dim \pi(p_n) = 1$ for all $\pi \in \hat{A}$ and $\Sigma p_n = 1$.*

Proof. If $\{\pi_m\}$ is a sequence dense in $\hat{A}$ then by [4, 4.5.3. Proposition] there exists for each m an element $x_m \in A^+$ and an open neighbourhood $\mathcal{O}_m$ of π_m such that $\pi(x_m)$ is a one-dimensional projection for $\pi \in \mathcal{O}_m$. If f_m denotes the characteristic function for the Baire set $\mathcal{O}_m \backslash (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{m-1})$ then $p_1 = \Sigma f_m \cdot x_m$ is a projection in $\mathscr{B}$ by [6, Proposition 4.6] and since $\cup\, \mathcal{O}_m = \hat{A}$ we have $\dim \pi(p_1) = 1$ for all $\pi \in \hat{A}$.

If $\tilde{A}$ denotes the C^*-algebra obtained when an identity is adjoined to A then we can choose a sequence $\{u_m\}$ dense in the unitary group of $\tilde{A}$.

Suppose now that we have already constructed $p_1 \dots p_n$ subject to the conditions of the lemma and put $q_0 = \Sigma p_i$. Then define

$$q_1 = (u_1^* p_1 u_1) \vee q_0 - q_0$$

$$q_{m+1} = (u_{m+1}^* p_1 u_{m+1}) \vee (q_0 + c\text{-supp}(q_1 + \cdots + q_m)) - (q_0 + c\text{-supp}(q_1 + \cdots + q_m)) .$$

Since for any projection $q \in \mathscr{B}$ we have

$$c\text{-supp}\, q = \bigvee_m u_m^* q u_m \in \mathscr{B}$$

it follows that $\{q_m\}$ is a sequence of projections in $\mathscr{B}$ with orthogonal central supports and $\dim \pi(q_m) \leqq 1$ for any $\pi \in \hat{A}$. Hence $p_{n+1} = \Sigma q_m \in \mathscr{B}$ is orthogonal to q_0 and $\dim \pi(p_{n+1}) \leqq 1$ for all $\pi \in \hat{A}$. If however $\pi(p_{n+1}) = 0$ for some $\pi \in \hat{A}$ then $\pi(u_m^* p_1 u_m) \leqq \pi(q_0)$ for all m hence $\pi(q_0) = 1$. If d is finite this implies $n = d$ and $q_0 = 1$ hence the lemma is proved. If d is infinite it is impossible since $\dim \pi(q_0) = n$. So we may assume $\dim \pi(p_{n+1}) = 1$ for all $\pi \in \hat{A}$ and continuing in this fashion we get a sequence $\{p_n\}$ with $\operatorname{card}\{p_n\} = d$.

If we have chosen $u_1 = 1$ then we can show that for all n

$$u_n^* p_1 u_n \leqq p_1 + \cdots + p_n .$$

Suppose this has been established for all $m \leqq n$ and put $q_0 = p_1 + \cdots + p_n$. Then in the construction for p_{n+1} given above we have $q_m = 0$ for all $m \leqq n$ hence

$$q_{n+1} = (u_{n+1}^* p_1 u_{n+1}) \vee q_0 - q_0$$

$$\Rightarrow u_{n+1}^* p_1 u_{n+1} \leqq q_0 + q_{n+1} \leqq p_1 + \cdots + p_{n+1} .$$

It follows that

$$1 = c\text{-supp}\, p_1 = \bigvee_n u_n^* p_1 u_n \leqq \Sigma p_n \leqq 1$$

and the lemma is proved.

For a topological space T and a Hilbert space H_d of dimension $d \leqq \aleph_0$ we let $\mathscr{B}(T, \mathscr{B}_d)$ denote the set of functions $x: T \to \mathscr{B}_d$, $\mathscr{B}_d$ denoting the bounded operators on H_d, such that for each $\xi \in H_d$ the function $t \to (x(t)\, \xi \mid \xi)$ is a bounded Baire function on T. It is easily verified that $\mathscr{B}(T, \mathscr{B}_d)$ is a Σ^*-algebra and that $x \in \mathscr{B}^R(T, \mathscr{B}_d)$ iff $x(t)$ is self-adjoint for all $t \in T$ i.e. iff $t \to (x(t)\, \xi_n \mid \xi_n)$ is a real bounded Baire function on T for some complete orthonormal basis $\{\xi_n\} \subset H_d$.

Proposition 7. (E. B. DAVIES). *If A is a separable C^*-algebra with continuous trace and homogeneous of degree $d \leqq \aleph_0$ then $\mathscr{B} = \mathscr{B}(\hat{A}, \mathscr{B}_d)$.*

Proof. Let $\{p_n\}$ be the set of projections constructed in Lemma 6. If $\{x_k\}$ is a sequence dense in A^R then for each n, m and k let y_{nmk} denote the pointwise normalized of $p_n x_k p_m + p_m x_k p_n$. We have $y_{nmk} \in \mathscr{B}^R$ by Lemma 4 hence

$$y_{nm} = \Sigma 3^{-k} y_{nmk} \in \mathscr{B}^R .$$

We notice that $\|\pi(y_{nm})\| \neq 0$ for all $\pi \in \hat{A}$ since $\|\pi(y_{nmk})\|$ is either 0 or 1 and since for each π there is a smallest k such that $\|\pi(p_n x_k p_m + p_m x_k p_n)\| \neq 0$. It follows that if v_{nm} denotes the pointwise normalized of y_{nm} we may in each H_π choose a basis such that

$$\pi(p_n) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\pi(p_m) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\pi(v_{nm}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By Theorem 1 we have $w_{nm} = [p_n, v_{nm}] = [v_{nm}, p_m] \in \mathscr{B}^R$ with

$$\pi(w_{nm}) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

for all $\pi \in \hat{A}$.

If we make the obvious identification of all H_π and their various bases and regard $\mathscr{B}$ as functions from $\hat{A}$ to $\mathscr{B}_d$ then if $\{\xi_n\}$ denote the chosen basis in H_d we have for all $x \in \mathscr{B}^R$

$$(\pi(x)\,\xi_n \,|\, \xi_m) + (\pi(x)\,\xi_m \,|\, \xi_n) = 2 tr\, \pi(v_{nm} x)\,.$$

It follows from [6, Proposition 5.1] that the function $\pi \to tr\, \pi(v_{nm} x)$ is a bounded real Baire function hence $x \in \mathscr{B}^R(\hat{A}, \mathscr{B}_d)$.

Conversely if $x \in \mathscr{B}^R(\hat{A}, \mathscr{B}_d)$ then for any n, m there exist bounded real Baire functions α_{nm} and β_{nm} on $\hat{A}$ such that

$$\pi(p_n x p_m + p_m x p_n) = \begin{pmatrix} 0 & \alpha_{nm}(\pi) + i\beta_{nm}(\pi) \\ \alpha_{nm}(\pi) - i\beta_{nm}(\pi) & 0 \end{pmatrix}.$$

By [6, Proposition 4.6.] we have

$$p_n x p_m + p_m x p_n = \alpha_{nm} \cdot v_{nm} + \beta_{nm} \cdot w_{nm} \in \mathscr{B}^R\,.$$

Since $\mathscr{B}$ is uniformly closed this implies

$$(\Sigma 2^{-n} p_n)\, x\, (\Sigma 2^{-n} p_n) \in \mathscr{B}^R$$

hence by Lemma 5, $x \in \mathscr{B}^R$ and the proposition follows.

Theorem 8. *If A is a C^*-algebra of type I then $\mathscr{B}$ is a Σ^*-algebra.*

Proof. Suppose first that A is separable. Then by [4, 4.5.5. Théorème] combined with [4, 3.6.3. Proposition] we can find a countable ascending chain of ideals $\{I_\alpha\}$ such that $I_\alpha = \overline{\bigcup_{\beta<\alpha} I_\beta}$ if α is a limit ordinal and $A_\alpha = I_\alpha / I_{\alpha-1}$ is a separable C^*-algebra with continuous trace and homogeneous of degree $d_\alpha \leqq \aleph_0$ if α is not a limit ordinal. To each I_α corresponds a central projection $p_\alpha \in \mathscr{B}$ such that $I_\alpha = p_\alpha A'' \cap A$; hence by Lemma 2 $\mathscr{B}_{I_\alpha} = p_\alpha \mathscr{B}$. If α is not a limit ordinal then the quotient map $\Phi : I_\alpha \to A_\alpha$ extends to a normal homomorphism of I_α'' and since $\Phi(\mathscr{B}_{I_\alpha}) = \mathscr{B}_{A_\alpha}$ by [6, Proposition 4.2.] we have the isomorphisms

$$(p_\alpha - p_{\alpha-1})\,\mathscr{B} = (1 - p_{\alpha-1})\,\mathscr{B}_{I_\alpha} = \mathscr{B}_{A_\alpha} = \mathscr{B}(\hat{A}_\alpha, \mathscr{B}_{d_\alpha})\,.$$

Since therefore $\mathscr{B}$ is the countable direct sum of Σ^*-algebras it is itself a Σ^*-algebra.

In the general case we proceed as follows: If $\{x_n\} \subset \mathscr{B}$ is a sequence converging weakly to some x then by [6, Lemma 4.5] there is a separable C^*-subalgebra B of A such that $\{x_n\}$ is contained in the monotone σ-closure of B. Since by Lemma 2 this closure is isomorphic to $\mathscr{B}_B$ and since B as a subalgebra of A is of type I by [4, 4.3.5. Proposition] we have $x \in \mathscr{B}$ and the theorem is proved.

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