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On weak sharp minima in vector optimization with applications to parametric problems

by

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Abstract: In the paper we discuss the concepts of weak sharp solutions to vector optimization problems. As an application we provide sufficient conditions for stability of solutions in perturbed vector optimization problems.

Keywords: vector optimization, weak sharp solutions, stability.

1. Introduction

Let X and Y be normed spaces and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. We consider vector optimization problems of the form

$$(VOP) \qquad \begin{array}{c} \mathcal{K} - \min f(x) \\ \text{subject to } x \in A, \end{array}$$

where $f : X \to Y$ and $A \subset X$ is a feasible set. By $E \subset Y$ we denote the set of all global efficient points to (VOP), i.e., $\alpha \in E$ iff $(f(A) - \alpha) \cap (-\mathcal{K}) = \{0\}$ and by $S \subset X$ we denote the set of all its global solutions, $S = A \cap f^{-1}(E)$.

The role of weak sharp minima in scalar optimization in relation to stability of parametric problems and error bounds is widely recognized, see, e.g., Attouch and Wets (1993); Auslander and Crouzeix (1988); Azé and Corvellec (2002); Bonnans and Shapiro (2000); Burke and Deng (2002).

In vector optimization several definitions of global weak sharp solutions has been proposed, see, e.g., Bednarczuk (2004, 2007); Deng and Yang (2004), for the linear case.

The aim of this paper is to discuss several concepts of (global) weak sharp solutions to problem (VOP) and their applications to stability of parametric problems. In Section 2 weak sharp solutions to (VOP) are presented and their basic properties are elucidated. In Section 3 weak sharp solutions are exploited to formulate sufficient conditions for stability of parametric problems.

2. Global weak sharp solutions

By B_X and $\bar{B}_X$ we denote open and closed unit balls in X, respectively. For any set $C \subset X$, $d(x,C) = \inf\{\|x-c\| \ : \ c \in C\}$. For any $\alpha \in Y$ put $S_\alpha := \{x \in A : \ f(x) = \alpha\}$.

DEFINITION 1 (see Bednarczuk, 2007) *Let $\alpha \in E$. We say that the solution set S to* (VOP) *is* (globally) S_α-weak sharp *if there exists a constant $\tau > 0$ such that*

$$f(x) - \alpha \notin \tau d(x, S_\alpha) B_Y - \mathcal{K} \quad \text{for all} \quad x \in A \setminus S_\alpha. \tag{1}$$

Optimality conditions for S_α weak sharpness in the local setting have been recently investigated by Studniarski (2007). If $\operatorname{int}\mathcal{K} \neq \emptyset$, a point $x_0 \in A$ is a *weak solution* to (VOP), $x_0 \in WS$, if $(f(A) - f(x_0)) \cap (-\operatorname{int}\mathcal{K}) = \emptyset$. If there exists $\alpha \in E$ such that S is S_α-weak sharp, then $S = WS$.

Let $\alpha \in E$. We define a set-valued mapping $\mathcal{E}^\alpha : R_+ \rightrightarrows X$ as

$$\mathcal{E}^\alpha(\varepsilon) := A \cap f^{-1}(\alpha + \varepsilon B_Y - \mathcal{K}). \tag{2}$$

Clearly, $\mathcal{E}^\alpha(0) = S_\alpha$ and $\operatorname{graph}\mathcal{E}^\alpha = \{(\varepsilon, x) \in R_+ \times A \ : \ f(x) \in \alpha + \varepsilon B_Y - \mathcal{K}\}$. There exist approaches to well-posedness of (VOP) via continuity properties of set-valued mappings similar to $\mathcal{E}^\alpha$ (see e.g. Bednarczuk, 2004, 2007; Miglierina and Molho, 2003, 2007; Zaffaroni, 2003).

PROPOSITION 1 *Let $\alpha \in E$ and let S be S_α-weak sharp with constant $\tau > 0$.*
(i) *If f is Lipschitz on A with constant L, then $\tau \leq L$.*
(ii) *The following condition holds:*

(C1) *there exists $\varepsilon_0 > 0$ such that for each $0 \leq \varepsilon \leq \varepsilon_0$*

$$A \cap f^{-1}(\alpha + \varepsilon B_Y - \mathcal{K}) \subset S_\alpha + \varepsilon \frac{1}{\tau} B_Y.$$

Proof. (i) If $f : X \to Y$ is Lipschitz on A with constant $L > 0$, i.e.

$$\|f(x) - f(x')\| \leq L\|x - x'\| \quad \text{for all} \quad x, x' \in A,$$

then $\|f(x) - \alpha\| \leq L\|x - x'\|$ for any x, $x' \in A$, $f(x') = \alpha$, and, consequently, $\|f(x) - \alpha\| \leq Ld(x, S_\alpha)$ for all $x \in A$. On the other hand, $f(x) - \alpha \notin \tau d(x, S_\alpha) B_Y - \mathcal{K}$ for $x \in A \setminus S_\alpha$. In particular, $\|f(x) - \alpha\| \geq \tau d(x, S_\alpha)$ for $x \in A \setminus S_\alpha$, which gives the required inequality.

(ii) Suppose, on the contrary, that $(C1)$ does not hold, i.e., there exist sequences $\varepsilon_n \to 0^+$ and $(x_n) \subset A$ such that

$$f(x_n) \in \alpha + \varepsilon_n B_Y - \mathcal{K} \quad \text{for} \quad n \geq 1,$$

and $d(x_n, S_\alpha) > \varepsilon_n \frac{1}{\tau}$. Hence, for $n \geq 1$, $x_n \notin S_\alpha$, $\tau d(x_n, S_\alpha) > \varepsilon_n$ and

$$f(x_n) \in \alpha + \tau d(x_n, S_\alpha) B_Y - \mathcal{K},$$

which contradicts S_α-weak sharpness of S. ∎

Condition $(C1)$ of Proposition 1 (ii) can be rephrased by saying that the set-valued mapping $\mathcal{E}^{\alpha}$ defined by (2), is upper Lipschitz at $0 \in \operatorname{dom} \mathcal{E}$ with constant $\frac{1}{\tau} > 0$, where a set-valued mapping $\Gamma : X \rightrightarrows Y$ is *upper Lipschitz* at $x_0 \in \operatorname{dom} \Gamma$ with constant $L > 0$ if there exists $t > 0$ such that $\Gamma(x) \subset \Gamma(x_0) + L\|x - x_0\| B_Y$ for $x \in B(x_0, t)$.

Recall that $\alpha \in E$ is a *(global) strict efficient point to* (VOP) (Bednarczuk, 2004) if there exists a constant $\gamma > 0$ such that

$$f(x) - \alpha \notin \gamma \|f(x) - \alpha\| B_Y - \mathcal{K} \quad \text{for} \quad x \in A \quad f(x) \neq \alpha. \tag{3}$$

As before, if f is Lipschitz on A with constant L we have $\|f(x) - \alpha\| \leq L\|x - x'\|$ for all $x \in A$ and $x' \in S_\alpha$ and consequently $\|f(x) - \alpha\| \leq L d(x, S_\alpha)$ for all $x \in A$.

If moreover, S is S_α-weak sharp with constant $\tau > 0$ we get

$$f(x) - \alpha \notin \frac{\tau}{L} \|f(x) - \alpha\| B_Y - \mathcal{K} \quad \text{for} \quad x \in A \setminus S_\alpha, \tag{4}$$

which means that $\alpha \in E$ is strict efficient with constant $\frac{\tau}{L}$.

In this way we proved the following proposition.

Proposition 2 *Let f be Lipschitz on A with constant $L > 0$. If S is S_α-weak sharp with constant $\tau > 0$, then $\alpha \in E$ is strict efficient with constant $\frac{\tau}{L}$.*

Definition 2 (see Bednarczuk, 2007) *Let $\alpha \in E$. We say that the solution set S to (VOP) is* α-weak sharp *if there exists a constant $\tau > 0$ such that*

$$f(x) - \alpha \notin \tau d(x, S) B_Y - \mathcal{K} \quad \textit{for all} \quad x \in A \setminus S. \tag{5}$$

If, for some $\alpha \in E$, the solution set S is S_α-weak sharp, then S is α-weak sharp.

Proposition 3 *Let $\alpha \in E$. If S is α-weak sharp with constant $\tau > 0$, the following condition holds:*
(C2) *there exists $\varepsilon_0 > 0$ such that for each $0 \leq \varepsilon \leq \varepsilon_0$*

$$A \cap f^{-1}(\alpha + \varepsilon B_Y - \mathcal{K}) \subset S + \varepsilon \frac{1}{\tau} B_Y.$$

Proof. Suppose, on the contrary, that $(C2)$ does not hold, i.e., there exist sequences $\varepsilon_n \to 0^+$ and $(x_n) \subset A$ such that $f(x_n) \in \alpha + \varepsilon_n B_Y - \mathcal{K}$ and $d(x_n, S) > \varepsilon_n \frac{1}{\tau}$ for $n \geq 1$. Hence, $x_n \notin S$, $\tau d(x_n, S) > \varepsilon_n$ and $f(x_n) \in \alpha + \tau d(x_n, S) B_Y - \mathcal{K}$, which contradicts α-weak sharpness of S. ■

Consider now linear multicriteria problems of the form

$$(LMP) \qquad \begin{array}{c} R^m_+ - \min Cx \\ \text{subject to } x \in A, \end{array}$$

where R^m_+ is a nonnegative orthant, $C : R^n \to R^m$ is a linear mapping and $A \subset R^n$ is polyhedral set. According to Deng and Yang (2004), WS is a set of *weak sharp solutions to* (LMP) if there exists a constant $\tau > 0$ such that

$$d(Cx, WE) \geq \tau d(x, WS) \quad \text{for} \quad x \in A, \tag{6}$$

where $WE = f(WS)$. Basing ourselves on this idea we define weak sharp solutions to (VOP).

DEFINITION 3 *We say that the solution set S to (VOP) is* (globally) weak sharp *if there exists a constant $\tau > 0$ such that*

$$d(f(x), E) \geq \tau d(x, S) \quad \textit{for all} \quad x \in A. \tag{7}$$

PROPOSITION 4 *Let $\tau > 0$ be given. If for any $\alpha \in E$ the set S is α-weak sharp with constant τ, then the solution set S is weak sharp with constant τ.*

Proof. By assumption, for any $\alpha \in E$,

$$f(x) - \alpha \notin \tau d(x, S)B_Y - \mathcal{K} \quad \text{for} \quad x \in A \setminus S.$$

In particular, $f(x) - \alpha \notin \tau d(x, S)B_Y$ for $x \in A \setminus S$ and any $\alpha \in E$, which gives the assertion. ■

3. Lipschitz continuities of efficient points

Consider now parametric vector optimization problems of the form

$$(VOP)_u \qquad \begin{array}{c} \mathcal{K} - \min f(x) \\ \text{subject to } x \in A(u), \end{array}$$

where the parameter u belongs to a normed space U. By $E(u)$ and $S(u)$ we denote the set of efficient points and the solution set to $(VOP)_u$, respectively.

In this section we exploit weak sharpness and S_α-weak sharpness to provide sufficient conditions for Lipschitzness of $E(u)$ and $S(u)$ near a given $u_0 \in U$. For other types of convergence of efficient points see e.g. Miglierina and Molho (2007).

In what follows the restrictions on behaviour of sets $A(u)$ around a given u_0 are expressed through continuity properites of the mapping $F : U \rightrightarrows X$, $F(u) = A(u)$, $F(u_0) = A$. Recall that a set-valued mapping $\Gamma : U \rightrightarrows X$ is *lower Lipschitz* at $u_0 \in \operatorname{dom}\Gamma$ if there exist constants $L > 0$ and $t > 0$ such that $\Gamma(u_0) \subset \Gamma(u) + L\|u - u_0\|B_Y$ for $u \in B(u_0, t)$. Γ is *Lipschitz* at $u_0 \in \operatorname{dom}\Gamma$ if Γ is upper and lower Lipschitz at u_0. Moreover, Γ is *Lipschitz* around $u_0 \in \operatorname{dom}\Gamma$ if there exist constants $L > 0$ and $t > 0$ such that $\Gamma(u) \subset \Gamma(u') + L\|u - u'\|B_Y$ for $u, u' \in B(u_0, t)$. The *domination property* (DP) holds for (VOP) if for any $x \in A$ there exists $\bar{x} \in S$ such that $f(\bar{x}) \in f(x) - \mathcal{K}$. Let us note that if $f : X \to R$, (DP) is satisfied provided $S \neq \emptyset$.

THEOREM 1 *Let $f : X \to Y$ be Lipschitz on X with constant $L_f > 0$. If*
(i) *$F : U \rightrightarrows X$ is Lipschitz at $u_0 \in \operatorname{dom} F$ with constants $L_c > 0$, $t > 0$,*
(ii) *(DP) holds for all $(VOP)_u$ with $u \in B(u_0, t)$,*
(iii) *there exists $\tau > 0$ such that for each $\alpha \in E$ the solution set S is S_α-weak sharp with constant τ, i.e. for each $\alpha \in E$,*

$$f(x) - \alpha \notin \tau d(x, S_\alpha) B_Y - \mathcal{K} \quad \text{for} \quad x \in A \setminus S_\alpha,$$

then

$$E \subset E(u) + (L_c L_f + \frac{2L_c L_f^2}{\tau}) \|u - u_0\| B_Y \quad \text{for} \quad u \in B(u_0, t).$$

If moreover, for $u \in B(u_0, t) \setminus \{u_0\}$ the sets $S(u)$ are weak sharp with constant τ, then

$$S \subset S(u) + (L_c + \frac{2L_c L_f}{\tau} + \frac{2L_c L_f^2}{\tau^2}) \|u - u_0\| B_X \quad \text{for} \quad u \in B(u_0, t).$$

*P*roof. By (i), (ii), $u_0 \in \text{int } \text{dom} E$. Take any $\alpha \in E$ and $u \in B(u_0, t)$. There exists $\bar{x} \in S$ such that $f(\bar{x}) = \alpha$. By (i), there exists $z \in A(u)$ such that $\|\bar{x} - z\| \le L_c \|u - u_0\|$. If $d(f(z), E(u)) \le 2L_c L_f \|u - u_0\|$, the conclusion follows. Otherwise, by (i), there exists $\bar{z} \in S(u)$ such that $f(\bar{z}) \in f(z) - \mathcal{K}$ and $\|f(z) - f(\bar{z})\| > 2L_c L_f \|u - u_0\|$. By (i), there exists $x \in A$ such that $\|x - \bar{z}\| \le L_c \|u - u_0\|$ and by the Lipschitzness of f

$$\|f(x) - f(\bar{x})\| \ge \|f(z) - f(\bar{z})\| - \|f(z) - f(\bar{x})\| - \|f(\bar{z}) - f(x)\| > 0,$$

and

$$\begin{aligned} f(x) - f(\bar{x}) &= (f(x) - f(\bar{z})) + (f(\bar{z}) - f(z)) + (f(z) - f(\bar{x})) \\ &\in 2L_f L_c \|u - u_0\| B_Y - \mathcal{K}. \end{aligned}$$

By (iii) and by Proposition 2, $f(x) - f(\bar{x}) \notin \frac{\tau}{L_f} \|f(x) - f(\bar{x})\| B_Y - \mathcal{K}$. This proves that $\tau \|f(x) - f(\bar{x})\| \le 2L_c L_f^2 \|u - u_0\|$ and consequently

$$\begin{aligned} \|f(\bar{x}) - f(\bar{z})\| &\le \|f(\bar{x}) - f(x)\| + \|f(x) - f(\bar{z})\| \\ &\le (L_f L_c + \tfrac{2L_f^2 L_c}{\tau}) \|u - u_0\| \end{aligned}$$

which proves the first assertion.

To prove the second assertion take any $x_0 \in S$ and $u \in B(u_0, t) \setminus \{u_0\}$. By the first assertion, there exists $z_0 \in S(u)$, $f(z_0) = \eta$, such that

$$f(x_0) - \eta \in (L_c L_f + \frac{2L_c L_f^2}{\tau}) \|u - u_0\| B_Y.$$

By (i), there exists $z \in A(u)$ such that $\|x_0 - z\| \leq L_c\|u - u_0\|$. If $d(z, S(u)) \leq L_c\|u - u_0\|$, the conclusion follows. Otherwise, since $S(u)$, $u \neq u_0$, is weak sharp, $f(z) - \eta \notin \tau d(z, S(u))B_Y$. Moreover,

$$f(z) - \eta = (f(z) - f(x_0)) + (f(x_0) - \eta) \in (2L_cL_f + \frac{2L_cL_f^2}{\tau})\|u - u_0\|B_Y.$$

Hence, $\tau d(z, S(u)) \leq (2L_cL_f + \frac{2L_cL_f^2}{\tau})\|u - u_0\|$ and

$$d(x_0, S(u)) \leq \|x_0 - z\| + d(z, S(u)) \leq (L_c + \frac{2L_cL_f}{\tau} + \frac{2L_cL_f^2}{\tau^2})\|u - u_0\|.$$

■

Let $u \in U$ and $\eta \in Y$. Put $S_\eta(u) = \{x \in A(u) \ : \ f(x) = \eta\}$.

THEOREM 2 *Let $f : X \to Y$ be Lipschitz on X with constant $L_f > 0$. If*
(i) *$F : U \rightrightarrows X$ is Lipschitz at $u_0 \in \operatorname{dom} F$ with constants $L_c > 0$ and $t > 0$,*
(ii) *(DP) holds for (VOP),*
(iii) *there exists $\tau > 0$ such that for $u \in B(u_0, t)$, $u \neq u_0$, and $\eta \in E(u)$ the sets $S(u)$ are $S_\eta(u)$-weak sharp with constant τ, i.e.*

$$f(x) - \eta \notin \tau d(x, S_\eta(u))B_Y - \mathcal{K} \quad \text{for} \quad x \in A(u) \setminus S_\eta(u),$$

then $E(u) \subset E + (L_fL_c + \frac{2L_cL_f^2}{\tau})\|u - u_0\|B_Y$ for $u \in B(u_0, t)$.

If, moreover, S is weak sharp, then

$$S(u) \subset S + (L_c + \frac{2L_cL_f}{\tau} + \frac{2L_cL_f^2}{\tau})\|u - u_0\|B_X \quad \text{for} \quad u \in B(u_0, t).$$

Proof. Note that by (ii), $E \neq \emptyset$. Take any $u \in B(u_0, t)$. If $E(u) = \emptyset$, the conclusion follows. Otherwise, take any $\eta \in E(u)$. There exists $z_0 \in S(u)$, $f(z_0) = \eta$. By (i), there exists $x \in A$ such that $\|z_0 - x\| \leq L_c\|u - u_0\|$. If $d(f(x), E) \leq 2L_cL_f\|u - u_0\|$, the conclusion follows.

Otherwise, by (ii), there is $x_0 \in S$, $f(x_0) = \alpha$, such that $f(x_0) \in f(x) - \mathcal{K}$ and $\|f(x) - \alpha\| > 2L_cL_f\|u - u_0\|$. By (i), there exists $z \in A(u)$ such that $\|z - x_0\| \leq L_c\|u - u_0\|$. By the Lipschitzness of f, $f(z) - \eta = f(z) - f(x_0) + f(x_0) - f(x) + f(x) - \eta \in 2L_cL_f\|u - u_0\|B_Y - \mathcal{K}$. Since

$$\|f(z) - \eta\| \geq \|f(x) - \alpha\| - \|f(x) - \eta\| - \|f(z) - \alpha\| > 0,$$

by (iii) and by Proposition 2, $f(z) - \eta \notin \frac{\tau}{L_f}\|f(z) - \eta\|B_Y - \mathcal{K}$. Consequently, $\|f(z) - \eta\| \leq \frac{2L_cL_f^2}{\tau}\|u - u_0\|$ and

$$\|\eta - \alpha\| \leq \|f(z) - \eta\| + \|f(z) - \alpha\| \qquad \leq (L_cL_f + \frac{2L_cL_f}{\tau})\|u - u_0\|.$$

To prove the second assertion, take any $z_0 \in S(u)$, $u \in B(u_0,t)$. By the first assertion of the theorem, there exists $x_0 \in S$, $f(x_0)=\alpha$, such that $f(z_0) - f(x_0) \in (L_cL_f + \frac{2L_cL_f}{\tau})\|u-u_0\|$. By (i), there exists $x \in A$ such that $\|x - z_0\| \leq L_c\|u-u_0\|$. If $d(x,S) \leq 2L_c\|u-u_0\|$, the conclusion follows. Otherwise, by (ii), there exists $x_0 \in S$, $f(x_0)=\alpha$, such that $f(x_0) \in f(x) - \mathcal{K}$ and $\|x-x_0\| > 2L_c\|u-u_0\|$. Hence,

$$f(x)-f(x_0) = f(x)-f(z_0)+f(z_0)-f(x_0) \in (2L_cL_f + \frac{2L_cL_f^2}{\tau})\|u-u_0\|.$$

Since S is weak sharp, $f(x)-f(x_0) \notin \tau d(x,S)B_Y$, which proves that $d(z_0,S) \leq (L_c + \frac{2L_cL_f}{\tau} + \frac{2L_cL_f^2}{\tau^2})\|u-u_0\|$. ■

The above theorems immediately lead to the following result.

THEOREM 3 *Let $f: X \to Y$ be Lipschitz on X with constant $L_f > 0$. Assume that*

(i) *the set valued mapping $F: U \rightrightarrows X$ is Lipschitz around $u_0 \in \operatorname{dom} F$ with constants $L_c > 0$ and $t > 0$,*

(ii) *(DP) holds for $(VOP)_u$, $u \in B(u_0,t)$,*

(iii) *there exists $\tau > 0$ such that for $u \in B(u_0,t)$ and $\eta \in E(u)$ the solution sets $S(u)$ to $(VOP)_u$ are $S_\eta(u)$-weak sharp with constant τ.*

Then

$$S(u) \subset S(u') + (L_c + \frac{2L_cL_f}{\tau} + \frac{2L_cL_f^2}{\tau^2})\|u'-u\|B_X \quad \textit{for} \quad u,u' \in B(u_0,t/2).$$

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