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ON WEAK SUPERCYCLICITY II

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Abstract. This paper considers weak supercyclicity for bounded linear operators on a normed space. On the one hand, weak supercyclicity is investigated for classes of Hilbert-space operators: (i) self-adjoint operators are not weakly supercyclic, (ii) diagonalizable operators are not weakly l-sequentially supercyclic, and (iii) weak l-sequential supercyclicity is preserved between a unitary operator and its adjoint. On the other hand, weak supercyclicity is investigated for classes of normed-space operators: (iv) the point spectrum of the normed-space adjoint of a power bounded supercyclic operator is either empty or is a singleton in the open unit disk, (v) weak l-sequential supercyclicity coincides with supercyclicity for compact operators, and (vi) every compact weakly l-sequentially supercyclic operator is quasinilpotent.

Keywords: supercyclic operator; weakly supercyclic operator; weakly l-sequentially supercyclic operator

MSC 2010: 47A16, 47B15

1. Introduction

The reason of this paper is to characterize weak supercyclicity, in particular, weak l-sequential supercyclicity for bounded linear operators on a normed space. Section 2 deals with notation, terminology and basic notions that will be required throughout the text. In Section 3 it is shown: (i) self-adjoint operators are not weakly supercyclic (Theorem 3.1), (ii) diagonalizable operators are not weakly l-sequentially supercyclic (Theorem 3.2), (iii) weak l-sequential supercyclicity is preserved between a unitary operator and its adjoint (Theorem 3.3), and (iv) the point spectrum of the normed-space adjoint of a power bounded supercyclic operator is either empty or is a singleton in the open unit disk (Theorem 3.4), and it is also shown when this happens for weakly l-sequentially supercyclic operators. The first result of Section 4 gives a first characterization for weakly l-sequentially supercyclic compact operators: they are supercyclic (Theorem 4.1)—does weak supercyclicity also coincide

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with weak l-sequential supercyclicity for compact operators? The section closes by giving a full spectral characterization for weakly l-sequentially supercyclic compact operators: they are quasinilpotent (Theorem 4.2).

2. Notation, terminology and basics

Let $\mathcal{X}$ be a nonzero complex normed space and let $\mathcal{X}^*$ be the dual of $\mathcal{X}$. A subspace of $\mathcal{X}$ is a *closed* linear manifold of $\mathcal{X}$. If $\mathcal{M}$ is a linear manifold of $\mathcal{X}$, then its closure $\mathcal{M}^-$ is a subspace. The normed algebra of all operators on $\mathcal{X}$ (i.e., of all bounded linear transformations of $\mathcal{X}$ into itself) will be denoted by $\mathcal{B}[\mathcal{X}]$. For any operator T on a normed space $\mathcal{X}$ let $\mathcal{N}(T) = T^{-1}\{0\} = \{x \in \mathcal{X} \colon Tx = 0\}$ be the kernel of T, which is a subspace of $\mathcal{X}$, and let $\mathcal{R}(T) = T(\mathcal{X})$ be the range of T, which is a linear manifold of $\mathcal{X}$. Let T^* in $\mathcal{B}[\mathcal{X}^*]$ stand for the normed-space adjoint of T. We use the same notation for the Hilbert-space adjoint of a Hilbert-space operator.

For each normed-space operator T the limit $r(T) = \lim_n \|T^n\|^{1/n}$ exists in $\mathbb{R}$ and is such that $0 \leqslant r(T) \leqslant \|T\|$. If an operator T on a normed space is such that $r(T) = 0$, then it is quasinilpotent. On the other hand, if T is such that $r(T) = \|T\|$, then it is normaloid. Let $\sigma_P(T) = \{\lambda \in \mathbb{C} \colon \mathcal{N}(\lambda I - T) \neq \{0\}\}$ be the point spectrum of T, the set of eigenvalues of T. An operator T on a normed space $\mathcal{X}$ is power bounded if $\sup_{n\geqslant 0} \|T^n\| < \infty$, and strongly stable or weakly stable if the $\mathcal{X}$-valued sequence $\{T^n x\}_{n\geqslant 0}$ converges to zero in the norm topology or in the weakly topology of $\mathcal{X}$,

$$T^n x \to 0 \quad \text{or} \quad T^n x \xrightarrow{w} 0,$$

which means $\|T^n x\| \to 0$ or $f(T^n x) \to 0$ for every $f \in \mathcal{X}^*$ for every $x \in \mathcal{X}$, respectively. If $\mathcal{X}$ is a Banach space, such that T lies in the Banach algebra $\mathcal{B}[\mathcal{X}]$, then let $\sigma(T) \subset \mathbb{C}$ stand for the spectrum of T (which is compact and nonempty). In this case, $r(T)$ coincides with the spectral radius of T; that is, $r(T) = \max_{\lambda\in\sigma(T)} |\lambda|$ (by the Gelfand-Beurling formula). Thus, if $\mathcal{X}$ is a Banach space, then $T \in \mathcal{B}[\mathcal{X}]$ is quasinilpotent if and only if $\sigma(T) = \{0\}$.

With the assumption that $\mathcal{X}$ is a normed space still in force, the orbit of a vector $y \in \mathcal{X}$ under an operator $T \in \mathcal{B}[\mathcal{X}]$ is the set

$$\mathcal{O}_T(y) = \bigcup_{n\geqslant 0} T^n y = \big\{T^n y \in \mathcal{X} \colon n \in \mathbb{N}_0\big\},$$

where $\mathbb{N}_0$ denotes the set of nonnegative integers—we write $\bigcup_{n\geqslant 0} T^n y$ for the set $\bigcup_{n\geqslant 0} T^n(\{y\}) = \bigcup_{n\geqslant 0} \{T^n y\}$. The orbit $\mathcal{O}_T(A)$ of a set $A \subseteq \mathcal{X}$ under T is likewise defined: $\mathcal{O}_T(A) = \bigcup_{n\geqslant 0} T^n(A) = \bigcup_{y\in A} \mathcal{O}_T(y)$. Let $\operatorname{span} A$ stand for the linear span of a set $A \subseteq \mathcal{X}$ and consider the projective orbit of a vector y under T, which is the orbit of the one-dimensional space spanned by the singleton $\{y\}$,

$$\mathcal{O}_T(\operatorname{span}\{y\}) = \bigcup_{n\geqslant 0} T^n(\operatorname{span}\{y\}) = \{\alpha T^n y \in \mathcal{X}\colon\ \alpha \in \mathbb{C},\ n \in \mathbb{N}_0\}.$$

The closure (in the norm topology of $\mathcal{X}$) of a set $A \subseteq \mathcal{X}$ is denoted by A^-, and the weak closure (in the weak topology of $\mathcal{X}$) is denoted by A^{-w}. Thus, A is dense or weakly dense if $A^- = \mathcal{X}$ or $A^{-w} = \mathcal{X}$, respectively. A set A is weakly sequentially closed if every A-valued weakly convergent sequence has its limit in A, and the weak sequential closure A^{-sw} of A is the smallest weakly sequentially closed subset of $\mathcal{X}$ including A, and A is weakly sequentially dense if $A^{-sw} = \mathcal{X}$. The weak limit set A^{-lw} of a set A is the set of all weak limits of weakly convergent A-valued sequences, and a set A is weakly l-sequentially dense if $A^{-lw} = \mathcal{X}$. In general, the inclusions $A^- \subseteq A^{-lw} \subseteq A^{-sw} \subseteq A^{-w}$ are proper (see, e.g. [30], pages 259, 260, [5]). However, if a set $A \subseteq \mathcal{X}$ is convex, then $A^- = A^{-w}$ (see, e.g. [10], Theorem V.1.4), and so if A is convex, then the above chain of inclusions becomes an identity.

A vector $y \in \mathcal{X}$ is supercyclic or weakly supercyclic for an operator $T \in \mathcal{B}[\mathcal{X}]$ if

$$\mathcal{O}_T(\operatorname{span}\{y\})^- = \mathcal{X} \quad \text{or} \quad \mathcal{O}_T(\operatorname{span}\{y\})^{-w} = \mathcal{X},$$

and it is weakly l-sequentially supercyclic or weakly sequentially supercyclic if

$$\mathcal{O}_T(\operatorname{span}\{y\})^{-lw} = \mathcal{X} \quad \text{or} \quad \mathcal{O}_T(\operatorname{span}\{y\})^{-sw} = \mathcal{X},$$

respectively. An operator $T \in \mathcal{B}[\mathcal{X}]$ is supercyclic, weakly l-sequentially supercyclic, weakly sequentially supercyclic, or weakly supercyclic if it has a supercyclic, a weakly l-sequentially supercyclic, a weakly sequentially supercyclic, or a weakly supercyclic vector, respectively. Thus,

$$\text{supercyclicity} \implies \begin{array}{c}\text{weak } l\text{-sequential}\\ \text{supercyclicity}\end{array} \implies \begin{array}{c}\text{weak sequential}\\ \text{supercyclicity}\end{array} \implies \begin{array}{c}\text{weak}\\ \text{supercyclicity.}\end{array}$$

So a vector $y \in \mathcal{X}$ is supercyclic or weakly l-sequentially supercyclic for an operator $T \in \mathcal{B}[\mathcal{X}]$ (i.e. $\mathcal{O}_T(\operatorname{span}\{y\})^- = \mathcal{X}$ or $\mathcal{O}_T(\operatorname{span}\{y\})^{-lw} = \mathcal{X}$) if and only if for every $x \in \mathcal{X}$ there is a $\mathbb{C}$-valued sequence $\{\alpha_i\}_{i\geqslant 0}$ (that depends on x and y, and consists of nonzero numbers) such that for some subsequence $\{T^{n_i}\}_{i\geqslant 0}$ of $\{T^n\}_{n\geqslant 0}$,

$$\alpha_i T^{n_i} y \to x \quad \text{or} \quad \alpha_i T^{n_i} y \xrightarrow{w} x,$$

respectively. Weak l-sequential supercyclicity was considered in [6] (and implicitly in [4]), and it was referred to as weak 1-sequential supercyclicity in [30]. Although there are reasons for such a terminology, we have changed it here to weak l-sequential supercyclicity, replacing the numeral "1" with the letter "l" for "limit". Any form of cyclicity implies separability for $\mathcal{X}$ (see, e.g. [23], Section 3).

The contribution to linear dynamics of the present paper in contrast with [4], [5], [6], [30] is the characterization of weak l-sequential supercyclicity for further classes of operators (including self-adjoint, diagonalizable, unitary, normal, hyponormal, and compact) as in Theorems 3.1, 3.2, 3.3, 3.4 and Theorems 4.1 and 4.2. These were carried out here on Banach spaces (or normed spaces when completeness was not necessary) or, in particular, on Hilbert spaces. The stronger notion of hypercyclicity has been investigated in Fréchet spaces, or F-spaces, or locally convex spaces (see, e.g. [2], [5], [7], [8], [15], [26]). Some of the above classes of operators may have a natural extention on some of these spaces, which perhaps might be worth investigating in light of the weaker notion of weak l-sequential supercyclicity. However, we refrain from going further than Banach spaces (or normed spaces) here to keep up with the focus on the main topic of the paper.

3. Adjointness and weak supercyclicity

The following proposition summarizes a few known results that will be often required throughout the next two sections, which are germane to Hilbert spaces. An operator T on a Hilbert space is self-adjoint or unitary if $T^* = T$ or $TT^* = T^*T = I$, respectively, where I stands for the identify operator. A unitary operator is absolutely continuous, singular-continuous, or singular-discrete if its scalar spectral measure is absolutely continuous, singular-continuous, or singular-discrete, respectively, with respect to normalized Lebesgue measure on the σ-algebra of Borel subsets of the unit circle. An operator is an isometry if $T^*T = I$ and a coisometry if its adjoint is an isometry. Thus, a unitary is an isometry and a coisometry, which means an invertible isometry. An operator is normal if $TT^* = T^*T$, hyponormal if $TT^* \leqslant T^*T$, and cohyponormal if its adjoint is hyponormal. These are all normaloid operators. Some extensions along the lines discussed in Proposition 3.1 below from hyponormal to further classes of normaloid operators, such as paranormal operators and beyond, have recently been considered in literature (see, e.g. [12], Corollary 3.1, [13], Theorem 2.7), but again we refrain from going further than hyponormal operators here to keep up with the focus on the main topic of the paper.

Proposition 3.1. *The following assertions hold for Hilbert-space operators.*

(a) *No hyponormal operator is supercyclic (no unitary operator is supercyclic).*

(b) *A hyponormal weakly supercyclic operator is a multiple of a unitary.*

(c) *There exist weakly l-sequentially supercyclic unitary operators.*

(d) *A weakly l-sequentially supercyclic unitary operator is singular-continuous.*

Proof. (a) See [9], Theorem 3.1 (for the unitary case see also [3] proof of Theorem 2.1).

(b) See [4], Theorem 3.1.

(c) See [4], Example 3.6 (also [4], pages 10, 12, [30], Proposition 1.1, Theorem 1.2).

(d) See [22], Theorem 4.2. □

Although a unitary operator can be weakly supercyclic, a self-adjoint cannot.

Theorem 3.1. *A self-adjoint operator on a Hilbert space is not weakly supercyclic.*

Proof. Since a weakly supercyclic hyponormal operator is a multiple of a unitary operator (cf. Proposition 3.1 (b)), if T is self-adjoint on a Hilbert space $\mathcal{H}$ and weakly supercyclic, then it is a self-adjoint multiple of a unitary, which implies that T^2 is a positive multiple of the identity, say, $T^2 = |\beta|^2 I$ and so $T^n = |\beta|^n I$ if n is even or $T^n = |\beta|^{n-1} T$ if n is odd. Thus, the projective orbit of any vector $z \in \mathcal{H}$ is included in a pair of one-dimensional subspaces

$$\begin{aligned}\mathcal{O}_T(\operatorname{span}\{z\}) &= \{\alpha T^n z \in \mathcal{H}\colon\ \alpha \in \mathbb{C},\ n \in \mathbb{N}_0\}\\ &\subseteq \{\alpha z \in \mathcal{H}\colon\ \alpha \in \mathbb{C}\} \cup \{\alpha T z \in \mathcal{H}\colon\ \alpha \in \mathbb{C}\} = \operatorname{span}\{z\} \cup \operatorname{span}\{Tz\},\end{aligned}$$

which is not dense in $\mathcal{H}$ in the weak topology if $\dim \mathcal{H} > 1$. Hence, a self-adjoint operator T (on a space of dimension greater than 1) is not weakly supercyclic. □

Normal operators are not supercyclic (neither are hyponormal) but can be weakly l-sequentially supercyclic (so can unitary), but diagonalizable operators cannot.

Theorem 3.2. *A diagonalizable operator on a Hilbert space is not weakly l-sequentially supercyclic.*

Proof. A diagonalizable operator T on a Hilbert space $\mathcal{H}$ is precisely an operator unitarily equivalent to a diagonal operator (see, e.g. [21], Proposition 3.A). So it is normal and therefore if it is weakly supercyclic, then it acts on a separable Hilbert space (i.e. $\mathcal{H}$ is separable), and it is a multiple of a unitary operator (cf. Proposition 3.1 (b)). Thus, such a unitary operator is unitarily equivalent to a unitary diagonal U on ℓ^2_+, which is discrete (i.e. singular-discrete). If, in addition, T is weakly l-sequentially supercyclic, then so is U, and hence U must be singular-continuous (cf. Proposition 3.1 (d)), which is a contradiction. Then a diagonalizable operator is not weakly l-sequentially supercyclic. □

Remark 3.1. It was asked in [22], Question 5.1, whether every weakly supercyclic unitary operator is singular-continuous. An affirmative answer ensures that Theorem 3.2 holds if weakly l-sequentially supercyclic is replaced by weakly supercyclic.

If T is an invertible supercyclic, then so is its inverse [3], Section 4, [27], Corollary 2.4. There are, however, invertible weakly supercyclic operators in $\mathcal{B}[\ell^p] = \mathcal{B}[\ell^p(\mathbb{Z})]$ for any $p \in [2, \infty)$ whose inverses are not weakly supercyclic, see [28], Corollary 2.5. For $p = 2$ this exhibits a Hilbert-space invertible operator whose inverse is not weakly supercyclic. Since for $p = 2$ such an operator is not unitary, the following question crops up: if a unitary operator is weakly supercyclic, is its inverse (i.e. its adjoint) weakly supercyclic? Recall that the adjoint of a supercyclic coisometry may not be supercyclic (example: a backward unilateral shift S^* is a supercyclic coisometry ([19], Theorem 3), while its adjoint, the unilateral shift S, being an isometry is not supercyclic ([19], page 564, also see [3], Proof of Theorem 2.1, [23], Lemma 4.1 (b)). The same happens with weak supercyclicity: the adjoint of a weakly supercyclic coisometry may not be weakly supercyclic (example: S is not weakly supercyclic by Proposition 3.1 (b), but S^* is weakly supercyclic, since it is supercyclic). However, if an isometry is invertible and weakly l-sequentially supercyclic, then it has a weakly l-sequentially supercyclic adjoint (i.e. inverse), as we show in Theorem 3.3 below.

Let $\mathbb{D}$ stand for the open unit disk (about the origin in the complex plane $\mathbb{C}$), let $\mathbb{D}^-$ (the closure of $\mathbb{D}$) stand for the closed unit disk, and let their boundary $\mathbb{T} = \partial\mathbb{D}$ stand for unit circle (about the origin).

Theorem 3.3. *A unitary operator on a Hilbert space is weakly l-sequentially supercyclic if and only if its adjoint is weakly l-sequentially supercyclic.*

P r o o f. We split the proof into 2 parts.

Part 1. Let μ be a positive measure on the σ-algebra $\mathcal{A}_{\mathbb{T}}$ of Borel subsets of the unit circle $\mathbb{T}$ and consider the Hilbert space $L^2(\mathbb{T}, \mu)$. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ denote the identity function, $\varphi(\gamma) = \gamma$ μ-a.e. for $\gamma \in \mathbb{T}$, and consider the multiplication operator $U_\mu\colon L^2(\mathbb{T}, \mu) \to L^2(\mathbb{T}, \mu)$ induced by φ, $U_\mu\psi = \varphi\psi$, which is given by

$$(U_\mu\psi)(\gamma) = \varphi(\gamma)\psi(\gamma) = \gamma\psi(\gamma) \quad \mu\text{-a.e. for } \gamma \in \mathbb{T},$$

so that $U_\mu^*\psi = \overline{\varphi}\psi$, which is given by

$$(U_\mu^*\psi)(\gamma) = \overline{\varphi}(\gamma)\psi(\gamma) = \overline{\gamma}\psi(\gamma) \quad \mu\text{-a.e. for } \gamma \in \mathbb{T},$$

for every $\psi \in L^2(\mathbb{T}, \mu)$. It is clear that U_μ is unitary. Let $C\colon L^2(\mathbb{T}, \mu) \to L^2(\mathbb{T}, \mu)$ denote the complex conjugate transformation (i.e. $C(\psi) = \overline{\psi}$), which has the following properties: it is a contraction (thus norm continuous), weakly continuous (in fact, $\langle C(\zeta_k-\zeta);\psi\rangle=\overline{\langle\zeta_k-\zeta;\overline{\psi}\rangle}$ for every $\zeta_k,\zeta,\psi\in L^2(\mathbb{T},\mu)$), an involution (i.e. $C^2=I$), additive, and conjugate homogeneous (i.e. $C(\alpha f)=\overline{\alpha}C(f)$).

Claim 1. $CU_\mu=U_\mu^*C$.

Proof. $C(U_\mu\psi)=\overline{(U_\mu\psi)}=\overline{\varphi\,\psi}=\overline{\varphi}\,\overline{\psi}=\overline{\varphi}\,C\,\psi)=U_\mu^*(C\psi)$ for any $\psi\in L^2(\mathbb{T},\mu)$. □

Claim 2. Let $\{U^{n_k}\}_{k\geqslant 0}$ be an arbitrary subsequence of $\{U^n\}_{n\geqslant 0}$, let $\{\alpha_k\}_{k\geqslant 0}$ be any sequence of scalars, and let φ,ψ be functions in $L^2(\mathbb{T},\mu)$. Then

$$\alpha_k U_\mu^{n_k}\varphi\xrightarrow{w}\psi\quad\text{if and only if}\quad\overline{\alpha}_k U_\mu^{*n_k}\overline{\varphi}\xrightarrow{w}\overline{\psi}.$$

Proof. Since C is weakly continuous, it follows by Claim 1 (since C is conjugate homogeneous) that if $\alpha_k U_\mu^{n_k}\varphi\xrightarrow{w}\psi$, then

$$\overline{\alpha}_k U_\mu^{*n_k}\overline{\varphi}=\overline{\alpha}_k U_\mu^{*n_k}(C\varphi)=\overline{\alpha}_k C(U_\mu^{n_k}\varphi)=C(\alpha_k U_\mu^{n_k}\varphi)\xrightarrow{w}C(\psi)=\overline{\psi}.$$

Dually, since C and the adjoint operation are involutions, the converse holds. □

Take an arbitrary $\psi\in L^2(\mathbb{T},\mu)$. If U_μ is weakly l-sequentially supercyclic, then there is a supercyclic vector $\varphi\in L^2(\mathbb{T},\mu)$ for U_μ, a sequence of nonzero numbers $\{\alpha_k(\varphi,\psi)\}_{k\geqslant 0}$, and a corresponding subsequence $\{U_\mu^{n_k}\}_{k\geqslant 0}$ of $\{U_\mu^n\}_{n\geqslant 0}$ such that

$$\alpha_k(\varphi,\psi)U_\mu^{n_k}\varphi\xrightarrow{w}\overline{\psi}.$$

According to Claim 2 this happens if and only if

$$\overline{\alpha}_k(\varphi,\psi)U_\mu^{*n_k}\overline{\varphi}\xrightarrow{w}\overline{\overline{\psi}}=\psi,$$

and so $\overline{\varphi}$ is a weakly l-sequentially supercyclic vector for U_μ^*, and hence U_μ^* is weakly l-sequentially supercyclic. Again the converse holds dually. Therefore,

$$U_\mu \text{ is weakly } l\text{-sequentially supercyclic if and only if so is its adjoint } U_\mu^*.$$

Part 2. Take a unitary operator U on a Hilbert space $\mathcal{H}$. If U is weakly supercyclic, then it is weakly cyclic, and so it is cyclic (i.e. if there exists a vector $y\in\mathcal{H}$ such that $\mathcal{O}_U(\operatorname{span}\{y\})^{-w}=\mathcal{H}$, then $\Bigl(\operatorname{span}\bigcup_n U^n y\Bigr)^-=(\operatorname{span}\mathcal{O}_U(y))^-=(\operatorname{span}\mathcal{O}_U(y))^{-w}=\mathcal{H}$ because $\operatorname{span}\mathcal{O}_U(y)$ is convex). Cyclicity implies star-cyclicity, which in turn implies separability for $\mathcal{H}$—since U is normal, star-cyclicity for U means: there exists a vector $y\in\mathcal{H}$ for which $\Bigl(\operatorname{span}\bigcup_n U^n U^{*n}y\Bigr)^{-w}=\mathcal{H}$—see, e.g. [21], pages 73 and 74. Star-cyclicity ensures, by the Spectral Theorem, that U is unitarily equivalent to a unitary multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ induced by the identity function $\varphi:\mathbb{T}\to\mathbb{T}$ (thus of multiplicity one), where the positive measure μ on $\mathcal{A}_{\mathbb{T}}$ is finite and supported on $\sigma(U)\subseteq\mathbb{T}$—see, e.g. [21], Part (a), proof of Theorem 3.11. If, in addition, the unitary U on $\mathcal{H}$ is weakly l-sequentially supercyclic, then so is the unitary multiplication operator U_μ on $L^2(\mathbb{T},\mu)$ (which is unitarily equivalent to it), and the result of Part 1 ensures that U_μ^* is weakly l-sequentially supercyclic, and so is the unitary U^* (which again is unitarily equivalent to U_μ^*). Dually, if U^* is weakly l-sequentially supercyclic, then so is U. □

Corollary 3.1. *A hyponormal (normal) operator is weakly l-sequentially supercyclic if and only if its adjoint is weakly l-sequentially supercyclic.*

Proof. Proposition 3.1 (b) and Theorem 3.3. □

If T is a power bounded operator on a Banach space, then $r(T)\leqslant 1$ (equivalently, $\sigma(T)\subseteq\mathbb{D}^-$) and so $\sigma_P(T)\subseteq\mathbb{D}^-$. As we will see in the proof of Theorem 4.2., if an operator T on a Banach space is supercyclic, then the point spectrum of its normed-space adjoint $\sigma_P(T^*)$ has at most one element. As a consequence of the forthcoming Theorem 3.4, if a supercyclic operator T is power bounded, then this possible unique element λ of $\sigma_P(T^*)$ is such that $|\lambda|<1$ (so that $\sigma_P(T^*)\subseteq\{\lambda\}\subset\mathbb{D}$).

To proceed we need the following definition. A normed space $\mathcal{X}$ is said to be of *type 1* if convergence in the norm topology for an arbitrary $\mathcal{X}$-valued sequence $\{x_k\}$ coincides with weak convergence plus convergence of the norm sequence $\{\|x_k\|\}$ (i.e. $x_k\to x\iff\{x_k\xrightarrow{w}x$ and $\|x_k\|\to\|x\|\}$—also called *Radon-Riesz space* and the *Radon-Riesz property*, respectively; see, e.g. [24], Definition 2.5.26). Hilbert spaces are Banach spaces of type 1 [16], Problem 20.

Theorem 3.4. *Let $T\in\mathcal{B}[\mathcal{X}]$ be a supercyclic (or weakly l-sequentially supercyclic) operator on a normed space $\mathcal{X}$. Suppose there exists a nonzero eigenvalue λ of T^* (i.e. $0\neq\lambda\in\sigma_P(T^*)$) and take any nonzero eigenvector $f\in\mathcal{X}^*$ of $T^*\in\mathcal{B}[\mathcal{X}^*]$ associated with λ (i.e. $0\neq f\in\mathcal{N}(\lambda I-T^*)$). Then for every supercyclic (or weakly l-sequentially supercyclic) vector y for T and every $x\in\mathcal{X}$ such that $f(x)\neq 0$ (i.e. $x\in\mathcal{X}\setminus\mathcal{N}(f)$), there exists a subsequence $\{n_k\}_{k\geqslant 0}$ of the integers $\{n\}_{n\geqslant 0}$ such that*

$$\frac{f(x)}{f(y)}\frac{1}{\lambda^{n_k}}T^{n_k}y\longrightarrow x\quad\textit{or}\quad\frac{f(x)}{f(y)}\frac{1}{\lambda^{n_k}}T^{n_k}y\xrightarrow{w}x$$

(i.e. we may set $\alpha_k(x,y)=(\lambda^{n_k})^{-1}f(x)/f(y)$). In particular,

$$\frac{1}{\lambda^{n_k}}T^{n_k}y\longrightarrow y\quad\textit{or}\quad\frac{1}{\lambda^{n_k}}T^{n_k}y\xrightarrow{w}y.$$

Moreover:

(a) *If T is power bounded and supercyclic, then $|\lambda| < 1$.*

(b) *If T is power bounded and weakly l-sequentially supercyclic on a type 1 normed space, and if $|f(y)| = \|f\| \lim_n \|T^n y\|$ for some weakly l-sequentially supercyclic vector y and some $0 \neq f \in \mathcal{N}(\lambda I - T^*)$, then $|\lambda| < 1$.*

Proof. Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator on a normed space $\mathcal{X}$ and consider its normed-space adjoint $T^* \in \mathcal{B}[\mathcal{X}^*]$. Let λ be a nonzero eigenvalue of T^* and take any nonzero eigenvector $f \in \mathcal{X}^*$ associated with the nonzero eigenvalue λ of T^* such that

$$(*) \qquad f(T^n x) = \lambda^n f(x)$$

for every $n \geqslant 0$ and every $x \in \mathcal{X}$. (Indeed, $f(T^n x) = (T^{n*} f)(x) = (T^{*n} f)(x) = (\lambda^n f)(x) = \lambda^n f(x)$.) Suppose T is supercyclic (or weakly l-sequentially supercyclic). Fix an arbitrary (nonzero) supercyclic (or weakly l-sequentially supercyclic) vector $y \in \mathcal{X}$ for T, and take an arbitrary vector $x \in \mathcal{X}$. Thus, there is a sequence of nonzero numbers $\{\alpha_k(y,x)\}_{k\geqslant 0}$ and a corresponding subsequence $\{T^{n_k}\}_{k\geqslant 0}$ of $\{T^n\}_{n\geqslant 0}$ (which depends on x and y) such that

$$\alpha_k(y,x) T^{n_k} y \longrightarrow x \quad \text{or} \quad \alpha_k(y,x) T^{n_k} y \xrightarrow{w} x.$$

So, according to $(*)$ (for the supercyclic case recall that f is continuous),

$$\alpha_k(y,x)\lambda^{n_k} f(y) = \alpha_k(y,x) f(T^{n_k} y) = f(\alpha_k(y,x) T^{n_k} y) \to f(x).$$

Observe that

$$(**) \qquad f(y) \neq 0.$$

(Indeed, by the above convergence if $f(y) = 0$, then $f(x) = 0$ for every $x \in \mathcal{X}$, which is a contradiction). Hence,

$$\alpha_k(y,x)\lambda^{n_k} \to \frac{f(x)}{f(y)},$$

and so if $f(x) \neq 0$ (which ensures $(\alpha_k(y,x)\lambda^{n_k})^{-1} \to f(y)/f(x)$), then

$$\frac{f(x)}{f(y)} \frac{1}{\lambda^{n_k}} T^{n_k} y \longrightarrow x \quad \text{or} \quad \frac{f(x)}{f(y)} \frac{1}{\lambda^{n_k}} T^{n_k} y \xrightarrow{w} x.$$

In particular, by setting $x = y$,

$$\frac{1}{\lambda^{n_k}} T^{n_k} y \longrightarrow y \quad \text{or} \quad \frac{1}{\lambda^{n_k}} T^{n_k} y \xrightarrow{w} y.$$

Moreover: (a) *If a power bounded operator T on a normed space $\mathcal{X}$ is supercyclic, then it is strongly stable*, see [3], Theorem 2.2. Thus, in this case $T^{n_k}y \to 0$ so $\lambda^{n_k} \to 0$ (since $(\lambda^{n_k})^{-1}T^{n_k}y \to y \neq 0$), which implies $|\lambda| < 1$.

(b) Suppose $|\lambda| = 1$. From $(*)$ and $(**)$,

$$|f(T^n y)| = |f(y)| \neq 0$$

for every $n \geqslant 0$, so

$$f(T^n y) \not\to 0 \quad \text{and} \quad 0 < \liminf_n |f(T^n y)|,$$

for every weakly l-sequentially supercyclic vector y and every nonzero eigenvector f associated with the eigenvalue λ. Then $T^n y \overset{w}{\not\to} 0$, so T is not weakly stable. However, according to [23], Theorem 6.2, if a power bounded operator T on a type 1 normed space $\mathcal{X}$ is weakly l-sequentially supercyclic, then either

(i) T is weakly stable, or

(ii) if y is any weakly l-sequentially supercyclic vector such that $T^n y \overset{w}{\not\to} 0$, then for every nonzero $f \in \mathcal{X}^*$ such that $f(T^n y) \not\to 0$ either

$\liminf_n |f(T^n y)| = 0$, or $\limsup_j |f(T^{n_k} y)| < \|f\| \limsup_j \|T^{n_j} y\|$ for some subsequence $\{n_j\}_{j\geqslant 0}$ of $\{n\}_{n\geqslant 0}$.

Consider a weakly l-sequentially supercyclic power bounded operator T on a type 1 normed space. By the above results if $|\lambda| = 1$, then $|f(y)| < \|f\| \limsup_{n_j} \|T^{n_j} y\|$ for some subsequence $\{n_j\}_{j\geqslant 0}$ of $\{n\}_{n\geqslant 0}$ for every weakly l-sequentially supercyclic vector y and every nonzero eigenvector f associated with the eigenvalue λ (since $|f(T^n y)| = |f(y)|$). Therefore if $|f(y)| = \|f\| \lim_n \|T^n y\|$ for some weakly l-sequentially supercyclic vector y and some nonzero eigenvector f associated with the eigenvalue λ, then $|\lambda| \neq 1$, and hence $|\lambda| < 1$ (since T is power bounded). □

Remark 3.2. Since isometries are weakly supercyclic only if they are unitaries (cf. Proposition 3.1 (b)), and since there exist weakly l-sequentially supercyclic unitaries (cf. Proposition 3.1 (c)), it follows by Theorem 3.4 that if T is a weakly l-sequentially supercyclic isometry on a Hilbert space (so it is unitary, and so is its adjoint), then $|\lambda| < 1$ whenever $\lambda \in \sigma_P(T^*)$ so that $\sigma_P(T^*) = \emptyset$. Actually, by Theorem 3.3 the unitary T^* is weakly l-sequentially supercyclic as well, and Proposition 3.1 (d) says that the weakly l-sequentially supercyclic unitaries T and T^* must be singular-continuous, and *cyclic* (in particular, weakly l-sequentially supercyclic) *singular-continuous unitaries have no eigenvalues*. Indeed, if a star-cyclic (equivalently, a cyclic) singular-continuous unitary has an eigenvalue, then there exists a unitary multiplication operator U_μ induced by the identity function with respect to some positive singular-continuous measure μ on $\mathcal{A}_{\mathbb{T}}$ (after the Spectral Theorem), which has an eigenvalue λ, and this implies $\gamma\psi(\gamma) = \lambda\psi(\gamma)$ μ-a.e. for $\gamma \in \mathbb{T}$ for some nonzero eigenvector ψ associated with the eigenvalue λ. So $\gamma = \lambda$ for every $\gamma \in \mathbb{T} \setminus \mathcal{N}(\psi) \in \mathcal{A}_{\mathbb{T}}$. Therefore since $\mu(\mathbb{T} \setminus \mathcal{N}(\psi)) \neq 0$, we get $\mu(\{\lambda\}) > 0$ which is a contradiction (because, being continuous, μ is null when acting on measurable singletons).

4. Compactness and weak supercyclicity

To begin with we need an auxiliary result on the range $\mathcal{R}(T)$ of an operator T.

Lemma 4.1. *If an operator T on a normed space $\mathcal{X}$ is weakly supercyclic, then*

$$\mathcal{R}(T)^{-} = \mathcal{R}(T)^{-wl} = \mathcal{R}(T)^{-w} = \mathcal{X}.$$

Proof. If a set $A \subseteq \mathcal{X}$ is convex, then $A^{-} = A^{-w}$, and so $A^{-} = A^{-wl} = A^{-w}$. Since a linear manifold is trivially convex,

$$\mathcal{R}(T)^{-} = \mathcal{R}(T)^{-wl} = \mathcal{R}(T)^{-w}.$$

But the projective orbit of any vector $u \in \mathcal{X}$ is included in the range of T,

$$\begin{aligned}\mathcal{O}_T(\operatorname{span}\{u\}) &= \{\alpha T^n u \in \mathcal{X}\colon\ \alpha \in \mathbb{C},\ n \in \mathbb{N}_0\} \\ &\subseteq \{z \in \mathcal{X}\colon\ z = Tx \quad \text{for some } x \in \mathcal{X}\} = \mathcal{R}(T).\end{aligned}$$

Thus, if T is weakly supercyclic, then $\mathcal{O}_T(\operatorname{span}\{y\})^{-w} = \mathcal{X}$ for some $y \in \mathcal{X}$, so $\mathcal{R}(T)^{-w} = \mathcal{X}$. □

Theorem 4.1 gives a first characterization for weakly l-sequentially supercyclic compact operators: they are supercyclic.

Theorem 4.1. *A compact operator on a normed space is weakly l-sequentially supercyclic if and only if it is supercyclic.*

Proof. Suppose T is a weakly l-sequentially supercyclic operator on a normed space $\mathcal{X}$. Take an arbitrary $x \in \mathcal{X}$ such that $x \in \mathcal{R}(T)^{-}$ according to Lemma 4.1. Thus, there exists an $\mathcal{X}$-valued sequence $\{x_k\}_{k\geqslant 0}$ such that

$$Tx_k \to x.$$

Since T is weakly l-sequentially supercyclic, there exists a nonzero vector $y \in \mathcal{X}$ such that for each x_k there exists a sequence of nonzero numbers $\{\alpha_j(x_k)\}_{j\geqslant 0}$ and a corresponding subsequence $\{T^{n_j}\}_{j\geqslant 0}$ of $\{T^n\}_{n\geqslant 0}$ such that

$$\alpha_j(x_k)T^{n_j}y \xrightarrow{w} x_k.$$

If in addition T is compact, then

$$\alpha_j(x_k)T^{n_j+1}y \to Tx_k$$

for every k (convergence in the norm topology, see e.g. [20], Problem 4.69). Thus

$$(*) \qquad \alpha_j(x_k)T^{n_j+1}y \underset{j}{\longrightarrow} Tx_k \underset{k}{\longrightarrow} x.$$

This ensures the existence of a sequence of nonzero numbers $\{\alpha_i(x)\}_{i\geqslant 0}$ such that

$$(**) \qquad \alpha_i(x)T^{n_i}y \to x$$

for some subsequence $\{T^{n_i}\}_{i\geqslant 0}$ of $\{T^n\}_{n\geqslant 0}$. Indeed, consider both convergences in $(*)$. Take an arbitrary $\varepsilon > 0$. Thus, there exists a positive integer k_ε such that $\|Tx_k - x\| \leqslant \varepsilon/2$ whenever $k \geqslant k_\varepsilon$. Moreover, for each k there exists a positive integer $j_{\varepsilon,k}$ such that $\|\alpha_j(x_k)T^{n_j+1}y - Tx_k\| \leqslant \varepsilon/2$ whenever $j \geqslant j_{\varepsilon,k}$. Therefore

$$j \geqslant j_{\varepsilon,k_\varepsilon} \implies \|\alpha_j(x_{k_\varepsilon})T^{n_j+1}y - x\| \leqslant \|\alpha_j(x_{k_\varepsilon})T^{n_j+1}y - Tx_{k_\varepsilon}\| + \|Tx_{k_\varepsilon} - x\| \leqslant \varepsilon.$$

For each integer $i \geqslant 1$ set $\varepsilon = 1/i$. Consequently, set $k(i) = k_\varepsilon = k_{1/i}$ and $j(i) = j_{\varepsilon,k_\varepsilon} = j_{1/i,k_i}$, so that $\alpha_j(x_{k_\varepsilon}) = \alpha_j(x_{k(i)})$. Thus, for every integer $i \geqslant 1$ there is another integer $j(i) \geqslant 1$ such that $\|\alpha_j(x_{k(i)})T^{n_j+1}y - x\| \leqslant 1/i$ whenever $j \geqslant j(i)$. Hence,

$$\|\alpha_{j(i)}(x_{k(i)})T^{n_{j(i)}+1}y - x\| \leqslant \frac{1}{i} \quad \text{for every integer } i \geqslant 0,$$

and so there exists a sequence of nonzero numbers $\{\alpha_{j(i)}(x_{k(i)})\}_{i\geqslant 0}$ for which

$$\alpha_{j(i)}(x_{k(i)})T^{n_{j(i)}+1}y \to x.$$

By setting $\alpha_i(x) = \alpha_{j(i)}(x_{k(i)})$ and $T^{n_i} = T^{n_{j(i)}+1}$ we get: there exists a sequence of nonzero numbers $\{\alpha_i(x)\}_{i\geqslant 0}$ and a subsequence $\{T^{n_i}\}_{i\geqslant 0}$ of $\{T^n\}_{n\geqslant 0}$ such that $(**)$ holds true. Thus, T is supercyclic (since x was taken to be an arbitrary vector in $\mathcal{X}$). Therefore if T is weakly l-sequentially supercyclic, then T is supercyclic. The converse is trivial. □

The next result gives an elementary proof that the classical Volterra operator $V \in \mathcal{B}[L^p[0,1]]$ given by $V(f)(s) = \int_0^s f(t)\,\mathrm{d}t$ for every $f \in L^p[0,1]$ for $p \geqslant 1$, is not weakly l-sequentially supercyclic. A previous nonelementary proof that the Volterra operator is not even weakly supercyclic was given in [25], Section 2.

Corollary 4.1. *The Volterra operator is not weakly l-sequentially supercyclic.*

Proof. It was shown in [14] that the Volterra operator is not supercyclic. It is well known that the Volterra operator is compact (see, e.g. [1], Example 7.8). Thus, the Volterra operator is not weakly l-sequentially supercyclic by Theorem 4.1. □

Question 4.1. Does weak supercyclicity coincide with weak l-sequential supercyclicity for compact operators on normed spaces?

Theorem 4.1 yields an immediate proof that a compact hyponormal (equivalently, a compact normal) operator is not weakly l-sequentially supercyclic.

Corollary 4.2. *A compact hyponormal operator is not weakly l-sequentially supercyclic.*

Proof. A hyponormal operator on a Hilbert space is never supercyclic (cf. Proposition 3.1 (a)). Thus, the claimed result follows from Theorem 4.1. □

Remark 4.1. The above result can be proved without using Theorem 4.1 neither Proposition 3.1 (a), but using the results in Proposition 3.1 (b), (d) as follows. Suppose a nonzero operator T on a Hilbert space is weakly supercyclic. If T is compact and hyponormal, then it is a compact nonzero multiple of a unitary U, since a weakly supercyclic hyponormal is a multiple of a unitary (cf. Proposition 3.1 (b)). Thus, T and so U are invertible compact, and hence they must act on a finite-dimensional space (since the collection of all compact operators on a normed space $\mathcal{X}$ is an ideal of $\mathcal{B}[\mathcal{X}]$, and the identity operator is not compact on an infinite-dimensional space). On the other hand, a weakly l-sequentially supercyclic unitary operator is singular-continuous (cf. Proposition 3.1 (d)), and so it must act on an infinite-dimensional space (since on a finite-dimensional space spectra are finite, where unitaries are singular-discrete). This leads to a contradiction. Thus, a compact hyponormal operator is not weakly l-sequentially supercyclic.

As it is well-known, a compact operator is hyponormal if and only if it is compact and normal (see, e.g. [20], Problem 6.23), which means a compact diagonalizable; equivalently, a countable weighted sum of projections (according to the Spectral Theorem). Thus, the result in Corollary 4.2 (compact hyponormal are not weakly l-sequentially supercyclic; and so not supercyclic) also follows from Theorem 3.2.

Theorem 4.2 fully characterizes weakly l-sequentially supercyclic compact operators: they are quasinilpotent.

Theorem 4.2. *A compact weakly l-sequentially supercyclic operator is quasinilpotent (acting on an infinite-dimensional Banach space).*

P r o o f. Take $T \in \mathcal{B}[\mathcal{X}]$, where $\mathcal{X}$ is a normed space, and let $T^{m*} \in \mathcal{B}[\mathcal{X}^*]$ be the normed-space adjoint of $T^m \in \mathcal{B}[\mathcal{X}]$ for an arbitrary nonnegative integer m. Suppose T is compact and weakly l-sequentially supercyclic (thus weakly supercyclic). Let $\sigma_P(T^*)$ be the point spectrum of T^*. Theorem 4.1 says that the compact T is supercyclic, and hence $\#\sigma_P(T^*) \leqslant 1$; that is, T^* has at most one eigenvalue. (This has been verified for supercyclic operators in a Hilbert space setting in [17], Proposition 3.1, and extended to a normed space setting in [3], Theorem 3.2). Since the operator T in $\mathcal{B}[\mathcal{X}]$ is compact, its normed-space adjoint T^* in $\mathcal{B}[\mathcal{X}^*]$ is compact as well (see, e.g. [29], Theorem 4.15). The dual $\mathcal{X}^*$ of a normed space $\mathcal{X}$ is a Banach space, and so the spectrum of T^* is nonempty. Since T^* is compact, $\sigma(T^*) \setminus \{0\} = \sigma_P(T^*) \setminus \{0\}$ (Fredholm alternative). Moreover, if $\mathcal{X}$ is infinite dimensional, then so is $\mathcal{X}^*$, and hence $0 \in \sigma(T^*)$ (i.e. zero lies in $\sigma(T^*)$ since an invertible compact operator must act on a finite-dimensional space). Summing up: if T is a weakly l-sequentially supercyclic compact operator on an infinite-dimensional normed space, then

$$\#\sigma_P(T^*) \leqslant 1, \quad \sigma(T^*) \setminus \{0\} = \sigma_P(T^*) \setminus \{0\}, \quad \text{and} \quad 0 \in \sigma(T^*).$$

Since $\#\sigma_P(T^*) \leqslant 1$, either $\sigma_P(T^*) = \{\lambda\}$ for $\lambda \neq 0$, or $\sigma_P(T^*) \subseteq \{0\}$.

(a) Suppose $\sigma_P(T^*) = \{\lambda\}$ for some $0 \neq \lambda \in \mathbb{C}$. Since $0 \in \sigma(T^*)$ and $\sigma(T^*) \setminus \{0\} = \sigma_P(T^*) \setminus \{0\}$, we get $\sigma(T^*) = \{0, \lambda\}$. If $\mathcal{X}$ is a Banach space, then $\sigma(T)$ is a compact nonempty set such that $\sigma(T) = \sigma(T^*) = \{0, \lambda\}$ (see, e.g. [10]—for Hilbert-space adjoint this becomes $\sigma(T) = \sigma(T^*)^* = \{0, \overline{\lambda}\}$, which does not alter the next argument). The spectrum of a weakly l-sequentially supercyclic operator T on a Banach space is such that all components of the spectrum meet one and the same circle about the origin of the complex plane for some finite (nonnegative) radius. (Again, this has been verified for supercyclic operators on a Hilbert space in [17], Proposition 3.1, and for weakly hypercyclic operators on a Banach space regarding the unit circle in [11], Theorem 3, and extended to weakly supercyclic operators on a Banach space in [4], Proposition 3.5). Then $\{\lambda\} \neq \{0\}$ cannot be a component of $\sigma(T)$, and hence $\sigma_P(T^*) \neq \{\lambda\}$ for $\lambda \neq 0$, leading to a contradiction.

(b) Thus, $\sigma_P(T^*) \subseteq \{0\}$ so $\sigma(T^*) = \sigma(T) = \{0\}$ and T is quasinilpotent. □

Remark 4.2. (a) A hypercyclic operator is not compact, and there is no supercyclic operator on a complex normed space with finite dimension greater than 1, see [18], Section 4. There are, however, compact supercyclic operators on a separable infinite-dimensional complex Banach space, see [18], Theorem 1 and Section 4.

(b) The Volterra operator is an example of a compact quasinilpotent operator that is not supercyclic (and so it is not weakly l-sequentially supercyclic) but, according to item (a) above and Theorems 4.1, 4.2, there exist quasinilpotent supercyclic operators. It was also shown in [27], Corollary 5.3, that if the adjoint of a bilateral or of a unilateral weighted shift on ℓ^2 or on ℓ^2_+ has a weighting sequence possessing a subsequence that goes to zero, then there is an infinite-dimensional subspace whose all nonzero vectors are supercyclic for it. In particular, this happens for weighted shifts with weighting sequences converging to zero, and so this happens for the adjoint of compact quasinilpotent weighted shifts.

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