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# ON WEAKLY CONFORMALLY SYMMETRIC MANIFOLDS

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**Abstract.** The object of the present paper is to study *weakly conformally symmetric manifolds*. Among others it is shown that an Einstein weakly conformally symmetric manifold reduces to a weakly symmetric manifold. Also several examples of weakly conformally symmetric manifolds with non-vanishing scalar curvature have been obtained.

## 1. Introduction

The notions of weakly symmetric and weakly projective symmetric manifolds were introduced by Tamássy and Binh [5], [2]. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called a weakly symmetric manifold if its curvature tensor $R$ of type (0,4) satisfies the condition

$$(\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V) + B(Y)R(X, Z, U, V) \\ + F(Z)R(Y, X, U, V) + D(U)R(Y, Z, X, V) + E(V)R(Y, Z, U, X) \quad (1.1)$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $A, B, F, D$ and $E$ are 1-forms (not simultaneously zero) and $\nabla$ denotes the operator of covariant differentiation with respect to the Riemannian metric $g$. The 1-forms are called the associated 1-forms of the manifold and an $n$-dimensional manifold of this kind is denoted by $(WS)_n$. On the analogy of $(WS)_n$, U. C. De and S. Bandyopadhyay [3] introduced the notion of weakly conformally symmetric manifolds and proved its existence by an example. A non-conformally flat Riemannian manifold $(M^n, g)$ $(n > 3)$ [This condition will be assumed throughout the paper as the conformal curvature tensor vanishes identically for $n = 3$] is said to be weakly conformally symmetric manifold if the conformal curvature tensor $C$ of type (0,4) satisfies the condition

$$(\nabla_X C)(Y, Z, U, V) = A(X)C(Y, Z, U, V) + B(Y)C(X, Z, U, V) \\ + F(Z)C(Y, X, U, V) + D(U)C(Y, Z, X, V) + E(V)C(Y, Z, U, X) \quad (1.2)$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where $A, B, F, D$ and $E$ are 1-forms (not simultaneously zero) and are called the associated 1-forms of the manifold. Such an

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$n$-dimensional manifold is denoted by $(WCS)_n$. Also in [3] the authors proved that in a $(WCS)_n$, the associated 1-forms $B = F$ and $D = E$ and hence the defining condition (1.2) of a $(WCS)_n$ reduces to the following form [3]:

$$(\nabla_X C)(Y,Z,U,V) = A(X)C(Y,Z,U,V) + B(Y)C(X,Z,U,V) + B(Z)C(Y,X,U,V) + D(U)C(Y,Z,X,V) + D(V)C(Y,Z,U,X). \quad (1.3)$$

This paper presents a study of $(WCS)_n$. In section 2, it is shown that if in a $(WCS)_n$, the Ricci tensor vanishes then it reduces to a $(WS)_n$. Also it is proved that in a $(WCS)_n$ satisfying the condition $A(C(Y,Z)U) + D(C(Y,Z)U) = 0$, the Ricci tensor is of Codazzi type if and only if the scalar curvature is constant. Section 3 is concerned with Einstein $(WCS)_n$ and proved that such a manifold is $(WS)_n$. In section 4 we study conformal transformation in a $(WCS)_n$ and we prove that if a $(WCS)_n$ is transformed into another $(WCS)_n$ under such a transformation then the transformation is homothetic. In [3] the authors gave an example of a $(WCS)_n$ which was of zero scalar curvature. Natural question arises whether there exists or not $(WCS)_n$ of non-zero scalar curvature? The last section provides the answer to this question affirmatively by several examples.

## 2. Some basic results of $(WCS)_n$

In this section, we derive some formulas, which will be needed to the study of a $(WCS)_n$. The Weyl conformal curvature tensor $C$ of type (0,4) is given by

$$C(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \quad (2.1)$$

where $R$, $S$, $r$ denotes respectively the Riemannian curvature tensor of type (0,4), the Ricci tensor of type (0,2) and the scalar curvature of the manifold.

From (2.1), it follows that

$$\sum_{i=1}^{n} C(e_i,Y,Z,e_i) = 0 = \sum_{i=1}^{n} C(Y,e_i,e_i,Z), \quad (2.2)$$

where $\{e_i : i = 1,2,\dots,n\}$ is an orthonormal basis of the tangent space at any point of the manifold.

From (2.1) it follows by virtue of Bianchi identity that

$$(\nabla_X C)(Y,Z,U,V) + (\nabla_Y C)(Z,X,U,V) + (\nabla_Z C)(X,Y,U,V) = -\frac{1}{n-2}[g(Y,V)\{(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)\} + g(Z,V)\{(\nabla_Y S)(X,U) - (\nabla_X S)(Y,U)\} + g(Z,U)\{(\nabla_X S)(Y,V) - (\nabla_Y S)(X,V)\} + g(Y,U)\{(\nabla_Z S)(X,V) - (\nabla_X S)(Z,V)\} + g(X,V)\{(\nabla_Z S)(Y,U)$$

$$- (\nabla_Y S)(Z,U)\} + g(X,U)\{(\nabla_Y S)(Z,V) - (\nabla_Z S)(Y,V)\}]$$
$$+ \frac{1}{(n-1)(n-2)}[dr(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}$$
$$+ dr(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}$$
$$+ dr(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}]. \quad (2.3)$$

If the Ricci tensor is of Codazzi type [4], i.e., if

$$(\nabla_X S)(Z,U) = (\nabla_Z S)(X,U),$$

then $dr(X) = 0$ for all $X$ and hence from (2.3) we obtain

$$(\nabla_X C)(Y,Z,U,V) + (\nabla_Y C)(Z,X,U,V) + (\nabla_Z C)(X,Y,U,V) = 0. \quad (2.4)$$

Hence we can state the following:

PROPOSITION 2.1. *If in a Riemannian manifold* $(M^n, g)$ $(n > 3)$*, the Ricci tensor is of Codazzi type, then the second Bianchi identity of the conformal curvature tensor* $C$ *has just the same form as that of the curvature tensor* $R$ *of* $M^n$*.*

Again if a Riemannian manifold satisfies the relation (2.4), then (2.3) yields

$$g(Y,V)\{(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)\} + g(Z,V)\{(\nabla_Y S)(X,U)$$
$$- (\nabla_X S)(Y,U)\} + g(Z,U)\{(\nabla_X S)(Y,V) - (\nabla_Y S)(X,V)\}$$
$$+ g(Y,U)\{(\nabla_Z S)(X,V) - (\nabla_X S)(Z,V)\} + g(X,V)\{(\nabla_Z S)(Y,U)$$
$$- (\nabla_Y S)(Z,U)\} + g(X,U)\{(\nabla_Y S)(Z,V) - (\nabla_Z S)(Y,V)\}$$
$$- \frac{1}{(n-1)}[dr(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}$$
$$+ dr(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}$$
$$+ dr(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}] = 0. \quad (2.5)$$

Putting $Y = V = e_i$ in (2.5) and taking summation over $i$, $1 \le i \le n$, we obtain

$$(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U) = \frac{1}{2(n-1)}[dr(X)g(Z,U) - dr(Z)g(X,U)]. \quad (2.6)$$

If $r$ is a constant then (2.6) yields that the Ricci tensor is of Codazzi type. This leads to the following:

PROPOSITION 2.2. *If a Riemannian manifold* $(M^n, g)$ $(n > 3)$ *with constant scalar curvature satisfies the relation (2.4) then the Ricci tensor is of Codazzi type.*

Setting $Y = V = e_i$ in (1.3) and then using (2.2), it follows that

$$B(C(X,Z)U) + D(C(X,U)Z) = 0 \quad (2.7)$$

for all $X, Z, U$. Thus we have the following:

Proposition 2.3. *In a $(WCS)_n$, the relation (2.7) holds.*

In view of (2.1), (2.7) yields

$$\begin{aligned} B(R(X,Z)U) + D(R(X,U)Z) - \frac{1}{n-2}[S(Z,U)\{B(X)+D(X)\} \\ + g(Z,U)\{B(QX)+D(QX)\} - S(X,U)B(Z) - S(X,Z)D(U) \\ - g(X,U)B(QZ) - g(X,Z)D(QU)] + \frac{r}{(n-1)(n-2)}[g(Z,U) \\ \{B(X)+D(X)\} - g(X,U)B(Z) - g(X,Z)D(U)] = 0, \end{aligned} \tag{2.8}$$

where $Q$ is the Ricci-operator i.e., $g(QX,Y) = S(X,Y)$.

Now, if the Ricci tensor vanishes i.e., if $S(X,Y) = 0$ for all $X, Y$ then (2.1) implies that $C(X,Y,Z,W) = R(X,Y,Z,W)$ and hence (1.2) reduces to (1.1). Also for $S(X,Y) = 0$, (2.8) yields

$$B(R(X,Z)U) + D(R(X,U)Z) = 0 \tag{2.9}$$

for all vector fields $X, Z, U$ on the manifold. Thus we can state the following:

Proposition 2.4. *A $(WCS)_n$ with vanishing Ricci tensor is a $(WS)_n$ and satisfies the relation (2.9).*

Again from (1.3) it follows that

$$(divC)(Y,Z)U = A(C(Y,Z)U) + D(C(Y,Z)U) \tag{2.10}$$

where *'div'* denotes the divergence. Also in a Riemannian manifold $(M^n, g)$ $(n > 3)$ we have

$$\begin{aligned} (divC)(X,Y)Z = \frac{n-3}{n-2}[\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} \\ - \frac{1}{2(n-1)}\{g(Y,Z)dr(X) - g(X,Z)dr(Y)\}]. \end{aligned} \tag{2.11}$$

In view of (2.7) and (2.10), it follows that

$$(divC)(Y,Z)U = A(C(Y,Z)U) - B(C(Y,U)Z). \tag{2.12}$$

A Riemannian manifold with *div* $C = 0$ is said to be of harmonic conformal curvature [1].

This leads to the following:

Proposition 2.5. *A $(WCS)_n$ is of harmonic conformal curvature if and only if $A(C(Y,Z)U) = B(C(Y,U)Z)$ for all vector fields $Y, Z, U$.*

We now suppose that a $(WCS)_n$ satisfies the relation

$$A(C(Y,Z)U) = B(C(Y,U)Z) \tag{2.12a}$$

for all vector fields $Y, Z, U$. Then (2.12) yields

$$(divC)(Y,Z)U = 0. \tag{2.13}$$

In view of (2.13) and (2.11) we obtain

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}\{g(Y,Z)dr(X) - g(X,Z)dr(Y)\}. \tag{2.14}$$

From (2.14), it follows that if the scalar curvature $r$ is constant, then

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$$

for all $X, Y, Z$, which means that the Ricci tensor is of Codazzi type [4]. Conversely, if the Ricci tensor is of Codazzi type, then (2.14) implies that $r$ is a constant. Hence we can state the following:

Proposition 2.6. *Let $(M^n, g)$ $(n > 3)$ be a $(WCS)_n$ satisfying the relation (2.12a). Then the Ricci tensor is of Codazzi type if and only if its scalar curvature is a constant.*

## 3. Einstein $(WCS)_n$

Let us consider a $(WCS)_n$, which is Einstein. Then we have

$$S(X,Y) = \frac{r}{n}g(X,Y), \tag{3.1}$$

which yields $(\nabla_Z S)(X,Y) = 0$ and hence $dr(X) = 0$ for all $X$. By virtue of (3.1), (2.1) takes the form

$$C(X,Y,Z,W) = R(X,Y,Z,W) - \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{3.2}$$

In view of (3.2), (1.3) reduces to the following:

$$\begin{aligned}
&(\nabla_X R)(Y,Z,U,V)\\
&\quad = A(X)[R(Y,Z,U,V) - \frac{r}{n(n-1)}\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}]\\
&\quad + B(Y)[R(X,Z,U,V) - \frac{r}{n(n-1)}\{g(Z,U)g(X,V) - g(X,U)g(Z,V)\}]\\
&\quad + B(Z)[R(Y,X,U,V) - \frac{r}{n(n-1)}\{g(X,U)g(Y,V) - g(Y,U)g(X,V)\}]\\
&\quad + D(U)[R(Y,Z,X,V) - \frac{r}{n(n-1)}\{g(Z,X)g(Y,V) - g(Y,X)g(Z,V)\}]\\
&\quad + D(V)[R(Y,Z,U,V) - \frac{r}{n(n-1)}\{g(Z,U)g(Y,X) - g(Y,U)g(Z,X)\}].
\end{aligned} \tag{3.3}$$

Applying Bianchi identity, it follows from (3.3) that

$$\begin{aligned} &A(X)R(Y,Z,U,V)+A(Y)R(Z,X,U,V)+A(Z)R(X,Y,U,V)\\ &\quad+B(Y)R(X,Z,U,V)+B(Z)R(Y,X,U,V)+B(X)R(Z,Y,U,V)\\ &\quad-\frac{r}{n(n-1)}[\{A(X)-2B(X)\}\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\}\\ &\qquad+\{A(Y)-2B(Y)\}\{g(X,U)g(Z,V)-g(Z,U)g(X,V)\}\\ &\qquad+\{A(Z)-2B(Z)\}\{g(Y,U)g(X,V)-g(X,U)g(Y,V)\}]=0. \qquad (3.4)\end{aligned}$$

Putting $Y=V=e_i$ in (3.4) and then taking summation over $i, 1\le i\le n$, we get

$$\begin{aligned} &[A(X)-B(X)]S(Z,U)-[A(Z)-B(Z)]S(X,U)+A(R(Z,X)U)\\ &\quad-B(R(Z,X)U)-\frac{(n-2)r}{n(n-1)}[\{A(X)-2B(X)\}g(Z,U)\\ &\qquad-\{A(Z)-2B(Z)\}g(X,U)]=0. \qquad (3.5)\end{aligned}$$

Again setting $Z=U=e_i$ in (3.5) and taking summation over $i, 1\le i\le n$, we obtain

$$r[A(X)-B(X)]-2[A(QX)-B(QX)]-\frac{(n-2)r}{n}[A(X)-2B(X)]=0. \qquad (3.6)$$

From (3.1) we have

$$A(QX)=\frac{r}{n}A(X) \text{ and } B(QX)=\frac{r}{n}B(X).$$

Using the above in (3.6) we obtain $rB(X)=0$, which yields $r=0$, since $B(X)\neq 0$. Consequently (3.1) implies that $S(X,Y)=0$ for all $X,Y$ and hence by Proposition 2.4., the manifold is a $(WS)_n$. Thus we can state the following:

THEOREM 3.1. *An Einstein $(WCS)_n$ is of vanishing scalar curvature and hence is a $(WS)_n$.*

## 4. Conformal transformation of $(WCS)_n$

Let $(M^n,g)(n>3)$ be an n-dimensional Riemannian manifold with the metric tensor $g$. Let $\rho$ be a positive function on $M$. Then $\overset{*}{g}=\rho^2 g$ defines a change of metric on $M$ which does not change the angle between two vectors at any point $p\in M$ and is called a conformal transformation. In particular, if the function $\rho$ is a positive constant, then the conformal transformation is said to be homothetic.

If $\overset{*}{\nabla}$ denotes the operator of covariant differentiation with respect to $\overset{*}{g}$, then we have [6]

$$\overset{*}{\nabla}_X Y=\nabla_X Y+\omega(Y)X+\omega(X)Y-g(X,Y)U \qquad (4.1)$$

for any vector fields $X,Y$ where $\omega$ is a 1-form defined by $\omega=d(log\rho)$ and $U$ is a vector field defined by

$$\omega(X)=g(X,U).$$

Since the conformal curvature tensor is invariant under conformal transformation, we have

$$\overset{*}{C}(X,Y,Z,W) = C(X,Y,Z,W), \tag{4.2}$$

where each object denoted by '$*$' is from $\overset{*}{M}$. Differentiating (4.2) covariantly and then using (4.1) we obtain

$$\begin{aligned}(\overset{*}{\nabla}_X\overset{*}{C})(Y,Z,W,V) &= (\nabla_X C)(Y,Z,W,V) - 2\omega(X)C(Y,Z,W,V)\\ &- \omega(Y)C(X,Z,W,V) - \omega(Z)C(Y,X,W,V) - \omega(W)C(Y,Z,X,V)\\ &+ g(X,Y)C(U,Z,W,V) + g(X,Z)C(Y,U,W,V)\\ &+ g(X,W)C(Y,Z,U,V) - C(Y,Z,W,X)U. \end{aligned} \tag{4.3}$$

We now suppose that both $M$ and $\overset{*}{M}$ are $(WCS)_n$. Then we have the relation (1.3) and

$$\begin{aligned}(\overset{*}{\nabla}_X\overset{*}{C})(Y,Z,W,V) &= \overset{*}{A}(X)\overset{*}{C}(Y,Z,W,V) + \overset{*}{B}(Y)\overset{*}{C}(X,Z,W,V)\\ &+ \overset{*}{B}(Z)\overset{*}{C}(Y,X,W,V) + \overset{*}{D}(W)\overset{*}{C}(Y,Z,X,V) + \overset{*}{D}(V)\overset{*}{C}(Y,Z,W,X), \end{aligned} \tag{4.4}$$

where $\overset{*}{A}$, $\overset{*}{B}$, $\overset{*}{D}$ are non-zero 1-forms such that $\overset{*}{g}(X,\rho_1) = \overset{*}{A}(X)$ and so on. By virtue of (4.2) and (4.4) we obtain from (4.3) that

$$\begin{aligned}&[\overset{*}{A}(X) - A(X)]C(Y,Z,W,V) + [\overset{*}{B}(Y) - B(Y)]C(X,Z,W,V) +\\ &[\overset{*}{B}(Z) - B(Z)]C(Y,X,W,V) + [\overset{*}{D}(W) - D(W)]C(Y,Z,X,V) +\\ &[\overset{*}{D}(V) - D(V)]C(Y,Z,W,X) + 2\omega(X)C(Y,Z,W,V) + \omega(Y)C(X,Z,W,V)\\ &+ \omega(Z)C(Y,X,W,V) + \omega(W)C(Y,Z,X,V) - g(X,Y)C(U,Z,W,V)\\ &- g(X,Z)C(Y,U,W,V) - g(X,W)C(Y,Z,U,V) - C(Y,Z,W,X)\omega(V) = 0. \end{aligned} \tag{4.5}$$

Putting $W = V = e_i$ in (4.5) and then taking summation over $i$, $1 \le i \le n$, we get

$$\omega(C(Y,Z)X) = 0 \text{ for all } Y, Z, X. \tag{4.6}$$

This implies that

$$C(Y,Z,U,V) = C(Y,Z,V,U) = C(U,V,Y,Z) = C(V,U,Y,Z) = 0.$$

Hence using these relations, (4.5) reduces to

$$\begin{aligned}&[\overset{*}{A}(X) - A(X)]C(Y,Z,W,V) + [\overset{*}{B}(Y) - B(Y)]C(X,Z,W,V)\\ &+ [\overset{*}{B}(Z) - B(Z)]C(Y,X,W,V) + [\overset{*}{D}(W) - D(W)]C(Y,Z,X,V)\\ &+ [\overset{*}{D}(V) - D(V)]C(Y,Z,W,X) + 2\omega(X)C(Y,Z,W,V) +\\ &\omega(Y)C(X,Z,W,V) + \omega(Z)C(Y,X,W,V) + \omega(W)C(Y,Z,X,V) = 0. \end{aligned} \tag{4.7}$$

We now first suppose that $\overset{*}{A}(X) = A(X)$, $\overset{*}{B}(X) = B(X)$ and $\overset{*}{D}(X) = D(X)$ for all $X$. Then (4.7) yields

$$2\omega(X)C(Y,Z,W,V) + \omega(Y)C(X,Z,W,V) + \omega(Z)C(Y,X,W,V) + \omega(W)C(Y,Z,X,V) = 0. \quad (4.8)$$

Setting $X = U$ in (4.8) and then using (4.6) we obtain

$$2\omega(U)C(Y,Z,W,V) = 0.$$

Since the manifold is non-conformally flat, we must have $\omega(U) = 0$, which yields $\|U\| = 0$ and hence $\rho =$ constant. Consequently, the transformation is homothetic.

Next we suppose that $\overset{*}{A}(X) \neq A(X)$, $\overset{*}{B}(X) \neq B(X)$ and $\overset{*}{D}(X) \neq D(X)$. Suppose, if possible, $\overset{*}{A}(X) - A(X) = \omega(X) \neq 0$. Then

$$\overset{*}{A}(U) - A(U) = \omega(U) \neq 0. \quad (4.9)$$

Putting $X = U$ in (4.7) and then using (4.6) and (4.9) we get

$$3\omega(U)C(Y,Z,W,V) = 0,$$

which is a contradiction as $\omega(U) \neq 0$ and $C \neq 0$. Thus we must have $\overset{*}{A}(X) = A(X)$ for all $X$. Similarly it can be shown that $\overset{*}{B} = B$ and $\overset{*}{D} = D$. Hence the associated 1-forms can not be different and the transformation is homothetic. This leads to the following:

THEOREM 4.1. *If a $(WCS)_n$ is transformed into another $(WCS)_n$ under a conformal transformation, then the associated 1-forms of the manifold are invariant and the transformation is homothetic.*

## 5. Some examples of $(WCS)_n$

EXAMPLE 5.1. Let $M^4 = \{(x^1, x^2, x^3, x^4) \in R^4 : 0 < x^3 < \frac{\pi}{2}\}$ be a manifold endowed with the metric

$$ds^2 = e^{x^1}(dx^1)^2 + e^{x^2}(dx^1)^2 + (dx^3)^2 + \sin^2 x^3 (dx^4)^2. \quad (5.1)$$

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, conformal curvature tensor are given by

$$\Gamma^1_{11} = \frac{1}{2} = \Gamma^2_{22}, \Gamma^3_{44} = -\sin x^3 \cos x^3, \Gamma^4_{34} = \cot x^3,$$
$$R_{3443} = -\sin^2 x^3, S_{33} = -1, S_{44} = -\sin^2 x^3, r = -2 \neq 0,$$
$$C_{1221} = -\frac{1}{3}e^{x^1+x^2}, C_{1331} = \frac{1}{6}e^{x^1}, C_{1441} = \frac{1}{6}e^{x^1}\sin^2 x^3,$$
$$C_{2332} = \frac{1}{6}e^{x^2}, C_{2442} = \frac{1}{6}e^{x^2}\sin^2 x^3, C_{3443} = -\frac{1}{3}\sin^2 x^3$$

and the components that can be obtained from these by the symmetry properties. The covariant derivatives of all components of the conformal curvature tensor vanish. In terms of local coordinate system, if we consider the components of the 1-forms $A$, $B$ and $D$ as

$$A(\partial_i) = A_i = 0 \text{ for } \ i = 1, 2, 3, 4,$$
$$B(\partial_i) = B_i = 0 \text{ for } \ i = 1, 2, 3, 4,$$
$$D(\partial_i) = D_i = 0 \text{ for } \ i = 1, 2, 3, 4,$$

where $\partial_i = \frac{\partial}{\partial x^i}$, then one can easily prove that $(M^4, g)$ is a conformally symmetric manifold, which is a trivial $(WCS)_4$. Thus we can state the following:

THEOREM 5.1. *Let $(M^4, g)$ be a Riemannian manifold endowed with the metric given in (5.1). Then $(M^4, g)$ is a weakly conformally symmetric manifold with non-vanishing scalar curvature, which is conformally symmetric.*

EXAMPLE 5.2. Let $M^4 = \{(x^1, x^2, x^3, x^4) \in R^4 : x^3 \neq 0\}$ be a manifold endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1 dx^2 + (dx^3)^2 + (dx^4)^2, \ (i, j = 1, 2, 3, 4), \quad (5.2)$$

where $f = a_0 + a_1 x^3 + e^{x^1}\{\frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} + \frac{(x^3)^6}{6!} + \cdots + \frac{(x^3)^{2n+2}}{(2n+2)!} + \ldots\}$, $a_0$, $a_1$ are non-constant functions of $x^1$ only. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, conformal curvature tensor and its covariant derivatives are given by

$$\Gamma^2_{11} = \frac{1}{2}f_{.1}, \Gamma^2_{13} = \frac{1}{2}f_{.3} = -\Gamma^3_{11},$$
$$R_{1331} = \frac{1}{2}f_{.33} = \frac{1}{2}e^{x^1}\cosh x^3 = S_{11}, r = 0,$$
$$C_{1331} = \frac{1}{4}f_{.33} = \frac{1}{4}e^{x^1}\cosh x^3 = -C_{1441},$$
$$C_{1331,1} = \frac{1}{4}e^{x^1}\cosh x^3, \quad (5.3)$$
$$C_{1331,3} = \frac{1}{2}e^{x^1}\sinh x^3, \quad (5.4)$$
$$C_{1441,1} = -\frac{1}{4}e^{x^1}\cosh x^3, \quad (5.5)$$
$$C_{1441,3} = -\frac{1}{2}e^{x^1}\sinh x^3 \quad (5.6)$$

and the components that can be obtained from these by the symmetry properties, where '.' denotes the partial differentiation with respect to the coordinates, ',' denotes the covariant differentiation with respect to the metric tensor $g$, and $r$ is the scalar curvature of the manifold whose value is zero here. Therefore, our $M^4$ with the considered metric is a Riemannian manifold which is neither conformally flat nor conformally symmetric and is of vanishing scalar curvature.

In terms of local coordinate system, let us consider the 1-forms $A, B, D$ as follows:

$$
\begin{aligned}
A(\partial_i) = A_i &= \begin{cases} \frac{3}{10} & \text{for } \ i=1, \\ \tanh x^3 & \text{for } \ i=3, \\ 0 & \text{otherwise }, \end{cases} \\
B(\partial_i) = B_i &= \begin{cases} \frac{1}{5} & \text{for } \ i=1, \\ 0 & \text{otherwise }, \end{cases} \\
D(\partial_i) = D_i &= \begin{cases} \frac{1}{2} & \text{for } \ i=1, \\ 0 & \text{otherwise }, \end{cases}
\end{aligned} \tag{5.7}
$$

where $\partial_i = \frac{\partial}{\partial x^i}$. In terms of local coordinate system, the defining condition (1.3) of a $(WCS)_n$ can be written as

$$C_{ijkl,p} = A_p C_{ijkl} + B_i C_{pjkl} + B_j C_{ipkl} + D_k C_{ijpl} + D_l C_{ijkp}, \tag{5.8}$$

which reduces with these 1-forms to the following equations:

$$
\begin{aligned}
C_{1331,i} &= A_i C_{1331} + B_1 C_{i331} + B_3 C_{1i31} + D_3 C_{13i1} + D_1 C_{133i}, && (5.9)\\
C_{1441,i} &= A_i C_{1441} + B_1 C_{i441} + B_4 C_{1i41} + D_4 C_{14i1} + D_1 C_{144i}, && (5.10)\\
C_{1332,i} &= A_i C_{1332} + B_1 C_{i332} + B_3 C_{1i32} + D_3 C_{13i2} + D_2 C_{133i}, && (5.11)\\
C_{1334,i} &= A_i C_{1334} + B_1 C_{i334} + B_3 C_{1i34} + D_3 C_{13i4} + D_4 C_{133i}, && (5.12)\\
C_{3112,i} &= A_i C_{3112} + B_3 C_{i112} + B_1 C_{3i12} + D_1 C_{31i2} + D_2 C_{311i}, && (5.13)\\
C_{3114,i} &= A_i C_{3114} + B_3 C_{i114} + B_1 C_{3i14} + D_1 C_{31i4} + D_4 C_{311i}, && (5.14)\\
C_{1442,i} &= A_i C_{1442} + B_1 C_{i442} + B_4 C_{1i42} + D_4 C_{14i2} + D_2 C_{144i}, && (5.15)\\
C_{1443,i} &= A_i C_{1443} + B_1 C_{i443} + B_4 C_{1i43} + D_4 C_{14i3} + D_3 C_{144i}, && (5.16)\\
C_{4112,i} &= A_i C_{4112} + B_4 C_{i112} + B_1 C_{4i12} + D_1 C_{41i2} + D_2 C_{411i}, && (5.17)
\end{aligned}
$$

where $i = 1, 2, 3, 4$, since for the cases other than (5.9)–(5.17), the components of each term of (5.8) either vanishes identically or the relation (5.8) holds trivially using the skew-symmetry property of $C$.

Now, from (5.3) and (5.7), it follows that, for $i = 1$, right hand side of (5.9) $= (A_1 + B_1 + D_1)C_{1331} = \frac{1}{4}e^{x^1} \cosh x^3 = C_{1331,1} =$ left hand side of (5.9). For $i = 2, 3, 4$, the relation (5.7) implies that both sides of equation (5.9) are equal. By the similar argument, it can be easily seen that the equations (5.10)–(5.17) holds. Thus, the manifold under consideration is weakly conformally symmetric manifold. Hence we can state the following:

THEOREM 5.2. *Let $(M^4, g)$ be a Riemannian manifold endowed with the metric given in (5.2). Then $(M^4, g)$ is a weakly conformally symmetric manifold with vanishing scalar curvature which is neither conformally symmetric nor conformally recurrent.*

EXAMPLE 5.3. Let $M^4 = \{(x^1, x^2, x^3, x^4) \in R^4 : x^3 \neq 0\}$ be a manifold endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = \sinh x^3[(dx^1)^2 + (dx^3)^2] + (dx^2)^2 + (dx^4)^2, \tag{5.18}$$

$(i, j = 1, 2, 3, 4)$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, conformal curvature tensor and its covariant derivatives are given by

$$\Gamma^1_{13} = \frac{1}{2}\coth x^3 = \Gamma^3_{33} = -\Gamma^3_{11}, R_{1331} = -\frac{1}{2\sinh x^3},$$
$$S_{11} = -\frac{1}{2\sinh^2 x^3} = S_{33}, r = -\frac{1}{\sinh^3 x^3} \neq 0,$$
$$C_{1331} = -\frac{1}{6\sinh x^3}, C_{2442} = -\frac{1}{6\sinh^3 x^3},$$
$$C_{1221} = C_{1441} = C_{2332} = C_{3443} = \frac{1}{12\sinh^2 x^3},$$
$$C_{1331,3} = \frac{\coth x^3}{2\sinh x^3}, \tag{5.19}$$
$$C_{2442,3} = \frac{\coth x^3}{2\sinh^3 x^3}, \tag{5.20}$$
$$C_{1221,3} = C_{1441,3} = C_{2332,3} = C_{3443,3} = -\frac{\coth x^3}{4\sinh^2 x^3} \tag{5.21}$$

and the components that can be obtained from these by the symmetry properties, where ‘,’ denotes the covariant differentiation with respect to the metric tensor and $S_{ij}$ denotes the components of the Ricci tensor and $r$ is the scalar curvature of the manifold. Therefore, the manifold $M^4$ with the considered metric is a Riemannian manifold, which is neither conformally flat nor conformally symmetric and is of non-vanishing scalar curvature.

We shall now show that this $(M^4, g)$ is a $(WCS)_4$, that is, it satisfies (1.3).

In terms of local coordinate system, we consider the components of the 1-forms $A$, $B$ and $D$ as follows:

$$A(\partial_i) = A_i = \begin{cases} -3\coth x^3 & \text{for } \ i = 3, \\ 0 & \text{otherwise}, \end{cases}$$
$$B(\partial_i) = B_i = 0 \text{ for } \ i = 1, 2, 3, 4, \tag{5.22}$$
$$D(\partial_i) = D_i = 0 \text{ for } \ i = 1, 2, 3, 4,$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

In terms of local coordinate system, the defining condition (1.3) of a $(WCS)_n$ can be written as (5.8), which reduces to the following equations:

$$C_{1221,i} = A_i C_{1221} + B_1 C_{i221} + B_2 C_{1i21} + D_2 C_{12i1} + D_1 C_{122i}, \tag{5.23}$$
$$C_{1331,i} = A_i C_{1331} + B_1 C_{i331} + B_3 C_{1i31} + D_3 C_{13i1} + D_1 C_{133i}, \tag{5.24}$$

$$C_{1441,i} = A_iC_{1441} + B_1C_{i441} + B_4C_{1i41} + D_4C_{14i1} + D_1C_{144i}, \tag{5.25}$$
$$C_{2332,i} = A_iC_{2332} + B_2C_{i332} + B_3C_{2i32} + D_3C_{23i2} + D_2C_{233i}, \tag{5.26}$$
$$C_{2442,i} = A_iC_{2442} + B_2C_{i442} + B_4C_{2i42} + D_4C_{24i2} + D_2C_{244i}, \tag{5.27}$$
$$C_{3443,i} = A_iC_{3443} + B_3C_{i443} + B_4C_{3i43} + D_4C_{34i3} + D_3C_{344i}, \tag{5.28}$$
$$C_{1332,i} = A_iC_{1332} + B_1C_{i332} + B_3C_{1i32} + D_3C_{13i2} + D_2C_{133i}, \tag{5.29}$$
$$C_{1334,i} = A_iC_{1334} + B_1C_{i334} + B_3C_{1i34} + D_3C_{13i4} + D_4C_{133i}, \tag{5.30}$$
$$C_{3112,i} = A_iC_{3112} + B_3C_{i112} + B_1C_{3i12} + D_1C_{31i2} + D_2C_{311i}, \tag{5.31}$$
$$C_{3114,i} = A_iC_{3114} + B_3C_{i114} + B_1C_{3i14} + D_1C_{31i4} + D_4C_{311i}, \tag{5.32}$$
$$C_{1442,i} = A_iC_{1442} + B_1C_{i442} + B_4C_{1i42} + D_4C_{14i2} + D_2C_{144i}, \tag{5.33}$$
$$C_{1443,i} = A_iC_{1443} + B_1C_{i443} + B_4C_{1i43} + D_4C_{14i3} + D_3C_{144i}, \tag{5.34}$$
$$C_{4112,i} = A_iC_{4112} + B_4C_{i112} + B_1C_{4i12} + D_1C_{41i2} + D_2C_{411i}, \tag{5.35}$$
$$C_{2334,i} = A_iC_{2334} + B_2C_{i334} + B_3C_{2i34} + D_3C_{23i4} + D_4C_{233i}, \tag{5.36}$$
$$C_{3221,i} = A_iC_{3221} + B_3C_{i221} + B_2C_{3i21} + D_2C_{32i1} + D_1C_{322i}, \tag{5.37}$$
$$C_{3224,i} = A_iC_{3224} + B_3C_{i224} + B_2C_{3i24} + D_2C_{32i4} + D_4C_{322i}, \tag{5.38}$$
$$C_{2443,i} = A_iC_{2443} + B_2C_{i443} + B_4C_{2i43} + D_4C_{24i3} + D_3C_{244i}, \tag{5.39}$$
$$C_{4221,i} = A_iC_{4221} + B_4C_{i221} + B_2C_{4i21} + D_2C_{42i1} + D_1C_{422i}, \tag{5.40}$$

where $i$ =1,2,3,4, since for the cases other than (5.23)–(5.40), the components of each term of (5.8) either vanishes identically or the relation (5.8) holds trivially using the skew-symmetry property of $C$.

Now using (5.21) and (5.22), it follows, for $i = 3$, that right hand side of (5.23) $= A_3C_{1221} = -\frac{\coth x^3}{4\sinh^2 x^3} = C_{1221,3} =$ left hand side of (5.23).

For $i = 1, 2, 4$, the relation (5.22) implies that both sides of equation (5.23) are equal. By the similar arguement, it can be easily seen that the equations (5.24)–(5.40) hold. Thus the manifold under consideration is weakly conformally symmetric. Hence we can state the following:

Theorem 5.3. *Let $(M^4, g)$ be a Riemannian manifold endowed with the metric given in (5.18). Then $(M^4, g)$ is a weakly conformally symmetric manifold with non-vanishing scalar curvature, which is neither conformally flat nor conformally symmetric but conformally recurrent.*

Example 5.4. Let $M = \{(x^1, x^2, x^3, \ldots, x^n) \in R^n : x^3 \neq 0\}$ be a manifold endowed with the metric

$$ds^2 = \sinh x^3[(dx^1)^2 + (dx^3)^2] + (dx^2)^2 + (dx^4)^2 + \sum_{k=5}^{n}(dx^k)^2. \tag{5.41}$$

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, conformal curvature tensor and its covariant derivatives are given by

$$\Gamma^1_{13} = \frac{1}{2}\coth x^3 = \Gamma^3_{33} = -\Gamma^3_{11},$$

$$
\begin{aligned}
R_{1331} &= -\frac{1}{2\sinh x^3}, S_{11} = -\frac{1}{2\sinh^2 x^3} = S_{33},\\
r &= -\frac{1}{\sinh^3 x^3} \neq 0, C_{1331} = -\frac{n-3}{2(n-1)\sinh x^3},\\
C_{2442} &= C_{2kk2} = C_{4kk4} = C_{mkkm} = -\frac{1}{(n-1)(n-2)\sinh^3 x^3},\\
C_{1221} &= C_{1441} = C_{1kk1} = C_{2332} = C_{3443}\\
&= C_{3kk3} = \frac{n-3}{2(n-1)(n-2)\sinh^2 x^3},
\end{aligned}
$$

$$
C_{1331,3} = \frac{3(n-3)\coth x^3}{2(n-1)\sinh x^3}, \tag{5.42}
$$

$$
C_{2442,3} = C_{2kk2,3} = C_{4kk4,3} = C_{mkkm,3} = \frac{3\coth x^3}{(n-1)(n-2)\sinh^3 x^3}, \tag{5.43}
$$

$$
\begin{aligned}
C_{1221,3} &= C_{1441,3} = C_{1kk1,3} = C_{2332,3} = C_{3443,3} = C_{3kk3,3}\\
&= -\frac{3(n-3)\coth x^3}{2(n-1)(n-2)\sinh^2 x^3} \text{ for } 5 \leq k \leq n, 5 \leq m \leq n, k \neq m,
\end{aligned} \tag{5.44}
$$

and the components that can be obtained from these by the symmetry properties, where ‘,’ denotes the covariant differentiation with respect to the metric tensor and $S_{ij}$ denotes the components of the Ricci tensor and $r$ is the scalar curvature of the manifold. Therefore, the manifold $M^n$ with the considered metric is a Riemannian manifold, which is neither conformally flat nor conformally symmetric and is of non-vanishing scalar curvature.

In terms of local coordinate system, we consider the components of the 1-forms $A$, $B$ and $D$ as follows:

$$
\begin{aligned}
A(\partial_i) = A_i &= \begin{cases} -3\coth x^3 & \text{for } \ i = 3,\\ 0 & \text{otherwise}, \end{cases}\\
B(\partial_i) = B_i &= 0 \text{ for } \ i = 1, 2, \ldots, n,\\
D(\partial_i) = D_i &= 0 \text{ for } \ i = 1, 2, \ldots, n,
\end{aligned} \tag{5.45}
$$

where $\partial_i = \frac{\partial}{\partial x^i}$. In terms of local coordinate system, the defining condition (1.3) of a $(WCS)_n$ can be written as (5.8), which reduces to the following equations:

$$C_{1221,i} = A_i C_{1221} + B_1 C_{i221} + B_2 C_{1i21} + D_2 C_{12i1} + D_1 C_{122i}, \tag{5.46}$$

$$C_{1331,i} = A_i C_{1331} + B_1 C_{i331} + B_3 C_{1i31} + D_3 C_{13i1} + D_1 C_{133i}, \tag{5.47}$$

$$C_{1441,i} = A_i C_{1441} + B_1 C_{i441} + B_4 C_{1i41} + D_4 C_{14i1} + D_1 C_{144i}, \tag{5.48}$$

$$C_{1kk1,i} = A_i C_{1kk1} + B_1 C_{ikk1} + B_k C_{1ik1} + D_k C_{1ki1} + D_1 C_{1kki}, \tag{5.49}$$

$$C_{2332,i} = A_i C_{2332} + B_2 C_{i332} + B_3 C_{2i32} + D_3 C_{23i2} + D_2 C_{233i}, \tag{5.50}$$

$$C_{2442,i} = A_i C_{2442} + B_2 C_{i442} + B_4 C_{2i42} + D_4 C_{24i2} + D_2 C_{244i}, \tag{5.51}$$

$$C_{2kk2,i} = A_i C_{2kk2} + B_2 C_{ikk2} + B_k C_{2ik2} + D_k C_{2ki2} + D_2 C_{2kki}, \tag{5.52}$$

$$C_{3443,i} = A_iC_{3443} + B_3C_{i443} + B_4C_{3i43} + D_4C_{34i3} + D_3C_{344i}, \quad (5.53)$$
$$C_{3kk3,i} = A_iC_{3kk3} + B_3C_{ikk3} + B_kC_{3ik3} + D_kC_{3ki3} + D_3C_{3kki}, \quad (5.54)$$
$$C_{4kk4,i} = A_iC_{4kk4} + B_4C_{ikk4} + B_kC_{4ik4} + D_kC_{4ki4} + D_4C_{4kki}, \quad (5.55)$$
$$C_{mkkm,i} = A_iC_{mkkm} + B_mC_{ikkm} + B_kC_{mikm} + D_kC_{mkim} + D_mC_{mkki}, \quad (5.56)$$
$$C_{1223,i} = A_iC_{1223} + B_1C_{i223} + B_2C_{1i23} + D_2C_{12i3} + D_3C_{122i}, \quad (5.57)$$
$$C_{1224,i} = A_iC_{1224} + B_1C_{i224} + B_2C_{1i24} + D_2C_{12i4} + D_4C_{122i}, \quad (5.58)$$
$$C_{122k,i} = A_iC_{122k} + B_1C_{i22k} + B_2C_{1i2k} + D_2C_{12ik} + D_kC_{122i}, \quad (5.59)$$
$$C_{2113,i} = A_iC_{2113} + B_2C_{i113} + B_1C_{2i13} + D_1C_{21i3} + D_3C_{211i}, \quad (5.60)$$
$$C_{2114,i} = A_iC_{2114} + B_2C_{i114} + B_1C_{2i14} + D_1C_{21i4} + D_4C_{211i}, \quad (5.61)$$
$$C_{211k,i} = A_iC_{211k} + B_2C_{i11k} + B_1C_{2i1k} + D_1C_{21ik} + D_kC_{211i}, \quad (5.62)$$
$$C_{1332,i} = A_iC_{1332} + B_1C_{i332} + B_3C_{1i32} + D_3C_{13i2} + D_2C_{133i}, \quad (5.63)$$
$$C_{1334,i} = A_iC_{1334} + B_1C_{i334} + B_3C_{1i34} + D_3C_{13i4} + D_4C_{133i}, \quad (5.64)$$
$$C_{133k,i} = A_iC_{133k} + B_1C_{i33k} + B_3C_{1i3k} + D_3C_{13ik} + D_kC_{133i}, \quad (5.65)$$
$$C_{3114,i} = A_iC_{3114} + B_3C_{i114} + B_1C_{3i14} + D_1C_{31i4} + D_4C_{311i}, \quad (5.66)$$
$$C_{311k,i} = A_iC_{311k} + B_3C_{i11k} + B_1C_{3i1k} + D_1C_{31ik} + D_kC_{311i}, \quad (5.67)$$
$$C_{1442,i} = A_iC_{1442} + B_1C_{i442} + B_4C_{1i42} + D_4C_{14i2} + D_2C_{144i}, \quad (5.68)$$
$$C_{1443,i} = A_iC_{1443} + B_1C_{i443} + B_4C_{1i43} + D_4C_{14i3} + D_3C_{144i}, \quad (5.69)$$
$$C_{144k,i} = A_iC_{144k} + B_1C_{i44k} + B_4C_{1i4k} + D_4C_{14ik} + D_kC_{144i}, \quad (5.70)$$
$$C_{411k,i} = A_iC_{411k} + B_4C_{i11k} + B_1C_{4i1k} + D_1C_{41ik} + D_kC_{411i}, \quad (5.71)$$
$$C_{2334,i} = A_iC_{2334} + B_2C_{i334} + B_3C_{2i34} + D_3C_{23i4} + D_4C_{233i}, \quad (5.72)$$
$$C_{233k,i} = A_iC_{233k} + B_2C_{i33k} + B_3C_{2i3k} + D_3C_{23ik} + D_kC_{233i}, \quad (5.73)$$
$$C_{3224,i} = A_iC_{3224} + B_3C_{i224} + B_2C_{3i24} + D_2C_{32i4} + D_4C_{322i}, \quad (5.74)$$
$$C_{322k,i} = A_iC_{322k} + B_3C_{i22k} + B_2C_{3i2k} + D_2C_{32ik} + D_kC_{322i}, \quad (5.75)$$
$$C_{2443,i} = A_iC_{2443} + B_2C_{i443} + B_4C_{2i43} + D_4C_{24i3} + D_3C_{244i}, \quad (5.76)$$
$$C_{244k,i} = A_iC_{244k} + B_2C_{i44k} + B_4C_{2i4k} + D_4C_{24ik} + D_kC_{244i}, \quad (5.77)$$
$$C_{422k,i} = A_iC_{422k} + B_4C_{i22k} + B_2C_{4i2k} + D_2C_{42ik} + D_kC_{422i}, \quad (5.78)$$
$$C_{344k,i} = A_iC_{344k} + B_3C_{i44k} + B_4C_{3i4k} + D_4C_{34ik} + D_kC_{344i}, \quad (5.79)$$
$$C_{433k,i} = A_iC_{433k} + B_4C_{i33k} + B_3C_{4i3k} + D_3C_{43ik} + D_kC_{433i}, \quad (5.80)$$
$$C_{1kk2,i} = A_iC_{1kk2} + B_1C_{ikk2} + B_kC_{1ik2} + D_kC_{1ki2} + D_2C_{1kki}, \quad (5.81)$$
$$C_{1kk3,i} = A_iC_{1kk3} + B_1C_{ikk3} + B_kC_{1ik3} + D_kC_{1ki3} + D_3C_{1kki}, \quad (5.82)$$
$$C_{1kk4,i} = A_iC_{1kk4} + B_1C_{ikk4} + B_kC_{1ik4} + D_kC_{1ki4} + D_4C_{1kki}, \quad (5.83)$$
$$C_{1kkm,i} = A_iC_{1kkm} + B_1C_{ikkm} + B_kC_{1ikm} + D_kC_{1kim} + D_mC_{1kki}, \quad (5.84)$$
$$C_{k11m,i} = A_iC_{k11m} + B_kC_{i11m} + B_1C_{ki1m} + D_1C_{k1im} + D_mC_{k11i}, \quad (5.85)$$
$$C_{2kk3,i} = A_iC_{2kk3} + B_2C_{ikk3} + B_kC_{2ik3} + D_kC_{2ki3} + D_3C_{2kki}, \quad (5.86)$$

$$C_{2kk4,i} = A_iC_{2kk4} + B_2C_{ikk4} + B_kC_{2ik4} + D_kC_{2ki4} + D_4C_{2kki}, \quad (5.87)$$
$$C_{2kkm,i} = A_iC_{2kkm} + B_2C_{ikkm} + B_kC_{2ikm} + D_kC_{2kim} + D_mC_{2kki}, \quad (5.88)$$
$$C_{k22m,i} = A_iC_{k22m} + B_kC_{i22m} + B_2C_{ki2m} + D_2C_{k2im} + D_mC_{k22i}, \quad (5.89)$$
$$C_{3kk4,i} = A_iC_{3kk4} + B_3C_{ikk4} + B_kC_{3ik4} + D_kC_{3ki4} + D_4C_{3kki}, \quad (5.90)$$
$$C_{3kkm,i} = A_iC_{3kkm} + B_3C_{ikkm} + B_kC_{3ikm} + D_kC_{3kim} + D_mC_{3kki}, \quad (5.91)$$
$$C_{k33m,i} = A_iC_{k33m} + B_kC_{i33m} + B_3C_{ki3m} + D_3C_{k3im} + D_mC_{k33i}, \quad (5.92)$$
$$C_{4kkm,i} = A_iC_{4kkm} + B_4C_{ikkm} + B_kC_{4ikm} + D_kC_{4kim} + D_mC_{4kki}, \quad (5.93)$$
$$C_{k44m,i} = A_iC_{k44m} + B_kC_{i44m} + B_4C_{ki4m} + D_4C_{k4im} + D_mC_{k44i}, \quad (5.94)$$
$$C_{mkk1,i} = A_iC_{mkk1} + B_mC_{ikk1} + B_kC_{mik1} + D_kC_{mki1} + D_1C_{mkki}, \quad (5.95)$$
$$C_{mkk2,i} = A_iC_{mkk2} + B_mC_{ikk2} + B_kC_{mik2} + D_kC_{mki2} + D_2C_{mkki}, \quad (5.96)$$
$$C_{mkk3,i} = A_iC_{mkk3} + B_mC_{ikk3} + B_kC_{mik3} + D_kC_{mki3} + D_3C_{mkki}, \quad (5.97)$$
$$C_{mkk4,i} = A_iC_{mkk4} + B_mC_{ikk4} + B_kC_{mik4} + D_kC_{mki4} + D_4C_{mkki}, \quad (5.98)$$
$$C_{mkkp,i} = A_iC_{mkkp} + B_mC_{ikkp} + B_kC_{mikp} + D_kC_{mkip} + D_pC_{mkki}, \quad (5.99)$$

for $5 \le k \le n$, $5 \le m \le n$, $5 \le p \le n$, $k \ne m \ne p$, where $i = 1, 2, \dots, n$, since for the cases other than (5.46)–(5.99), the components of each term of (5.8) either vanishes identically or the relation (5.8) holds trivially using the skew-symmetry property of $C$.

Now using (5.44) and (5.45), it follows, for $i = 3$, that right hand side of (5.46) $= A_3C_{1221} = -\frac{3(n-3)\coth x^3}{2(n-1)(n-2)\sinh^2 x^3} = C_{1221,3} =$ left hand side of (5.46).

For $i = 1, 2, 4, \dots, n$, the relation (5.45) implies that both sides of equation (5.46) are equal. By the similar argument, it can be easily seen that the equations (5.47)–(5.99) hold. Thus the manifold under consideration is weakly conformally symmetric. Hence we can state the following:

Theorem 5.4. *Let $(M^n, g)(n \ge 4)$ be a Riemannian manifold endowed with the metric given in (5.41). Then $(M^n, g)$ is a weakly conformally symmetric manifold with non-vanishing scalar curvature, which is neither conformally flat nor conformally symmetric but conformally recurrent.*

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