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ON WEAKLY PROJECTIVE SYMMETRIC MANIFOLDS

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Abstract. The object of the present paper is to study *weakly projective symmetric manifolds* and its decomposability with several non-trivial examples. Among others it is shown that in a decomposable weakly projective symmetric manifold both the decompositions are weakly Ricci symmetric.

1. Introduction

The notions of weakly symmetric and weakly projective symmetric manifolds were introduced by Tamássy and Binh [6] and later Binh [1] studied decomposable weakly symmetric manifolds. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called a weakly symmetric manifold if its curvature tensor R of type (0,4) satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \\ & + C(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\ & + E(V)R(Y,Z,U,X)\end{aligned} \tag{1.1}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where A, B, C, D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. Moreover, it is to be noted that in a $(WS)_n$, $B = C$ and $D = E$ [2] and hence the defining condition (1.1) of a $(WS)_n$ reduces to

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \\ & + B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\ & + D(V)R(Y,Z,U,X),\end{aligned} \tag{1.2}$$

where A, B and D are 1-forms (not simultaneously zero). A non-projectively flat Riemannian manifold $(M^n, g)(n > 2)$ (this condition will be assumed

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throughout the paper) is said to be a *weakly projective symmetric manifold* if its projective curvature tensor P of type (0,4) satisfies the condition

$$\begin{aligned}(\nabla_X P)(Y,Z,U,V) = {} & A(X)P(Y,Z,U,V) + B(Y)P(X,Z,U,V) \\ & + C(Z)P(Y,X,U,V) + D(U)P(Y,Z,X,V) \\ & + E(V)P(Y,Z,U,X)\end{aligned} \tag{1.3}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where A, B, C, D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g ([6]). Such an n-dimensional manifold is denoted by $(WPS)_n$.

Also in 1993 Tamássy and Binh [7] introduced the notion of a weakly Ricci symmetric manifold. A Riemannian manifold $(M^n, g)(n > 2)$ is called weakly Ricci symmetric manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(Z,X) + D(Z)S(Y,X), \tag{1.4}$$

where A, B and D are three non-zero 1-forms and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. Such an n-dimensional manifold is denoted by $(WRS)_n$.

The aim of the present paper is to study a $(WPS)_n$. Section 2 deals with some basic results of $(WPS)_n$. In [6] Tamássy and Binh found out the necessary and sufficient condition for a weakly symmetric manifold to be a weakly projective symmetric manifold with the same associated 1-forms, namely, they obtained the following result:

Theorem A. *A Riemannian manifold $(M^n, g)(n \geq 4)$ is weakly symmetric and also weakly projective symmetric with the same associated 1-forms A, B, C, D and associated vector field $F \neq 0$ if and only if the Ricci tensor S vanishes.*

In a $(WS)_n$, the associated 1-forms $B = C$ and $D = E$. However, in a $(WPS)_n$ this is not true, in general. In section 2 of the paper it is shown that in a $(WPS)_n$, the associated 1-forms $B = C$ but $D \neq E$ and hence the defining condition (1.3) of a $(WPS)_n$ reduces to

$$\begin{aligned}(\nabla_X P)(Y,Z,U,V) = {} & A(X)P(Y,Z,U,V) + B(Y)P(X,Z,U,V) \\ & + B(Z)P(Y,X,U,V) + D(U)P(Y,Z,X,V) \\ & + E(V)P(Y,Z,U,X),\end{aligned} \tag{1.5}$$

where A, B, D and E are 1-forms (not simultaneously zero). It is proved that in a $(WPS)_n$, if the Ricci tensor is of Codazzi type, then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L defined by $g(X, L) = \alpha(X)$. Also it is shown that in a $(WPS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_2 defined by $g(X, \rho_2) = T(X) = B(X) + E(X)$ for all X.

Section 3 deals with an Einstein $(WPS)_n$ and it is proved that an Einstein $(WPS)_n$ is a $(WS)_n$ provided that $A+B+D$ is not everywhere zero. Also it is shown that if the vector L defined by $g(X,L)=\alpha(X)$ is a concurrent vector field in an Einstein $(WPS)_n$, then it reduces to a $(WS)_n$.

Section 4 is devoted to the study of decomposable $(WPS)_n$ which is generally called the product $(WPS)_n$ and it is shown that in such a manifold one of the decomposition is Ricci symmetric and the other is projectively flat. Shaikh and Jana [5] already proved that every $(WS)_n$ is not a $(WRS)_n$, in general. In this paper it is shown that if a Riemannian manifold (M^n,g) is a decomposable $(WPS)_n$ such that $M=M_1^p\times M_2^{n-p}(2\le p\le n-2)$, then M_1 is $(WRS)_p$ and M_2 is $(WRS)_{n-p}$. The last section deals with several non-trivial examples of $(WPS)_n$ and also of decomposable $(WPS)_n$.

2. Some basic results of $(WPS)_n$

In this section, we derive some formulas, which will be needed to the study of a $(WPS)_n$. The projective curvature tensor P of type (0,4) is given by

(2.1)
$$P(Y,Z,U,V)=R(Y,Z,U,V)-\frac{1}{n-1}[S(Z,U)g(Y,V)-S(Y,U)g(Z,V)],$$

where S is the Ricci tensor of type (0,2) of the manifold. Let

$$\{e_i : i=1,2,\dots,n\}$$

be an orthonormal basis of the tangent space at any point of the manifold. Then the Ricci tensor S of type (0,2) and the scalar curvature r are given by the following:

$$S(X,Y)=\sum_{i=1}^{n}R(e_i,X,Y,e_i) \text{ and } r=\sum_{i=1}^{n}S(X,Y)=\sum_{i=1}^{n}g(Qe_i,e_i),$$

where Q is the Ricci-operator i.e., $g(QX,Y)=S(X,Y)$.

Now, from (2.1), we have the following:

$$\sum_{i=1}^{n}P(e_i,Z,U,e_i)=0=\sum_{i=1}^{n}P(e_i,e_i,U,V)=\sum_{i=1}^{n}P(Y,Z,e_i,e_i), \tag{2.2}$$

and

$$\sum_{i=1}^{n}P(Y,e_i,e_i,V)=\frac{n}{n-1}[S(Y,V)-\frac{r}{n}g(Y,V)]. \tag{2.3}$$

Also from (2.1), it follows that

$$\begin{aligned} &\text{(i)} && P(Y,Z,U,V)=-P(Z,Y,U,V),\\ &\text{(ii)} && P(Y,Z,U,V)\ne -P(Y,Z,V,U),\\ &\text{(iii)} && P(Y,Z,U,V)\ne P(U,V,Y,Z),\\ &\text{(iv)} && P(X,Y,Z,U)+P(Y,Z,X,U)+P(Z,X,Y,U)=0. \end{aligned} \tag{2.4}$$

Proposition 2.1. *The defining condition of a* $(WPS)_n$ *can always be expressed in the form (1.5).*

Proof. Interchanging Y and Z in (1.3) we get

$$\begin{aligned}(\nabla_X P)(Z,Y,U,V) = {} & A(X)P(Z,Y,U,V) + B(Z)P(X,Y,U,V)\\ & + C(Y)P(Z,X,U,V) + D(U)P(Z,Y,X,V)\\ & + E(V)P(Z,Y,U,X).\end{aligned} \tag{2.5}$$

Adding (1.3) and (2.5) we obtain by virtue of (2.4)(i) that

$$\gamma(Y)P(X,Z,U,V) + \gamma(Z)P(X,Y,U,V) = 0, \tag{2.6}$$

where $\gamma(X) = B(X) - C(X)$ for all X.

If we choose a vector field ρ such that $\gamma(\rho) \neq 0$, then putting $Y = Z = \rho$ in (2.6) we get $P(X,\rho,U,V) = 0$.

Again setting $Z = \rho$ in (2.6) we obtain $P(X,Y,U,V) = 0$ for all vector fields X, Y, U and V, which contradicts to our assumption that the manifold is not projectively flat. Hence, we must have $\gamma(X) = 0$ for all X and consequently $B(X) = C(X)$ for all X.

But, in view of (2.4)(ii), it follows that the relation $D = E$ does not hold in a $(WPS)_n$. Hence, the defining condition of a $(WPS)_n$ can be written as (1.5). This proves the proposition. □

Proposition 2.2. *In a Riemannian manifold* $(M^n, g)(n > 2)$*, the Ricci tensor is of Codazzi type if and only if*

$$(\nabla_X P)(Y,Z,U,V) + (\nabla_Y P)(Z,X,U,V) + (\nabla_Z P)(X,Y,U,V) = 0. \tag{2.7}$$

Proof. We first suppose that in a Riemannian manifold $(M^n, g)(n > 2)$, the Ricci tensor S is of Codazzi type [3]. Then we have

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z) = (\nabla_Z S)(X,Y) \tag{2.8}$$

for all vector fields X, Y, Z on the manifold.

From (2.1) it follows by virtue of Bianchi identity that

$$\begin{aligned}&(\nabla_X P)(Y,Z,U,V) + (\nabla_Y P)(Z,X,U,V) + (\nabla_Z P)(X,Y,U,V) =\\ &-\frac{1}{n-1}[g(Y,V)\{(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)\}\\ &+ g(Z,V)\{(\nabla_Y S)(X,U) - (\nabla_X S)(Y,U)\}\\ &+ g(X,V)\{(\nabla_Z S)(Y,U) - (\nabla_Y S)(Z,U)\}].\end{aligned} \tag{2.9}$$

Using (2.8) in (2.9) we obtain (2.7).

Conversely, if a Riemannian manifold satisfies the relation (2.7), then (2.9) yields

$$\begin{aligned}&g(Y,V)\{(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)\}\\ &\quad + g(Z,V)\{(\nabla_Y S)(X,U) - (\nabla_X S)(Y,U)\}\\ &\quad + g(X,V)\{(\nabla_Z S)(Y,U) - (\nabla_Y S)(Z,U)\} = 0.\end{aligned} \tag{2.10}$$

Putting $Y = V = e_i$ in (2.10) and taking summation over $i, 1 \leq i \leq n$, we obtain $(\nabla_X S)(Z, U) = (\nabla_Z S)(X, U)$ for all $X, Z, U \in \chi(M^n)$ and hence the Ricci tensor is of Codazzi type. This proves the proposition. $\square$

In view of (1.5), the relation (2.7) reduces to the following:

$$\begin{aligned} \alpha(X)P(Y,Z,U,V) + \alpha(Y)P(Z,X,U,V) + \alpha(Z)P(X,Y,U,V) \\ + E(V)[P(Y,Z,U,X) + P(Z,X,U,Y) + P(X,Y,U,Z)] = 0 \end{aligned} \tag{2.11}$$

where $\alpha(X) = A(X) - 2B(X)$ for all X.

Setting $Y = V = e_i$ in (2.11) and taking summation over $i, 1 \leq i \leq n$, we get

$$\alpha(P(Z,X)U) = 0. \tag{2.12}$$

Again putting $X = U = e_i$ in (2.12) and taking summation over $i, 1 \leq i \leq n$, we get

$$\alpha(QZ) = \frac{r}{n}\alpha(Z), \tag{2.13}$$

that is, $S(Z, L) = \frac{r}{n} g(Z, L)$. This leads to the following:

Proposition 2.3. *If in a $(WPS)_n$, the Ricci tensor is of Codazzi type then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L defined by $g(X, L) = \alpha(X)$ for all X.*

Next, by virtue of (1.5), the relation (2.9) takes the form

$$\begin{aligned} &\alpha(X)P(Y,Z,U,V) + \alpha(Y)P(Z,X,U,V) + \alpha(Z)P(X,Y,U,V) \\ &\quad + E(V)[P(Y,Z,U,X) + P(Z,X,U,Y) + P(X,Y,U,Z)] \\ &= -\frac{1}{n-1}[g(Y,V)\{(\nabla_X S)(Z,U) - (\nabla_Z S)(X,U)\} \\ &\quad + g(Z,V)\{(\nabla_Y S)(X,U) - (\nabla_X S)(Y,U)\} \\ &\quad + g(X,V)\{(\nabla_Z S)(Y,U) - (\nabla_Y S)(Z,U)\}]. \end{aligned} \tag{2.14}$$

Setting $Y = V = e_i$ in (2.14) and taking summation over $i, 1 \leq i \leq n$, we get

$$\begin{aligned} \alpha(R(Z,X)U) - \frac{1}{n-1}[\alpha(Z)S(X,U) - \alpha(X)S(Z,U)] \\ = \frac{n-2}{n-1}[(\nabla_Z S)(X,U) - (\nabla_X S)(Z,U)]. \end{aligned} \tag{2.15}$$

Putting $X = U = e_i$ in (2.15) and taking summation over $i, 1 \leq i \leq n$, we obtain

$$\frac{n-2}{2n} dr(Z) = \alpha(QZ) - \frac{r}{n}\alpha(Z). \tag{2.16}$$

If the manifold is of constant scalar curvature then (2.16) reduces to (2.13) and hence we can state the following:

Proposition 2.4. *If a $(WPS)_n$ is of constant scalar curvature, then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L defined by $g(X, L) = \alpha(X)$ for all X.*

Again using (2.1) the equation (1.5) becomes

$$
\begin{aligned}
&(\nabla_X R)(Y, Z, U, V) \\
&\quad - \frac{1}{n-1}[(\nabla_X S)(Z, U)g(Y, V) - (\nabla_X S)(Y, U)g(Z, V)] \\
&= A(X)[R(Y, Z, U, V) - \frac{1}{n-1}\{S(Z, U)g(Y, V) - S(Y, U)g(Z, V)\}] \\
&\quad + B(Y)[R(X, Z, U, V) - \frac{1}{n-1}\{S(Z, U)g(X, V) - S(X, U)g(Z, V)\}] \\
&\quad + B(Z)[R(Y, X, U, V) - \frac{1}{n-1}\{S(X, U)g(Y, V) - S(Y, U)g(X, V)\}] \\
&\quad + D(U)[R(Y, Z, X, V) - \frac{1}{n-1}\{S(Z, X)g(Y, V) - S(Y, X)g(Z, V)\}] \\
&\quad + E(V)[R(Y, Z, U, X) - \frac{1}{n-1}\{S(Z, U)g(Y, X) - S(Y, U)g(Z, X)\}].
\end{aligned}
\tag{2.17}
$$

Setting $Y = V = e_i$ in (2.17) and taking summation over $i, 1 \leq i \leq n$, we have

$$
\begin{aligned}
&B(R(X, Z)U) - E(R(U, X)Z) = \\
&= -\frac{1}{n-1}[B(X)S(Z, U)B(Z)S(X, U) + E(X)S(Z, U) - E(QU)g(Z, X)].
\end{aligned}
\tag{2.18}
$$

Also putting $Z = U = e_i$ in (2.18) and taking summation over $i, 1 \leq i \leq n$, we obtain

$$T(QX) = \frac{r}{n}T(X), \tag{2.19}$$

which can be written as

$$S(X, \rho_2) = \frac{r}{n}g(X, \rho_2), \tag{2.20}$$

where $g(X, \rho_2) = T(X) = B(X) + E(X)$ for all X.
This leads to the following:

Proposition 2.5. *In a $(WPS)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_2 defined by $g(X, \rho_2) = T(X)$ for all X.*

3. Einstein $(WPS)_n$

Let us consider a $(WPS)_n$, which is an Einstein manifold. Then we have

$$S(X, Y) = \frac{r}{n}g(X, Y), \tag{3.1}$$

from which it follows that

$$dr(X) = 0 \text{ and } (\nabla_Z S)(X, Y) = 0 \text{ for all } X, Y, Z. \tag{3.2}$$

By virtue of (3.1) and (3.2), it follows from (2.1) that

$$P(Y,Z,U,V) = R(Y,Z,U,V) - \frac{r}{n(n-1)}[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)] \tag{3.3}$$

and

$$(\nabla_X P)(Y,Z,U,V) = (\nabla_X R)(Y,Z,U,V). \tag{3.4}$$

In view of (3.3) and (3.4), (1.5) reduces to

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) = {} & A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) \\ & + B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) \\ & + E(V)R(Y,Z,U,X) - \frac{r}{n(n-1)}[A(X) \\ & \quad \{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\} \\ & + B(Y)\{g(Z,U)g(X,V) - g(X,U)g(Z,V)\} \\ & + B(Z)\{g(X,U)g(Y,V) - g(Y,U)g(X,V)\} \\ & + D(U)\{g(Z,X)g(Y,V) - g(Y,X)g(Z,V)\} \\ & + E(V)\{g(Z,U)g(Y,X) - g(Y,U)g(Z,X)\}].\end{aligned} \tag{3.5}$$

Using (1.2) in (3.5) we obtain

$$\begin{aligned}[E(V) - D(V)]R(Y,Z,U,X) = {} & \frac{r}{n(n-1)}[A(X)\{g(Z,U)g(Y,V) \\ - g(Y,U)g(Z,V)\} + B(Y)&\{g(Z,U)g(X,V) - g(X,U)g(Z,V)\} \\ + B(Z)\{g(X,U)g(Y,V) - g(Y,U)&g(X,V)\} + D(U)\{g(Z,X)g(Y,V) \\ - g(Y,X)g(Z,V)\} + E(V)&\{g(Z,U)g(Y,X) - g(Y,U)g(Z,X)\}].\end{aligned} \tag{3.6}$$

Setting $X = U = e_i$ in (3.6) and then taking summation over $i, 1 \leq i \leq n$, we get

$$\begin{aligned} r[A(Z)g(Y,V) - A(Y)g(Z,V) - (n-1)B(Y)g(Z,V) \\ + (n-1)B(Z)g(Y,V) + D(Z)g(Y,V) - D(Y)g(Z,V)] = 0.\end{aligned} \tag{3.7}$$

Further setting $Z = V = e_i$ in (3.7) and taking summation over $i, 1 \leq i \leq n$, we have

$$r[A(Y) + (n-1)B(Y) + D(Y)] = 0. \tag{3.8}$$

Again contracting (3.6) over X and Y we obtain

$$\begin{aligned}\frac{r}{n}[E(V) - D(V)]g(Z,U) = {} & \frac{r}{n(n-1)}[A(V)g(Z,U) \\ & - A(U)g(Z,V) + B(V)g(Z,U) \\ & - B(U)g(Z,V) - (n-1)D(U)g(Z,V) \\ & + (n-1)E(V)g(Z,U)].\end{aligned} \tag{3.9}$$

Now, setting $Z = V = e_i$ in (3.9) and taking summation over $i, 1 \leq i \leq n$, we get

$$r[A(U) + B(U) + (n-1)D(U)] = 0 \text{ for all } U. \tag{3.10}$$

Replacing U by Y in the above equation we have

$$r[A(Y) + B(Y) + (n-1)D(Y)] = 0. \tag{3.11}$$

Also setting $Y = V = e_i$ in (3.6) and taking summation over $i, 1 \leq i \leq n$, we get

$$\begin{aligned} E(R(X,U)Z) - D(R(X,U)Z) = \frac{r}{n(n-1)}&[(n-1)A(X)g(Z,U) \\ &+ B(X)g(Z,U) - B(Z)g(X,U) + (n-1)B(Z)g(X,U) \\ &+ (n-1)D(U)g(Z,X) + E(X)g(Z,U) - E(U)g(Z,X)], \end{aligned}$$

which yields, on further contraction with respect to Z and U, that

$$r[nA(X) + 2B(X) + 2D(X)] = 0 \text{ for all } X.$$

Interchanging X and Y in the above equation we have

$$r[nA(Y) + 2B(Y) + 2D(Y)] = 0. \tag{3.12}$$

Adding (3.8), (3.11) and (3.12) we obtain

$$r = 0 \text{ if } A(Y) + B(Y) + D(Y) \neq 0 \text{ for all } Y.$$

This leads to the following:

Theorem 3.1. *An Einstein $(WPS)_n$ is a $(WS)_n$ provided that $A + B + D$ is not everywhere zero on the manifold.*

Definition 3.1. In a Riemannian manifold a vector field W is said to be parallel if it satisfies the following condition:

$$\nabla_X W = 0 \text{ for all } X. \tag{3.13}$$

Let us now consider an Einstein $(WPS)_n$ in which the vector field L defined by $g(X, L) = A(X) - 2B(X)$ is a parallel vector field. Then we have

$$\nabla_X L = 0 \text{ for all } X. \tag{3.14}$$

Therefore, using Ricci identity we get

$$R(X, Y, L, U) = 0,$$

which yields

$$S(Y, L) = 0. \tag{3.15}$$

From (3.15) and (2.20) , it follows that $r = 0$ if $\|L\|^2 \neq 0$.

Again if $r = 0$ then (3.1) implies that $S(X, Y) = 0$ and consequently the manifold is a $(WS)_n$. Thus, we can state the following:

Theorem 3.2. *If in an Einstein $(WPS)_n$ the vector field L defined by $g(X, L) = \alpha(X)$ is a parallel vector field, then it is a $(WS)_n$ provided that $\|L\|^2 \neq 0$.*

Definition 3.2. A vector field F on a Riemannian manifold is said to be concurrent [4] if $(\nabla_X F) = kX$, where k is a constant.

In particular if $k = 0$ then F is said to be a parallel vector field.

Next, we suppose that in an Einstein $(WPS)_n$ the vector field L defined by $g(X, L) = \alpha(X) = A(X) - 2B(X)$ is a concurrent vector field.

Then we have

$$\nabla_X L = kX, \tag{3.16}$$

where k is a constant.

Making use of Ricci identity we have $R(X, Y, L, U) = 0$, which implies that

$$S(Y, L) = 0 \text{ for all } Y. \tag{3.17}$$

Now, (3.17) yields $r = 0$ provided that $\|L\|^2 \neq 0$.

Thus, arguing as in the case of parallel vector field we obtain that the manifold under consideration is a $(WS)_n$. Hence, we can state the following:

Theorem 3.3. *If in an Einstein $(WPS)_n$ the vector field L defined by*

$$g(X, L) = \alpha(X)$$

is a concurrent vector field, then it is a $(WS)_n$ provided that $\|L\|^2 \neq 0$.

4. Decomposable $(WPS)_n$

A Riemannian manifold (M^n, g) is said to be decomposable [8] if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-2$, that is, in some coordinate neighbourhood of the Riemannian manifold (M^n, g), the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \tilde{g}_{ab}dx^a dx^b + \overset{*}{g}_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{4.1}$$

where $\tilde{g}_{ab}$ are functions of $x^1, x^2, \cdots, x^p (p < n)$ denoted by $\tilde{x}$ and $\overset{*}{g}_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, \cdots, x^n$ denoted by $\overset{*}{x}$; $a, b, c, \cdots$ run from 1 to p and $\alpha, \beta, \gamma, \cdots$ run from $p+1$ to n. The two parts of (4.1) are the metrics of $M_1^p (p \geq 2)$ and $M_2^{n-p} (n-p \geq 2)$ which are called the decomposition of the manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$.

Let (M^n, g) be a Riemannian manifold such that $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq n-2$. Here throughout this section each object denoted by a 'tilde' is assumed to be from M_1 and each object denoted by a 'star' is assumed to be from M_2.

Let $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$, $\tilde{U}$, $\tilde{V} \in \chi(M_1)$ and $\overset{*}{X}$, $\overset{*}{Y}$, $\overset{*}{Z}$, $\overset{*}{U}$, $\overset{*}{V} \in \chi(M_2)$, then we have the following relations:

$$\begin{aligned}
R(\overset{*}{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = 0 &= R(\tilde{X}, \overset{*}{Y}, \tilde{Z}, \overset{*}{U}) = R(\tilde{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),\\
(\nabla_{\overset{*}{X}} R)(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{V}) = 0 &= (\nabla_{\tilde{X}} R)(\tilde{Y}, \overset{*}{Z}, \tilde{U}, \overset{*}{V}) = (\nabla_{\overset{*}{X}} R)(\tilde{Y}, \overset{*}{Z}, \tilde{U}, \overset{*}{V}),\\
R(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) &= \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}); R(\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}) = \overset{*}{R}(\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),\\
S(\tilde{X}, \tilde{Y}) &= \tilde{S}(\tilde{X}, \tilde{Y}); S(\overset{*}{X}, \overset{*}{Y}) = \overset{*}{S}(\overset{*}{X}, \overset{*}{Y}),\\
(\nabla_{\tilde{X}} S)(\tilde{Y}, \tilde{Z}) &= (\tilde{\nabla}_{\tilde{X}} S)(\tilde{Y}, \tilde{Z}); (\nabla_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}) = (\overset{*}{\nabla}_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}),
\end{aligned}$$

and $r = \tilde{r} + \overset{*}{r}$, where r, $\tilde{r}$, and $\overset{*}{r}$ are the scalar curvature of M, M_1, M_2 respectively. Let us consider a Riemannian manifold (M^n, g) which is a decomposable $(WPS)_n$. Then $M^n = M_1^p \times M_2^{n-p}$, $(2 \le p \le n-2)$.

Now, from (2.1), we have

$$P(\overset{*}{Y}, \tilde{Z}, \tilde{U}, \tilde{V}) = 0 = P(\tilde{Y}, \overset{*}{Z}, \overset{*}{U}, \overset{*}{V}), \tag{4.2}$$

$$P(\overset{*}{Y}, \tilde{Z}, \tilde{U}, \overset{*}{V}) = -\frac{1}{n-1} S(\tilde{Z}, \tilde{U}) g(\overset{*}{Y}, \overset{*}{V}), \tag{4.3}$$

$$P(\overset{*}{Y}, \overset{*}{Z}, \tilde{U}, \tilde{V}) = 0 \tag{4.4}$$

and

$$P(\overset{*}{Y}, \tilde{Z}, \overset{*}{U}, \tilde{V}) = \frac{1}{n-1} S(\overset{*}{Y}, \overset{*}{U}) g(\tilde{Z}, \tilde{V}). \tag{4.5}$$

Again from (1.5), we have

$$\begin{aligned}
(\nabla_{\tilde{X}} P)(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{V}) = {} & A(\tilde{X}) P(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{V}) + B(\tilde{Y}) P(\tilde{X}, \tilde{Z}, \tilde{U}, \tilde{V})\\
& + B(\tilde{Z}) P(\tilde{Y}, \tilde{X}, \tilde{U}, \tilde{V}) + D(\tilde{U}) P(\tilde{Y}, \tilde{Z}, \tilde{X}, \tilde{V})\\
& + E(\tilde{V}) P(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{X}).
\end{aligned} \tag{4.6}$$

Changing $\tilde{X}$ by $\overset{*}{X}$ in (4.6) we get

$$A(\overset{*}{X}) P(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{V}) = 0. \tag{4.7}$$

Similarly we have

$$B(\overset{*}{Y}) P(\tilde{X}, \tilde{Z}, \tilde{U}, \tilde{V}) = 0, \tag{4.8}$$

$$D(\overset{*}{U}) P(\tilde{Y}, \tilde{Z}, \tilde{X}, \tilde{V}) = 0 \tag{4.9}$$

and

$$E(\overset{*}{V}) P(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{X}) = 0. \tag{4.10}$$

Now, putting $\tilde{X} = \overset{*}{X}$, $\tilde{Y} = \overset{*}{Y}$ in (4.6) we get

$$D(\tilde{U})S(\overset{*}{Y},\overset{*}{X})g(\tilde{Z},\tilde{V}) - E(\tilde{V})S(\tilde{Z},\tilde{U})g(\overset{*}{Y},\overset{*}{X}) = 0. \tag{4.11}$$

Similarly, putting $\tilde{Y} = \overset{*}{Y}$, $\tilde{V} = \overset{*}{V}$ in (4.6) we obtain

$$(\nabla_{\tilde{X}}S)(\tilde{Z},\tilde{U}) = A(\tilde{X})S(\tilde{Z},\tilde{U}) + B(\tilde{Z})S(\tilde{X},\tilde{U}) + D(\tilde{U})S(\tilde{Z},\tilde{X}). \tag{4.12}$$

Also putting $\tilde{X} = \overset{*}{X}$, $\tilde{Y} = \overset{*}{Y}$, $\tilde{U} = \overset{*}{U}$ in (4.6) we have

$$(\nabla_{\overset{*}{X}}S)(\overset{*}{Y},\overset{*}{U}) = A(\overset{*}{X})S(\overset{*}{Y},\overset{*}{U}) + B(\overset{*}{Y})S(\overset{*}{X},\overset{*}{U}) + D(\overset{*}{U})S(\overset{*}{Y},\overset{*}{X}). \tag{4.13}$$

In the similar way, from (4.6), we have the following:

$$D(\overset{*}{U})S(\tilde{Y},\tilde{X})g(\overset{*}{Z},\overset{*}{V}) - E(\overset{*}{V})S(\overset{*}{Z},\overset{*}{U})g(\tilde{Y},\tilde{X}) = 0, \tag{4.14}$$

$$A(\tilde{X})P(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0, \tag{4.15}$$

$$B(\tilde{Y})P(\overset{*}{X},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0, \tag{4.16}$$

$$D(\tilde{U})P(\overset{*}{Y},\overset{*}{Z},\overset{*}{X},\overset{*}{V}) = 0, \tag{4.17}$$

$$E(\tilde{V})P(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{X}) = 0 \tag{4.18}$$

and

$$\begin{aligned}(\nabla_{\overset{*}{X}}P)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = {}& A(\overset{*}{X})P(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) + B(\overset{*}{Y})P(\overset{*}{X},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) \\ & + B(\overset{*}{Z})P(\overset{*}{Y},\overset{*}{X},\overset{*}{U},\overset{*}{V}) + D(\overset{*}{U})P(\overset{*}{Y},\overset{*}{Z},\overset{*}{X},\overset{*}{V}) \\ & + E(\overset{*}{V})P(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{X}).\end{aligned} \tag{4.19}$$

From (4.7)–(4.10) we have two cases. Namely,

(I) $A = B = D = E = 0$ on M_2,

(II) M_1 is projectively flat.

At first, we consider the case (I). Then from (4.19) we have

$$(\nabla_{\overset{*}{X}}P)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0,$$

that is,

$$\begin{aligned}&(\nabla_{\overset{*}{X}}R)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) \\ &\quad - \frac{1}{n-1}[(\nabla_{\overset{*}{X}}S)(\overset{*}{Z},\overset{*}{U})g(\overset{*}{Y},\overset{*}{V}) - (\nabla_{\overset{*}{X}}S)(\overset{*}{Y},\overset{*}{U})g(\overset{*}{Z},\overset{*}{V})] = 0.\end{aligned} \tag{4.20}$$

Setting $\overset{*}{Z}=\overset{*}{V}=\overset{*}{e}_\alpha$ in (4.20) and taking summation over α, $p+1 \le \alpha \le n$, we obtain

$$(\nabla_{\overset{*}{X}}S)(\overset{*}{Y},\overset{*}{U}) = 0, \tag{4.21}$$

which implies that M_2 is a Ricci symmetric manifold.

Secondly, we discuss the case of (II).

Since, M_1 is projectively flat, it is a manifold of constant curvature. Hence we can state the following:

Theorem 4.1. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(WPS)_n$ then the following holds:*

(I) *In the case of $A = B = D = E = 0$ on M_2, the manifold M_2 is Ricci symmetric.*
(II) *When M_1 is projectively flat, it is a manifold of constant curvature.*

Similarly, we have from (4.15) — (4.18) that

Theorem 4.2. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(WPS)_n$ then the following holds:*

(I) *In the case of $A = B = D = E = 0$ on M_1, the manifold M_1 is Ricci symmetric.*
(II) *When M_2 is projectively flat, it is a manifold of constant curvature.*

Next, we consider the contraction with respect to $\overset{*}{X}$ and $\overset{*}{Y}$ in (4.11) and obtain

$$D(\tilde{U}) \overset{*}{r} g(\tilde{Z}, \tilde{V}) - (n-p)E(\tilde{V})S(\tilde{Z}, \tilde{U})] = 0, \tag{4.22}$$

which yields

$$E(Q\tilde{U}) = r_1 D(\tilde{Y}), \tag{4.23}$$

where

$$r_1 = \frac{p}{n-p} \overset{*}{r} . \tag{4.24}$$

Similarly, from (4.14) we have

$$E(Q \overset{*}{U}) = r_2 E(\overset{*}{U}), \tag{4.25}$$

where

$$r_2 = \frac{n-p}{p} \tilde{r}. \tag{4.26}$$

Thus, we can state the following:

Theorem 4.3. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(WPS)_n$ then the 1-forms D and E are related by $E(Q\tilde{U}) = r_1 D(\tilde{Y})$ on M_1, where $r_1 = \frac{p}{n-p} \overset{*}{r}$ and $E(Q \overset{*}{U}) = r_2 E(\overset{*}{U})$ on M_2, where $r_2 = \frac{n-p}{p} \tilde{r}$.*

Again from (4.12) and (4.13) we can state the following:

Theorem 4.4. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(WPS)_n$ then the manifold M_1 is a $(WRS)_p$ and the manifold M_2 is a $(WRS)_{n-p}$.*

5. Some Examples of $(WPS)_n$ and decomposable $(WPS)_n$

Example 5.1. Let $M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : 0 < x^1 < \frac{\pi}{2}, 0 < x^2 < \frac{\pi}{2}\}$ be a manifold endowed with the metric

$$ds^2 = (dx^1)^2 + \sin^2 x^1 (dx^2)^2 + \sin^2 x^1 \sin^2 x^2 (dx^3)^2 + (dx^4)^2. \tag{5.1}$$

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, projective curvature tensor are given by

$$\begin{aligned}
\Gamma^1_{22} &= -\frac{1}{2}\sin(2x^1), \Gamma^1_{33} = -\frac{1}{2}\sin(2x^1)\sin^2 x^2,\\
\Gamma^2_{33} &= -\frac{1}{2}\sin(2x^2), \Gamma^2_{12} = \cot x^1 = \Gamma^3_{13}, \Gamma^3_{23} = \cot x^2,\\
R_{1221} &= -\sin^2 x^1, R_{1331} = -\sin^2 x^1 \sin^2 x^2, R_{2332} = -\sin^4 x^1 \sin^2 x^2,\\
S_{11} &= -2, S_{22} = -2\sin^2 x^1, S_{33} = -2\sin^2 x^1 \sin^2 x^2, r = -6 \neq 0,\\
P_{1221} &= -\frac{1}{3}\sin^2 x^1 = -P_{1212}, P_{1331} = -\frac{1}{3}\sin^2 x^1 \sin^2 x^2 = -P_{1313},\\
P_{2332} &= -\frac{1}{3}\sin^4 x^1 \sin^2 x^2 = -P_{2323}, P_{1441} = -\frac{2}{3},\\
P_{2424} &= -\frac{2}{3}\sin^2 x^1, P_{3434} = -\frac{2}{3}\sin^2 x^1 \sin^2 x^2
\end{aligned}$$

and the components that can be obtained from these by the symmetry properties. The covariant derivatives of all components of projective curvature tensor are vanish. In terms of local coordinate system we consider the components of the 1-forms A, B, D and E as follows:

$$A_i = B_i = D_i = E_i = 0 \text{ for } i = 1, 2, 3, 4.$$

Then one can easily prove that (M^4, g) is a $(WPS)_4$ with non-vanishing scalar curvature.

Thus, we can state the following:

Theorem 5.1. *Let (M^4, g) be a Riemannian manifold endowed with the metric given in (5.1). Then (M^4, g) is a weakly projective symmetric manifold with non-vanishing scalar curvature, which is projectively symmetric.*

Example 5.2. Let $M^n = \{(x^1, x^2, x^3, \cdots, x^n) \in \mathbb{R}^n : 0 < x^1 < \frac{\pi}{2}, 0 < x^2 < \frac{\pi}{2}\}$ be a manifold endowed with the metric

$$ds^2 = (dx^1)^2 + \sin^2 x^1 (dx^2)^2 + \sin^2 x^1 \sin^2 x^2 (dx^3)^2 + \sum_{k=4}^{n} (dx^k)^2. \tag{5.2}$$

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, projective curvature tensor are given by

$$
\begin{aligned}
\Gamma^1_{22} &= -\frac{1}{2}\sin(2x^1), \Gamma^1_{33} = -\frac{1}{2}\sin(2x^1)\sin^2 x^2,\\
\Gamma^2_{33} &= -\frac{1}{2}\sin(2x^2), \Gamma^2_{12} = \cot x^1 = \Gamma^3_{13}, \Gamma^3_{23} = \cot x^2,\\
R_{1221} &= -\sin^2 x^1, R_{1331} = -\sin^2 x^1 \sin^2 x^2, R_{2332} = -\sin^4 x^1 \sin^2 x^2,\\
S_{11} &= -2, S_{22} = -2\sin^2 x^1, S_{33} = -2\sin^2 x^1 \sin^2 x^2, r = -6 \neq 0,\\
P_{1221} &= -\frac{n-3}{n-1}\sin^2 x^1 = P_{1212}, P_{1331} = -\frac{n-3}{n-1}\sin^2 x^1 \sin^2 x^2 = P_{1313},\\
P_{2332} &= -\frac{n-3}{n-1}\sin^4 x^1 \sin^2 x^2 = P_{2323}, P_{1kk1} = \frac{2}{n-1},\\
P_{2kk2} &= \frac{2}{n-1}\sin^2 x^1, P_{3kk3} = \frac{2}{n-1}\sin^2 x^1 \sin^2 x^2, \text{ for } 4 \leq k \leq n.
\end{aligned}
$$

and the components that can be obtained from these by the symmetry properties.

In view of the above all the covariant derivatives of the projective curvature tensor are vanish.

Therefore, our M^n with the considered metric g in (5.2) is a Riemannian manifold of non-vanishing scalar curvature.

In terms of local coordinate system we consider the components of the 1-forms A, B, D and E as follows:

$$A_i = B_i = D_i = E_i = 0 \text{ for } i = 1, 2, \cdots, n.$$

Then (M^n, g) is a $(WPS)_n$ and hence we can state the following:

Theorem 5.2. *Let $(M^n, g)(n \geq 4)$ be a Riemannian manifold equipped with the metric given in (5.2). Then $(M^n, g)(n \geq 4)$ is a weakly projective symmetric manifold with non-vanishing scalar curvature, which is projectively symmetric.*

Now, we consider the two manifolds (M_1^4, ds_1^2) and (M_2^{n-4}, ds_2^2) where $M_1^4 = \mathbb{R}^4$ and $M_2^{n-4} = \mathbb{R}^{n-4}$,

$$(5.3)\quad ds_1^2 = (dx^1)^2 + \sin^2 x^1[(dx^2)^2 + \sin^2 x^2 (dx^3)^2] + (dx^4)^2, ds_2^2 = \sum_{k=5}^{n}(dx^k)^2.$$

Then (M^n, ds^2) is obviously a decomposable manifold where ds^2 is given by (5.2). Hence we can state the following:

Theorem 5.3. *Let $(M^n, g)(n \geq 4)$ be a Riemannian manifold endowed with the metric given in (5.2). Then $(M^n, g)(n \geq 4)$ is a decomposable weakly projective symmetric manifold $(M_1^4, ds_1^2) \times (M_2^{n-4}, ds_2^2)$ with non-vanishing scalar curvature, where ds_1^2 and ds_2^2 are given in (5.3).*

Example 5.3. Let $M = \mathbb{R}^4$ be a manifold endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = (1+2\gamma)[(dx^1)^2 + (dx^3)^2] + (dx^2)^2 + (dx^4)^2, \tag{5.4}$$

(i,j = 1,2,3,4), where $\gamma = \frac{e^{x^1}}{K^2} \neq \frac{1}{4}$ and K is a non-zero constant.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, projective curvature tensor and its covariant derivatives are given by

$$\Gamma^1_{11} = \frac{\gamma}{1+2\gamma} = \Gamma^3_{13} = -\Gamma^1_{33}, R_{1331} = \frac{\gamma}{1+2\gamma},$$

$$S_{11} = \frac{\gamma}{(1+2\gamma)^2} = S_{33}, r = \frac{2\gamma}{(1+2\gamma)^3} \neq 0,$$

$$P_{1212} = P_{1414} = P_{3434} = \frac{\gamma}{3(1+2\gamma)^2},$$

$$P_{2332} = -\frac{\gamma}{3(1+2\gamma)^2}, P_{1331} = \frac{2}{3}\frac{\gamma}{1+2\gamma} = -P_{1313},$$

$$P_{1212,1} = P_{1414,1} = P_{3434,1} = \frac{\gamma(1-4\gamma)}{3(1+2\gamma)^3}, \tag{5.5}$$

$$P_{2332,1} = -\frac{\gamma(1-4\gamma)}{3(1+2\gamma)^3}, \tag{5.6}$$

$$P_{1331,1} = \frac{2\gamma(1-4\gamma)}{3(1+2\gamma)^2} = -P_{1313,1} \tag{5.7}$$

and the components that can be obtained from these by the symmetry properties, where ',' denotes the covariant differentiation with respect to the metric tensor and S_{ij} denotes the components of the Ricci tensor and r is the scalar curvature of the manifold. Therefore, the manifold M^4 with the considered metric is a Riemannian manifold, which is neither projectively flat nor projectively symmetric and is of non-vanishing scalar curvature.

We shall now show that this (M^4, g) is a $(WPS)_4$, that is, it satisfies (1.5).

In terms of local coordinate system, we consider the components of the 1-forms A, B, D and E as follows:

$$A(\partial_i) = A_i = \begin{cases} \frac{1-4\gamma}{1+2\gamma}, & \text{for } i = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.8}$$
$$B_i = D_i = E_i = 0 \text{ for } i = 1, 2, 3, 4,$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

In our M^4 with the considered 1-forms, (1.5) reduces to the following equations:

$$P_{121l,i} = A_i P_{121l} + B_1 P_{i21l} + B_2 P_{1i1l} + D_1 P_{12il} + E_l P_{121i}, \tag{5.9}$$

$$P_{12t2,i} = A_i P_{12t2} + B_1 P_{i2t2} + B_2 P_{1it2} + D_t P_{12i2} + E_2 P_{12ti}, \tag{5.10}$$

$$P_{1u12,i} = A_i P_{1u12} + B_1 P_{iu12} + B_u P_{1i12} + D_1 P_{1ui2} + E_2 P_{1u1i}, \tag{5.11}$$

(5.12) $P_{t212,i} = A_i P_{t212} + B_t P_{i212} + B_2 P_{ti12} + D_1 P_{t2i2} + E_2 P_{t21i}$,

(5.13) $P_{133l,i} = A_i P_{133l} + B_1 P_{i33l} + B_3 P_{1i3l} + D_3 P_{13il} + E_l P_{133i}$,

(5.14) $P_{13w1,i} = A_i P_{13w1} + B_1 P_{i3w1} + B_3 P_{1iw1} + D_w P_{13i1} + E_1 P_{13wi}$,

(5.15) $P_{1w31,i} = A_i P_{1w31} + B_1 P_{iw31} + B_w P_{1i31} + D_3 P_{1wi1} + E_1 P_{1w3i}$,

(5.16) $P_{t331,i} = A_i P_{t331} + B_t P_{i331} + B_3 P_{ti31} + D_3 P_{t3i1} + E_1 P_{t33i}$,

(5.17) $P_{131s,i} = A_i P_{131s} + B_1 P_{i31s} + B_3 P_{1i1s} + D_1 P_{13is} + E_s P_{131i}$,

(5.18) $P_{13t3,i} = A_i P_{13t3} + B_1 P_{i3t3} + B_3 P_{1it3} + D_t P_{13i3} + E_3 P_{13ti}$,

(5.19) $P_{1p13,i} = A_i P_{1p13} + B_1 P_{ip13} + B_p P_{1i13} + D_1 P_{1pi3} + E_3 P_{1p1i}$,

(5.20) $P_{t313,i} = A_i P_{t313} + B_t P_{i313} + B_3 P_{ti13} + D_1 P_{t3i3} + E_3 P_{t31i}$,

(5.21) $P_{14lu,i} = A_i P_{14lu} + B_1 P_{i4lu} + B_4 P_{1ilu} + D_l P_{14iu} + E_u P_{14li}$,

(5.22) $P_{14t4,i} = A_i P_{14t4} + B_1 P_{i4t4} + B_4 P_{1it4} + D_t P_{14i4} + E_4 P_{14ti}$,

(5.23) $P_{1114,i} = A_i P_{1114} + B_1 P_{i114} + B_1 P_{1i14} + D_1 P_{11i4} + E_4 P_{111i}$,

(5.24) $P_{t414,i} = A_i P_{t414} + B_t P_{i414} + B_4 P_{ti14} + D_1 P_{t4i4} + E_4 P_{t41i}$,

(5.25) $P_{233t,i} = A_i P_{233t} + B_2 P_{i33t} + B_3 P_{2i3t} + D_3 P_{23it} + E_t P_{233i}$,

(5.26) $P_{23w2,i} = A_i P_{23w2} + B_2 P_{i3w2} + B_3 P_{2iw2} + D_w P_{23i2} + E_2 P_{23wi}$,

(5.27) $P_{2w32,i} = A_i P_{2w32} + B_2 P_{iw32} + B_w P_{2i32} + D_3 P_{2wi2} + E_2 P_{2w3i}$,

(5.28) $P_{s332,i} = A_i P_{s332} + B_s P_{i332} + B_3 P_{si32} + D_3 P_{s3i2} + E_2 P_{s33i}$,

(5.29) $P_{343t,i} = A_i P_{343t} + B_3 P_{i43t} + B_4 P_{3i3t} + D_3 P_{34it} + E_t P_{343i}$,

(5.30) $P_{34w4,i} = A_i P_{34w4} + B_3 P_{i4w4} + B_4 P_{3iw4} + D_w P_{34i4} + E_4 P_{34wi}$,

(5.31) $P_{3334,i} = A_i P_{3334} + B_3 P_{i334} + B_3 P_{3i34} + D_3 P_{33i4} + E_4 P_{333i}$,

(5.32) $P_{v434,i} = A_i P_{v434} + B_v P_{i434} + B_4 P_{vi34} + D_3 P_{v4i4} + E_4 P_{v43i}$,

where $i = 1,2,3,4; l = 1,2,3,4; t = 2,3,4; u = 1,3,4; w = 1,2,4; v = 1,3; s = 3,4; p = 1,4; v = 2,4$. Since, for the cases other than (5.9) – (5.32), the components of each term of (1.5) either vanishes identically or the relation (1.5) holds trivially using the skew-symmetry property of P.

Now, using (5.5) and (5.8), it follows, for $i = 1$, that right hand side of (5.9)(for $l = 2$) $= (A_1 + B_1 + D_1)P_{1212} = \frac{\gamma(1-4\gamma)}{3(1+2\gamma)^3} = P_{1212,1} =$ left hand side of (5.9)(for $l = 2$).

For $i = 2,3,4$, the relation (5.8) implies that both sides of equation (5.9) are equal. By the similar argument, it can be easily seen that the equations (5.10) – (5.32) hold. Thus, the manifold under consideration is weakly projective symmetric.

Hence, we can state the following:

Theorem 5.4. *Let (M^4, g) be a Riemannian manifold endowed with the metric given in (5.4). Then (M^4, g) is a weakly projective symmetric manifold with non-vanishing scalar curvature, which is neither projectively flat nor projectively symmetric but projectively recurrent.*

Example 5.4. Let $M = \mathbb{R}^n$ be a manifold endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = 2\gamma[(dx^1)^2 + (dx^3)^2] + \delta_{ij}dx^i dx^j, \quad (i,j = 1,2,\cdots,n), \tag{5.33}$$

where $\gamma = \frac{e^{x^1}}{K^2} \neq \frac{1}{4}$ and K is a non-zero constant.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, projective curvature tensor and its covariant derivatives are given by

$$\begin{aligned}
\Gamma^1_{11} &= \frac{\gamma}{1+2\gamma} = \Gamma^3_{13} = -\Gamma^1_{33}, R_{1331} = \frac{\gamma}{1+2\gamma},\\
S_{11} &= \frac{\gamma}{(1+2\gamma)^2} = S_{33}, r = \frac{2\gamma}{(1+2\gamma)^3} \neq 0,\\
P_{1212} &= P_{1414} = P_{3434} = \frac{\gamma}{(n-1)(1+2\gamma)^2},\\
P_{2332} &= -\frac{\gamma}{(n-1)(1+2\gamma)^2}, P_{1331} = \frac{n-2}{n-1}\frac{\gamma}{1+2\gamma} = -P_{1313},\\
P_{1k1k} &= P_{3k3k} = \frac{\gamma}{(n-1)(1+2\gamma)^2} \text{ for } 5 \leq k \leq n,
\end{aligned}$$

$$P_{1212,1} = P_{1414,1} = P_{3434,1} = \frac{\gamma(1-4\gamma)}{(n-1)(1+2\gamma)^3}, \tag{5.34}$$

$$P_{2332,1} = -\frac{\gamma(1-4\gamma)}{(n-1)(1+2\gamma)^3}, \tag{5.35}$$

$$P_{1331,1} = \frac{(n-2)\gamma(1-4\gamma)}{(n-1)(1+2\gamma)^2} = -P_{1313,1}, \tag{5.36}$$

$$P_{1k1k,1} = P_{3k3k,1} = \frac{\gamma(1-4\gamma)}{(n-1)(1+2\gamma)^3} \text{ for } 5 \leq k \leq n \tag{5.37}$$

and the components that can be obtained from these by the symmetry properties, where ‘,’ denotes the covariant differentiation with respect to the metric tensor and S_{ij} denotes the components of the Ricci tensor and r is the scalar curvature of the manifold. Therefore, the manifold M^n with the considered metric is a Riemannian manifold, which is neither projectively flat nor projectively symmetric and is of non-vanishing scalar curvature.

We shall now show that this (M^n, g) is a $(WPS)_n$, that is, it satisfies (1.5).

In terms of local coordinate system, if we consider the components of the 1-forms A, B, D and E as

$$\begin{aligned}
A(\partial_i) = A_i &= \begin{cases} \frac{1-4\gamma}{1+2\gamma}, & \text{for } i = 1,\\ 0, & \text{otherwise}, \end{cases}\\
B_i = D_i &= E_i = 0 \text{ for } i = 1,2,\cdots,n,
\end{aligned} \tag{5.38}$$

where $\partial_i = \frac{\partial}{\partial x^i}$.
In our M^n with the considered 1-forms, (1.5) reduces to the following equations:

$$P_{121l,i} = A_i P_{121l} + B_1 P_{i21l} + B_2 P_{1i1l} + D_1 P_{12il} + E_l P_{121i}, \tag{5.39}$$
$$P_{12t2,i} = A_i P_{12t2} + B_1 P_{i2t2} + B_2 P_{1it2} + D_t P_{12i2} + E_2 P_{12ti}, \tag{5.40}$$
$$P_{1u12,i} = A_i P_{1u12} + B_1 P_{iu12} + B_u P_{1i12} + D_1 P_{1ui2} + E_2 P_{1u1i}, \tag{5.41}$$
$$P_{t212,i} = A_i P_{t212} + B_t P_{i212} + B_2 P_{ti12} + D_1 P_{t2i2} + E_2 P_{t21i}, \tag{5.42}$$
$$P_{133l,i} = A_i P_{133l} + B_1 P_{i33l} + B_3 P_{1i3l} + D_3 P_{13il} + E_l P_{133i}, \tag{5.43}$$
$$P_{13w1,i} = A_i P_{13w1} + B_1 P_{i3w1} + B_3 P_{1iw1} + D_w P_{13i1} + E_1 P_{13wi}, \tag{5.44}$$
$$P_{1w31,i} = A_i P_{1w31} + B_1 P_{iw31} + B_w P_{1i31} + D_3 P_{1wi1} + E_1 P_{1w3i}, \tag{5.45}$$
$$P_{t331,i} = A_i P_{t331} + B_t P_{i331} + B_3 P_{ti31} + D_3 P_{t3i1} + E_1 P_{t33i}, \tag{5.46}$$
$$P_{131s,i} = A_i P_{131s} + B_1 P_{i31s} + B_3 P_{1i1s} + D_1 P_{13is} + E_s P_{131i}, \tag{5.47}$$
$$P_{13t3,i} = A_i P_{13t3} + B_1 P_{i3t3} + B_3 P_{1it3} + D_t P_{13i3} + E_3 P_{13ti}, \tag{5.48}$$
$$P_{1p13,i} = A_i P_{1p13} + B_1 P_{ip13} + B_p P_{1i13} + D_1 P_{1pi3} + E_3 P_{1p1i}, \tag{5.49}$$
$$P_{t313,i} = A_i P_{t313} + B_t P_{i313} + B_3 P_{ti13} + D_1 P_{t3i3} + E_3 P_{t31i}, \tag{5.50}$$
$$P_{14lu,i} = A_i P_{14lu} + B_1 P_{i4lu} + B_4 P_{1ilu} + D_l P_{14iu} + E_u P_{14li}, \tag{5.51}$$
$$P_{14t4,i} = A_i P_{14t4} + B_1 P_{i4t4} + B_4 P_{1it4} + D_t P_{14i4} + E_4 P_{14ti}, \tag{5.52}$$
$$P_{111q,i} = A_i P_{111q} + B_1 P_{i11q} + B_1 P_{1i1q} + D_1 P_{11iq} + E_q P_{111i}, \tag{5.53}$$
$$P_{t414,i} = A_i P_{t414} + B_t P_{i414} + B_4 P_{ti14} + D_1 P_{t4i4} + E_4 P_{t41i}, \tag{5.54}$$
$$P_{233t,i} = A_i P_{233t} + B_2 P_{i33t} + B_3 P_{2i3t} + D_3 P_{23it} + E_t P_{233i}, \tag{5.55}$$
$$P_{23w2,i} = A_i P_{23w2} + B_2 P_{i3w2} + B_3 P_{2iw2} + D_w P_{23i2} + E_2 P_{23wi}, \tag{5.56}$$
$$P_{2w32,i} = A_i P_{2w32} + B_2 P_{iw32} + B_w P_{2i32} + D_3 P_{2wi2} + E_2 P_{2w3i}, \tag{5.57}$$
$$P_{s332,i} = A_i P_{s332} + B_s P_{i332} + B_3 P_{si32} + D_3 P_{s3i2} + E_2 P_{s33i}, \tag{5.58}$$
$$P_{343t,i} = A_i P_{343t} + B_3 P_{i43t} + B_4 P_{3i3t} + D_3 P_{34it} + E_t P_{343i}, \tag{5.59}$$
$$P_{34w4,i} = A_i P_{34w4} + B_3 P_{i4w4} + B_4 P_{3iw4} + D_w P_{34i4} + E_4 P_{34wi}, \tag{5.60}$$
$$P_{333q,i} = A_i P_{333q} + B_3 P_{i33q} + B_3 P_{3i3q} + D_3 P_{33iq} + E_q P_{333i}, \tag{5.61}$$
$$P_{v434,i} = A_i P_{v434} + B_v P_{i434} + B_4 P_{vi34} + D_3 P_{v4i4} + E_4 P_{v43i}, \tag{5.62}$$
$$P_{1k1l,i} = A_i P_{1k1l} + B_1 P_{ik1l} + B_k P_{1i1l} + D_1 P_{1kil} + E_l P_{1k1i}, \tag{5.63}$$
$$P_{1kmk,i} = A_i P_{1kmk} + B_1 P_{ikmk} + B_k P_{1imk} + D_m P_{1kik} + E_k P_{1kmi}, \tag{5.64}$$
$$P_{1y1k,i} = A_i P_{1y1k} + B_1 P_{iy1k} + B_y P_{1i1k} + D_1 P_{1yik} + E_k P_{1y1i}, \tag{5.65}$$
$$P_{tk1k,i} = A_i P_{tk1k} + B_t P_{ik1k} + B_k P_{ti1k} + D_1 P_{tkik} + E_k P_{tk1i}, \tag{5.66}$$
$$P_{3k3l,i} = A_i P_{3k3l} + B_3 P_{ik3l} + B_k P_{3i3l} + D_3 P_{3kil} + E_l P_{3k3i}, \tag{5.67}$$
$$P_{3kwk,i} = A_i P_{3kwk} + B_3 P_{ikwk} + B_k P_{3iwk} + D_w P_{3kik} + E_k P_{3kwi}, \tag{5.68}$$
$$P_{3z3k,i} = A_i P_{3z3k} + B_3 P_{iz3k} + B_z P_{3i3k} + D_3 P_{3zik} + E_k P_{3z3i}, \tag{5.69}$$
$$P_{tk3k,i} = A_i P_{tk3k} + B_t P_{ik3k} + B_k P_{ti3k} + D_3 P_{tkik} + E_k P_{tk3i}, \tag{5.70}$$

where $i = 1, 2, \cdots, n$; $l = 1, 2, \cdots, n$; $t = 2, 3, \cdots, n$; $u = 1, 3, 4, \cdots, n$; $w = 1, 2, 4, \cdots, n$; $v = 2, 4, \cdots, n$; $s = 3, 4, \cdots, n$; $p = 1, 4, \cdots, n$; $q = 4, 5, \cdots, n; m = 2, 3, 4$; $y = 1, 3, 4$; $z = 1, 2, 3, 4$. Since, for the cases other than (5.39) – (5.70), the components of each term of (1.5) either vanishes identically or the relation (1.5) holds trivially using the skew-symmetry property of P.

Then it can be easily shown that the manifold under consideration is weakly projective symmetric and hence we can state the following:

Theorem 5.5. *Let (M^n, g) be a Riemannian manifold endowed with the metric given in (5.33). Then (M^n, g) is a weakly projective symmetric manifold with non-vanishing scalar curvature, which is neither projectively flat nor projectively symmetric but projectively recurrent.*

Let (M_3^4, g_3) be a Riemannian manifold in Example 5.3. and $(\mathbb{R}^{n-4}, g_0)$ be an $(n-4)$ dimensional Euclidean space with standard metric g_0. Then (M^n, g) in Example 5.4. is a product manifold of (M_3^4, g_3) and $(\mathbb{R}^{n-4}, g_0)$. Thus, we can state the following:

Theorem 5.6. *Let $(M^n, g)(n \geq 4)$ be a Riemannian manifold endowed with the metric given in (5.33). Then $(M^n, g)(n \geq 4)$ is a decomposable weakly projective symmetric manifold $(M_3^4, g_3) \times (\mathbb{R}^{n-4}, g_0)$ with non-vanishing scalar curvature, which is neither projectively flat nor projectively symmetric but projectively recurrent.*

Example 5.5. Let $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : 0 < x^4 < 1\}$ be a manifold endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^4)^2] + (dx^2)^2 + (dx^3)^2, \tag{5.71}$$

$(i, j = 1, 2, 3, 4)$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, projective curvature tensor and its covariant derivatives are given by

$$\Gamma^1_{14} = \Gamma^4_{44} = \frac{2}{3x^4} = -\Gamma^4_{11}, R_{1441} = -\frac{2}{3(x^4)^{\frac{2}{3}}},$$

$$S_{11} = S_{44} = -\frac{2}{3(x^4)^2},$$

$$r = -\frac{4}{3(x^4)^{\frac{10}{3}}} \neq 0, P_{1212} = P_{1313} = -\frac{2}{9(x^4)^2},$$

$$P_{1441} = -\frac{4}{9(x^4)^{\frac{2}{3}}} = -P_{1414}, P_{2442} = P_{3443} = \frac{2}{9(x^4)^2},$$

$$P_{1212,4} = P_{1313,4} = \frac{20}{27(x^4)^3}, \tag{5.72}$$

$$P_{1441,4} = \frac{40}{27(x^4)^{\frac{5}{3}}} = -P_{1414,4}, \tag{5.73}$$

$$P_{2442,4} = -\frac{20}{27(x^4)^3} = P_{3443,4} \tag{5.74}$$

and the components that can be obtained from these by the symmetry properties, where ',' denotes the covariant differentiation with respect to the metric tensor. Therefore, the manifold M^4 with the considered metric is a Riemannian manifold, which is neither projectively flat nor projectively symmetric and is of non-vanishing scalar curvature.

We shall now show that this (M^4, g) is a $(WPS)_4$, that is, it satisfies (1.5).

In terms of local coordinate system we consider the components of the 1-forms A, B, D and E as follows:

$$A(\partial_i) = A_i = \begin{cases} -\frac{10}{3x^4}, & \text{for } \ i = 4 \\ 0, & \text{otherwise} \ , \end{cases} \tag{5.75}$$
$$B_i = D_i = E_i = 0 \text{ for } i = 1, 2, 3, 4,$$

where $\partial_i = \frac{\partial}{\partial x^i}$.
In our M^4 with the considered 1-forms, (1.5) reduces to the following equations:

$$P_{121l,i} = A_i P_{121l} + B_1 P_{i21l} + B_2 P_{1i1l} + D_1 P_{12il} + E_l P_{121i}, \tag{5.76}$$
$$P_{12t2,i} = A_i P_{12t2} + B_1 P_{i2t2} + B_2 P_{1it2} + D_t P_{12i2} + E_2 P_{12ti}, \tag{5.77}$$
$$P_{1u12,i} = A_i P_{1u12} + B_1 P_{iu12} + B_u P_{1i12} + D_1 P_{1ui2} + E_2 P_{1u1i}, \tag{5.78}$$
$$P_{t212,i} = A_i P_{t212} + B_t P_{i212} + B_2 P_{ti12} + D_1 P_{t2i2} + E_2 P_{t21i}, \tag{5.79}$$
$$P_{131u,i} = A_i P_{131u} + B_1 P_{i31u} + B_3 P_{1i1u} + D_1 P_{13iu} + E_u P_{131i}, \tag{5.80}$$
$$P_{13t3,i} = A_i P_{13t3} + B_1 P_{i3t3} + B_3 P_{1it3} + D_t P_{13i3} + E_3 P_{13ti}, \tag{5.81}$$
$$P_{1p13,i} = A_i P_{1p13} + B_1 P_{ip13} + B_p P_{1i13} + D_1 P_{1pi3} + E_3 P_{1p1i}, \tag{5.82}$$
$$P_{t313,i} = A_i P_{t313} + B_t P_{i313} + B_3 P_{ti13} + D_1 P_{t3i3} + E_3 P_{t31i}, \tag{5.83}$$
$$P_{144l,i} = A_i P_{144l} + B_1 P_{i44l} + B_4 P_{1i4l} + D_4 P_{14il} + E_l P_{144i}, \tag{5.84}$$
$$P_{14v1,i} = A_i P_{14v1} + B_1 P_{i4v1} + B_4 P_{1iv1} + D_v P_{14i1} + E_1 P_{14vi}, \tag{5.85}$$
$$P_{1v41,i} = A_i P_{1v41} + B_1 P_{iv41} + B_v P_{1i41} + D_4 P_{1vi1} + E_1 P_{1v4i}, \tag{5.86}$$
$$P_{t441,i} = A_i P_{t441} + B_t P_{i441} + B_4 P_{ti41} + D_4 P_{t4i1} + E_1 P_{t44i}, \tag{5.87}$$
$$P_{141t,i} = A_i P_{141t} + B_1 P_{i41t} + B_4 P_{1i1t} + D_1 P_{14it} + E_t P_{141i}, \tag{5.88}$$
$$P_{14q4,i} = A_i P_{14q4} + B_1 P_{i4q4} + B_4 P_{1iq4} + D_q P_{14i4} + E_4 P_{14qi}, \tag{5.89}$$
$$P_{1114,i} = A_i P_{1114} + B_1 P_{i114} + B_1 P_{1i14} + D_1 P_{11i4} + E_4 P_{111i}, \tag{5.90}$$
$$P_{t414,i} = A_i P_{t414} + B_t P_{i414} + B_4 P_{ti14} + D_1 P_{t4i4} + E_4 P_{t41i}, \tag{5.91}$$
$$P_{244t,i} = A_i P_{244t} + B_2 P_{i44t} + B_4 P_{2i4t} + D_4 P_{24it} + E_t P_{244i}, \tag{5.92}$$
$$P_{24v2,i} = A_i P_{24v2} + B_2 P_{i4v2} + B_4 P_{2iv2} + D_v P_{24i2} + E_2 P_{24vi}, \tag{5.93}$$
$$P_{2v42,i} = A_i P_{2v42} + B_2 P_{iv42} + B_v P_{2i42} + D_4 P_{2vi2} + E_2 P_{2v4i}, \tag{5.94}$$
$$P_{s442,i} = A_i P_{s442} + B_s P_{i442} + B_4 P_{si42} + D_4 P_{s4i2} + E_2 P_{s44i}, \tag{5.95}$$
$$P_{344t,i} = A_i P_{344t} + B_3 P_{i44t} + B_4 P_{3i4t} + D_4 P_{34it} + E_t P_{344i}, \tag{5.96}$$

$$P_{34v3,i} = A_i P_{34v3} + B_3 P_{i4v3} + B_4 P_{3iv3} + D_v P_{34i3} + E_3 P_{34vi}, \tag{5.97}$$

$$P_{3v43,i} = A_i P_{3v43} + B_3 P_{iv43} + B_v P_{3i43} + D_4 P_{3vi3} + E_3 P_{3v4i}, \tag{5.98}$$

$$P_{4443,i} = A_i P_{4443} + B_4 P_{i443} + B_4 P_{4i43} + D_4 P_{44i3} + E_3 P_{444i}, \tag{5.99}$$

where $i = 1,2,3,4; l = 1,2,3,4; t = 2,3,4; u = 1,3,4; p = 1,4; v = 1,2,3; q = 2,3; s = 3,4$, since for the cases other than (5.76) – (5.99), the components of each term of (1.5) either vanishes identically or the relation (1.5) holds trivially using the skew-symmetry property of P.

Now, using (5.72) and (5.75), it follows for $i = 4$ that, right hand side of (5.76)(for $l = 2$) $= A_4 P_{1212} = \frac{20}{27(x^4)^3} = P_{1212,4} =$ left hand side of (5.76)(for $l = 2$).

For $i = 1,2,3$, the relation (5.39) implies that both sides of equation (5.40) are equal. By the similar argument, it can be easily seen that the equation (5.41) – (5.63) holds. Thus, the manifold under consideration is weakly projective symmetric manifold.

Hence, we can state the following:

Theorem 5.7. *Let (M^4, g) be a Riemannian manifold endowed with the metric given in (5.75). Then (M^4, g) is a weakly projective symmetric manifold with non-vanishing scalar curvature, which is neither projectively flat nor projectively symmetric but projectively recurrent.*

Example 5.6. Let $M = \{(x^1, x^2, x^3, \cdots, x^n) \in \mathbb{R}^n : 0 < x^4 < 1\}$ be a manifold endowed with the metric

$$ds^2 = [(x^4)^{\frac{4}{3}} - 1][(dx^1)^2 + (dx^4)^2] + \delta_{ab} dx^a dx^b, \tag{5.100}$$

where δ_{ab} is the Kronecker delta and a, b run from 1 to n. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, Ricci tensor, scalar curvature, projective curvature tensor and its covariant derivatives are given by

$$\Gamma^1_{14} = \Gamma^4_{44} = \frac{2}{3x^4} = -\Gamma^4_{11}, R_{1441} = -\frac{2}{3(x^4)^{\frac{2}{3}}},$$

$$S_{11} = S_{44} = -\frac{2}{3(x^4)^2}, r = -\frac{4}{3(x^4)^{\frac{10}{3}}} \neq 0,$$

$$P_{1212} = P_{1313} = -\frac{2}{3(n-1)(x^4)^2}, P_{1441} = -\frac{2(n-2)}{3(n-1)(x^4)^{\frac{2}{3}}} = -P_{1414},$$

$$P_{2442} = P_{3443} = \frac{2}{3(n-1)(x^4)^2}, P_{1k1k} = P_{4k4k} = -\frac{2}{3(n-1)(x^4)^2},$$

$$P_{1212,4} = P_{1313,4} = \frac{20}{9(n-1)(x^4)^3}, \tag{5.101}$$

$$P_{1441,4} = \frac{20(n-2)}{9(n-1)(x^4)^{\frac{5}{3}}} = -P_{1414,4}, \tag{5.102}$$

$$P_{2442,4} = -\frac{20}{9(n-1)(x^4)^3} = P_{3443,4}, \tag{5.103}$$

$$P_{1k1k,4} = P_{4k4k,4} = \frac{20}{9(n-1)(x^4)^3} \text{ for } 5 \leq k \leq n. \tag{5.104}$$

If we consider the components of the 1-forms A, B, D and E as

$$A(\partial_i) = A_i = \begin{cases} -\frac{10}{3x^4}, & \text{for } i = 4, \\ 0, & \text{otherwise,} \end{cases}$$

$$B_i = D_i = E_i = 0 \text{ for } i = 1, 2, \cdots, n,$$

where $\partial_i = \frac{\partial}{\partial x^i}$, then it can be easily shown that M^n is a $(WPS)_n$, which is not projectively symmetric.

Hence, we can state the following:

Theorem 5.8. *Let $(M^n, g)(n \geq 4)$ be a Riemannian manifold endowed with the metric given in (5.100). Then (M^n, g) is a weakly projective symmetric manifold with non-vanishing scalar curvature, which is not projectively symmetric but projectively recurrent.*

Let (M_5^4, g_5) be a Riemannian manifold in Example 5.5. and $(\mathbb{R}^{n-4}, g_4)$ be an $(n-4)$ dimensional Euclidean space with standard metric g_4. Then (M^n, g) in Example 5.6. is a product manifold of (M_5^4, g_5) and $(\mathbb{R}^{n-4}, g_4)$. Thus, we can state the following:

Theorem 5.9. *Let $(M^n, g)(n \geq 4)$ be a Riemannian manifold endowed with the metric given in (5.100). Then $(M^n, g)(n \geq 4)$ is a decomposable weakly projective symmetric manifold $(M_5^4, g_5) \times (\mathbb{R}^{n-4}, g_4)$ with non-vanishing scalar curvature, which is not projectively symmetric but projectively recurrent.*

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