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ON WEAKLY STABLE YANG-MILLS FIELDS OVER POSITIVELY PINCHED MANIFOLDS AND CERTAIN SYMMETRIC SPACES

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Abstract

In this paper it is proved that for $n\geq 5$ there exists a constant $\delta(n)$ with $1/4<\delta(n)<1$ such that any weakly stable Yang-Mills connection over a simple connected compact Riemannian manifold M of dimension n with $\delta(n)$-pinched sectional curvatures is always flat. The pinching constants are possible to compute by elementary functions. Moreover we give some remarks on stability of Yang-Mills connections over certain symmetric spaces.

Introduction.

Let M be an n-dimensional compact Riemannian manifold with a metric g and G be a compact Lie group with the Lie algebra g. Let E be a Riemannian vector bundle over M with structure group G, and let $\mathcal{C}_E$ denote the space of G-connections on E, which is an affine space modeled on the vector space $\Omega^1(g_E)$ of smooth 1-forms with values in the adjoint bundle g_E of E. The Yang-Mills functional $\mathcal{YM}: \mathcal{C}_E \to \boldsymbol{R}$ is

$$\mathcal{YM}(\nabla)=\frac{1}{2}\int_M \|F^\nabla\|^2 dvol,$$

for each $\nabla\in\mathcal{C}_E$, where F^∇ is the curvature form of the connection ∇. Note that F^∇ is a smooth section of $\Omega^2(g_E)$. The Yang-Mills connection $\nabla\in\mathcal{C}_E$ is a critical point of $\mathcal{YM}$. A Yang-Mills connection ∇ is called *weakly stable* if, for each $\nabla^t\in\mathcal{C}_E$ with $\nabla=\nabla^0$,

$$(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0}\geqq 0 .$$

M is called *Yang-Mills unstable* (cf. [K-O-T]) if, for every vector bundle (E, G) over M, any weakly stable Yang-Mills connection on E is always fiat. First Simons proved that the Euclidean n-sphere S^n for $n\geqq 5$ is Yang-Mills unstable ([B-L]). Ever since several persons have investigated the instability of Yang-Mills fields over various Riemannian manifolds; convex hypersurfaces, submani-

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folds, compact symmetric spaces (cf. [Ka], [K-O-T], [Pa1], [Sh], [Ta], [We]). In [K-O-T] it was shown that the Cayley projective plane $P_2(\mathrm{Cay})$ and the compact symmetric space of exceptional type E_6/F_4 are Yang-Mills unstable.

In this paper we first establish the instability theorem for Yang-Mills fields over a simply connected compact Riemannian manifold with sufficiently pinched sectional curvatures. Okayasu [Ok] used the construction and results of Ruh, Grove and Karcher ([Ru], [G-K-R1], [G-K-R2]) to show the instability of harmonic maps into a Riemannian manifold with sufficiently pinched sectional curvatures. By using the same idea, the second named author [Pa2] showed an instability theorem for harmonic maps from a Riemannian manifold with sufficiently pinched sectional curvatures to an arbitrary Riemannian manifold. We will also use it. Next we shall prove some results on weakly stable Yang-Mills fields over certain symmetric spaces. Some of them were stated in [K-O-T] without proof. They supplement results of Laquer [La] which determined the stablity of canonical connections over simply connected compact irreducible spaces. Moreover we prove that a weakly stable Yang-Mills field satisfying a certain condition over a quaternionic projective space $P_m(\boldsymbol{H})$ is a B_2-connection in a sense of [Ni], or equivalently a self-dual connection in a sense of [C-S], and hence it minimizes the Yang-Mills functional.

1. Preliminaries on Yang-Mills fields.

Let $\nabla\in\mathcal{C}_E$. For any $B\in\Omega^1(\mathfrak{g}_E)$, set $\nabla^t=\nabla+tB\in\mathcal{C}_E$. The second variational formula formula for the Yang-Mills functional is given as follows ([B-L]);

$$(1.1)\qquad (d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0}=\mathfrak{I}^{\nabla}(B, B)$$
$$=\int_M(\mathcal{S}_0^{\nabla}(B), B)dvol$$
$$=\int_M\{(\mathcal{S}^{\nabla}(B), B)-(\delta^{\nabla}B, \delta^{\nabla}B)\}dvol,$$

where $\mathcal{S}_0^{\nabla}(B)=\delta^{\nabla}d^{\nabla}B+\mathfrak{F}^{\nabla}(B)$ and $\mathcal{S}^{\nabla}(B)=\Delta^{\nabla}(B)+\mathfrak{F}^{\nabla}(B)$. Here d^{∇} and δ^{∇} denote the exterior covariant differentiation induced by the connection $\nabla\in\mathcal{C}_E$ and its adjoint differential operator, and $\mathfrak{F}^{\nabla}$ is a symmetric bundle endomorphism of $T^*M\otimes\mathfrak{g}_E$ defined by $(\mathfrak{F}^{\nabla}(b))(X)=\sum_{i=1}^n[F^{\nabla}(e_i, X), b(e_i)]$ for $b\in T_x^*M\otimes(\mathfrak{g}_E)_x$ and $X\in T_xM$, where $\{e_i\}$ is an orthonormal basis of T_xM.

Let $\{\omega^i\}$ be the dual frame of a local orthonormal frame field $\{e_i\}$ in M, Throughout this paper we use the summation convention. Set $B=B_i\omega^i$ and $F^{\nabla}=(1/2)F_{ij}\omega^i\wedge\omega^j$. Then we have

$$d^{\nabla}B=(\nabla_iB_j-\nabla_jB_i)\omega^i\wedge\omega^j,$$
$$\delta^{\nabla}d^{\nabla}B=(\nabla_j\nabla_iB_j-\nabla_j\nabla_iB_i)\omega^i,$$
$$\mathfrak{F}^{\nabla}(B)=[F_{ij}, B_i]\omega^j,$$

$$\|F^{\nabla}\|^2=(F_{ij}, F_{ij})/2 .$$

And (1.1) becomes

$$(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0}$$
$$=\int_M\{(\nabla_j\nabla_i B_j, B_i)-(\nabla_j\nabla_j B_i, B_i)+([F_{ij}, B_i], B_j)\}dvol.$$

Let D be a Riemannian connection of M and let R denote the curvature tensor field of D; $R(e_i, e_j)e_k=R_{ijkl}e_l$. The Ricci tensor field Ric of M is defined by $R_{ij}=R_{ikkj}$. The scalar curvature R of M is defined by $R=R_{ii}$. The Ricci identities are as follows:

$$D_kD_jX^i-D_jD_kX^i=R_{kjli}X^l \qquad \text{for } X=X^ie_i,$$
$$\nabla_l\nabla_kF_{ij}-\nabla_k\nabla_lF_{ij}=-F_{mj}R_{lkij}-F_{im}R_{lkjm}+[F_{lk}, F_{ij}],$$

The curvature form F^{∇} always satisfies the Bianchi identity $d^{\nabla}F^{\nabla}=0$, or equivalently

$$\nabla_kF_{ij}+\nabla_iF_{jk}+\nabla_jF_{ki}=0 . \tag{1.2}$$

The Yang-Mills equation is $\delta^{\nabla}F^{\nabla}=0$, namely

$$\nabla_jF_{ij}=0 . \tag{1.3}$$

Let $\nabla\in\mathcal{C}_E$. Assume that $\varphi=(1/2)\varphi_{ij}\omega^i\wedge\omega^j\in\Omega^2(g_E)$ is harmonic with respect to ∇, that is, $d^{\nabla}\varphi=0$ and $\delta^{\nabla}\varphi=0$. Note that if ∇ is a Yang-Mills connection, we can take $\varphi=F^{\nabla}$. Let $V\in C^{\infty}(TM)$ with $V=V^ie_i$. Set $B=i_V\varphi=B_i\omega_i\in\Omega^1(g_E)$. Here $B_i=V^j\varphi_{ji}$. Then by the harmonicity of φ and the Bochner-Weitzenböck formula (cf. [B-L]) we compute

$$(\mathcal{S}^{\nabla}(B))(X)=\varphi(D^*DV, X)-2\sum_{i=1}^{n}(\nabla_{e_i}\varphi)(D_{e_i}V, X) \tag{1.4}$$
$$+\varphi(V, \mathrm{Ric}(X))-\{\varphi\circ(\mathrm{Ric}\wedge I-2\mathcal{R})\}(V, X)$$
$$-\sum_{i=1}^{n}\{[F^{\nabla}(e_i, V), \varphi(e_i, X)]+[F^{\nabla}(e_i, X), \varphi(e_i, V)]\},$$

where $D^*DV=-\sum_{i=1}^{n}D^2V(e_i, e_i)$, and $\mathcal{R}$ denotes the curvature operator of (M, g) acting on $\wedge^2TM$. We define a quadratic form Q_{φ} on $C^{\infty}(TM)$ as

$$Q_{\varphi}(V)=(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0}=\int_M q_{\varphi}(V)dvol,$$

where $\nabla^t=\nabla+t(i_V\varphi)\in\mathcal{C}_E$. By straightforward computations we have

(1.5) $$q_{\varphi}(V)=D_jD_iV^kV^l(\varphi_{kj}, \varphi_{li})-D_jD_jV^kV^l(\varphi_{ki}, \varphi_{li})$$
$$+D_jV^kV^l(\nabla_i\varphi_{kj}\varphi_{li})-2D_jV^kV^l(\nabla_j\varphi_{ki}, \varphi_{li})$$
$$+V^kV^l([F^{\nabla}_{jk}, \varphi_{ij}]+[F^{\nabla}_{ji}, \varphi_{kj}], \varphi_{li})$$
$$+V^kV^l\{R_{ikmj}(\varphi_{mj}, \varphi_{li})-R_{jikm}(\varphi_{mj}, \varphi_{li})+R_{km}(\varphi_{im}, \varphi_{li})\}.$$

2. The constructionn of Ruh for a δ-pinched manifold.

We recall the idea and construction of Ruh ([Ru], [G-K-R1], [G-K-R2]). Let (M, g) be an n-dimensional simply connected compact Riemannian manifold with δ-pinched sectional curvature, namely $\delta<K\leqq 1$. We fix a normalized Riemannian metric $g_0=\{(1+\delta)/2\}g$ on M. Then we have $2\delta/(1+\delta)<K_{g_0}\leqq 2/(1+\delta)$. Consider a vector bundle $\Xi=TM\oplus\varepsilon(M)$ with a fibre metric $\langle\, ,\rangle$ over M. Here $\varepsilon(M)$ is a trivial line bundle with a fiber metric and it is orthogonal to TM. Let e denote a smooth section of lengh 1 in $\varepsilon(M)$. Now we define a metric connection D'' in Ξ as follows;

$$D''_XY=D_XY-g_0(X, Y)e\,,$$
$$D''_Xe=X$$

for $X, Y\in C^{\infty}(TM)$. It was proved that if δ is sufficiently close to 1, there exists a flat connection D' in Ξ close to D'' ([G-K-R1]). Define

$$\|D'-D''\|:=\underset{x\in M}{Max}\{\|D'_XY-D''_XY\|\,;\, X\in T_xM,\ g_0(X, X)=1,\ Y\in\Xi_x,\ \|Y\|=1\}.$$

Note that it is a half of that one in [G-K-R2]. Set

$$k_1(\delta)=(4/3)(1-\delta)\delta^{-1}\{1+(\delta^{1/2}\sin(1/2)\pi\delta^{-1/2})^{-1}\},$$
$$k_2(\delta)=\{(1+\delta)/2\}^{-1}k_1(\delta),$$
$$k_3(\delta)=k_2(\delta)[1+\{1-(1/24)\pi^2k_2(\delta)^2\}^{-2}]^{1/2}.$$

[G-K-M2] proved that $\|D'-D''\|\leqq k_3(\delta)/2$. The curvature form R'' of the connection D'' is

(2.1) $$R''(X, Y)Z=R(X, Y)Z-\langle Y, Z\rangle X+\langle X, Z\rangle Y\,,$$

(2.2) $$R''(X, Y)e=0$$

for $X, Y, Z\in T_xM$.

3. Trace formula for second variations of Yang-Mills fields over a δ-pinched manifold.

Assume that M is a simply connected compact Riemannian manifold with δ-pinched sectional curvatures. Let $P=\{v\in C^\infty(\Xi)\,;\,D'v=0\}$, which is linerly isometric to $\boldsymbol{R}^{n+1}$. For each $v\in P$, we denote by $V=v^T$ the TM-component of v in Ξ. Set $\mathcal{V}=\{V\in C^\infty(TM)\,;\,V=v^T$ for some $v\in P\}$, which has a natural inner product so that it is linearly isometric to P. Choose an orthonormal basis $\{V_\alpha\}_{\alpha=0,\dots,n}$ of $\mathcal{V}$. Set $V_\alpha=(v_\alpha)^T$. Then $\sum_{\alpha=0}^n V_\alpha^k V_\alpha^l=\delta^{kl}$. In this section we compute the trace $\mathrm{Tr}_{\mathcal{V}}Q_\varphi=\sum_{\alpha=0}^n Q_\varphi(V_\alpha)$ of Q_φ on $\mathcal{V}$ relative to the inner product.

A straightforward computation shows

LEMMA 3.1.

$$D_jV^k=\langle D''_{e_j}v,\, e_k\rangle-\langle v,\, e\rangle\delta_{jk}\,. \tag{3.1}$$

$$D_jD_iV^k=\langle (D''^2v)(e_i,\, e_j),\, e_k\rangle-\delta_{jk}\langle D''_{e_i}v,\, e\rangle-\delta_{ik}\langle D''_{e_j}v,\, e\rangle-\delta_{ik}\langle v,\, e_j\rangle\,. \tag{3.2}$$

LEMMA 3.2.

$$\int_M\{D_jD_iV^kV^l(\varphi_{kj},\,\varphi_{li})+D_jV^kV^l(\nabla_i\varphi_{kj},\,\varphi_{li})\}dvol$$
$$=\int_M\{R_{jimk}V^mV_l(\varphi_{kj},\,\varphi_{li})-D_jV^kD_iV^l((\varphi_{kj},\,\varphi_{li})\}dvol. \tag{3.3}$$

$$\int_M-2D_jV_\alpha^kV_\alpha^l(\nabla_j\varphi_{ki},\,\varphi_{li})dvol \tag{3.4}$$
$$=\int_M\{-2D_kD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\,\varphi_{li})-2D_jV_\alpha^kD_kV_\alpha^l(\varphi_{ij},\,\varphi_{li})$$
$$-D_iD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\,\varphi_{kl})-D_jV_\alpha^kD_iV_\alpha^l(\varphi_{ij},\,\varphi_{kl})$$
$$-2D_iD_jV_\alpha^kV_\alpha^l(\varphi_{jk},\,\varphi_{li})-2D_jV_\alpha^kD_iV_\alpha^l(\varphi_{jk},\,\varphi_{li})\}dvol.$$

Proof. (3.3) is due to the Ricci identity and the divergence theorem. We show (3.4). By $d^\nabla\varphi=0$, we have

$$-2D_jV_\alpha^kV_\alpha^l(\nabla_j\varphi_{ki},\,\varphi_{li}) \tag{3.5}$$
$$=2D_jV_\alpha^kV_\alpha^l(\nabla_k\varphi_{ij},\,\varphi_{li})+2D_jV_\alpha^kV_\alpha^l(\nabla_i\varphi_{jk},\,\varphi_{li}),$$

By using the divergence theorem, we get

$$\int_M 2D_jV_\alpha^kV_\alpha^l(\nabla_i\varphi_{jk},\,\varphi_{li})dvol$$

$$=\int_M \{-2D_iD_jV^k_\alpha V^l_\alpha(\varphi_{jk},\ \varphi_{li})-2D_jV^k_\alpha D_iV^l_\alpha(\varphi_{jk},\ \varphi_{li})\}dvol.$$

We compute

$$2D_jV^k_\alpha V^l_\alpha(\nabla_k\varphi_{ij},\ \varphi_{li})$$
$$=2D_k\{D_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \varphi_{li})\}-2D_kD_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \varphi_{li})$$
$$-2D_jV^k_\alpha D_kV^l_\alpha(\varphi_{ij},\ \varphi_{li})-2D_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \nabla_k\varphi_{li}).$$

Since

$$D_jV^k_\alpha V^l_\alpha=-V^k_\alpha D_jV^l_\alpha\,, \tag{3.6}$$

we have

$$D_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \nabla_k\varphi_{li})=D_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \nabla_l\varphi_{ik}).$$

Hence by Bianchi identity we get

$$-2D_jV^k_\alpha V^k_\alpha(\varphi_{ij},\ \nabla_k\varphi_{li})=D_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \nabla_i\varphi_{kl}).$$

Thus by using the divergence theorem we obtain

$$\int_M 2D_jV^k_\alpha V^l_\alpha(\nabla_k\varphi_{ij},\ \varphi_{li})dvol$$
$$=\int_M\{-2D_kD_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \varphi_{li})-2D_jV^k_\alpha D_kV^l_\alpha(\varphi_{ij},\ \varphi_{li})$$
$$-D_iD_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \varphi_{kl})-D_jV^k_\alpha D_iV^l_\alpha(\varphi_{ij},\ \varphi_{kl})\}dvol.$$

q. e. d.

By (1.5), (3.3) and (3.4), we get

$$\begin{aligned}\mathrm{Tr}_{\mathcal{V}}Q_\varphi=\int_M\{&-D_jV^k_\alpha D_iV^l_\alpha(\varphi_{kj},\ \varphi_{li})-D_jD_jV^k_\alpha V^l_\alpha(\varphi_{ki},\ \varphi_{li})\\ &-2D_kD_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \varphi_{li})-2D_jV^k_\alpha D_kV^l_\alpha(\varphi_{ij},\ \varphi_{li})\\ &-D_iD_jV^k_\alpha V^l_\alpha(\varphi_{ij},\ \varphi_{kl})-D_jV^k_\alpha D_iV^l_\alpha(\varphi_{ij},\ \varphi_{kl})\\ &-2D_iD_jV^k_\alpha V^l_\alpha(\varphi_{jk},\ \varphi_{li})-2D_jV^k_\alpha D_iV^l_\alpha(\varphi_{jk},\ \varphi_{li})\\ &+R_{jilk}(\varphi_{kj},\ \varphi_{li})+R_{ikmj}(\varphi_{mj},\ \varphi_{ki})\\ &-R_{jikm}(\varphi_{mj},\ \varphi_{ki})+R_{km}(\varphi_{im},\ \varphi_{ki})\}dvol.\end{aligned}\tag{3.7}$$

LEMMA 3.3.

$$\begin{aligned}&-2D_iD_jV^k_\alpha V^l_\alpha(\varphi_{jk},\ \varphi_{li})\\ =&D_jV^k_\alpha D_iV^l_\alpha(\varphi_{jk},\ \varphi_{li})+D_iV^k_\alpha D_jV^l_\alpha(\varphi_{jk},\ \varphi_{li})\\ &+R_{jimk}V^m_\alpha V^l_\alpha(\varphi_{jk},\ \varphi_{li}),\end{aligned}\tag{3.8}$$

(3.9) $$-D_iD_jV_\alpha^kV_\alpha^l(\varphi_{ij}, \varphi_{kl})=-(1/2)R_{ijmk}V_\alpha^mV_\alpha^l(\varphi_{ij}, \varphi_{kl}).$$

Proof. (3.9) is due to the Ricci identity. We show (3.8). Differentiating covariantly (3.6), we have

(3.10) $$D_iD_jV_\alpha^kV_\alpha^l+V_\alpha^kD_iD_jV_\alpha^l+D_jV_\alpha^kD_iV_\alpha^l+D_iV_\alpha^kD_jV_\alpha^l=0.$$

(3.8) follows from (3.10) and the Ricci identity. q.e.d.

LEMMA 3.4.

(3.11) $$-D_jD_jV_\alpha^kV_\alpha^l(\varphi_{ki}, \varphi_{li})=\langle D''_{e_j}v_\alpha, D''_{e_i}v_\beta\rangle V_\beta^kV_\alpha^l(\varphi_{ki}, \varphi_{li}) +\{2\langle D''_{e_k}v_\alpha, e\rangle+\langle v_\alpha, e_k\rangle\}V_\alpha^l(\varphi_{ki}, \varphi_{li}).$$

Proof. From $\langle v_\alpha, v_\beta\rangle=\delta_{\alpha\beta}$, we have

(3.12) $$\langle(D''^2v_\alpha)(e_i, e_j), v_\beta\rangle+\langle(D''^2v_\beta)(e_i, e_j), v_\alpha\rangle =-\langle D''_{e_i}v_\alpha, D''_{e_j}v_\beta\rangle-\langle D''_{e_j}v_\alpha, D''_{e_i}v_\beta\rangle.$$

Using (3.2) and (3.12), we obtain (3.11). q.e.d.

LEMMA 3.5.

(3.13) $$\int_M -2D_kD_jV_\alpha^kV_\alpha^l(\varphi_{ij}, \varphi_{li})dvol$$
$$\begin{aligned}
=\int_M[&2\langle D''_{e_j}v_\alpha, e\rangle V_\alpha^l(\varphi_{ij}, \varphi_{li})\\
&+2\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_j}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{li})\\
&+2\{(2-(n/2))\langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle-(1/4)\langle R''(e_l, e_k)e_k, e_l\rangle\\
&\quad-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_k\rangle\langle v_\alpha, e_l\rangle\\
&\quad-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_l\rangle\langle v_\alpha, e_k\rangle\\
&\quad-(1/2)\langle D''_{e_k}v_\alpha, e\rangle V_\alpha^k+(1/2)\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_l}v_\alpha, e_l\rangle\}\|\varphi\|^2\\
&-2\langle R''(e_k, e_j)e_l, e_k\rangle(\varphi_{ij}, \varphi_{li})+2(n+1)\langle D''_{e_j}v_\alpha, e\rangle V_\alpha^l(\varphi_{ij}, \varphi_{li})\\
&+2\langle v_\alpha, e_j\rangle V_\alpha^l(\varphi_{ij}, \varphi_{li})]dvol.
\end{aligned}$$

Proof. By (3.2), we have

(3.14) $$\begin{aligned}
&-2D_kD_jV_\alpha^kV_\alpha^l(\varphi_{ij}, \varphi_{li})\\
&=-2\{\langle(D''^2v_\alpha)(e_j, e_k), e_k\rangle-(n+1)\langle D''_{e_j}v_\alpha, e\rangle\\
&\quad-\langle v_\alpha, e_j\rangle\}V_\alpha^l(\varphi_{ij}, \varphi_{li}).
\end{aligned}$$

By using the Ricci identity we get

$$\text{(3.15)} \qquad \langle (D''^2 v_\alpha)(e_j, e_k), e_k \rangle V^l_\alpha(\varphi_{ij}, \varphi_{li})$$
$$= \{\langle (D''^2 v_\alpha)(e_k, e_j), e_k\rangle + \langle R''(e_k, e_j)v_v, e_k\rangle\} V^l_\alpha(\varphi_{ij}, \varphi_{li}) .$$

We compute

$$\text{(3.16)} \qquad \langle (D''^2 v_\alpha)(e_k, e_j), e_k\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li})$$
$$= D_j\{\langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li})\} - \langle D''_{e_j} v_\alpha, e\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li})$$
$$- \langle D''_{e_k} v^v, e_k\rangle\langle D''_{e_l} v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{li}) - \langle D''_{e_k} v_\alpha, e_k\rangle\langle v_\alpha, e\rangle(\varphi_{ij}, \varphi_{ij})$$
$$- \langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha(\varphi_{ij}, \nabla_j \varphi_{li}) .$$

By the Bianchi identity we get

$$\text{(3.17)} \qquad -\langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha(\varphi_{ij}, \nabla_j\varphi_{li}) = (1/4)\langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha D_l \|\varphi\|^2 .$$

We compute

$$\text{(3.18)} \qquad \langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha D_l\|\varphi\|^2$$
$$= D_l\{\langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha \|\varphi\|^2\} - \langle (D''^2 v_\alpha)(e_k, e_l), e_k\rangle V^l_\alpha\|\varphi\|^2$$
$$+ \langle D''_{e_k} v_\alpha, e\rangle V^k_\alpha\|\varphi\|^2 - \langle D''_{e_k} v_\alpha, e_k\rangle\langle D''_{e_l} v_\alpha, e_l\rangle\|\varphi\|^2$$
$$+ n\langle D''_{e_k} v_\alpha, e_k\rangle\langle v_\alpha, e\rangle\|\varphi\|^2 .$$

By using (3.12) and the Ricci identity we get

$$\text{(3.19)} \qquad \langle (D''^2 v_\alpha)(e_k, e_l), e_k\rangle V^l_\alpha$$
$$= -(1/2)\{\langle R''(e_l, e_k)e_k, e_l\rangle + \langle D''_{e_k} v_\alpha, D''_{e_l} v_\beta\rangle V^k_\alpha V^l_\alpha$$
$$+ \langle D''_{e_k} v_\alpha, D''_{e_l} v_\beta\rangle V^l_\beta V^k_\alpha\} .$$

Hence, by the divergence theorem, (3.13) follows from (3.14), (3.15), (3.16), (3.17), (3.16) and (3.19). q.e.d.

Therefore, by (2.1), (3.8), (3.11) and (3.13), (3.7) reduces to the following trace formula.

$$\text{(3.20)} \qquad \mathrm{Tr}_{CV}\, Q_\varphi$$
$$= \int_M [2\{5 - 2n + (n(n-1) - R)/4\}\|\varphi\|^2 + R_{jl}(\varphi_{ij}, \varphi_{il})$$
$$+ \langle D''_{e_i} v_\alpha, D''_{e_i} v_\beta\rangle V^k_\beta V^l_\alpha(\varphi_{ki}, \varphi_{li}) - 2\langle D''_{e_k} v_\alpha, e_k\rangle\langle D''_{e_j} v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{il})$$

$$+2\{(2-(n/2))\langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle$$
$$-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_k\rangle\langle v_\alpha, e_l\rangle$$
$$-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_l\rangle\langle v_\alpha, e_k\rangle$$
$$-(1/2)\langle D''_{e_k}v_\alpha, e\rangle V^k_\alpha+(1/2)\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_l}v_\alpha, e_l\rangle\}\|\varphi\|^2$$
$$-2(n+1)\langle D''_{e_j}v_\alpha, e\rangle V^l_\alpha(\varphi_{ij}, \varphi_{il})-8\langle D''_{e_j}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle(\varphi_{ij}, \varphi_{ik})$$
$$+2\langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_k}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{il})-\langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_i}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{kl})$$
$$+\langle D''_{e_l}v_\alpha, e_j\rangle\langle D''_{e_i}v_\alpha, e_k\rangle(\varphi_{ij}, \varphi_{kl})]dvol.$$

4. Instability theorem for Yang-Millf fields over a δ-pinched Riemannian manifold.

Note that if $\delta=1$, then $D'=D''$, hence (3.20) becomes

$$\mathrm{Tr}_{\mathcal{V}}\, Q_\varphi=2(4-n)\int_M\|\varphi\|^2.$$

Since the sectional curvatures of M are δ-pinched, we have

$$2\{5-2n+(1/4)(n(n-1)-R)\}\|\varphi\|^2+R_{jl}(\varphi_{ij}, \varphi_{il})$$
$$\leqq 2[5-2n+(1/4)n(n-1)\{1-2\delta/(1+\delta)\}+2(n-1)/(1+\delta)]\|\varphi\|^2,$$

We can make estimates for each other term of (3.20) as follows:

$$\langle D''_{e_l}v_\alpha, D''_{e_i}v_\beta\rangle V^j_\beta V^l_\alpha(\varphi_{kl}, \varphi_{li})\leqq(n/2)k_3(\delta)^2\|\varphi\|^2,$$
$$-2\langle D''_{e_l}v_\alpha, e_k\rangle\langle D''_{e_j}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{il})\leqq n(n+1)k_3(\delta)^2\|\varphi\|^2,$$
$$(2-(n/2))\langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle\leqq n(n/4-1)k_3(\delta),$$
$$-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_k\rangle\langle v_\alpha, e_l\rangle\leqq(n^2/16)k_3(\delta)^2,$$
$$-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_l\rangle\langle v_\alpha, e_k\rangle\leqq(n^2/16)k_3(\delta)^2,$$
$$-(1/2)\langle D''_{e_k}v_\alpha, e\rangle V^k_\alpha\leqq(n/4)k_3(\delta),$$
$$(1/2)\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_l}v_\alpha, e_l\rangle\leqq(n^2/8)k_3(\delta)^2,$$
$$-2(n+1)\langle D''_{e_j}v_\alpha, e\rangle V^l_\alpha(\varphi_{ij}, \varphi_{il})\leqq 2(n+1)k_3(\delta)\|\varphi\|^2,$$
$$-8\langle D''_{e_j}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle(\varphi_{ij}, \varphi_{ik})\leqq 8k_3(\delta)\|\varphi\|^2,$$
$$2\langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_k}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{il})\leqq nk_3(\delta)\|\varphi\|^2,$$
$$\langle D''_{e_l}v_\alpha, e_j\rangle\langle D''_{e_i}v_\alpha, e_k\rangle(\varphi_{ij}, \varphi_{kl})$$
$$-\langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_i}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{kl})\leqq k_3(\delta)\|\varphi\|^2.$$

Hence we get

$$\operatorname{Tr}_{\mathcal{V}} Q_{\varphi} \leqq 2[5-2n+(1/4)n(n-1)\{1-2\delta/(1+\delta)\}+2(n-1)/(1+\delta) + (1/4)(n^2+n+20)k_3(\delta)+(1/4)(3n^2+5n+2)k_3(\delta)^2]\int_M \|\varphi\|^2. \tag{4.1}$$

Therefore we obtain

THEOREM 4.1. *If* $n \geqq 5$ *and*

$$5-2n+(1/4)n(n-1)\{1-2\delta/(1+\delta)\}+2(n-1)/(1+\delta) + (1/4)(n^2+n+20)k_3(\delta)+(1/4)(3n^2+5n+2)k_3(\delta)^2<0, \tag{4.2}$$

then M is Yang-Mills unstable.

COROLLARY 4.2. *For* $n \geqq 5$, *there exists a constant* $\delta(n)$, *which depends only on n, with* $1/4<\delta(n)<1$ *such that any n-dimensional simply connected compact Riemannian manifold M with* $\delta(n)$*-pinched sectional curvatures is Yang-Mills unstable.*

Remark. As n tends to the infinity, the right hand side of (4.2) divided by $(1/4)(3n^2+5n+2)$ tends to $(1/3)\{1-2\delta/(1+\delta)\}+(1/3)k_3(\delta)+k_3(\delta)^2>0$. In our argument it is not possible to find a pinching constant δ independent of the dimension of the base manifold M such that M is Yang-Mills unstable.

5. Trace formula for second variations of Yang-Mills fields over submanifolds in Euclidean space.

Assume that M is isometrically immersed in a Euclidean space $\boldsymbol{R}^N$. Let Φ denote the immersion. We may assume that $\Phi(M)$ is not contained in any hyperplane of $\boldsymbol{R}^N$. Set $\mathcal{V}=\{U \in C^\infty(TM);\, U=\operatorname{grad} f_u \text{ for some } u \in \boldsymbol{R}^N\}$. Here f_u denotes the hight function on M defined by $f_u(u)=\langle \Phi(x), u\rangle$. Suppose that ∇ is a connection on a Riemannian vector bundle (E, G) over M and $\varphi \in \Omega^2(g_E)$ is harmonic with respect to ∇. Then we recall

PROPOSITION 5.1 ([K-O-T]). *For* $U=\operatorname{grad} f_u \in \mathcal{V}$,

$$\begin{aligned} \mathcal{S}^\nabla(i_U\varphi)(X) = &-\{\varphi\circ(\operatorname{Ric}\wedge I-2R)\}(U, X) \\ &+n\varphi(A_\eta(U), X)+\varphi(U, \operatorname{Ric}(X))-\varphi(\operatorname{Ric}(U), X) \\ &-\sum_{i=1}^n \{[F^\nabla(e_i, U), \varphi(e_i, X)]+[F^\nabla(e_i, X), \varphi(e_i, U)]\} \\ &-2\sum_{i,j=1}^n \langle B(e_i, e_j), u\rangle(\nabla_{e_j}\varphi)(e_i, X)-n\sum_{i=1}^n \langle D^\perp{}_{e_i}\eta, \mu\rangle\varphi(e_i, X). \end{aligned} \tag{5.1}$$

(5.2) $$\operatorname{tr}_{\mathcal{U}} Q_\varphi = 2\int_M (\varphi\circ\{(n/2)(A_\eta \wedge I) - \mathrm{Ric}\wedge I + 2\mathcal{R}\},\ \varphi)dvol,$$

where $\mathcal{R}$, B, A, η and $D^\perp$ denote the curvature operator of M acting on $\wedge^2 TM$, the second fundamental form, the shape operator, the mean curvature and the normal connection of Φ, respectively.

Consider a compact Riemannian homogeneous space with irreducible isotropy representation M.

LEMMA 5.2. *If ∇ is a weakly stable Yang-Mills connection, then we have*

(5.3) $$\sum_{i=1}^n \{[F^\nabla(e_i, Y),\ \varphi(e_i, X)] + [F^\nabla(e_i, X),\ \varphi(e_i, Y)]\} = 0$$

for every $X, Y \in T_x M$.

Proof. Let K be the group of isometries of M and let k be its Lie algebra of Killing vector fields on M. Since M has irreducible isotropy representation, we can fix a K-invariant inner product on k which induces the K-invariant Riemannian metric of M. By [B-L, (10.4) Lemma], for each $V \in k$

$$\mathcal{S}_0^\nabla(i_V\varphi)(X) = -\sum_{i=1}^n \{[F^\nabla(e_i, V),\ \varphi(e_i, X)] + [F^\nabla(e_i, X),\ \varphi(e_i, V)]\}.$$

Hence $\operatorname{tr}_k Q_\varphi = 0$. Since ∇ is weakly stable, we have $\mathfrak{I}^\nabla(i_V\varphi, i_V\varphi) = 0$ for all $V \in k$. For any $B \in \Omega^1(g_E)$,

$$0 \leqq \mathfrak{I}^\nabla(i_V\varphi + tB,\ i_V\varphi + tB) = 2t\mathfrak{I}^\nabla(i_V\varphi, B) + t^2\mathfrak{I}^\nabla(B, B),$$

hence $\mathfrak{I}^\nabla(i_V\varphi, B) = 0$. Thus $\mathcal{S}_0^\nabla(i_V\varphi) = 0$ for all $V \in k$. q.e.d.

Consider $\Phi: M \to S^{N-1}(\sqrt{n/\lambda_1}) \subset \boldsymbol{R}^N$ be the first standard minimal immersion of M (cf. [K-O-T]). Since M is an Einstein manifold and Φ is a minimal immersion onto a sphere of radius $\sqrt{n/\lambda_1}$, if $\varphi = F^\nabla$, then (5.1) becomes

(5.4) $$\mathcal{S}^\nabla(i_U\varphi)(X) = [\varphi\circ\{(\lambda_1 - 2c)I + 2\mathcal{R}\}](U, X) - 2\sum_{i,j=1}^n \langle B(e_i, e_j), u\rangle (\nabla_{e_j}\varphi)(e_i, X),$$

where c and λ_1 denote the Einstein constant of M and the first eigenvalue of the Laplace-Beltrami operator of M acting on functions, respectively.

Assume that M is a compact irreducible symmetric space. Let

(5.5) $$\overset{2}{\wedge} T_x M = h_0 + h_1 + \cdots + h_p$$

be the orthogonal decomposition into eigenspaces of $\mathcal{R}$, where h_0 is the eigenspace with eigenvalue 0 and h_s is the eigenspace with eigenvalue $\mu_s > 0$. We decompose $\varphi=\varphi_0+\varphi_1+\cdots+\varphi_p$ along (5.5). Note that $\nabla\varphi=0$ if and only if $\nabla\varphi_s=0$ for each $s=0, \cdots, p$. Assume that $\nabla\varphi=0$. If ∇ is weakly stable Yang-Mills field, then by (5.3) we have

$$S^{\nabla}(i_V\varphi_s)=(\lambda_1-2c+2\mu_s)(i_V\varphi_s) \quad \text{for each } s=0, \cdots, p. \tag{5.6}$$

6. Remarks on Yang-Mills fields over compact symmetric spaces.

First we remark on the stability of the canonical connections over compact globally Riemannian symmetric spaces. Laquer [La] determined the indices and nullities of the canonical connection on the standard principal bundle of each simply connected compact irreducible symmetric spaces. We denote by $i(\nabla)$ and $n(\nabla)$ the index and nullity of a Yang-Mills connection ∇ (cf. [B-L] for their definitions).

THEOREM 6.1 ([La]). *Let $M=K/H$ be a simply connected compact irreducible symmetric space associated with a symmetric pair (K, H) and let ∇ the canonical connection of the principal bundle $K\to K/H$.*

(1) *If M is a group manifold, then $i(\nabla)=1$ and $n(\nabla)=0$.*

(2) *If $M=S^n$ $(n\geqq 5)$, $P_2(\mathrm{Cay})$, E_6/F_4, then $i(\nabla)=n+1$, 26, 54 and $n(\nabla)=0$, respectively.*

(3) *If $M=P_m(H)$ $(m\geqq 1)$, then $i(\nabla)=0$, $n(\nabla)=10$ $(m=1)$ or $m(2m+3)$ $(m\geqq 2)$.*

(4) *If M is otherwise, then $i(\nabla)=n(\nabla)=0$.*

We should note that the values $i(\nabla)$ for $M=S^n$ $(n\geqq 5)$, $P_2(\mathrm{Cay})$, E_6/F_4 and $n(\nabla)$ for $M=P_m(H)$ $(m\geqq 2)$ are equal to the dimension of the first eigenspace of the Laplace-Beltrami operator of M acting on functions, and $n(\nabla)$ for $M=P_1(H)=S^4$ is equal to its twice. It is known that, in the cases of $M=S^n$, $P_m(H)$, $P_2(\mathrm{Cay})$, the space of all gradient vector fields for the first eigenfunctions on M coincides with the space of all proper infinitesimal conformal transformations or projective transformations on M.

We observe the case when M is a non-simply connected, compact irreducible symmetric space. From [La] we see that if M is a group manifold, then $i(\nabla)=1$, $n(\nabla)=0$. Suppose that M is not a group manifold. We easily check that if the canonical connection of the universal covering $\tilde{M}$ of M has $i(\nabla)=n(\nabla)=0$, then the canonical connection of M also has $i(\nabla)=n(\nabla)=0$. When $\tilde{M}=S^n$, by virtue of [B-L, (9.1) Theorem], we have $i(\nabla)=n(\nabla)=0$. From the theory of symmetric spaces (cf. [He]) we know that if $\tilde{M}=P_n(H)$ or $P_2(\mathrm{Cay})$, then $\tilde{M}=M$, and if $\tilde{M}=E_6/F_4$, then $M=E_6/F_4\cdot Z_3$. We show that the canonical connection of $M=E_6/F_4\cdot \boldsymbol{Z}_3$ has $i(\nabla)=n(\nabla)=0$. From Theorem 6.1 we see $n(\nabla)=0$. First we recall the realization of E_6/F_4 and $E_6/F_4\cdot\boldsymbol{Z}_3$ (cf. [Yo]). Consider the Jordan algebra $\mathfrak{X}=\{u\in M(3, \mathrm{Cay});\, u^*=u\}$ of (real) dimension 27. Let $R^{54}=C^{27}=\mathfrak{X}^C$ be the complexification of $\mathfrak{X}$ with a natural real inner product $\langle\,,\rangle$. Let $S^{53}=\{u\in R^{54};\, \langle u, u\rangle=3\}$, a hypersphere of $\mathfrak{X}^C$. Set $\tilde{M}=\{u\in S^{53};\, \det(u)=1\}$ and let

Φ denote the embedding $\tilde{M}\rightarrow S^{53}\subset \boldsymbol{R}^{54}$.

PROPOSITION 6.2. (1) *$\tilde{M}$ is isometric to a simply connected compact irreducible symmetric space E_6/F_4* (cf. [Yo]).

(2) *The embedding Φ is the first standard minimal immersion of $\tilde{M}=E_6/F_4$* (cf. [Oh]).

Now we define a finite group Γ acting freely and isometrically on $\boldsymbol{R}^{54}-\{0\}$ and $\tilde{M}$ by

$$\Gamma=\{1,\ \sigma,\ \sigma^2\}\cong \boldsymbol{Z}_3\,,$$

$$\sigma(u)=e^{(2/3)\pi\sqrt{-1}}u \qquad \text{for each } u\in \boldsymbol{R}^{54}\,.$$

Then the quotient $M=\tilde{M}/\Gamma$ is isometric to the symmetric space $E_6/F_4\cdot \boldsymbol{Z}_3$.

Set $K=E_6$, $H=F_4$ and $N=54$. Let R^∇ be the curvature form of the cononical connection ∇ for (K, H). Then we have

$$\overset{2}{\wedge} T_x\tilde{M}=\mathrm{so}(T_x\tilde{M})=h_0+h_1\,,$$

where h_1 is isometric to the Lie algebra of F_4, which is the holonomy algebra of $\tilde{M}$. Since $\lambda_1-2c+2\mu_1<0$ by virtue of the result of [K-O-T], from (5.4) we see that

$$\Theta=\{i_U R^\nabla\,;\,U=\mathrm{grad}\, f_u \quad \text{for some } u\in \boldsymbol{R}^N\}$$

is an eigenspaces of $\mathcal{S}^\nabla$ of dimension 54 with a negative eigenvalue. From Theorem 6.1 we see $i(\nabla)=\dim\Theta$. In order to show that the canonical connection of M has $i(\nabla)=0$, it suffices to show that if $i_U R^\nabla\in\Theta$ is invariant by Γ, then $U=0$. It follows from the following two lemmas.

LEMMA 6.3. *Let $V\in C^\infty(TM)$. If*

$$\gamma(i_V R^\nabla)=i_V R^\nabla \qquad \textit{for each } \gamma\in\Gamma\,,$$

then $\gamma_* V=V$ for each $\gamma\in\Gamma$.

Proof. For any $X\in T_xM$,

$$R^\nabla(V_x,\ X)=\gamma(i_V R^\nabla)(X)=\gamma(R^\nabla(V_{\gamma^{-1}(x)},\ \gamma_*^{-1}X))$$
$$=R^\nabla(\gamma_* V_{\gamma^{-1}(x)},\ X)\,,$$

hence $R^\nabla(\gamma_* V_{\gamma^{-1}(x)}-V_x,\ X)=0$. If we let the canonical decomposition $k=h+m$ at $x\in\tilde{M}$ and we use the identification $m=T_xM$, then $R^\nabla(X, Y)=-\mathrm{ad}_m[X, Y]$ (cf. [K-N]). Thus $\mathrm{ad}_m[\gamma_* V_{\gamma^{-1}(x)}-V_x,\ X]=0$ for each $X\in m$. Since $h=[m, m]$ and k is semisimple, $\gamma_* V_{\gamma^{-1}(x)}-V_x=0$. q.e.d.

LEMMA 6.4. *Let $U=\mathrm{grad}\, f_u\in C^\infty(TM)$ for some $u\in\boldsymbol{R}^N$. If $\gamma\in\Gamma-\{1\}$ and*

$\gamma_* U=U$, then $u=0$.

Proof. For each $x\in\tilde{M}$ and $X\in T_xM$,

$$\langle\gamma_* U, X\rangle=\langle U, \gamma_*^{-1}X\rangle=\langle\gamma^{-1}(X), u\rangle=\langle X, \gamma(u)\rangle=\langle U, X\rangle=\langle X, u\rangle,$$

hence $\langle X, \gamma(u)-u\rangle=0$. Thus $\langle x, \gamma(u)-u\rangle$ is constant in $x\in\tilde{M}$. Since $\Phi(\tilde{M})$ is not contained in any hyperplane of $\boldsymbol{R}^N$, we have $\gamma(u)=u$. Since Γ acts freely on $\boldsymbol{R}^N-\{0\}$, we get $u=0$. q.e.d.

Next we remark on weakly stable Yang-Mills fields over a quaternionic projective space $M=P_m(H)$. Generally let M be a quaternionic Kähler manifold. The $Sp(m)\cdot Sp(1)$-structure induces the orthogonal decomposition

$$\bigwedge^2 T^*M=W_0+W_1+W_2,$$

where $(W_0)_x$, $(W_1)_x\cong sp(1)$, $(W_2)_x\cong sp(m)$ are irreducible $Sp(m)\cdot Sp(1)$-modules. The curvature form $F^\nabla=F_0^\nabla+F_1^\nabla+F_2^\nabla$ of a connection ∇ on the vector bundle E over M splits into components F_i^∇ to $End(E)\otimes W_i$ at each point. A connection ∇ with $F^\nabla=F_2^\nabla$ (resp. $F^\nabla=F_1^\nabla$) is called a B_2-connection (resp. A_1'-connection) as in [Ni], or a *self-dual* connection (resp. an *anti-self-dual* connection) as in [C-S]. They are Yang-Mills connections which minimizes the Yang-Mills functional ([C-S], [Ni]).

PROPOSITION 6.5. *Let E be a Riemannian vector bundle over $P_m(\boldsymbol{H})$. If ∇ is a weakly stable Yang-Mills connection on E satisfying $F_1^\nabla=0$, then ∇ is a B_2-connection (self-dual).*

Proof. We may suppose that g is an $Sp(m+1)$-invariant Riemannian metric on $P_m(\boldsymbol{H})=Sp(m+1)/Sp(m)\times Sp(1)$ induced by the Killing form of the Lie algebra of $Sp(m+1)$. From [K-O-T] we know

$$\begin{aligned}&\mathfrak{R}=\mathfrak{R}_0+\mathfrak{R}_1+\mathfrak{R}_2,\\&\mathfrak{R}_0=0,\\&\mathfrak{R}_1=(m/2(m+2))I,\\&\mathfrak{R}_2=(1/2(m+2))I.\end{aligned}\tag{6.1}$$

Hence by virtue of (5.2), we get

$$\begin{aligned}&\mathrm{Tr}_U Q_{F^\nabla}\\&=2\int_M(F^\nabla\circ\{2\mathfrak{R}-(1/(m+2))I\}, F^\nabla)dvol\\&=2\left\{-1/(m+2)\int_M(F_0^\nabla, F_0^\nabla)dvol+(m-1)/(m+2)\int_M(F_1^\nabla, F_1^\nabla)dvol\right\},\end{aligned}$$

Proposition 6.5 follows from this equation. q.e.d.

From the proof of Proposition 6.5, we see that if ∇ satisfies the assumption, then

$$\sum_{i,j=1}^{n} \langle B(e_i, e_j), u\rangle(\nabla_{e_j} F^{\nabla})(e_i, X)=0, \tag{6.2}$$

for all $u\in \boldsymbol{R}^N$ and all $X\in T_xM$. Using the properties of the second fundamental form of Φ and the curvature tensor field of $P_m(\boldsymbol{H})$, we can check that (6.2) implies that the restriction of F^{∇} to every quaternionic projective line $P_1(\boldsymbol{H})\subset P_m(\boldsymbol{H})$ is always a Yang-Mills field. Hence by (5.6) and (6.1) we obtain that, for any B_2-connection ∇ over $P_m(\boldsymbol{H})$ and any infinitesimal projective transformation U on $P_m(\boldsymbol{H})$, we have $\mathcal{S}^{\nabla}(i_U F^{\nabla})=0$. This means the existence of an infinitesimal action of the projective transformation group of $P_m(\boldsymbol{H})$ on the space of all B_2-connections over $P_m(\boldsymbol{H})$. In fact, it is known that the projective transformation group of $P_m(\boldsymbol{H})$ acts on the moduli space of all B_2-connections on E.

By (5.4), (5.6) and (6.1) we obtain that the indices $i(\nabla)$ and the nullity $n(\nabla)$ of the canonical connection of $M=S^n$ $(n\geqq 5)$, $P_2(\mathrm{Cay})$ and E_6/F_4 come from $\mathrm{span}_{\boldsymbol{R}}\{i_U R^{\nabla}; U\in\mathfrak{U}\}$, and the nullities for $M=P_1(\boldsymbol{H})=S^4$ and $P_m(\boldsymbol{H})$ $(m\geqq 2)$ come from $\mathrm{span}_{\boldsymbol{R}}\{i_U R_1^{\nabla}, i_U R_2^{\nabla}; U\in\mathfrak{U}\}$ and $\mathrm{span}_{\boldsymbol{R}}\{i_U R_2^{\nabla}; U\in\mathfrak{U}\}$, respectively. We do not know whether each weakly stable canonical connection over a compact symmetric space minimizes the Yang-Mills functional. And it is interesting to investigate relationships of Yang-Mills fields with holonomy groups and the classification of vector bundles with Yang-Mills connections satisfying $\nabla F^{\nabla}=0$ over compact symmetric spaces. From results of [B-L, p. 211] and [K-O-T] we can find gap phenomena for Yang-Mills fields over every compact irreducible symmetric space which is not locally Hermitian symmetric. The classification of such Yang-Mills connections may also be useful to establish accurately isolation theorems for Yang-Mills fields over compact symmetric spaces.

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