

## ON WEIERSTRASS POINTS OF RIEMANN SURFACES ASSOCIATED WITH $\Gamma(n, 2n)$

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**Introduction.** Let  $\Gamma$  be a congruence subgroup of  $SL(2, \mathbf{R})$ , and let  $R(\Gamma)$  be the Riemann surface associated with  $\Gamma$ , i.e.,  $R(\Gamma)$  is the canonical compactification of  $\Gamma \backslash H$ , where  $H$  is the upper half plane of  $\mathbf{C}$ . If the genus of  $R(\Gamma)$  is not less than two, then we can ask the following problems:

Problem 1. Is  $R(\Gamma)$  hyperelliptic?

Problem 2. Are the cusps of  $R(\Gamma)$  Weierstrass points?

Historically, Problems 1 and 2 are completely solved for  $\Gamma = \Gamma(n)$  by H. Petersson [8] and by B. Schoeneberg [9] respectively. In the case of  $\Gamma_0(n)$ , partial solutions are given by J. Lehener and M. Newmann [6] and A.O.L. Atkin [2].

The purpose of this note is to answer both problems in the case of  $\Gamma = \Gamma(n, 2n)$  (as for the definition of  $\Gamma(n, 2n)$ , see Definition 1). Our results are the following:

1.  $R(\Gamma(n, 2n))$  is non-hyperelliptic for any positive integer  $n \geq 4$  (see Theorem 4).

2. Every cusp of  $R(\Gamma(n, 2n))$  is a Weierstrass point for any even integer  $n \geq 4$ . But there is an example, where the opposite situation may occur if  $n$  is odd (see Theorem 6 and Remark 1).

Notation.  $SL(2, \mathbf{R})$  (resp.  $SL(2, \mathbf{Z})$ ) is the special linear group of degree two over the real number field  $\mathbf{R}$  (resp. the rational integer ring  $\mathbf{Z}$ ), and  $PSL(2, \mathbf{R})$  is the projective special linear group of degree two over  $\mathbf{R}$ . When  $\Delta$  is a subgroup of  $SL(2, \mathbf{R})$ , the image of  $\Delta$  under the canonical homomorphism  $SL(2, \mathbf{R}) \rightarrow PSL(2, \mathbf{R})$  is denoted by  $\bar{\Delta}$ .  $H^*$  means the disjoint union of the upper half plane  $H$ , the rational numbers  $\mathbf{Q}$  and  $\{\infty\}$ .

Let  $\Gamma$  be a congruence subgroup of  $SL(2, \mathbf{Z})$ , and let  $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be an element of  $\Gamma$  (or  $\bar{\Gamma}$ ). When  $z$  is a point on  $H^*$ ,  $\sigma(z)$  means  $(az+b)/(cz+d)$ , where  $a/0$  ( $a \neq 0$ ) means  $\infty$  and  $(a\infty+b)/(c\infty+d)$  means  $a/c$ . For a point  $z$  on  $H^*$ ,  $\Gamma_z$  (resp.  $\bar{\Gamma}_z$ ) means the isotropy subgroup of  $\Gamma$  (resp.  $\bar{\Gamma}$ ) at  $z$ . We denote the canonical projection  $H^* \rightarrow R(\Gamma)$  by  $\pi_\Gamma$ , and the genus of  $R(\Gamma)$  by  $g(\Gamma)$ .

§ 1. Genera of  $R(\Gamma(n, 2n))$

We start with the definition of the congruence subgroup  $\Gamma(n, 2n)$  for a positive integer  $n$ . These groups prove themselves useful when one discusses the theory of moduli of abelian varieties by means of the theory of theta functions in the case of general dimension. But we treat only the one dimensional case.

DEFINITION 1. Let  $n$  be a positive integer. We define,

$$\Gamma(n, 2n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(n) \mid ab \equiv cd \equiv 0 \pmod{2n} \right\},$$

The following lemma is an easy consequence of the definition (cf. Igusa [3]).

LEMMA 1. (0)  $\Gamma(n, 2n)$  is a congruence subgroup of  $SL(2, \mathbf{Z})$ .

$$(1) \quad \Gamma(n, 2n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(n) \mid ac \equiv bd \equiv 0 \pmod{2n} \right\}.$$

(2) If  $n$  is an even integer, then

$$\Gamma(n, 2n) = \left\{ I_2 + n \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(n) \mid b \equiv c \equiv 0 \pmod{2} \right\}.$$

(3) If  $n$  is an odd integer, then

$$\Gamma(n, 2n) = \Gamma(n) \cap \Gamma(1, 2).$$

We continue to investigate the group  $\Gamma(n, 2n)$ .

LEMMA 2. (1) (Igusa [3]) Assume that  $n$  is an even integer. Then,

(a)  $\Gamma(n, 2n)$  is a normal subgroup of  $\Gamma(n)$ ,

(b)  $\Gamma(n)/\Gamma(n, 2n) \simeq (\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2\mathbf{Z})$ , and this isomorphism is induced by

$$I_2 + n \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (b \pmod{2}, c \pmod{2}),$$

$$(c) \quad [\bar{\Gamma}(1) : \bar{\Gamma}(n, 2n)] = \begin{cases} 24 & (\text{if } n=2) \\ 2n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) & (\text{if } n \geq 4), \end{cases}$$

where  $p|n$  means that  $p$  runs over all prime numbers dividing  $n$ .

(2) Assume that  $n$  is an odd integer. Then,

(a)  $\Gamma(n, 2n)$  is a normal subgroup of  $\Gamma(1, 2)$ , but it is not a normal subgroup of  $\Gamma(n)$ ,

(b)  $\Gamma(n, 2n) = \Gamma(2n) + \Gamma(2n) \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}$ , where the right hand side of the equality means the coset decomposition of  $\Gamma(n, 2n) \pmod{\Gamma(2n)}$ ,

$$(c) \quad [\bar{\Gamma}(1) : \bar{\Gamma}(n, 2n)] = \begin{cases} 3 & (\text{if } n=1) \\ \frac{3}{2} n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) & (\text{if } n \geq 3). \end{cases}$$

PROOF. (1) All of them are the special cases of results in Igusa's paper [3].

(2) (a) Since  $\Gamma(n, 2n) = \Gamma(n) \cap \Gamma(1, 2)$ ,  $\Gamma(n, 2n)$  is a normal subgroup of  $\Gamma(1, 2)$ . It can easily be verified that  $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$  belongs to  $\Gamma(n)$  and that  $\begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}$  belongs to  $\Gamma(n, 2n)$  but that

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1-n-n^2 & n+2n^2+n^3 \\ -n & 1+n+n^2 \end{pmatrix}$$

does not belong to  $\Gamma(n, 2n)$ , so  $\Gamma(n, 2n)$  is not a normal subgroup of  $\Gamma(n)$ .

(b) Let  $G$  be the subgroup of  $\Gamma(1)$  generated by  $\Gamma(2n)$  and  $\begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}$ . Then we see that the coset decomposition of  $G$  mod.  $\Gamma(2n)$  is given by  $G = \Gamma(2n) + \Gamma(2n) \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}$  and that the inclusion relations  $\Gamma(n) \supseteq \Gamma(n, 2n) \supset G \supseteq \Gamma(2n)$  holds. Since  $[\Gamma(n) : \Gamma(2n)] = 6$ , we have  $\Gamma(n, 2n) = G$ .

(c) It is clear that  $[\bar{\Gamma}(n, 2n) : \bar{\Gamma}(2n)] = 2$ . Combining this equality with the known formula :

$$[\bar{\Gamma}(1) : \bar{\Gamma}(n)] = \begin{cases} 6 & (\text{if } n=2) \\ \frac{1}{2} n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) & (\text{if } n \geq 3), \end{cases}$$

we obtain our formula. Q.E.D.

The following proposition is the main purpose in this section.

PROPOSITION 3. (1) *Let  $n$  be an even integer. Then  $R(\Gamma(n, 2n))$  has no elliptic point. The number of cusps on  $R(\Gamma(n, 2n))$  is equal to 6 if  $n=2$ , and is equal to  $n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right)$  if  $n \geq 4$ . The genus of  $R(\Gamma(n, 2n))$  is given by the following formula :*

$$g(\Gamma(n, 2n)) = \begin{cases} 0 & (\text{if } n=2) \\ 1 + \frac{1}{6} n^2 (n-3) \prod_{p|n} \left(1 - \frac{1}{p^2}\right) & (\text{if } n \geq 4). \end{cases}$$

(2) *Let  $n$  be an odd integer. Then  $R(\Gamma(n, 2n))$  has no elliptic point if  $n \geq 3$ , and has exactly one elliptic point, whose order is two, if  $n=1$ . In the case of  $n=1$ ,  $R(\Gamma(1, 2))$  has exactly two cusps which are represented by  $\infty$  and 1. In the case of  $n \geq 3$ ,  $R(\Gamma(n, 2n))$  has  $\frac{1}{2} n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right)$  cusps lying over  $\pi_{\Gamma(1,2)}(\infty)$  and  $\pi_{\Gamma(1,2)}(1)$  respectively. Hence the number of the cusps on  $R(\Gamma(n, 2n))$  is equal to  $n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right)$ . The genus of  $R(\Gamma(n, 2n))$  is given by the following formula :*

$$g(\Gamma(n, 2n)) = \begin{cases} 0 & (\text{if } n=1) \\ 1 + \frac{1}{8} n^2 (n-4) \prod_{p|n} \left(1 - \frac{1}{p^2}\right) & (\text{if } n \geq 3). \end{cases}$$

PROOF. (1) Since  $\Gamma(n)$  ( $n \geq 2$ ) has no elliptic element,  $\Gamma(n, 2n)$  has no elliptic element. Since  $[\bar{\Gamma}(1)_\infty : \bar{\Gamma}(n, 2n)_\infty] = 2n$ , the number of cusps on  $R(\Gamma(n, 2n))$  is equal to  $[\bar{\Gamma}(1) : \bar{\Gamma}(n, 2n)]/2n$ . Our formula comes from the genus formula of Riemann surfaces associated with modular groups (cf. G. Shimura [10] Proposition 1.40).

(2) Since  $R(\Gamma(2))$  is a covering Riemann surface of genus 0 over  $R(\Gamma(1, 2))$ ,  $R(\Gamma(1, 2))$  is of genus 0. Suppose that  $R(\Gamma(1, 2))$  has an elliptic point of order 3. Then  $\Gamma(1, 2)$  must have an element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  whose trace is 1 or  $-1$ . On the other hand,  $ab \equiv cd \equiv 0 \pmod{2}$ . These contradict to  $ad - bc = 1$ , so  $R(\Gamma(1, 2))$  has no elliptic point of order 3. By the similar argument, we see that  $\infty$  and 1 on  $H^*$  are not  $\Gamma(1, 2)$ -equivalent. Let  $\nu_\infty$  be the number of the cusps on  $R(\Gamma(1, 2))$ , and  $\nu_2$  the number of elliptic points of order 3. Then we have  $\nu_\infty \geq 2$  and  $\nu_2 \geq 1$ . On the other hand, by the genus formula, we get the equation  $0 = 1 + 3/12 - \nu_2/4 - \nu_\infty/12$ . Therefore we have  $\nu_\infty = 2$  and  $\nu_2 = 1$ .

Next, we shall calculate the number of cusps on  $R(\Gamma(n, 2n))$  in the case of  $n \geq 3$ . We put,

$$\begin{aligned} \nu_\infty^{(\infty)} &= \text{the number of cusps on } R(\Gamma(n, 2n)) \text{ lying over } \pi_{\Gamma(1, 2)}(\infty) \\ &= [\bar{\Gamma}(1, 2) : \bar{\Gamma}(n, 2n)] / [\bar{\Gamma}(1, 2)_\infty : \bar{\Gamma}(n, 2n)_\infty], \\ \nu_\infty^{(1)} &= \text{the number of cusps on } R(\Gamma(n, 2n)) \text{ lying over } \pi_{\Gamma(1, 2)}(1) \\ &= [\bar{\Gamma}(1, 2) : \bar{\Gamma}(n, 2n)] / [\bar{\Gamma}(1, 2)_1 : \bar{\Gamma}(n, 2n)_1]. \end{aligned}$$

Since  $\bar{\Gamma}(n, 2n)_\infty = \left\{ \begin{pmatrix} 1 & 2nm \\ 0 & 1 \end{pmatrix} \mid m \in \mathbf{Z} \right\}$ , we have  $\nu_\infty^{(\infty)} = \frac{1}{2} n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right)$ .

Let  $\sigma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ . Then  $\sigma(1) = \infty$ , so  $\sigma(\Gamma(n, 2n)_1)\sigma^{-1} = (\sigma\Gamma(n, 2n)\sigma^{-1})_\infty$ .

By Lemma 2, for any odd integer  $n$ , we see that

$$\begin{aligned} \sigma\Gamma(n, 2n)\sigma^{-1} &= \sigma(\Gamma(2n) + \Gamma(2n)) \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix} \sigma^{-1} \\ &= \Gamma(2n) + \Gamma(2n) \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence  $\overline{(\sigma\Gamma(n, 2n)\sigma^{-1})_\infty} = \left\{ \begin{pmatrix} 1 & nm \\ 0 & 1 \end{pmatrix} \mid m \in \mathbf{Z} \right\}$ , and also

$$[\bar{\Gamma}(1, 2)_1 : \bar{\Gamma}(n, 2n)_1] = [(\overline{\sigma\Gamma(1, 2)\sigma^{-1}})_\infty : \overline{(\sigma\Gamma(n, 2n)\sigma^{-1})_\infty}] = n.$$

Therefore  $\nu_\infty^{(1)} = \frac{1}{2} n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right)$ . Finally, by the genus formula we get our formula.

*Q.E.D.*

§ 2. The answer to the first problem

Let  $R$  be a Riemann surface of genus  $g$ , and let  $P$  be a point on  $R$ . We say that a positive integer  $m$  is a gap at  $P$ , if there exists no meromorphic function on  $R$  with a pole of order  $m$  at  $P$  and holomorphic at any other point. It is known that for any  $P$  there are exactly  $g$  gaps and that except for finitely many points the gap sequence coincides with  $\{1, 2, \dots, g\}$ . A point on  $R$  whose gap sequence differs from  $\{1, 2, \dots, g\}$  is called a Weierstrass point.

If the genus  $g$  is greater than or equal to two, then at least  $2g+2$  Weierstrass points exist, the number of the Weierstrass points on  $R$ , say  $w$ , satisfies the following inequalities:

$$2g+2 \leq w \leq (g-1)g(g+1),$$

and  $w$  is equal to  $2g+2$  if and only if  $R$  is hyperelliptic.

The purpose in this section is to prove the following theorem. Our proof goes in the same way as Peterson's proof for the principal congruence case [8].

**THEOREM 4.**  $R(\Gamma(n, 2n))$  is a non-hyperelliptic Riemann surface for any integer  $n \geq 4$ .

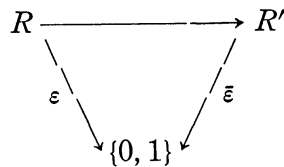
Before proving our theorem, we state some remarks. In the case of  $n \geq 4$ , there exists a Weierstrass points on  $R(\Gamma(n, 2n))$  because their genera are greater than two.

**DEFINITION 2.** Let  $R$  be a Riemann surface of genus  $g \geq 2$ .

$\varepsilon = \varepsilon_R: R \rightarrow \{0, 1\}$  is defined by the following:

$$\varepsilon(P) = \begin{cases} 0 & \text{if } P \text{ is not a Weierstrass point,} \\ 1 & \text{if } P \text{ is a Weierstrass point.} \end{cases}$$

Let  $G$  be an automorphism group of  $R$ , and let  $R \rightarrow R' = G \backslash R$  be the canonical covering map. Then there exists a function  $\bar{\varepsilon} = \bar{\varepsilon}_{R,G}: R' \rightarrow \{0, 1\}$  such that the following diagram is commute:



**PROOF OF THEOREM 4.** The first, we assume that  $n$  is an odd integer. We put,

$$\begin{aligned} \pi &= \pi_{\Gamma(1,2)}, \\ \mu &= \mu(n) = [\bar{\Gamma}(1,2): \bar{\Gamma}(n, 2n)] = \frac{1}{2} n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right), \end{aligned}$$

$$\begin{aligned} \nu_{\infty}^{(\infty)} &= \nu_{\infty}^{(\infty)}(n) = \text{the number of cusps on } R(\Gamma(n, 2n)) \text{ lying over } \pi(\infty), \\ \nu_{\infty}^{(1)} &= \nu_{\infty}^{(1)}(n) = \text{the number of cusps on } R(\Gamma(n, 2n)) \text{ lying over } \pi(1), \\ \nu &= \nu(n) = \frac{1}{2} n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right). \end{aligned}$$

By Proposition 3, we have  $\nu_{\infty}^{(\infty)}(n) = \nu_{\infty}^{(1)}(n) = \nu(n)$ . Since  $\bar{\Gamma}(n, 2n)$  is a normal subgroup of  $\bar{\Gamma}(1, 2)$ , we have

$$R(\Gamma(1, 2)) = (\bar{\Gamma}(1, 2) / \bar{\Gamma}(n, 2n)) \backslash R(\Gamma(n, 2n)).$$

Futhermore, we put

$$\begin{aligned} \varepsilon &= \varepsilon_{R(\Gamma(n, 2n))}, \\ \bar{\varepsilon} &= \bar{\varepsilon}_{R(\Gamma(n, 2n)), \bar{\Gamma}(1, 2) / \bar{\Gamma}(n, 2n)}. \end{aligned}$$

Then it is easy to show that the number of the Weierstrass points on  $R(\Gamma(n, 2n))$  is given by

$$\bar{\varepsilon}(\pi(\sqrt{-1})) \frac{\mu}{2} + \bar{\varepsilon}(\pi(\infty)) \nu + \bar{\varepsilon}(\pi(1)) \nu + \sum_P' \bar{\varepsilon}(P) \mu \quad (1)$$

where  $\sum_P'$  means that  $P$  runs over all points on  $R(\Gamma(1, 2))$  except for  $\pi(1)$ ,  $\pi(\infty)$  and  $\pi(\sqrt{-1})$ . Suppose that  $R(\Gamma(n, 2n))$  is hyperelliptic. Then (1) is equal to  $2g(\Gamma(n, 2n)) + 2$ . Therefore

$$8 = \nu(n) \cdot (nx + 2y + 2z + 2nw + 4 - n) \quad (2)$$

must have an integral solution on  $x, y, z$  and  $w$ . But  $1 / \prod_{p|n} \left(1 - \frac{1}{p^2}\right) \leq \zeta(2)$  where  $\zeta(s)$  is Riemann's zeta function. Hence  $\nu(n) \geq n^2 / 2\zeta(2) = 3n^2 / \pi^2$ . Hence, in the case of  $n \geq 7$ , we have  $\nu(n) > 8$ . In the case of  $n = 5$ , we see that  $\nu(5) = 12$ . Therefore  $\nu(n) > 8$  for all our cases. This is a contradiction.

Next, let  $n$  be an even integer, Our proof is similar to the first case. We put,

$$\begin{aligned} \pi &= \pi_{\Gamma(1)}, \\ \mu &= \mu(n) = [\bar{\Gamma}(1) : \bar{\Gamma}(n, 2n)] = 2n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right), \\ \nu &= \nu(n) = n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right), \\ \varepsilon &= \varepsilon_{R(\Gamma(n, 2n))}, \\ \bar{\varepsilon} &= \bar{\varepsilon}_{R(\Gamma(n, 2n)), \bar{\Gamma}(1) / \bar{\Gamma}(n, 2n)}. \end{aligned}$$

Then the number of the Weierstrass points on  $R(\Gamma(n, 2n))$  is given by

$$\bar{\varepsilon}(\pi(\sqrt{-1})) \cdot \frac{\mu}{2} + \bar{\varepsilon}(\pi(e^{2\pi\sqrt{-1/3}})) \cdot \frac{\mu}{3} + \bar{\varepsilon}(\pi(\infty)) \cdot \nu + \sum'_P \bar{\varepsilon}(P) \cdot \mu \tag{3}$$

where  $\sum'_P$  means that  $P$  runs over all points on  $R(\Gamma(1))$  except for  $\pi(\infty)$ ,  $\pi(\sqrt{-1})$  and  $\pi(e^{2\pi\sqrt{-1/3}})$ . Suppose that  $R(\Gamma(n, 2n))$  is hyperelliptic. Then

$$12 = \nu(n) \cdot (3nx + 2ny + 3z + 6nw + 3 - n) \tag{4}$$

must have an integral solution on  $x, y, z$  and  $w$ . By the similar discussion to the first proof, we have  $\nu(n) > 12$  if  $n \geq 6$ . In the case of  $n=4$ , since  $\nu(4)=12$ ,  $2=12x + 8y + 3z + 24w$  must have an integral solution on  $x, y, z$  and  $w$ . But it is impossible. In any case, (4) has no integral solution. This is a contradiction. *Q.E.D.*

**COROLLARY 5.** *On  $R(\Gamma(4, 8))$ , its cusps coincide with its Weierstrass points, and the gap sequence at any Weierstrass points is  $\{1, 2, 5\}$ .*

**PROOF.** By the formula (3) in the proof of Theorem 4, the number of the Weierstrass points on  $R(\Gamma(4, 8))$  is given by

$$4(12\bar{\varepsilon}(\pi(\sqrt{-1})) + 8\bar{\varepsilon}(\pi(e^{2\pi\sqrt{-1/3}})) + 3\bar{\varepsilon}(\pi(\infty)) + 24\sum'_P \bar{\varepsilon}(P)).$$

Since the genus of  $R(\Gamma(4, 8))$  is 3, the number of the Weierstrass points is at most 24. Therefore  $\bar{\varepsilon}(\pi(\infty))$  is 1 and  $\bar{\varepsilon}(P)$  is 0 for any other point  $P$ . Since the number of the Weierstrass points is 12, all gap sequences are  $\{1, 2, 5\}$ . *Q.E.D.*

### § 3. The answer to the second problem

The main purpose in this section is to prove the following theorem.

**THEOREM 6.** *Let  $n$  be an even integer greater than or equal to 4. Then any cusp of  $R(\Gamma(n, 2n))$  is a Weierstrass point. Furthermore the Weierstrass points coincide with the cusps provided that  $n=4$ .*

Before starting our proof, we state two lemmas without proofs.

**LEMMA 7.** (Accola [1], Komiya [4]) *Let  $R \rightarrow R'$  be an abelian covering of Riemann surfaces of type  $(p, p)$  where  $p$  is a prime integer, and let  $R$  and  $R'$  have the genera  $g$  and  $g'$  respectively, and the genera of all intermediate Riemann surfaces between  $R$  and  $R'$  be  $g_1, \dots, g_{p+1}$ . Then we have the following formula:*

$$\sum_{j=1}^{p+1} g_j = g + pg'.$$

**LEMMA 8.** (Schoeneberg [9]) *Let  $R$  be a Riemann surface of genus  $g \geq 2$ , and  $\sigma$  be an automorphism of order  $n$ . Suppose that  $P$  is a fixed point of  $\sigma$  on  $R$ , and*

that the gap sequence at  $P$  is  $\{n_1, \dots, n_g\}$ . Then the genus of  $\langle \sigma \rangle \backslash R$  coincides with the cardinality of the set

$$\{j | n_j \equiv 0 \pmod{n} \quad 1 \leq j \leq g\}.$$

Here  $\langle \sigma \rangle$  means the automorphism group of  $R$  generated by  $\sigma$ .

PROOF OF THEOREM 6. In the case of  $n=4$ , the statement of the theorem was already proven by Corollary 5. So, from now on, we assume that  $n$  is an even integer greater than 4. Since each element of  $\bar{\Gamma}(1)/\bar{\Gamma}(n, 2n)$  induces a permutation on the set of all cusps on  $R(\Gamma(n, 2n))$  and the group of these permutations is transitive, it suffices to prove that some cusp is a Weierstrass point. By Lemma 2, we see that  $R(\Gamma(n, 2n)) \rightarrow R(\Gamma(n))$  is an abelian covering of type  $(2, 2)$ . We put,

$$\sigma_1 = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix},$$

$$H_i = \Gamma(n, 2n) + \Gamma(n, 2n)\sigma_i \quad (i=1, 2, 3).$$

Then we see the following diagram and relations:

$$\begin{array}{c} \text{Galois group of } R(\Gamma(n, 2n)) \text{ over } R(\Gamma(n)) \\ \parallel \\ \bar{\Gamma}(n)/\bar{\Gamma}(n, 2n) \\ \swarrow \quad \downarrow \quad \searrow \\ H_1/\bar{\Gamma}(n, 2n) \quad H_2/\bar{\Gamma}(n, 2n) \quad H_3/\bar{\Gamma}(n, 2n) \\ \swarrow \quad \downarrow \quad \searrow \\ \{1\} \end{array}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} H_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = H_3, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} H_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = H_3.$$

Hence  $R(H_i)$  ( $i=1, 2, 3$ ) have the same genus as one another, say  $g'$ . By Lemma 7, we get the equation  $3g' = g(\Gamma(n, 2n)) + 2g(\Gamma(n))$ , so

$$g' = 1 + \frac{1}{12} n^2 (n-4) \prod_{p|n} \left(1 - \frac{1}{p^2}\right).$$

Furthermore, we see that the automorphism  $\bar{\sigma}_i \pmod{\bar{\Gamma}(n, 2n)}$  is of order 2 and that it has a fixed point which is a cusp. Therefore, by Lemma 8, it suffices to



show that the inequality

$$g(\Gamma(n, 2n)) - 2g' = \frac{1}{6}n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) - 1 \geq 2$$

holds. It is clear that the above inequality holds if  $n \geq 6$ . Finally,

$$\begin{aligned} & 2g(\Gamma(n, 2n)) + 2 - (\text{the number of the cusps on } R(\Gamma(n, 2n))) \\ &= 4 + \frac{1}{3}n^2(n-6) \prod_{p|n} \left(1 - \frac{1}{p^2}\right) > 0 \end{aligned}$$

under the condition  $n \geq 6$ . This implies that there exists a Weierstrass point which is not a cusp under the condition  $n \geq 6$ . Q.E.D.

Finally we shall state two remarks. The first remark is a counter-example against the conclusion of Theorem 6 for an odd integer case, and the second remark concerns itself with Komiya and Kuribayashi's result.

REMARK 1. If  $n$  is an odd integer, then the conclusion of Theorem 6 is not always true. In fact, we saw that  $\sigma = \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}$  acts on  $R(\Gamma(2n))$  as an automorphism of order 2 having a fixed point  $\pi_{\Gamma(2n)}(1)$  and that  $R(\Gamma(n, 2n)) = \langle \sigma \rangle \backslash R(\Gamma(2n))$ . On the other hand, the gap sequence at  $\pi_{\Gamma(10)}(1)$  on  $R(\Gamma(10))$  is

$$\{1, 2, \dots, 8, 9, 11, 13, 17, 19\} \quad (\text{Lewittes [7]}).$$

Hence the gap sequence at  $\pi_{\Gamma(5,10)}(1)$  on  $R(\Gamma(5,10))$  is

$$\{1, 2, 3, 4\}.$$

Therefore  $\pi_{\Gamma(5,10)}(1)$  is not a Weierstrass point, but it is a cusp.

REMARK 2. Recently, A. Kuribayashi and K. Komiya discovered that there are exactly two Riemann surfaces up to isomorphism which are of genus 3 having exactly 12 Weierstrass points [5]. Our  $R(\Gamma(4, 8))$  is of genus 3 and has exactly 12 Weierstrass points. It coincides with

$$X^4 + Y^4 + Z^4 = 0 \quad \text{in } \mathbf{P}^2 \quad (\#)$$

in their paper (cf. Igusa [3]).

On the other hand, we see that  $\Gamma(4, 8)$  acts on  $H$  fixed point free. Furthermore, we saw that the cusps of  $R(\Gamma(4, 8))$  coincides with the Weierstrass points and that  $\Gamma(4, 8)$  is a normal subgroup of  $\Gamma(1)$ . Therefore the automorphism group of the curve (#) is isomorphic to  $\bar{\Gamma}(1)/\bar{\Gamma}(4, 8)$ .

**References**

- [ 1 ] Accola, R.D.M., Riemann surfaces with automorphisms, Proc. of the Amer. Math. Soc. **21**, (1969), 477-482.
- [ 2 ] Atkin, A.O.L., Weierstrass points at cusps of  $\Gamma_0(n)$ , Ann. of Math. **85**, (1967), 42-44.
- [ 3 ] Igusa, J., On the graded ring of theta constants, I, II, Amer. J. Math. **86**, (1964), 219-246; *ibid.* **88**, (1966), 221-236.
- [ 4 ] Komiya, K., Genus formulas in elementary abelian extension of function field, Memoirs of Yamanashi Univ. **19**, (1969), 1-3.
- [ 5 ] Kuribayashi, A. and Komiya, K., On Weierstrass points of non-hyperelliptic compact Riemann surfaces, Hiroshima Math. J. **7**, (1977), 743-768.
- [ 6 ] Lehner, J. and Newmann, M., Weierstrass points of  $\Gamma_0(n)$ , Ann. of Math. **79**, (1964), 360-368.
- [ 7 ] Lewittes, J., Gaps at Weierstrass points for the modular group, Bull. Amer. Math. Soc. **69**, (1963), 578-582.
- [ 8 ] Petersson, H., Zwei Bemerkungen über die Weierstrasspunkte der Kongruenzgruppen, Arch. Math. **2**, (1950), 246-250.
- [ 9 ] Schoeneberg, B., Über die Weierstrass-Punkte in den Körpern der elliptischen Modul-funktionen, Abh. Math. Sem. Hamburg **17**, (1951), 104-111.
- [10] Shimura, G., Introduction to the arithmetic theory of automorphic functions, Iwanami-Shoten and Princeton Univ. Press, (1971).

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