On Weighted Exponential Distribution and its Length Biased Version

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Abstract

In this paper we consider the weighted exponential distribution proposed by Gupta and Kundu (2009) and discuss its various reliability properties. We further consider the length biased version of the weighted exponential distribution, and discuss different properties and inferential issues. The maximum likelihood estimators of the unknown parameters of the proposed length biased weighted exponential distribution has been addressed. One data set has been analyzed for illustrative purposes.

Key Words and Phrases: maximum likelihood estimators; increasing failure rate; mean residual life; length biased distribution; moment generating function.

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1 Introduction

Azzalini (1985) proposed a novel approach to introduce an additional parameter to a normal distribution. This additional parameter incorporates skewness to the symmetric normal distribution. This distribution is well known in the statistical literature as the skew-normal distribution, and it has received considerable attention in the last two decades. Kundu and Gupta (2009) introduced a new class of weighted exponential distribution using the idea of Azzalini (1985) and it can be defined as follows: A random variable X is said to have a weighted exponential distribution with the shape parameter α and scale parameter λ if it has the following probability density function (PDF)

$$f_X(x, \alpha, \lambda) = \frac{\alpha + 1}{\alpha} \lambda e^{-\lambda x} \left(1 - e^{-\alpha\lambda x}\right), \quad \text{if} \quad x > 0, \tag{1}$$

and zero otherwise. From now on a random variable with the PDF (1) will be denoted by WE(α, λ). The WE distribution of Gupta and Kundu (2009) has several desirable properties. Although exponential distribution is not a member of this family of distributions, exponential distribution can be obtained as limiting distribution from the WE class. Recently, this model has received some attention in the statistical literature because of its flexibility and simplicity, see for example Shakhatreh (2012), Roy and Adnan (2012), Al-Mutairi et al. (2011), Farahani and Khorram (2014) and the references cited therein.

The main aim of this paper is two fold. First we consider the WE model and discuss several reliability properties of the model. Further we consider the length biased version of the WE model. Length biased model plays an important role in different area of statistical applications. If Y is a positive random variable with the PDF $f_Y(y)$ for $y > 0$, and with finite mean μ, then length biased (LB) or size biased version of Y is a random variable T, with the PDF;

$$f_T(t) = \frac{t f_Y(t)}{\mu}; \quad t > 0. \tag{2}$$

The distribution of T is the LB version of the distribution of Y. The mean of the original distribution has been incorporated in the LB version of the PDF so that it becomes a valid density function. Therefore, the LB distribution does not introduce any extra parameter in the model. The LB distribution has been used quite extensively in different areas such as biometry, ecology, environmental sciences, reliability and survival analysis. An extensive review of the LB distributions and their applications can be found in Gupta and Kirmani (1990). In this paper we develop the length biased weighted exponential (LBWE) distribution and discuss several properties and related inferential issues. One data analysis has been performed for illustrative purposes.

Rest of the paper is organized as follows. In Section 2, we briefly discuss the WE model and develop several reliability properties. In Section 3 we introduce the LBWE distribution and discuss several properties. The inferential issues and the data analysis are discussed in Section 4. The three-parameter LBWE is proposed in Section 5, and finally we conclude the paper in Section 6.

2 Weighted Exponential Model and its Properties

2.1 Weighted Exponential Model

A random variable X follows WE(α, λ) if it has the PDF (1). The PDF of WE distribution is unimodal and it has increasing hazard function for all values of α. Since the hazard function is always an increasing function this is suitable for modelling lifetime data when wear-out or ageing is present. If $\lambda = 1$, the moment generating function (MGF) of X can be written for $-1 < t < 1$ as

$$M_X(t) = \left(1 - \frac{t}{1+\alpha}\right)^{-1} (1-t)^{-1}. \tag{3}$$

Using the MGF, the mean (μ), variance (σ^2), coefficient of variation (CV) and skewness (β_1) can be obtained as

$$\mu = 1 + \frac{1}{1+\alpha}, \ \sigma^2 = 1 + \frac{1}{(1+\alpha)^2}, \ \mathrm{CV} = \sqrt{\left(1 - \frac{2(1+\alpha)}{(2+\alpha)^2}\right)}, \ \beta_1 = \sqrt{\frac{4((1+\alpha)^3+1)^2}{((1+\alpha)^2+1)^3}}.$$

Both the CV and skewness are functions of α, and the CV increases from $1/\sqrt{2}$ to 1, whereas skewness increases from $\sqrt{2}$ to 2. The following representation can be very useful for generating WE random variable and also developing several other properties also. Suppose $X \sim \mathrm{WE}(\alpha, \lambda)$, then

$$X = U + V, \tag{4}$$

here both U and V are exponential random variables with mean $1/\lambda$ and $1/\lambda(1+\beta)$, respectively, and they are independently distributed. For various other properties and for different physical interpretations, the readers are referred to the original paper of Gupta and Kundu (2009).

2.2 Different Reliability Properties of WE Distribution

The following theorem shows that WE distribution has the ILR (increasing in likelihood ratio) property. Let us recall that a positive random variable X with PDF $f(x)$, is said to be ILR if $\ln f(x)$ is concave in x.

Theorem 1: If $X \sim \mathrm{WE}(\alpha, \lambda)$, then X has the ILR property.

Proof: Note that X has ILR property, if and only if the probability density function of X is log concave. Now, the log density function of X is given by

$$\ln f_X(x; \alpha, \lambda) \quad = \quad \ln\left(\frac{\alpha+1}{\alpha}\lambda\right) - \lambda x + \ln\left(1 - e^{-\alpha\lambda x}\right). \tag{5}$$

Differentiating $\ln f_X(x; \alpha, \lambda)$ with respect to x, we get

$$\frac{d}{dx}\left[\ln f_X(x; \alpha, \lambda)\right] \quad = \quad -\lambda + \frac{\alpha\lambda e^{-\alpha\lambda x}}{1 - e^{-\alpha\lambda x}}, \tag{6}$$

which is a decreasing function in x. Hence, X has ILR property. ∎

The following theorem shows that WE distribution preserves the likelihood ratio ordering. Let us recall that a random variable Y_1 is said to be larger than another random variable Y_2 in likelihood ratio ordering (written as $Y_1 \geq_{LR} Y_2$) if, for all $x \geq 0$, $f_{Y_1}(x)/f_{Y_2}(x)$ is an increasing function in x.

THEOREM 2: Let $X_1 \sim \text{WE}(\alpha_1, \lambda_1)$ and $X_2 \sim \text{WE}(\alpha_2, \lambda_2)$, then $X_1 \geq_{LR} (\leq_{LR}) X_2$ provided

(i) $\lambda_2 \geq (\leq) \lambda_1$,

(ii) $\dfrac{e^{\alpha_2\lambda_2 x} - 1}{\alpha_2\lambda_2} \geq (\leq) \dfrac{e^{\alpha_1\lambda_1 x} - 1}{\alpha_1\lambda_1}, \qquad \text{for all} \quad x > 0.$

PROOF: $X_1 \geq_{LR} (\leq_{LR}) X_2$ if and only if $f_{X_1}(x, \alpha_1, \lambda_1)/f_{X_2}(x, \alpha_2, \lambda_2)$ is an increasing (decreasing) function of x. Now,

$$\begin{aligned}\frac{f_{X_1}(x; \alpha_1, \lambda_1)}{f_{X_2}(x; \alpha_2, \lambda_2)} &= \frac{\alpha_2(\alpha_1 + 1)\lambda_1 \left(1 - e^{-\alpha_1\lambda_1 x}\right)}{\alpha_1(\alpha_2 + 1)\lambda_2 \left(1 - e^{-\alpha_2\lambda_2 x}\right)} e^{(\lambda_2 - \lambda_1)x} \\ &= A(x), \ \textit{say}.\end{aligned}$$

$A(x)$ is an increasing (decreasing) function of x, if $\lambda_2 \geq (\leq) \lambda_1$ and $A_1(x) = \dfrac{1 - e^{-\alpha_1\lambda_1 x}}{1 - e^{-\alpha_2\lambda_2 x}}$ is an increasing (decreasing) function of x. Differentiating $A_1(x)$ with respect to x, we get

$$\begin{aligned}A_1'(x) &= \frac{\alpha_1\lambda_1 e^{-\alpha_1\lambda_1 x}(1 - e^{-\alpha_2\lambda_2 x}) - \alpha_2\lambda_2 e^{-\alpha_2\lambda_2 x}(1 - e^{-\alpha_1\lambda_1 x})}{(1 - e^{-\alpha_2\lambda_2 x})^2} \\ &\overset{\text{sign}}{=} \alpha_1\lambda_1(e^{\alpha_2\lambda_2 x} - 1) - \alpha_2\lambda_2(e^{\alpha_1\lambda_1 x} - 1) \\ &\geq (\leq) \ 0,\end{aligned}$$

provided $\dfrac{e^{\alpha_2\lambda_2 x} - 1}{\alpha_2\lambda_2} \geq (\leq) \dfrac{e^{\alpha_1\lambda_1 x} - 1}{\alpha_1\lambda_1}$, for all $x > 0$. ∎

Note that $\alpha_1 = \alpha_2$, then $X_1 \geq_{LR} (\leq_{LR}) X_2$ if $\lambda_2 \geq (\leq) \lambda_1$. The following example shows that if condition (i) of Theorem 2 not satisfied then the theorem may not hold.

Example 1: Let $X_1 \sim$ WE(1,4) and $X_2 \sim$ WE(3,2). The PDF of X_1 is given by

$$f_{X_1}(x;1,4)) \quad = \quad 8e^{-4x}(1-e^{-4x}), \quad x>0,$$

and the PDF of X_2 is given by

$$f_{X_2}(x;3,2) \quad = \quad \frac{8}{3}e^{-2x}(1-e^{-6x}), \quad x>0.$$

Clearly, condition (ii) of the Theorem 2 holds but not condition (i). Now, for all $x>0$

$$\begin{aligned} \frac{f_{X_1}(x;1,4)}{f_{X_2}(x;3,2)} &= \frac{3e^{-4x}(1-e^{-4x})}{e^{-2x}(1-e^{-6x})} \\ &= p_1(x)(\text{say}). \end{aligned}$$

We see that $p_1(1)=0.39954$, $p_1(2)=0.054927$, which shows that $\frac{f_{X_1}(x;1,4)}{f_{X_2}(x,3,2)}$ is not an increasing function of $x>0$. Hence, $X \geq_{LR} Y$ does not hold. ∎

The following examples shows that if the condition (ii) of Theorem 2 does not satisfy, then the theorem may or may not hold.

Example 2: Let $X_1 \sim$ WE(3,3) and and $X_2 \sim$ WE(2,4). The PDF of X_1 is given by

$$f_{X_1}(x;3,3) \quad = \quad 4e^{-3x}(1-e^{-9x}), \quad x>0,$$

and the PDF of X_2 is

$$f_{X_2}(x) \quad = \quad 6e^{-4x}(1-e^{-8x}), \quad x>0.$$

Clearly, condition (i) of the Theorem 2 holds, but not condition (ii). Now, for all $x>0$

$$\begin{aligned} \frac{f_{X_1}(x;3,3)}{f_{X_2}(x;2,4)} &= \frac{2e^{-3x}(1-e^{-9x})}{3e^{-4x}(1-e^{-8x})} \\ &= \frac{2(1-u^9)}{3u(1-u^8)} = p_2(u) \quad (\text{say}), \end{aligned}$$

where $u=e^{-x}$. Observe that

$$\frac{d}{du}p_2(u) = \frac{9u^8-8u^9-1}{u^2(1-u^8)^2}.$$

Since $9u^8 - 8u^9 - 1$ is an increasing function of $u \in (0,1)$, with maximum value less than 0, $p_2(u)$ is a decreasing function in u. Therefore, $\dfrac{f_{X_1}(x;3,3)}{f_{X_2}(x;2,4)}$ is an increasing function of x. Hence, $X_1 \geq_{LR} X_2$. ∎

EXAMPLE 3: Let $X_1 \sim$ WE(8, 1/4) and $X_2 \sim$ WE(1/4,1/2). The PDF of X_1 and X_2 are given by

$$f_{X_1}(x;8,1/4) \quad = \quad \frac{9}{32}e^{-x/4}(1-e^{-2x}), \qquad x>0,$$

and

$$f_{X_2}(x) \quad = \quad \frac{5}{2}e^{-x/2}(1-e^{-x/8}), \qquad x>0,$$

respectively. Clearly, condition (i) of the Theorem 2 holds but not condition (ii). Now, for all $x > 0$

$$\begin{aligned} \frac{f_{X_1}(x,8,1/4)}{f_{X_2}(x,1/4,1/2)} &= \frac{9e^{-x/4}(1-e^{-2x})}{80e^{-x/2}(1-e^{-x/8})} \\ &= p_3(x), \quad \text{(say)}. \end{aligned}$$

We see that $p_3(1) = 1.0630$, $p_3(3) = 0.75970$, and $p_3(6) = 0.95558$, which shows that $p_3(x)$ is not a monotone function of x. Hence, $X_1 \geq_{LR} X_2$ does not hold. ∎

The following theorem shows that the WE distribution preserves the up likelihood ratio ordering. Let us recall that a random variable Y_1 is said to be smaller than another random variable Y_2 in up likelihood ratio ordering (written as $Y_1 \leq_{LR\uparrow} Y_2$) if, for all $x \geq 0$, $f_{Y_1}(x+t)/f_{Y_2}(x)$ is an increasing function in x.

THEOREM 3: Let $X_1 \sim$ WE(α_1,λ_1) and $X_2 \sim$ WE(α_2,λ_2). Then $X_1 \leq_{LR\uparrow} X_2$ provided

(i) $\lambda_1 \geq \lambda_2$,

(ii) $\dfrac{e^{\alpha_1\lambda_1 x}-1}{\alpha_1\lambda_1} \geq \dfrac{e^{\alpha_2\lambda_2 x}-1}{\alpha_2\lambda_2}$, for all $x > 0$,

(iii) X_1 has ILR property.

Proof: Note that $X_1 \leq_{LR\uparrow} X_2$ if and only if $f_{X_1}(t+x;\alpha_1,\lambda_1)/f_{X_2}(x;\alpha_2,\lambda_2)$ is a decreasing function in $x>0$ for all $t>0$. Now,

$$\frac{f_{X_1}(t+x;\alpha_1,\lambda_1)}{f_{X_2}(x;\alpha_2,\lambda_2)} = \frac{f_{X_1}(x;\alpha_1,\lambda_1)f_{X_1}(t+x;\alpha_1,\lambda_1)}{f_{X_2}(x;\alpha_2,\lambda_2)f_{X_1}(x;\alpha_1,\lambda_1)}.$$

Since, X is ILR, then $f_{X_1}(t+x;\alpha_1,\lambda_1)/f_{X_1}(x;\alpha_1,\lambda_1)$ is a decreasing function in $x>0$ for all $t>0$. Again, we see that

$$\frac{f_{X_1}(x;\alpha_1,\lambda_1)}{f_{X_2}(x;\alpha_2,\lambda_2)} = \frac{\alpha_2(\alpha_1+1)\lambda_1\left(1-e^{-\alpha_1\lambda_1 x}\right)}{\alpha_1(\alpha_2+1)\lambda_2\left(1-e^{-\alpha_2\lambda_2 x}\right)}e^{(\lambda_2-\lambda_1)x}.$$

$\frac{f_{X_1}(x;\alpha_1,\lambda_1)}{f_{X_2}(x;\alpha_2,\lambda_2)}$ is a decreasing function in x if $\lambda_1 \geq \lambda_2$ and $A_2(x) = \dfrac{1-e^{-\alpha_1\lambda_1 x}}{1-e^{-\alpha_2\lambda_2 x}}$ is a decreasing function in x. Differentiating $A_2(x)$ with respect to x, we get

$$\begin{aligned} A_2'(x) &= \frac{\alpha_1\lambda_1 e^{-\alpha_1\lambda_1 x}\left(1-e^{-\alpha_2\lambda_2 x}\right) - \alpha_2\lambda_2 e^{-\alpha_2\lambda_2 x}\left(1-e^{-\alpha_1\lambda_1 x}\right)}{\left(1-e^{-\alpha_2\lambda_2 x}\right)^2} \\ &\overset{\text{sign}}{=} \alpha_1\lambda_1(e^{-\alpha_1\lambda_1 x}-1) - \alpha_2\lambda_2(e^{-\alpha_2\lambda_2 x}-1) \leq 0, \end{aligned}$$

provided $\dfrac{e^{\alpha_1\lambda_1 x}-1}{\alpha_1\lambda_1} \geq \dfrac{e^{\alpha_2\lambda_2 x}-1}{\alpha_2\lambda_2}$, for all $x>0$. ∎

Note that when $\alpha_1 = \alpha_2$, then $X_1 \leq_{LR\uparrow} X_2$ if $\lambda_1 \geq \lambda_2$ and X is ILR. The following example shows that if condition (i) of Theorem 3 does not satisfy, then the theorem may not hold.

Example 4: Let $X_1 \sim$ WE(2,1) and $X_2 \sim$ WE(1,3). Then the PDF of X_1 and X_2 are

$$f_{X_1}(x;2,1) = 3e^{-2(x+t)}(1-e^{-4(x+t)}), \quad x>0,$$

and

$$f_{X_2}(x;1,3) = 6e^{-3x}(1-e^{-3x}), \quad x>0,$$

respectively. Clearly, condition (ii) of the Theorem 3 holds, but not condition (i). Now, for all $x > 0$ and $t > 0$,

$$\frac{f_{X_1}(x+t;2,2)}{f_{X_2}(x;1,3)} = \frac{e^{-2(x+t)}(1-e^{-4(x+t)})}{2e^{-3x}(1-e^{-3x})} = p_4(x,t), \quad \text{(say)}.$$

We see that $p_4(0.1,1) = 0.569995$, $p_4(0.4,1) = 0.287848$, $p_4(1.5,1) = 0.613316$, which shows that $p_4(x,t)$ is not a monotone function of x for fixed t. Hence, $X_1 \leq_{LR\uparrow} X_2$ does not hold. ∎

The following examples shows that if condition (ii) of Theorem 3 does not satisfy, then the theorem may or may not hold.

Example 5: Let $X_1 \sim$ WE(1,3) and $X_2 \sim$ WE(2,2). Then the PDF of X_1 is given by

$$f_{X_1}(x;1,3) = 6e^{-3x}(1-e^{-3x}), \quad x > 0,$$

and the PDF of X_2 is given by

$$f_{X_2}(x;2,2) = 3e^{-2x}(1-e^{-4x}), \quad x > 0.$$

Clearly, condition (i) of the Theorem 3 holds, but not condition (ii). Now, for all $x > 0$ and $t > 0$

$$\begin{aligned}\frac{f_{X_1}(x+t;1,3)}{f_{X_2}(x,2,2)} &= \frac{2e^{-3(x+t)}(1-e^{-3(x+t)})}{e^{-2x}(1-e^{-4x})} \\ &= \frac{2u^3v^3(1-u^3v^3)}{1-u^4} \\ &= p_5(u,v), \quad \text{(say)},\end{aligned}$$

where $u = e^{-x}$ and $v = e^{-t}$. It easily follows that for fixed $0 \leq v \leq 1$, $p_5(u,v)$ is an increasing function of $0 \leq u \leq 1$. Hence, $X_1 \leq_{LR\uparrow} X_2$. ∎

Example 6: Let $X_1 \sim$ WE(1/4, 1/2) and $X_2 \sim$ WE(6, 1/3). Then the PDF of X_1 is given by

$$f_{X_1}(x;1/4,1/2) = \frac{5}{2}e^{-x/2}(1-e^{-x/8}), \quad x > 0,$$

and the PDF of X_2 is given by

$$f_{X_2}(x; 6, 1/3) = \frac{7}{18} e^{-x/3}(1 - e^{-2x}), \quad x > 0,$$

Clearly, condition (i) of the Theorem 3 holds, but not condition (ii). Now, for all $x > 0$ $t > 0$

$$\begin{aligned} \frac{f_{X_1}(x+t, \frac{1}{4}, \frac{1}{2})}{f_{X_2}(x, 6, \frac{1}{3})} &= \frac{45e^{-(x+t)/2}(1 - e^{-(x+t)/8})}{7e^{-x/3}(1 - e^{-2x})} \\ &= p_6(x,t), \quad \text{(say)}. \end{aligned}$$

We see that $p_6(0.5, 1) = 0.97028$, $p_6(1, 1) = 0.84440$, $p_6(3, 1) = 0.93285$, which shows that $p_6(x,t)$ is not a monotone function of x for fixed t, hence $X_1 \leq_{LR\uparrow} X_2$ does not hold. ∎

The following theorem shows that the WE distribution preserves the down likelihood ratio ordering. Note that a random variable Y_1 is said to be larger than another random variable Y_2 in down likelihood ratio ordering (written as $Y_1 \geq_{LR\downarrow} Y_2$) if, for all $t \geq 0$, $f_{Y_1}(x)/f_{Y_2}(x+t)$ is an increasing function in x.

Theorem 4: Let $X_1 \sim \text{WE}(\alpha_1, \lambda_1)$ and $X_2 \sim \text{WE}(\alpha_2, \lambda_2)$. Then $X_1 \geq_{LR\downarrow} X_2$ provided

(i) $\lambda_1 \leq \lambda_2$,

(ii) $\dfrac{e^{\alpha_1\lambda_1 x} - 1}{\alpha_1\lambda_1} \leq \dfrac{e^{\alpha_2\lambda_2 x} - 1}{\alpha_2\lambda_2}$, for all $x > 0$,,

(iii) X is ILR.

Proof: The proof can be obtained along the same line as the proof of Theorem 3, hence it is omitted. ∎

When $\alpha_1 = \alpha_2$, then $X_1 \geq_{LR\downarrow} X_2$ if $\lambda_1 \leq \lambda_2$ and X is ILR. The following example shows that if condition (i) of Theorem 4 does not satisfy, then the theorem may not hold.

Example 7: Let us consider Example 1. Clearly, condition (ii) of the Theorem 4 holds, but not condition (i). Now, for all $x > 0$, $t > 0$

$$\begin{aligned}\frac{f_X(x,1,4)}{f_Y(x+t,3,2)} &= \frac{3e^{-4x}(1-e^{-4x})}{e^{-2(x+t)}(1-e^{-6(x+t)})}\\ &= p_7(x,t),\ \ say.\end{aligned}$$

We see that $p_7(0.1,1) = 5.99149$, $p_7(0.5,1) = 7.05208$, $p_7(1,1) = 2.9451$, which shows that $p_7(x,t)$ is not a monotone function of x for a fixed t. Thus, $X_1 \geq_{LR\downarrow} X_2$ does not hold. Hence, condition (i) of Theorem 4 cannot be dropped. ∎

The following example shows that if condition (ii) of Theorem 4 does not satisfy, then the theorem may or may not hold.

Example 8: Let us consider Example 2. Clearly, condition (i) of the Theorem 4 satisfies, but not condition (ii). Now, for all $x > 0$, $t > 0$

$$\frac{f_X(x,3,3)}{f_Y(x+t,4,2)} = \frac{2e^{-3x}(1-e^{-9x})}{3e^{-4(x+t)}(1-e^{-8(x+t)})} = \frac{2(1-u^9)}{uv^4(1-u^8v^8)} = p_8(u,v), \quad \text{(say)}, \tag{7}$$

where $u = e^{-x}$ and $v = e^{-t}$. Since for fixed $0 \leq v \leq 1$, $p_5(u,v)$ is a decreasing function in $0 \leq u \leq 1$, $X_1 \geq_{LR\downarrow} X_2$. ∎

Example 9: Let us consider Example 3. Clearly, condition (i) of the Theorem 4 satisfies, but not condition (ii). Now, for all $x > 0$, $t > 0$

$$\begin{aligned}\frac{f_X(x,8,\frac{1}{4})}{f_Y(x+t,\frac{1}{4},\frac{1}{2})} &= \frac{9e^{-x/4}(1-e^{-2x})}{80e^{-(x+t)/2}(1-e^{-(x+t)/8})}\\ &= p_9(x),\ \ say.\end{aligned}$$

We see that $p_9(1,0.5) = 0.93805$, $p_9(2.5,0.5) = 0.857198$, and $p_9(4,0.5) = 0.91239$, which shows that $p_9(x,0.5)$ is not a monotone function in x for a fixed t, hence, $X_1 \geq_{LR\downarrow} X_2$ does not hold. ∎

The following theorem shows that the WE distribution preserves the up hazard rate ordering. A random variable X_1 is said to be smaller than another random variable X_2 in up hazard rate order (written as $X_1 \leq_{HR\uparrow} X_2$) if, $X_1 - x \leq_{HR} X_2$, for all $x \geq 0$, or equivalently, if $\bar{F}_{X_1}(x+t)/\bar{F}_{X_2}(x)$ is a decreasing function in x.

THEOREM 5: Let $X_1 \sim \text{WE}(\alpha_1, \lambda_1)$ and $X_2 \sim \text{WE}(\alpha_2, \lambda_2)$. Then $X_1 \leq_{HR\uparrow} X_2$ provided

(i) $\lambda_1 \geq \lambda_2$,

(ii) $\frac{(\alpha_1+1)e^{\alpha_1\lambda_1 x}-1}{\alpha_1\lambda_1} \geq \frac{(\alpha_2+1)e^{\alpha_2\lambda_2 x}-1}{\alpha_2\lambda_2}$, for all $x > 0$,

(iii) X is IFR.

PROOF: Let us recall that $X_1 \leq_{HR\uparrow} X_2$ if and only if $\bar{F}_X(t+x)/\bar{F}_Y(x)$ is a decreasing function in $x > 0$ for all $t > 0$. Now,

$$\frac{\bar{F}_{X_1}(t+x)}{\bar{F}_{X_2}(x)} = \frac{\bar{F}_{X_1}(x)\bar{F}_{X_1}(t+x)}{\bar{F}_{X_2}(x)\bar{F}_{X_1}(x)}.$$

Since, X_1 is IFR, then $\bar{F}_{X_1}(t+x)/\bar{F}_{X_1}(x)$ is a decreasing function in $x > 0$ for all $t > 0$. Again, we see that

$$\frac{\bar{F}_{X_1}(x)}{\bar{F}_{X_2}(x)} = \frac{\alpha_2\left(\alpha_1 + 1 - e^{-\alpha_1\lambda_1 x}\right)}{\alpha_1\left(\alpha_2 + 1 - e^{-\alpha_2\lambda_2 x}\right)} e^{(\lambda_2-\lambda_1)x}.$$

$\frac{\bar{F}_{X_1}(x)}{\bar{F}_{X_2}(x)}$ is a decreasing function in x if $\lambda_1 \geq \lambda_2$ and $A_3(x) = \dfrac{\alpha_1 + 1 - e^{-\alpha_1\lambda_1 x}}{\alpha_2 + 1 - e^{-\alpha_2\lambda_2 x}}$ is a decreasing function in x. Differentiating $A_3(x)$ with respect to x, we get

$$\begin{aligned} A_3'(x) &= \frac{\alpha_1\lambda_1 e^{-\alpha_1\lambda_1 x}\left(\alpha_2 + 1 - e^{-\alpha_2\lambda_2 x}\right) - \alpha_2\lambda_2 e^{-\alpha_2\lambda_2 x}\left(\alpha_1 + 1 - e^{-\alpha_1\lambda_1 x}\right)}{\left(\alpha_2 + 1 - e^{-\alpha_2\lambda_2 x}\right)^2} \\ &\overset{\text{sign}}{=} \alpha_1\lambda_1[(\alpha_2+1)e^{\alpha_2\lambda_2 x} - 1] - \alpha_2\lambda_2[(\alpha_1+1)e^{\alpha_1\lambda_1 x} - 1] \\ &\leq 0, \end{aligned}$$

provided $\frac{(\alpha_1+1)e^{\alpha_1\lambda_1 x}-1}{\alpha_1\lambda_1} \geq \frac{(\alpha_2+1)e^{\alpha_2\lambda_2 x}-1}{\alpha_2\lambda_2}$, for all $x > 0$. ∎

When $\alpha_1 = \alpha_2$, then $X_1 \leq_{HR\uparrow} X_2$ if $\lambda_1 \geq \lambda_2$ and X is IFR. The following example shows that if condition (i) of Theorem 5 does not hold, then the theorem may not hold.

Example 10: Let $X_1 \sim$ WE(1,2) and $X_2 \sim$ WE(2,3). Then the survival function of X_1 is given by

$$\bar{F}_{X_1}(x) \;=\; 2e^{-2x}(1-\frac{1}{2}e^{-2x}), \quad x>0,$$

and the survival function of X_2 is given by

$$\bar{F}_{Y}(x) \;=\; \frac{3}{2}e^{-3x}(1-\frac{1}{3}e^{-6x}), \quad x>0,$$

Clearly, condition (ii) of the Theorem 5 holds, but not condition (i). Now, for all $x, t > 0$

$$\begin{aligned}\frac{\bar{F}_{X_1}(x+t;1,2)}{\bar{F}_{X_2}(x;2,3)} &= \frac{4e^{-2(x+t)}(1-\frac{1}{2}e^{-2(x+t)})}{3e^{-3x}(1-\frac{1}{3}e^{-6x})}\\ &= p_{10}(x,t), \quad \text{(say)}.\end{aligned}$$

Observe $p_{10}(0,1) = 0.23404$, $p_{10}(0.1,1) = 0.21703$, $p_{10}(1.5,1) = 0.48191$, which shows that $p_{10}(x,1)$ is not a monotone function in x, hence, $X_1 \leq_{HR\uparrow} X_2$ does not hold. ∎

The following examples shows that if condition (ii) of Theorem 5 does not satisfy then the theorem may or may not hold.

Example 11: Let us consider Example 4. Then the survival function of X_1 is given by

$$\bar{F}_{X_1}(x) \;=\; 2e^{-3x}(1-\frac{1}{2}e^{-3x}), \quad x>0,$$

and the survival function of X_2 is given by

$$\bar{F}_{X_2}(x) \;=\; \frac{3}{2}e^{-2x}(1-\frac{1}{3}e^{-4x}), \quad x>0.$$

Clearly, condition (i) of the Theorem 5 holds true, but not condition (ii). Now, for all $x, t > 0$

$$\begin{aligned}\frac{\bar{F}_{X_1}(x+t,1,3)}{\bar{F}_{X_2}(x,2,2)} &= \frac{4e^{-3(x+t)}(1-\frac{1}{2}e^{-3(x+t)})}{3e^{-2x}(1-\frac{1}{3}e^{-4x})}\\ &= \frac{4uv^3(1-\frac{u^3v^3}{2})}{3(1-\frac{u^4}{3})}\\ &= p_{11}(u,v), \quad \text{(say)},\end{aligned}$$

where $u = e^{-x}$ and $v = e^{-t}$. Differentiating $p_{11}(u, v)$ with respect to u for all $v > 0$, we have

$$\begin{aligned} p'_{11}(u,v) &\overset{\text{sign}}{=} \left[4v^3\left(1-\frac{u^3v^3}{2}-6u^3v^6\right)\right](3-u^4)+16u^4v^3-8u^7v^6 \\ &\geq 0. \end{aligned}$$

Thus, $p_{11}(u, v)$ is an increasing function in u for all $v > 0$. Hence, $X_1 \leq_{HR\uparrow} X_2$. ∎

Example 12: Let us consider Example 6. Then the survival function of X_1 is given by

$$\bar{F}_{X_1}(x) = 5e^{-x/2}(1-\frac{4}{5}e^{-x/8}), \quad x > 0,$$

and the survival function of X_2 is given by

$$\bar{F}_{X_2}(x) = \frac{7}{6}e^{-x/3}(1-\frac{1}{7}e^{-2x}), \quad x > 0.$$

Clearly, condition (i) of the Theorem 5 holds true, but not condition (ii). Now, for all $x, t > 0$

$$\begin{aligned} \frac{\bar{F}_{X_1}(x+t, 1/4, 1/2)}{\bar{F}_{X_2}(x, 6, 1/3)} &= \frac{30e^{-(x+t)/2}(1-\frac{4}{5}e^{-(x+t)/8})}{7e^{-x/3}(1-\frac{1}{7}e^{-2x})} \\ &= p_{12}(x,t), \ \textit{say}. \end{aligned}$$

We see that $p_{12}(1, 1) = 0.84583$, $p_{12}(1.3, 1) = 0.845966$, $p_{12}(3, 1) = 0.81193$, which shows that $p_{12}(x, t)$ is not a monotone function in x for fixed t. Hence, $X_1 \leq_{HR\uparrow} X_2$ does not hold. ∎

The following theorem shows that the univariate weighted exponential distribution preserves the down hazard rate ordering. A random variable X_1 is said to be larger than another random variable X_2 in down hazard rate order (written as $X_1 \geq_{HR\downarrow} X_2$) if, $X_1 \geq_{HR} [X_2 - x | X_2 > x]$, for all $x \geq 0$, or equivalently, if $\bar{F}_{X_1}(x)/\bar{F}_{X_2}(x + t)$ is an increasing function in x, for all $t \geq 0$.

Theorem 6: Let $X_1 \sim \text{WE}(\alpha_1, \lambda_1)$ and $X_2 \sim \text{WE}(\alpha_2, \lambda_2)$. Then $X_1 \geq_{HR\downarrow} X_2$ provided

(i) $\lambda_1 \leq \lambda_2$,

(ii) $\dfrac{(\alpha_1+1)e^{\alpha_1\lambda_1 x}-1}{\alpha_1\lambda_1} \leq \dfrac{(\alpha_2+1)e^{\alpha_2\lambda_2 x}-1}{\alpha_2\lambda_2}$, for all $x>0$,

(iii) X_1 is IFR.

PROOF: It has been omitted ∎

When $\alpha_1=\alpha_2$, then $X_1 \geq_{HR\downarrow} X_2$ if $\lambda_1 \leq \lambda_2$ and X_1 is IFR. The following example shows that if condition (i) of Theorem 6 does not satisfy, then the theorem may not hold.

EXAMPLE 13: Let $X_1 \sim$ WE(1,4) and $X_2 \sim$ WE(3,2). Then the survival function of X_1 is given by

$$\bar{F}_{X_1}(x) \;=\; 2e^{-4x}(1-\frac{1}{2}e^{-4x}), \quad x>0,$$

and the survival function of X_2 is given by

$$\bar{F}_{X_2}(x) \;=\; \frac{4}{3}e^{-2x}(1-\frac{1}{4}e^{-6x}), \quad x>0,$$

Clearly, condition (ii) of the Theorem 6 has satisfied but not condition (i). Now, for all $x,t>0$

$$\begin{aligned}\frac{\bar{F}_{X_1}(x;1,4)}{\bar{F}_{X_2}(x+t;3,2)} &= \frac{6e^{-4x}(1-\frac{1}{2}e^{-4x})}{4e^{-2(x+t)}(1-\frac{1}{4}e^{-6(x+t)})}\\ &= p_{13}(x,t), \quad \text{(say)}\end{aligned}$$

We see that $p_{13}(0,1)=5.5450$, $p_{13}(0.1,1)=6.03512$, $p_{13}(0.5,1)=3.80163$, which shows that $p_{13}(x,1)$ is not a monotone function of x for a given t. Hence, $X_1 \geq_{HR\downarrow} X_2$ does not hold. ∎

The following examples show that if condition (ii) of Theorem 6 does not satisfy, then the theorem may or may not hold.

EXAMPLE 14: Let us consider Example 2. Then the survival function of X_1 is given by

$$\bar{F}_{X_1}(x) \;=\; \frac{4}{3}e^{-3x}(1-\frac{1}{4}e^{-9x}), \quad x>0,$$

and the survival function of X_2 is given by

$$\bar{F}_{X_2}(x) = \frac{3}{2}e^{-4x}(1-\frac{1}{3}e^{-8x}), \quad x>0.$$

Clearly, condition (i) of the Theorem 5 does not satisfy, but not condition (ii). Now, for all $x,t>0$

$$\begin{aligned}\frac{\bar{F}_{X_1}(x,3,3)}{\bar{F}_{X_2}(x+t,2,4)} &= \frac{8e^{-3x}(1-\frac{1}{4}e^{-9x})}{9e^{-4(x+t)}(1-\frac{1}{3}e^{-8(x+t)})}\\ &= \frac{8(1-\frac{u^9}{4})}{9uv^4(1-\frac{u^8v^8}{3})}\\ &= p_{14}(u,v), (\text{say}),\end{aligned}$$

where $u=e^{-x}$ and $v=e^{-t}$. Differentiating $p_{14}(u,v)$ with respect to u for all $v>0$, we have

$$\begin{aligned}p'_{14}(u,v) &\overset{\text{sign}}{=} -\frac{9}{4}u^9v^4\left(1-\frac{u^8v^8}{3}\right)-v^4\left(1-\frac{u^8v^8}{3}\right)\left(1-\frac{u^9}{4}\right)\\ &\quad +\frac{8}{3}u^8v^12\left(1-\frac{u^9}{4}\right)\\ &\leq 0.\end{aligned}$$

Thus, $p_{14}(u,v)$ is a decreasing function in u for all $v>0$. Hence, $X_1 \geq_{HR\downarrow} X_2$. ∎

Example 15: Let $X_1 \sim$ WE(1/3,1/4) and $X_2 \sim$ WE(1/5,4). Then the survival function of X_1 is given by

$$\bar{F}_{X_1}(x) = 4e^{-x/4}(1-\frac{3}{4}e^{-x/12}), \quad x>0,$$

and the survival function of X_2 is given by

$$\bar{F}_{X_2}(x) = 6e^{-4x}(1-\frac{5}{6}e^{-4x/5}), \quad x>0,$$

Clearly, condition (i) of the Theorem 6 has satisfied but not condition (ii). Now, for all $x,t>0$

$$\begin{aligned}\frac{\bar{F}_{X_1}(x;1/3,1/4)}{\bar{F}_{X_2}(x+t;1/5,4)} &= \frac{2e^{-x/4}(1-\frac{3}{4}e^{-x/12})}{3e^{-4(x+t)}(1-\frac{5}{6}e^{-4(x+t)/5})}\\ &= p_{15}(x,t), \quad (\text{say}).\end{aligned}$$

We see that $p_{15}(0,1) = 0.34211$, $p_{15}(1,1) = 0.31901$, $p_{15}(2,1) = 0.33813$, which shows that $p_{15}(x,1)$ is not a monotone function of x for a given t. Hence, $X_1 \geq_{HR\downarrow} X_2$ does not hold. ■

3 Length Biased Weighted Exponential Distribution

Suppose $X \sim \text{WE}(\alpha, \lambda)$, then the length biased version of the random variable X will be denoted by the random variable T which has the PDF for $t > 0$ as

$$f_T(t; \alpha, \lambda) = \frac{(\lambda(\alpha+1))^2}{\alpha(\alpha+2)} t e^{-\lambda t}(1 - e^{-\lambda\alpha t}), \tag{8}$$

and it will be denoted LBWE(α, λ). In this section, we discuss different properties of T, and for that without loss of generality, it is assumed that $\lambda = 1$. We will denote $f_T(t; \alpha, 1) = f_T(t; \alpha)$ only. It easily follows that the PDF of T is always log-concave and $f_T(t, \alpha) \to 0$ as $t \to 0$ or $t \to \infty$. Hence $f_T(t, \alpha)$ is always unimodal. Again, as $\alpha \to \infty$, then the PDF of T converges to the PDF of a gamma distribution with shape parameter 2, and as $\alpha \to 0$ then it converges to gamma distribution with shape parameter 3. The distribution function of T is given by

$$G(t, \alpha) = P(T \leq t) = 1 - \frac{(1+\alpha)^2}{\alpha(2+\alpha)}(1+t)e^{-t} + \frac{1}{\alpha(2+\alpha)}(1 + (1+\alpha)t)e^{-(1+\alpha)t}, \quad t > 0.$$

The corresponding hazard rate and mean residual life functions are given by, respectively,

$$r_T(t, \alpha) = \frac{(1+\alpha)^2 t(1 - e^{-\alpha t})}{(1+\alpha)^2(1+t) - (1 + (1+\alpha)t)e^{-\alpha t}},$$

and

$$m_T(t, \alpha) = \frac{2 + t - \frac{1}{(1+\alpha)^3}\left(2 + (1+\alpha)t\right) e^{-\alpha t}}{1 + t - \frac{1}{(1+\alpha)^2}\left(1 + (1+\alpha)t\right) e^{-\alpha t}}.$$

Since the PDF of T is always log-concave, thus hazard rate function is increasing function in x for all $\alpha > 0$. Again, the mean residual life function is decreasing in t for all $\alpha > 0$.

The moment generating function (MGF) of T for $-1 < t < 1$ is

$$
\begin{aligned}
M_T(t) = E(e^{tT}) &= \frac{(1+\alpha)^2}{\alpha(2+\alpha)}\left[\frac{1}{(1-t)^2} - \frac{1}{(1+\alpha-t)^2}\right] \\
&= \left(1 - \frac{2t}{2+\alpha}\right)(1-t)^{-2}\left(1 - \frac{t}{1+\alpha}\right)^{-2}.
\end{aligned}
$$

The above $M_T(t)$ can be written as follows:

$$
M_T(t) = \frac{2}{2+\alpha}(1-t)^{-1}\left(1 - \frac{t}{1+\alpha}\right)^{-2} + \frac{\alpha}{2+\alpha}(1-t)^{-2}\left(1 - \frac{t}{1+\alpha}\right)^{-2}.
$$

Hence T has the following representation:

$$
T = \begin{cases} U_0 + W & \text{with probability } \frac{\alpha}{2+\alpha} \\ U_1 + W & \text{with probability } \frac{2}{2+\alpha}, \end{cases}
$$

here U_0 follows a gamma distribution with the shape parameter 2 and scale parameter 1, U_1 follows an exponential distribution with the parameter 1 and W follows also a gamma distribution with the scale parameter $1+\alpha$ and shape parameter 2. Further, U_i and W are independently distributed for $i = 0$ and 1. This representation can be used quite conveniently to generate samples from a LBWE distribution. Further, if $T_1, T_2, \ldots, T_n$ are independent identically distributed random variables from LBWE$(\alpha, 1)$, then the distribution of $S = T_1 + \ldots + T_n$ can be written as follows:

$$
S = U_i + W \quad \text{with probability } \; p_k; \quad \text{for } \; k = 0, \ldots, n,
$$

where $p_k = \binom{n}{k}\left(\frac{2}{2+\alpha}\right)^k\left(\frac{\alpha}{2+\alpha}\right)^{n-k}$, and U_k follows a gamma distribution with the shape parameter $2n-k$ and scale parameter 1, and W is same as before. Further U_k and W are independently distributed for $k = 0, \ldots, n$.

The LBWE model can be obtained as a hidden truncation model similarly as the Azzalini's skew-normal model as it was observed by Arnold and Beaver (2000). Suppose the random variables Z and Y have the following joint PDF

$$
f_{Z,Y}(z,y) = \lambda^3 z^2 e^{-\lambda z(1+y)}; \quad z > 0, \quad y > 0. \tag{9}
$$

Consider a new random variable $T = Z$, given that $Y \le \alpha$. It easily follows that the PDF of T is (8). Therefore, T can be interpreted as a hidden truncation model as follows. Suppose Z and Y are two random variables with the joint PDF (9). We do not observe Y, but we observe Z only, provided $Y \le \alpha$. Then the observed sample has the PDF (8).

From the MGF of T, using $\ln M_T(t) = \ln\left(1 - \frac{2t}{2+\alpha}\right) - 2\ln(1-t) - 2\ln\left(1 - \frac{t}{1+\alpha}\right)$, we obtain

$$\begin{aligned}
\mu_T = E(T) &= 2\left(1 + \frac{1}{1+\alpha} - \frac{1}{2+\alpha}\right), \\
\sigma_T^2 = Var(T) &= 2\left(1 + \frac{1}{(1+\alpha)^2} - \frac{2}{(2+\alpha)^2}\right), \\
E(T-\mu_T)^3 &= 4\left(1 + \frac{1}{(1+\alpha)^3} - \frac{2^2}{(2+\alpha)^3}\right), \\
E(T-\mu_T)^k &= 2(k-1)!\left[1 + \frac{1}{(1+\alpha)^k} - \frac{2^{k-1}}{(2+\alpha)^k}\right], \quad \text{for} \quad k = 3, 4, \ldots.
\end{aligned}$$

The coefficient of variation (CV) and skewness of T can be obtained as

$$CV_T = \sqrt{\frac{1}{2}\left[1 - \frac{3(1+\alpha)^2}{(1+(1+\alpha)(2+\alpha))^2}\right]}$$

and

$$\gamma_T = \sqrt{\frac{2[4 - 3\alpha^2(2+\alpha) + (1+\alpha)^3(2+\alpha)^3]^2}{[2 - \alpha^2 + (1+\alpha)^2(2+\alpha)^2]^3}},$$

respectively. Both the mean and variance are decreasing functions of α and they decrease from 3 to 2. The coefficient of variation is increasing in α, and it increases from $1/\sqrt{3}$ to $1/\sqrt{2}$ but the skewness is not a monotone function of α.

Now we consider the stress-strength parameter of length biased weighted exponential distribution. Suppose that $T_1 \sim$ LBWE(α_1, λ_1) and $T_2 \sim$ LBWE(α_2, λ_2) and they are independently distributed, then

$$\begin{aligned}
P(T_1 < T_2) &= \frac{\lambda_2(1+\alpha_2)^2}{\lambda_1\alpha_2(2+\alpha_1)}\left[\frac{1}{\lambda_2^2} - \frac{1}{(\alpha_2+\lambda_2)^2}\right] - \frac{(1+\alpha_1)^2(1+\alpha_2)^2}{\alpha_1\alpha_2\lambda_2(2+\alpha_1)(2+\alpha_2)}\left[\frac{3\lambda_1+\lambda_2}{(\lambda_1+\lambda_2)^3}\right. \\
&\quad \left. - \frac{3\lambda_1 + \lambda 2 + \alpha_2\lambda_2}{(\lambda_1 + \lambda 2 + \alpha_2\lambda_2)^3}\right] + \frac{\lambda_2(1+\alpha_2)^2}{\alpha_1\alpha_2\lambda_1(2+\alpha_1)(2+\alpha_2)}\left[\frac{1}{((1+\alpha_1)\lambda_1+\lambda_2)^2}\right.
\end{aligned}$$

$$+\frac{2\lambda_1(1+\alpha_1)}{((1+\alpha_1)\lambda_1+\lambda_2)^3}+\frac{1}{((1+\alpha_1)\lambda_1+\lambda_2+\alpha_2\lambda_2)^2}\Bigg]$$

4 Maximum Likelihood Estimators & Data Analysis

4.1 Maximum Likelihood Estimators

In this subsection we discuss the maximum likelihood estimators (MLEs) of the unknown parameters, and derive their asymptotic properties. Just for brevity, we make a re-parameterization $\beta = \alpha\lambda$. We mainly discuss the MLEs of β and λ here, which are equivalent to α and λ. Suppose $\{x_1, \ldots, x_n\}$ is a random sample from a LBWE(α, λ), then based on the re-parameterization $\alpha\lambda = \beta$, the log-likelihood function becomes

$$\begin{aligned} L(\beta,\lambda) &= 2n\ln(\beta+\lambda) - n\ln\beta - n\ln(\beta+2\lambda) + 2n\ln\lambda + \sum_{i=1}^{n}\ln x_i - \lambda\sum_{i=1}^{n}x_i \\ &\quad + \sum_{i=1}^{n}\ln\left(1-e^{-\beta x_i}\right). \end{aligned} \tag{10}$$

Therefore, the maximum likelihood estimators (MLEs) of the unknown parameters can be obtained by maximizing (10) with respect β and λ. The two normal equations take the following forms:

$$\frac{\partial l(\beta,\lambda)}{\partial\lambda} = \frac{2n}{\beta+\lambda} - \frac{2n}{\beta+2\lambda} + \frac{n}{\lambda} - \sum_{i=1}^{n}x_i = 0, \tag{11}$$

$$\frac{\partial l(\beta,\lambda)}{\partial\beta} = \frac{2n}{\beta+\lambda} - \frac{n}{\lambda} + \sum_{i=1}^{n}\frac{x_i e^{-\beta x_i}}{1-e^{-\beta x_i}} = 0. \tag{12}$$

Clearly, (12) and (11) cannot be solved explicitly, some numerical technique, like Newton-Raphson method can be used to solve the two non-linear equations simultaneously. Alternatively, any two-dimensional optimization technique may be used to maximize (10) directly to compute the MLEs of λ and β. The Fisher information matrix $\Sigma = ((\sigma_{ij}))$ of λ and β can be obtained as follows:

$$\sigma_{11} = \frac{2}{(\beta+\lambda)^2} - \frac{4}{(\beta+2\lambda)^2} + \frac{1}{\lambda^2} \tag{13}$$

$$\sigma_{12} = \sigma_{21} = \frac{2}{(\beta+\lambda)^2} - \frac{2}{(\beta+2\lambda)^2} \tag{14}$$

$$\sigma_{22} = \frac{2}{(\beta+\lambda)^2} - \frac{1}{\beta^2} - \frac{1}{(\beta+2\lambda)^2} + \frac{\lambda(\beta+\lambda)}{\beta^4}\, u\left(\frac{\lambda}{\beta}\right), \tag{15}$$

where

$$u\left(\frac{\lambda}{\beta}\right) = \int_0^1 (\ln(1-z))^2 (1-z)^{\frac{\lambda}{\beta}} z^{-1} dz.$$

Since LBWE is a regular family, we have the following result:

$$\sqrt{n}\left((\hat{\lambda}-\lambda), (\hat{\beta}-\beta)\right) \to N_2(0, \Sigma^{-1}),$$

4.2 Data Analysis

We analyze one real data mainly for illustrative purpose, to show how the propose LBWE model behaves in practice. The data set is obtained from Lawless (1982, pp 228), and it represents the number of million revolution before failure for each of the 23 ball bearings. It is as follows: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40. Before progressing further we have subtracted 10.25 from each data point.

Preliminary data analysis suggests that it is coming from a skewed distribution, and the scaled TTT plot suggests that the empirical hazard function is an increasing function. The histogram and the scaled TTT plots are provided in Figure 1 and Figure 2, respectively. The preliminary data analysis suggests that the two-parameter LBWE distribution may be used to analyze this data set.

To get an idea about the initial estimates of β and λ, we use the grid search method with a grid length 0.01, and we obtain the initial estimates as 0.02 and 0.04, respectively. We use these initial estimates in the Newton-Raphson algorithm, and we obtain the final estimates of β and λ as 0.0257 and 0.0411, respectively. The associated 95% confidence

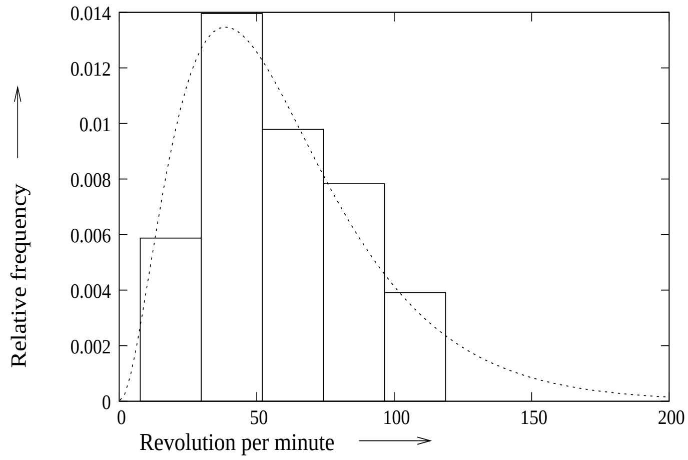

Figure 1: Relative histogram of the ball-bearing data set and the estimated PDF of the fitted LBWE

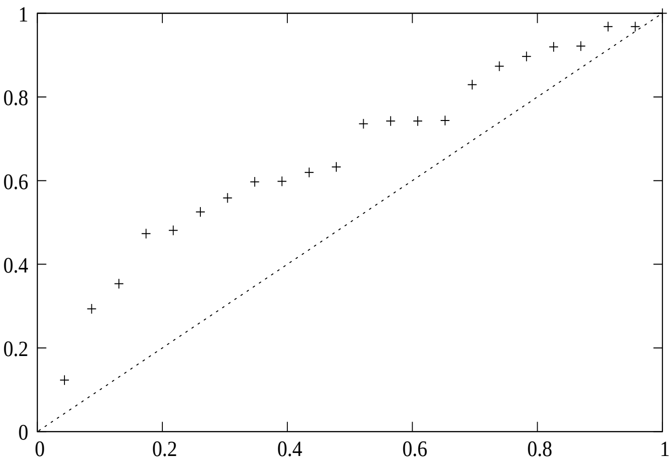

Figure 2: Scaled TTT plot of the ball-bearing data set.

intervals become 0.0257$\mp$0.0071 and 0.0411$\mp$0.0098, respectively. Now the natural question is how good is the model? The Kolmogorov-Smirnov (KS) statistic and the associated p value become 0.0854 and 0.9960 respectively. Therefore, it provides an excellent fit. We have provided the empirical survival function and the fitted survival function in Figure 3.

5 Three-Parameter LBWE Distribution

In this section we introduce the location parameter to the two-parameter LBWE distribution mainly for data analysis purpose. We call this new distribution as the three-parameter

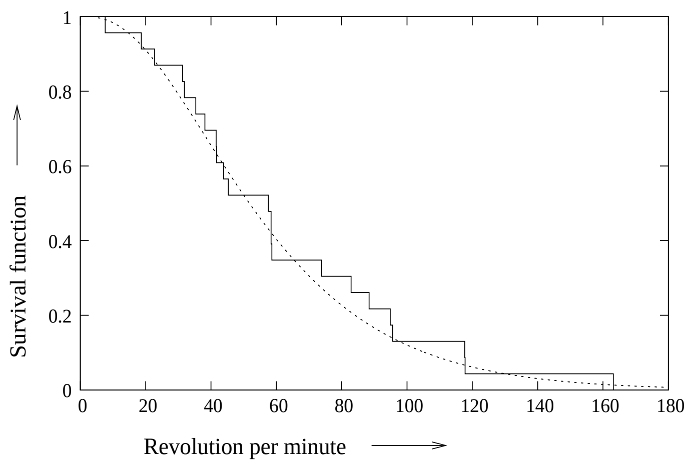

Figure 3: Scaled TTT plot of the ball-bearing data set.

LBWE distribution. A random variable T is said to have a three-parameter LBWE distribution if it has the following PDF:

$$f_T(t;\alpha,\lambda,\mu) = \frac{(\lambda(\alpha+1))^2}{\alpha(\alpha+2)}(t-\mu)e^{-\lambda(t-\mu)}(1-e^{-\lambda\alpha(t-\mu)}); \quad t>\mu, \tag{16}$$

for $\alpha > 0$, $\lambda > 0$ and $-\infty < \mu < \infty$. It will be denoted by LBWE(α,λ,μ). Clearly, the properties of the three-parameter LBWE distribution are similar to the properties of the two-parameter LBWE distribution. Now we will discuss the maximum likelihood estimators of the unknown parameters based on a random sample $\{x_1,\ldots,x_n\}$ from LBWE(α,λ,μ). Similarly as before, we make the re-parameterization $\beta = \alpha\lambda$, and we will be discussing the MLEs of β, λ and μ only. The log-likelihood function becomes:

$$\begin{aligned} L(\beta,\lambda,\mu) &= 2n\ln(\beta+\lambda) - n\ln\beta - n\ln(\beta+2\lambda) + 2n\ln\lambda + \sum_{i=1}^{n}\ln(x_i-\mu) - \lambda\sum_{i=1}^{n}(x_i-\mu) \\ &\quad + \sum_{i=1}^{n}\ln\left(1-e^{-\beta(x_i-\mu)}\right). \end{aligned} \tag{17}$$

The three normal equations take the following forms:

$$\frac{\partial L(\beta,\lambda,\mu)}{\partial\lambda} = \frac{2n}{\beta+\lambda} - \frac{2n}{\beta+2\lambda} + \frac{n}{\lambda} - \sum_{i=1}^{n}(x_i-\mu) = 0, \tag{18}$$

$$\frac{\partial L(\beta,\lambda,\mu)}{\partial\beta} = \frac{2n}{\beta+\lambda} - \frac{n}{\lambda} + \sum_{i=1}^{n}\frac{(x_i-\mu)e^{-\beta(x_i-\mu)}}{1-e^{-\beta(x_i-\mu)}} = 0, \tag{19}$$

$$\frac{\partial L(\beta,\lambda,\mu)}{\partial \mu} = -\sum_{i=1}^{n}\frac{1}{x_i-\mu}+n\lambda-\frac{\beta e^{-\beta(x_i-\mu)}}{1-e^{-\beta(x_i-\mu)}}=0. \qquad (20)$$

To compute the MLEs we need to solve (18), (19) and (20), simultaneously. We have used the same data set, and computed the MLEs as $\widehat{\beta} = 0.0317$, $\widehat{\lambda} = 0.0404$ and $\widehat{\mu} = 10.9998$. The associated log-likelihood value becomes -113.0198. In this case the KS statistic and the associated p value become 0.0849 and 0.9964, respectively. Gupta and Kundu (1999) fitted three-parameter Weibull, gamma and generalized exponential distributions to this same data set. For comparison purposes, we have presented below the associated log-likelihood values and KS statistics for each case. It is clear that the three-parameter LBWE distribution provides a very good fit to the given data set.

Table 1: Log-likelihood values and the KS statistics for different three parameter distributions.

Distribution	log-likelihood	KS
Gamma	-112.8501	0.107
Weibull	-112.9767	0.118
GE	-112.7666	0.103
LBWE	-113.0198	0.085

6 Conclusions

In this paper first we consider the weighted exponential distribution originally proposed by Gupta and Kundu (2009) and discussed its different reliability properties. We then consider two-parameter length biased weighted exponential distribution. We discussed different properties of the proposed LBWE distribution and provided some inferential results. We further consider three-parameter LBWE model mainly for data analysis purposes. We analyze one data set using the three-parameter LBWE model, and it is observed that it provides a better fit than some of the existing three-parameter models based on KS statistics. Therefore, the proposed model may be used for data analysis purposes for some situation in practice.

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