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On well-posedness and small data global existence for an interface damped free boundary fluid–structure model

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Abstract

We address a fluid–structure system which consists of the incompressible Navier–Stokes equations and a damped linear wave equation defined on two dynamic domains. The equations are coupled through transmission boundary conditions and additional boundary stabilization effects imposed on the free moving interface separating the two domains. Given sufficiently small initial data, we prove the global-in-time existence of solutions by establishing a key energy inequality which in addition provides exponential decay of solutions.

Keywords: Navier–Stokes equations, fluid–structure interaction, long time behaviour, global solutions, damped wave equation
Mathematics Subject Classification: 35R35, 35Q30, 76D05

1. Introduction

In this paper, we consider a coupled system of PDEs modelling the interaction of an incompressible viscous fluid with an elastic structure on a free moving interface when subjected to additional boundary stabilization effects. Well-posedness of the free boundary model was first established in [CS1], while other local-in-time existence of solutions results, with and without damping, have been addressed in several more recent works [IKLT, KT1, KT2]. In this paper, we establish global-in-time existence and exponential decay of the solutions to the system given sufficiently small initial data, subject to boundary stabilization terms.

The model in consideration is formulated in Lagrangian coordinates on the initial domain and consists of the Navier–Stokes equations and a damped wave equation, with additional boundary stabilization terms incorporated in the transmission boundary conditions at the free moving interface. (The methods can be easily adapted to the case of equations of linear elasticity with damping as in [KTZ3] for instance.) Standard energy estimates on time derivatives, usually sufficient to obtain local-in-time results, are insufficient by themselves for controlling the wave potential energy due to the coupling dynamics and the quasilinear nature of the variable coefficients Stokes system. For this reason, a combination of equipartition and flux multipliers techniques developed especially to address energy decay and stabilization of waves is used to control the growth of potential energy in the elastic component. The higher regularity requirement stemming from the presence of the variable coefficients requires several levels of these estimates to obtain the key energy inequality (5.15) from which global existence and exponential decay can be inferred.

One of the main obstacles for obtaining the decay is that the resulting *a priori* estimates allow norms to grow exponentially in time. However, the terms with exponential increase appear as super-quadratic on the right side and are thus controlled by the nonnegative terms on the left which are only quadratic. We note that the exponential decay of the norms is essential for obtaining that the Lagrangian coefficients are close to the identity for all time. As shown in [ZZ], the uniform decay of solutions cannot be expected even in the case of coupling of the linear heat equation with the linear undamped wave equation. In such a case one obtains the so-called *strong* stability, which can be quantified by at most rational decay rates, obtained for smoother initial data taken from the domain of the generator. The above negative result was also known in the case of linear coupled system consisting of the *Stokes* and the wave equation. In fact, the presence of the pressure in the equation changes the picture substantially. It is shown in [AT1] that even strong stability *fails* for the linear Stokes-undamped wave system unless the domain Ω_e satisfies special geometric condition (guaranteed by partial flatness of the domain). For instance, the case of spherical domains Ω_e provides a known counterexample to strong stability [AT1]. In view of the above, we do not expect exponential decay without (i) the static damping term βw in the wave equation (without the effect of this term, the elastic body is expected to shift and rotate) and (ii) without either velocity internal damping $\alpha > 0$ or boundary stabilizing term $\gamma > 0$. On a positive side, strong stability was shown in [AT1] for the Stokes-undamped wave model defined on the domain Ω_e that is partially flat and with the initial data satisfying an additional compatibility condition whose aim is to eliminate zero eigenvalue from the spectrum of the generator (a phenomenon specific to the presence of the pressure and therefore not present in the treatment of heat and wave equation alone). As for uniform or exponential decay rates, these hold for the Stokes-wave system with both static $\beta > 0$ and dynamic $\alpha > 0$ damping active, as shown in [AT2]. The above results motivate the framework for our study of global existence of free boundary interaction with damped wave equation.

In our analysis, we have considerably benefited from the wealth of tools used to study stabilization and control of damped hyperbolic dynamics [LT, LTr1, LTr2], and more recent works on coupled systems where equipartition of energy tools were employed successfully [LL1, LL2]. We note that the incorporation of the stabilization term with $\gamma \geqslant 0$ in the velocity matching condition can serve as a regularization of the physical model, and provide a possible tool for establishing existence of solutions to the internally damped wave equation in the limiting physical case $\gamma = 0$. However, establishing the exponential decay result for the limiting case requires new estimates and possibly further assumptions, and we hope to address it in a future work. The corresponding results in the linear case and the static interface case have been already obtained in [AT2, LL2].

The paper is organized as follows. In section 2, we formulate the mathematical model and the assumptions and then state the main theorem. In section 3, we provide certain preliminary properties on the Lagrangian fluid flow map and the variable coefficients in the Navier–Stokes equation, as well as standard elliptic and Stokes estimates utilized in later sections. In section 4, we derive the energy and equipartition estimates at several levels, and in section 5, we collect these estimates to obtain the key energy inequality (5.15) and show that it leads to the desired global existence and exponential decay. In section 6, we construct solutions of the fluid–structure model by an iteration method. As shown in [KT1, KT2] (see also [CS1]), it is sufficient to construct local solutions for a linear problem, i.e. the problem with the coefficients $a(x, t)$ given and smooth, satisfying the postulated compatibility conditions. The construction of solutions for the linear problem is obtained as follows. First, we address the problem where a is smooth and independent of t. The solution to this problem is obtained by a Galerkin procedure. The main difficulty in using a Galerkin procedure in this situation is that it is not known whether there exists a basis consisting of functions with matching regularity on the common boundary. To overcome this difficulty, we take advantage of the Neumann boundary conditions in order to find a Galerkin formulation which does not necessitate such matching basis. Then we prove that the system is indeed equivalent to the original set of equations. After establishing the local existence of solutions for the time-independent coefficients, we then obtain the existence of solutions with coefficients $a = a(x, t)$ depending on x and t by a perturbation (fixed point) technique.

2. The main results

We consider the free boundary fluid–structure system which models the motion of an elastic body moving and interacting with an incompressible viscous fluid (see [CS1, CS2, KT1, KT2, B, BG1]). This parabolic–hyperbolic system couples the Navier–Stokes equation

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \qquad \text{in } \Omega_f(t) \tag{2.1}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega_f(t) \tag{2.2}$$

and a damped wave equation

$$w_{tt} - \Delta w + \alpha w_t + \beta w = 0 \qquad \text{in } \Omega_e \tag{2.3}$$

for $\alpha, \beta > 0$. The Navier–Stokes equation is posed in the Eulerian framework and in a dynamic domain $\Omega_f(t)$, with $\Omega_f(0) = \Omega_f$, while the wave equation is posed in the domain Ω_e. The geometry is such that $\partial\Omega_e = \Gamma_c$ is the common boundary of the domains, and $\partial\Omega_f = \Gamma_c \cup \Gamma_f$. Both domains Ω_f and Ω_e are assumed bounded and smooth (see [CS1, KT1, KT2] for more details). The interaction is captured by natural velocity and stress matching conditions on the free moving interface between the fluid and the elastic body.

It is more convenient to consider the system formulated in the Lagrangian coordinates (see [CS1, KT2]). With $\eta \colon \Omega_f \to \Omega_f(t)$ the position function, the incompressible Navier–Stokes equation may be written as

$$v_t^i - \partial_j(a_l^j a_l^k \partial_k v^i) + \partial_k(a_i^k q) = 0 \qquad \text{in } \Omega_f \times (0, T), \qquad i = 1, 2, 3 \tag{2.4}$$

$$a_i^k \partial_k v^i = 0 \qquad \text{in } \Omega_f \times (0, T), \tag{2.5}$$

where $v(x, t)$ and $q(x, t)$ denote the Lagrangian velocity and the pressure of the fluid over the initial domain Ω_f, i.e. $v(x, t) = \eta_t(x, t) = u(\eta(x, t), t)$ and $q(x, t) = p(\eta(x, t), t)$ in Ω_f.

The matrix a with ij entry a^i_j is defined by $a(x,t) = (\nabla\eta(x,t))^{-1}$ in Ω_f, i.e. $\partial_m\eta_i a^m_j = \delta_{ij}$ for all $i, j = 1, 2, 3$. The elastic equation for the displacement function $w(x,t) = \eta(x,t) - x$ is formulated in the Lagrangian framework as

$$w^i_{tt} - \Delta w^i + \alpha w^i_t + \beta w^i = 0 \qquad \text{in } \Omega_e \times (0,T), \quad i = 1,2,3 \tag{2.6}$$

over the initial domain Ω_e. We thus seek a solution (v, w, q, a, η) to the system (2.4)–(2.6), where the coefficients a^i_j for $i, j = 1, 2, 3$ and η are determined from

$$a_t = -a : \nabla v : a \qquad \text{in } \Omega_f \times (0,T) \tag{2.7}$$

$$\eta_t = v \qquad \text{in } \Omega_f \times (0,T) \tag{2.8}$$

with the initial conditions $a(x,0) = I$ and $\eta(x,0) = x$ in Ω_f; here, the symbol : denotes matrix multiplication. On the interface Γ_c between Ω_f and Ω_e, we assume transmission boundary condition

$$w^i_t = v^i - \gamma\partial_j w^i N_j \qquad \text{on } \Gamma_c \times (0,T), \tag{2.9}$$

where $\gamma > 0$, and the matching of stresses

$$\partial_j w^i N_j = a^j_l a^k_l \partial_k v^i N_j - a^k_i q N_k \qquad \text{on } \Gamma_c \times (0,T), \tag{2.10}$$

while on the outside fluid boundary Γ_f, we assume the non-slip boundary condition

$$v^i = 0 \qquad \text{on } \Gamma_f \times (0,T) \tag{2.11}$$

for $i = 1, 2, 3$, where $N = (N_1, N_2, N_3)$ is the unit outward normal with respect to Ω_e. Note that we are working with the (Eulerian) stress ∇u; the modifications for the more physical stress $\frac{1}{2}(\nabla u + \nabla u^T)$ are notationally more challenging and follow [KTZ3]. We supplement the system (2.4)–(2.6) with the initial conditions $v(x,0) = v_0(x)$ and $(w(x,0), w_t(x,0)) = (0, w_1(x))$ on Ω_f and Ω_e, respectively. We also use the classical spaces $H = \{v \in L^2(\Omega_f) : \operatorname{div} v = 0, v \cdot N|_{\Gamma_f} = 0\}$ and $V = \{v \in H^1(\Omega_f) : \operatorname{div} v = 0, v|_{\Gamma_f} = 0\}$. Based on v_0, we determine the initial pressure by solving the problem

$$\begin{aligned} \Delta q_0 &= -\partial_i v^k_0 \partial_k v^i_0 \qquad \text{in } \Omega_f \\ \nabla q_0 \cdot N &= \Delta v_0 \cdot N \qquad \text{on } \Gamma_f \\ -q_0 &= -\partial_j v^i_0 N_j N_i + \partial_j w^i_0 N_j N_i \qquad \text{on } \Gamma_c. \end{aligned} \tag{2.12}$$

Our main result provides global-in-time existence for fluid–structure system with damping, given small data. Namely, the following assertion holds.

Theorem 2.1. *Let $\alpha, \beta, \gamma > 0$. Assume that $v_0 \in V \cap H^4(\Omega_f)$, $w_0 \in H^3(\Omega_e)$ and $w_1 \in H^2(\Omega_e)$ are sufficiently small and that they satisfy the compatibility conditions*

$$\begin{aligned} w_1 &= v_0 - \gamma\nabla w_0 \cdot N, \\ \Delta w_0 - \alpha w_1 - \beta w_0 + \gamma\nabla w_1 \cdot N &= \Delta v_0 - \nabla q_0, \\ \Delta w_1 - \alpha w_{tt}(0) - \beta w_1 + \gamma\nabla w_{tt}(0) \cdot N &= \Delta v_t(0) - \nabla q_t(0) \end{aligned} \tag{2.13}$$

on Γ_c,

$$\begin{aligned} \frac{\partial w_0}{\partial N} \cdot \tau &= \frac{\partial v_0}{\partial N} \cdot \tau, \\ \frac{\partial w_1}{\partial N} \cdot \tau &= \frac{\partial}{\partial N}\big(\Delta v_0 - \nabla q_0\big) \cdot \tau, \\ \frac{\partial w_{tt}(0)}{\partial N} \cdot \tau &= \frac{\partial}{\partial N}(\Delta v_t(0) - \nabla q_t(0)) \cdot \tau \end{aligned} \tag{2.14}$$

also on Γ_c*, for tangential vectors* τ*, and*

$$v_0 = 0,$$
$$\Delta v_0 - \nabla q_0 = 0,$$
$$-\partial_t a_k^j(0)\partial_{jk} v_0^i - \partial_j(\partial_t a_j^k(0)\partial_k v^i(0)) - \Delta\partial_t v^i(0) + \partial_t a_i^k(0)\partial_k q(0) + \partial_{it} q(0) = 0 \qquad (2.15)$$

on Γ_f*. Then there exists a unique global smooth solution* (v, w, q, a, η) *which satisfies*

$$v \in L^\infty([0,\infty); H^3(\Omega_f)) \qquad (2.16)$$
$$v_t \in L^\infty([0,\infty); H^2(\Omega_f)) \qquad (2.17)$$
$$v_{tt} \in L^\infty([0,\infty); L^2(\Omega_f)) \qquad (2.18)$$
$$\nabla v_{tt} \in L^2([0,\infty); L^2(\Omega_f)) \qquad (2.19)$$
$$\partial_t^j w \in C([0,\infty); H^{3-j}(\Omega_e)), \qquad j = 0, 1, 2, 3 \qquad (2.20)$$

with $q \in L^\infty([0,\infty); H^2(\Omega_f))$*,* $q_t \in L^\infty([0,\infty); H^1(\Omega_f))$*,* $a, a_t \in L^\infty([0,\infty); H^2(\Omega_f))$*,* $a_{tt} \in L^\infty([0,\infty); H^1(\Omega_f))$*,* $a_{ttt} \in L^2([0,\infty); L^2(\Omega_f))$ *and* $\eta|_{\Omega_f} \in C([0,\infty); H^3(\Omega_f))$*.*

When $\alpha = 0$ *the result remains valid provided the star-shaped condition*

$$(x - x_0)\cdot N(x) > 0, \qquad x \in \Gamma_c \qquad (2.21)$$

for some $x_0 \in \Omega_e$ *is imposed.*

In section 4, we present *a priori* estimates for the system. In section 5, we gather all the *a priori* estimates and show how they lead to global existence of solutions. In section 6, we carry out a complete construction of solutions based on *a priori* estimates in the earlier sections.

Remark 2.2. Note that we need to derive $q_t(0)$, $w_{ttt}(0)$ and $v_{tt}(0)$ from the system (2.4)–(2.11). Indeed, we have

$$w_{tt}(0) = \Delta w_0 - \alpha w_1(0) - \beta w_0$$
$$w_{ttt}(0) = \Delta w_1 - \alpha w_{tt}(0) - \beta w_1$$
$$v_{tt}(0) = \Delta v_t(0) - a_t(0)\Delta v_0 - \nabla q_t(0)$$

and $q_t(0)$ is determined as a solver of the elliptic problem

$$\Delta q_t(0) = \operatorname{div}(\Delta v_t(0) + a_t(0)\Delta v_0) \qquad \text{in } \Omega_f$$
$$\nabla q_t(0)\cdot N = \Delta v_t(0)\cdot N + a_t(0)\Delta v_0\cdot N \quad \text{on } \Gamma_f$$
$$-q_t(0) = \frac{\partial w_1}{\partial N} - \frac{\partial v_t(0)}{\partial N} \qquad \text{on } \Gamma_c,$$

with $v_t(0) = \Delta v_0 - \nabla q_0$ and $a_t(0) = -\nabla v_0$.

Also note that the first condition in (2.14) is interpreted as

$$\partial_j w_0^i N_j \tau_i = \partial_j v_0^i N_j \tau_i \qquad (2.22)$$

for all tangent vectors τ.

The proof of theorem 2.1 shows that the assumption $v_0 \in V \cap H^4(\Omega_f)$ may be replaced with $v(0) \in V \cap H^3(\Omega_f)$, $v_t(0) \in V$ and $v_{tt}(0) \in H$.

The proof of theorem 2.1 is given in sections 5 and 6.

3. Preliminary results

In this section, we provide formal *a priori* estimates on the time derivatives of the unknown functions needed in the proof of theorem 2.1. We begin with an auxiliary result providing bounds on the coefficients of the matrix a. In the whole paper, the symbol C denotes a sufficiently large constant depending on the domains Ω_e and Ω_f as well as on the parameters α, β and γ.

Lemma 3.1 ([IKLT]). *Assume that* $\|\nabla v\|_{L^\infty([0,T];H^2(\Omega_f))} \leqslant M$. *Let* $p \in [1, \infty]$ *and* $i, j = 1, 2, 3$. *With* $T \in [0, 1/CM]$, *where* C *is a sufficiently large constant, the following statements hold:*

(i) $\|\nabla\eta\|_{H^2(\Omega_f)} \leqslant C$ *for* $t \in [0, T]$;
(ii) $\|a\|_{H^2(\Omega_f)} \leqslant C$ *for* $t \in [0, T]$;
(iii) $\|a_t\|_{L^p(\Omega_f)} \leqslant C\|\nabla v\|_{L^p(\Omega_f)}$ *for* $t \in [0, T]$;
(iv) $\|\partial_i a_t\|_{L^p(\Omega_f)} \leqslant C\|\nabla v\|_{L^{p_1}(\Omega_f)}\|\partial_i a\|_{L^{p_2}(\Omega_f)} + C\|\nabla\partial_i v\|_{L^p(\Omega_f)}$ *for* $i = 1, 2, 3$ *and* $t \in [0, T]$ *where* $1 \leqslant p, p_1, p_2 \leqslant \infty$ *are such that* $1/p = 1/p_1 + 1/p_2$;
(v) $\|\partial_{ij} a_t\|_{L^2(\Omega_f)} \leqslant C\|\nabla v\|^{1/2}_{H^1(\Omega_f)}\|\nabla v\|^{1/2}_{H^2(\Omega_f)} + C\|\nabla v\|_{H^2(\Omega_f)}$ *for* $i, j = 1, 2, 3$ *and* $t \in [0, T]$;
(vi) $\|a_{tt}\|_{L^2(\Omega_f)} \leqslant C\|\nabla v\|_{L^2(\Omega_f)}\|\nabla v\|_{L^\infty(\Omega_f)} + C\|\nabla v_t\|_{L^2(\Omega_f)}$ *and* $\|a_{tt}\|_{L^3(\Omega_f)} \leqslant C\|v\|^2_{H^2(\Omega_f)} + C\|\nabla v_t\|_{L^3(\Omega_f)}$ *for* $t \in [0, T]$;
(vii) $\|a_{ttt}\|_{L^2(\Omega_f)} \leqslant C\|\nabla v\|^3_{H^1(\Omega_f)} + C\|\nabla v_t\|_{L^2(\Omega_f)}\|\nabla v\|_{L^\infty(\Omega_f)} + C\|\nabla v_{tt}\|_{L^2(\Omega_f)}$ *for* $t \in [0, T]$;
(viii) for every $\epsilon \in (0, 1/2]$ *and all* $t \leqslant T^* = \min\{\epsilon/CM^2, T\}$, *we have*

$$\|\delta_{jk} - a^j_l a^k_l\|^2_{H^2(\Omega_f)} \leqslant \epsilon, \qquad j, k = 1, 2, 3 \tag{3.1}$$

and

$$\|\delta_{jk} - a^j_k\|^2_{H^2(\Omega_f)} \leqslant \epsilon, \qquad j, k = 1, 2, 3. \tag{3.2}$$

In particular, the form $a^j_l a^k_l \xi^i_j \xi^i_k$ *satisfies the ellipticity estimate*

$$a^j_l a^k_l \xi^i_j \xi^i_k \geqslant \frac{1}{C}|\xi|^2, \qquad \xi \in \mathbb{R}^3 \times \mathbb{R}^3 \tag{3.3}$$

for all $t \in [0, T^*]$ *and* $x \in \Omega_f$, *provided* $\epsilon \leqslant 1/C$ *with* C *sufficiently large.*

This lemma was established in [IKLT, lemma 3.1].

From [IKLT], we also recall *a priori* estimates for the variable coefficient Stokes system.

Lemma 3.2 ([IKLT]). *Assume that* v *and* q *are solutions to the system*

$$v^i_t - \partial_j(a^j_l a^k_l \partial_k v^i) + \partial_k(a^k_i q) = 0 \qquad \text{in } \Omega_f \tag{3.4}$$

$$a^k_i \partial_k v^i = 0 \qquad \text{in } \Omega_f \tag{3.5}$$

$$v = 0 \qquad \text{on } \Gamma_f \tag{3.6}$$

$$a^j_l a^k_l \partial_k v^i N_j - a^k_i q N_k = \partial_j w^i N_j \qquad \text{on } \Gamma_c \tag{3.7}$$

for given coefficients $a^i_j \in L^\infty(\Omega_f)$ *with* $i, j = 1, 2, 3$ *satisfying lemma 3.1 with a sufficiently small constant* $\epsilon = 1/C$. *Then the estimate*

$$\|v\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)} \leqslant C\|v_t\|_{H^s(\Omega_f)} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)} \tag{3.8}$$

holds for $s = 0, 1$ *and for all* $t \in (0, T)$*. Moreover, the time derivatives* v_t *and* q_t *satisfy*

$$\|v_t\|_{H^2(\Omega_f)} + \|q_t\|_{H^1(\Omega_f)}$$
$$\leqslant C\|v_{tt}\|_{L^2(\Omega_f)} + C\left\|\frac{\partial w_t}{\partial N}\right\|_{H^{1/2}(\Gamma_c)} + C\|v\|^{1/2}_{H^2(\Omega_f)}\|v\|^{1/2}_{H^3(\Omega_f)}\left(\|v\|_{H^2(\Omega_f)} + \|q\|_{H^1(\Omega_f)}\right) \quad (3.9)$$

for all $t \in (0, T)$*, where* $T \leqslant 1/CM$ *for a sufficiently large constant* C*.*

From here on, for simplicity, we omit specifying the domains Ω_f and Ω_e in the norms involving the velocity v and the displacement w. Thus, for example, we write $\|v_0\|_{H^3}$ and $\|w_0\|_{H^3}$ for $\|v_0\|_{H^3(\Omega_f)}$ and $\|w_0\|_{H^3(\Omega_e)}$. However, we continue specifying the boundary domains Γ_c and Γ_f.

Now, let w be a solution to the wave equation (2.6) satisfying the condition (2.9) on the common boundary Γ_c. Then, we may write

$$\frac{\partial w}{\partial N} = \frac{1}{\gamma}(v - w_t). \quad (3.10)$$

Hence, we obtain the elliptic estimate from $\Delta w = w_{tt} + \alpha w_t + \beta w$ with Neumann boundary data

$$\|w\|_{H^3} \leqslant C\|w_{tt}\|_{H^1} + C\alpha\|w_t\|_{H^1} + C\beta\|w\|_{H^1} + C\gamma^{-1}\|(v - w_t)\|_{H^{3/2}(\Gamma_c)}$$
$$\leqslant C\|w_{tt}\|_{H^1} + C\alpha\|w_t\|_{H^1} + C\beta\|w\|_{H^1} + C\gamma^{-1}\|v\|_{H^2} + C\gamma^{-1}\|w_t\|_{H^2} \quad (3.11)$$

for all $t \in (0, T)$. Differentiating (2.6) and (2.9) in time, we also have by the ellipticity of $\Delta w_t = w_{ttt} + \alpha w_{tt} + \beta w_t$ with $\partial w_t/\partial N = \gamma^{-1}(v_t - w_{tt})$

$$\|w_t\|_{H^2} \leqslant C\|w_{ttt}\|_{L^2} + C\alpha\|w_{tt}\|_{L^2} + C\beta\|w_t\|_{L^2} + C\gamma^{-1}\|v_t\|_{H^1} + C\gamma^{-1}\|w_{tt}\|_{H^1} \quad (3.12)$$

for all $t \in (0, T)$.

From (3.8) with $s = 1$, (3.10) and (3.12), we conclude that the Stokes type estimate

$$\|v\|_{H^3} + \|q\|_{H^2} \leqslant C\|v_t\|_{H^1} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{3/2}(\Gamma_c)} \leqslant C\|v_t\|_{H^1} + C\gamma^{-1}\|v\|_{H^2} + C\gamma^{-1}\|w_t\|_{H^2}$$
$$\leqslant C\|v_t\|_{H^1} + C\gamma^{-1}\|v\|_{H^2}$$
$$+ C\gamma^{-1}\big(\|w_{ttt}\|_{L^2} + \alpha\|w_{tt}\|_{L^2} + \beta\|w_t\|_{L^2}$$
$$+ \gamma^{-1}\|v_t\|_{H^1} + \gamma^{-1}\|w_{tt}\|_{H^1}\big) \quad (3.13)$$

holds for all $t \in (0, T)$, where $T \leqslant 1/CM$.

Using (3.8) with $s = 0$ and (3.10), we also have

$$\|v\|_{H^2} + \|q\|_{H^1} \leqslant C\|v_t\|_{L^2} + C\gamma^{-1}\|v\|_{H^1} + C\gamma^{-1}\|w_t\|_{H^1}. \quad (3.14)$$

By (3.9), (3.12) and (3.14), we also obtain

$$\|v_t\|_{H^2} + \|q_t\|_{H^1}$$
$$\leqslant C\|v_{tt}\|_{L^2} + C\|w_t\|_{H^2} + C\|v\|^{1/2}_{H^2}\|v\|^{1/2}_{H^3}\big(\|v\|_{H^2} + \|q\|_{H^1}\big)$$
$$\leqslant C\|v_{tt}\|_{L^2} + C\|w_{ttt}\|_{L^2} + C\alpha\|w_{tt}\|_{L^2} + C\beta\|w_t\|_{L^2} + C\gamma^{-1}\|v_t\|_{H^1} + C\gamma^{-1}\|w_{tt}\|_{H^1}$$
$$+ C\|v\|^{1/2}_{H^3}\left(\|v_t\|_{L^2} + \gamma^{-1}\|v\|_{H^1} + \gamma^{-1}\|w_t\|_{H^1}\right)^{3/2} \quad (3.15)$$

for all $t \in (0, T)$, where $T \leqslant 1/CM$.

4. Global in time solutions

In this section, we establish *a priori* estimates for the global in time existence of the unique smooth solution to the damped fluid–structure system (2.4)–(2.6) provided the initial data are sufficiently small.

Assume that

$$\|v_0\|_{H^3}^2, \|v_t(0)\|_{H^1}^2, \|v_{tt}(0)\|_{L^2}^2, \|w_0\|_{H^3}^2, \|w_1\|_{H^2}^2 \leqslant \epsilon, \tag{4.1}$$

where $\epsilon > 0$ is a small parameter.

We need several auxiliary estimates involving different levels of energy.

4.1. First level estimates

First, denote by

$$E(t) = \tfrac{1}{2}\left(\|v(t)\|_{L^2}^2 + \beta\|w(t)\|_{L^2}^2 + \|w_t(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2\right) \tag{4.2}$$

the energy of the system.

Lemma 4.1. *The energy inequality*

$$E(t) + \int_0^t D(s)\,\mathrm{d}s \leqslant E(0) \tag{4.3}$$

holds for all $t \in [0, T]$, *where*

$$D(t) = \frac{1}{C}\|\nabla v(t)\|_{L^2}^2 + \alpha\|w_t(t)\|_{L^2}^2 + \gamma\left\|\frac{\partial w}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2 \tag{4.4}$$

denotes the dissipative term.

Proof of lemma 4.1 (sketch). In order to obtain (4.3), we take the L^2-inner product of (2.4) with v^i and (2.6) with w_t^i, respectively, and sum in i. Adding the resulting equalities and using the divergence-free condition (2.5) and boundary conditions (2.9)–(2.11) with (3.3) in lemma 3.1 then gives the result. □

The next useful lemma asserts the equipartition of the energy for the wave equation.

Lemma 4.2. *We have*

$$\begin{aligned}\frac{\alpha}{2}\|w(t)\|_{L^2}^2 &+ \int_0^t \|\nabla w\|_{L^2}^2\,\mathrm{d}s + \beta\int_0^t \|w\|_{L^2}^2\,\mathrm{d}s \\ &\leqslant \int_0^t \|w_t\|_{L^2}^2\,\mathrm{d}s + CE(t) + CE(0) + \int_0^t\int_{\Gamma_c} w\cdot\frac{\partial w}{\partial N}\,\mathrm{d}\sigma\,\mathrm{d}s\end{aligned} \tag{4.5}$$

for all $t \in [0, T]$.

Proof of lemma 4.2. Multiplying the wave equation (2.6) with w and integrating in the space variable leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_e} w_t\cdot w - \int_{\Omega_e}|w_t|^2 + \int_{\Omega_e}\partial_j w\cdot\partial_j w - \int_{\Gamma_c}\partial_j w\cdot wN_j + \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_e}|w|^2 + \beta\int_{\Omega_e}|w|^2 = 0. \tag{4.6}$$

Integrating also in the time variable then yields

$$\frac{\alpha}{2}\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 \,\mathrm{d}s + \beta \int_0^t \|w\|_{L^2}^2 \,\mathrm{d}s$$
$$= \int_0^t \|w_t\|_{L^2}^2 \,\mathrm{d}s - \int_{\Omega_e} w_t \cdot w \,\Big|_0^t + \frac{\alpha}{2}\|w(0)\|_{L^2}^2 + \int_0^t \int_{\Gamma_c} \partial_j w \cdot w N_j \,\mathrm{d}\sigma \,\mathrm{d}s \tag{4.7}$$

and the lemma follows. □

Lemma 4.3. *For any* $\alpha, \beta, \gamma > 0$*, there exists* $C > 0$ *such that*

$$E(t) + \int_0^t E(s)\,\mathrm{d}s \leqslant CE(0) \tag{4.8}$$

where $C = C_{\alpha,\beta,\gamma}$*. If* Ω_e *is star-shaped, then the above inequality is valid for all* $\alpha \geqslant 0$*. In particular,* $E(t) \leqslant CE(0)$*.*

Proof of lemma 4.3. First we start with a general Ω_e. From lemma 4.2 and using the inequality

$$\int_{\Gamma_c} w \cdot \frac{\partial w}{\partial N} \,\mathrm{d}\sigma(x) \leqslant \epsilon \|w\|_{L^2(\Gamma_c)}^2 + C_\epsilon \left\| \frac{\partial w}{\partial N} \right\|_{L^2(\Gamma_c)}^2 \tag{4.9}$$

with $\epsilon = \min\{1/C, \beta/C\}$, where C is sufficiently large, it follows that

$$\frac{\alpha}{2}\|w(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|\nabla w\|_{L^2}^2 \,\mathrm{d}s + \frac{\beta}{2}\int_0^t \|w\|_{L^2}^2 \,\mathrm{d}s$$
$$\leqslant \int_0^t \|w_t\|_{L^2}^2 \,\mathrm{d}s + CE(t) + CE(0) + C\int_0^t \int_{\Gamma_c} \left|\frac{\partial w}{\partial N}\right|^2 \mathrm{d}\sigma \,\mathrm{d}s$$
$$\leqslant CE(t) + CE(0) + C\left(\frac{1}{\alpha} + \frac{1}{\gamma}\right)\int_0^t D(s)\,\mathrm{d}s. \tag{4.10}$$

Multiplying (4.10) with a small constant and adding to the energy equation (4.3) leads to

$$E(t) + \int_0^t E(s)\,\mathrm{d}s \leqslant CE(0) \tag{4.11}$$

for all $t \in [0, T]$ with a constant C which does not depend on T, but depends on α, β and γ. (Note that the constant blows up as $\alpha \to 0$.)

Now, we consider the case when Ω_e is star-shaped. In this case we use a *flux multiplier*, which is used in boundary stabilization-controllability of waves. At this point, we could have taken $\alpha = 0$; however, since the approach works also for a sufficiently small positive α and since we believe that the result is of independent interest, we assume that $\alpha \geqslant 0$. (Note that combined with the first part of the proof, in the case of the star-shaped domain, the inequality (4.8) holds for all $\alpha \geqslant 0$.) The constants C_0 and $C_0(\epsilon)$ used in this derivation only depend on the domains but not on parameters α, β and γ.

With $x_0 \in \Omega_e$ fixed, denote $h(x) = x - x_0$ for $x \in \Omega_e$. Taking the L^2-inner product of (2.6) with $h_k \partial_k w^i$ and summing in i, we obtain the identity

$$-\left(\frac{n}{2} - 1\right)\int_0^t \|\nabla w\|_{L^2}^2 \,\mathrm{d}s - \frac{n\beta}{2}\int_0^t \|w\|_{L^2}^2 \,\mathrm{d}s + \frac{n}{2}\int_0^t \|w_t\|_{L^2}^2 \,\mathrm{d}s - \frac{1}{2}\int_0^t \int_{\Gamma_c} |w_t|^2 h \cdot N \,\mathrm{d}\sigma \,\mathrm{d}s$$
$$+ \frac{1}{2}\int_0^t \int_{\Gamma_c} |\nabla w|^2 h \cdot N \,\mathrm{d}\sigma \,\mathrm{d}s - \int_0^t \int_{\Gamma_c} \frac{\partial w}{\partial N} h \cdot \nabla w \,\mathrm{d}\sigma \,\mathrm{d}s + \frac{\beta}{2}\int_0^t \int_{\Gamma_c} |w|^2 h \cdot N \,\mathrm{d}\sigma \,\mathrm{d}s$$
$$= -(w_t, h \cdot \nabla w)|_0^t - \alpha \int_0^t (w_t, h \cdot \nabla w)\,\mathrm{d}s, \tag{4.12}$$

which is valid for any solution to the wave equation (without the boundary conditions). Here we used

$$\int_0^t \int_{\Omega_e} w_{tt}^i h_k \partial_k w^i \, dx \, ds = \frac{n}{2} \int_0^t \|w_t\|_{L^2}^2 \, ds - \frac{1}{2} \int_0^t \int_{\Gamma_c} |w_t|^2 h \cdot N \, d\sigma(x) \, ds + (w_t, h \cdot \nabla w) \Big|_0^t \quad (4.13)$$

with

$$- \int_0^t \int_{\Omega_e} \Delta w^i h_k \partial_k w^i \, dx \, ds = -\left(\frac{n}{2} - 1\right) \int_0^t \|\nabla w\|_{L^2}^2 \, ds + \frac{1}{2} \int_0^t \int_{\Gamma_c} |\nabla w|^2 h \cdot N \, d\sigma(x) \, ds - \int_0^t \left(\frac{\partial w}{\partial N}, h \cdot \nabla w\right)_{\Gamma_c} ds, \quad (4.14)$$

and

$$\beta \int_0^t \int_{\Omega_e} w^i h_k \partial_k w^i = -\frac{n\beta}{2} \int_0^t \|w\|_{L^2}^2 \, ds + \frac{\beta}{2} \int_0^t \int_{\Gamma_c} |w|^2 h \cdot N \, d\sigma(x) \, ds. \quad (4.15)$$

From (4.12), we obtain

$$\frac{n}{2} \int_0^t \left(\|w_t\|_{L^2}^2 - \|\nabla w\|_{L^2}^2 - \beta \|w\|_{L^2}^2\right) ds + \int_0^t \|\nabla w\|_{L^2}^2 \, ds - \frac{1}{2} \int_0^t \int_{\Gamma_c} |w_t|^2 h \cdot N \, d\sigma(x) \, ds + \frac{1}{2} \int_0^t \int_{\Gamma_c} \left(|\nabla w|^2 + \beta |w|^2\right) h \cdot N \, d\sigma(x) \, ds$$
$$\leqslant \epsilon \int_0^t \int_{\Gamma_c} |h \cdot \nabla w|^2 \, d\sigma(x) \, ds + C_\epsilon \int_0^t \int_{\Gamma_c} \left|\frac{\partial w}{\partial N}\right|^2 d\sigma(x) \, ds + CE(t) + CE(0) + \epsilon \alpha \int_0^t \|h \cdot \nabla w\|_{L^2}^2 \, ds + C_0(\epsilon) \alpha \int_0^t \|w_t\|_{L^2}^2 \, ds, \quad (4.16)$$

where $\epsilon > 0$ is a small parameter. By the star-shaped condition, we have $h \cdot N \geqslant \gamma_0$ for some $\gamma_0 > 0$. Taking ϵ small, this leads to

$$\frac{n}{2} \int_0^t \left(\|w_t\|_{L^2}^2 - \|\nabla w\|_{L^2}^2 - \beta \|w\|_{L^2}^2\right) ds + \frac{1}{2} \int_0^t \|\nabla w\|_{L^2}^2 \, ds + \frac{\gamma_0}{4} \int_0^t \int_{\Gamma_c} \left(|\nabla w|^2 + \beta |w|^2\right) d\sigma(x) \, ds$$
$$\leqslant \frac{1}{2} \int_0^t \int_{\Gamma_c} |w_t|^2 h \cdot N \, d\sigma(x) \, ds + C \int_0^t \int_{\Gamma_c} \left|\frac{\partial w}{\partial N}\right|^2 d\sigma(x) \, ds + C_0 \alpha \int_0^t \|w_t\|_{L^2}^2 \, ds + CE(t) + CE(0). \quad (4.17)$$

Now, multiplying (4.5) with $n/2 - \epsilon$, where $\epsilon \in (0, 1)$ is a small parameter, and omitting the first term on the left-hand side, gives

$$\left(\frac{n}{2} - \epsilon\right) \int_0^t \|\nabla w\|_{L^2}^2 \, ds + \beta \left(\frac{n}{2} - \epsilon\right) \int_0^t \|w\|_{L^2}^2 \, ds$$
$$\leqslant \left(\frac{n}{2} - \epsilon\right) \int_0^t \|w_t\|_{L^2}^2 \, ds + \frac{\gamma_0}{8} \beta \int_0^t \int_{\Gamma_c} |w|^2 \, d\sigma(x) \, ds + C \int_0^t \int_{\Gamma_c} \left|\frac{\partial w}{\partial N}\right|^2 d\sigma \, ds + CE(t) + CE(0). \quad (4.18)$$

By adding the last two inequalities, we obtain

$$\left(\frac{1}{2}-\epsilon\right)\int_0^t \|\nabla w\|_{L^2}^2\,\mathrm{d}s+\epsilon\int_0^t \|w_t\|_{L^2}^2\,\mathrm{d}s+\frac{\gamma_0}{8}\int_0^t\int_{\Gamma_c}\left(|\nabla w|^2+\beta|w|^2\right)\mathrm{d}\sigma(x)\,\mathrm{d}s$$
$$\leqslant \beta\epsilon\int_0^t \|w\|_{L^2}^2\,\mathrm{d}s+C\int_0^t\int_{\Gamma_c}|w_t|^2\,\mathrm{d}\sigma(x)\,\mathrm{d}s+C\int_0^t\int_{\Gamma_c}\left|\frac{\partial w}{\partial N}\right|^2\mathrm{d}\sigma(x)\,\mathrm{d}s$$
$$+C_0\alpha\int_0^t \|w_t\|_{L^2}^2\,\mathrm{d}s+CE(t)+CE(0). \tag{4.19}$$

Choosing $\epsilon>0$ sufficiently small, using the Poincaré inequality

$$\|w\|_{L^2}^2\leqslant C\|\nabla w\|_{L^2}^2+C\int_{\Gamma_c}|w|^2\,\mathrm{d}\sigma(x), \tag{4.20}$$

and assuming that $\alpha\leqslant\epsilon/C_0$ with a sufficiently large C_0 so that the fourth term on the right-hand side of (4.19) may be absorbed in the second term on the left-hand side, we obtain

$$\int_0^t\left(\|\nabla w\|_{L^2}^2+\beta\|w\|_{L^2}^2+\|w_t\|_{L^2}^2\right)\mathrm{d}s$$
$$\leqslant C\int_0^t\int_{\Gamma_c}\left(|w_t|^2+\left|\frac{\partial w}{\partial N}\right|^2\right)\mathrm{d}\sigma(x)\,\mathrm{d}s+CE(t)+CE(0). \tag{4.21}$$

Now, we use the condition (2.9) and obtain

$$\int_0^t\left(\|\nabla w\|_{L^2}^2+\beta\|w\|_{L^2}^2+\|w_t\|_{L^2}^2\right)\mathrm{d}s$$
$$\leqslant C\int_0^t\int_{\Gamma_c}\left(|v|^2+(\gamma^2+1)\left|\frac{\partial w}{\partial N}\right|^2\right)\mathrm{d}\sigma(x)\,\mathrm{d}s+CE(t)+CE(0). \tag{4.22}$$

Multiplying (4.22) by a small constant and adding the resulting inequality to (4.3) with $t=T$, we obtain

$$E(t)+\int_0^t E(s)\,\mathrm{d}s\leqslant CE(0) \tag{4.23}$$

and the proof is complete. □

Remark 4.4. If the solution exists for all time, and if a stays sufficiently close to the identity matrix so that a is uniformly elliptic, then lemma 4.3 implies the exponential decay rate for the energy $E(t)$, which is in the case of a star-shaped domain Ω_e independent of $\alpha\geqslant 0$ but depends on $\beta>0$ and $\gamma>0$. Indeed, any nonnegative measurable function E satisfying

$$E(t_2)+\int_{t_1}^{t_2}E(s)\,\mathrm{d}s\leqslant CE(t_1) \tag{4.24}$$

decays exponentially with the rate depending on the constant on the right-hand side of (4.24). When $\alpha>0$ then the first part of the proof of lemma 4.3 proves the desired conclusion. When $\alpha=0$ the second part of the proof asserts the same conclusion under a geometric star-shaped assumption. We have retained the parameter $\alpha\geqslant 0$ through the proof of the second part as this allows for further generalizations to variable coefficients $\alpha(x)\geqslant 0$ without a uniform bound from below qualifying for the arguments in the first part of the proof. □

4.2. Second level estimates

It is clear from the previous subsection that the value of the constant $\beta > 0$ does not play a role in the global existence, it only influences the size of the constant. Thus, for simplicity of notation, we set

$$\beta = 1 \tag{4.25}$$

from here on.

We next introduce the second level energy

$$E_1(t) = \frac{1}{2}\left(\|v_t(t)\|_{L^2}^2 + \|w_t(t)\|_{L^2}^2 + \|w_{tt}(t)\|_{L^2}^2 + \|\nabla w_t(t)\|_{L^2}^2\right) \tag{4.26}$$

of the system with the corresponding dissipation

$$D_1(t) = \frac{1}{C}\|\nabla v_t(t)\|_{L^2}^2 + \alpha\|w_{tt}(t)\|_{L^2}^2 + \gamma\left\|\frac{\partial w_t}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2. \tag{4.27}$$

In order to obtain the integral inequality for $E_1(t)$, we differentiate the full system in time. We obtain

$$v_{tt}^i - \partial_t\partial_j(a_l^j a_l^k \partial_k v^i) + \partial_t\partial_k(a_i^k q) = 0 \qquad \text{in } \Omega_f \times (0,T), \quad i = 1,2,3 \tag{4.28}$$

$$a_i^k \partial_k v_t^i + \partial_t a_i^k \partial_k v^i = 0 \qquad \text{in } \Omega_f \times (0,T) \tag{4.29}$$

and

$$w_{ttt}^i - \Delta w_t^i + \alpha w_{tt}^i + w_t^i = 0 \qquad \text{in } \Omega_e \times (0,T), \quad i = 1,2,3 \tag{4.30}$$

since β was set to 1.

Lemma 4.5. *The energy inequality*

$$E_1(t) + \int_0^t D_1(s)\,\mathrm{d}s \leqslant E_1(0) + \int_0^t (R_1(s), v_t(s))\,\mathrm{d}s \tag{4.31}$$

holds for all $t \in [0,T]$, where

$$\int_0^t (R_1(s), v_t(s))\,\mathrm{d}s = -\int_0^t\int_{\Omega_f} \partial_t(a_l^j a_l^k)\partial_k v^i \partial_j v_t^i\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Omega_f} \partial_t a_i^k q \partial_k v_t^i\,\mathrm{d}x\,\mathrm{d}s$$
$$-\int_0^t\int_{\Omega_f} \partial_t a_i^k \partial_t q \partial_k v^i\,\mathrm{d}x\,\mathrm{d}s. \tag{4.32}$$

Proof of lemma 4.5. We take the L^2-inner product of (4.28) with v_t^i and of (4.30) with w_{tt}^i, respectively. Summing in i and adding the two estimates, we obtain

$$\frac{1}{2}\|v_t(t)\|_{L^2}^2 + \frac{1}{2}\|w_t(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla w_t(t)\|_{L^2}^2 + \frac{1}{2}\|w_{tt}(t)\|_{L^2}^2$$
$$+\frac{1}{C}\int_0^t \|\nabla v_t\|_{L^2}^2\,\mathrm{d}s + \alpha\int_0^t \|w_{tt}\|_{L^2}^2\,\mathrm{d}s + \gamma\int_0^t\int_{\Gamma_c}\left|\frac{\partial w_t}{\partial N}\right|^2\mathrm{d}\sigma(x)\,\mathrm{d}s$$
$$\leqslant \frac{1}{2}\|v_t(0)\|_{L^2}^2 + \frac{1}{2}\|w_t(0)\|_{L^2}^2 + \frac{1}{2}\|\nabla w_t(0)\|_{L^2}^2 + \frac{1}{2}\|w_{tt}(0)\|_{L^2}^2$$
$$-\int_0^t\int_{\Omega_f} \partial_t(a_l^j a_l^k)\partial_k v^i \partial_j v_t^i\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Omega_f} \partial_t(a_i^k q)\partial_k v_t^i\,\mathrm{d}x\,\mathrm{d}s, \tag{4.33}$$

where we utilized the boundary conditions

$$v_t^i - \gamma \partial_j w_t^i N_j = w_{tt}^i \qquad \text{on } \Gamma_c \times (0,T) \tag{4.34}$$

$$\partial_t (a_l^j a_l^k \partial_k v^i) N_j - \partial_t (a_i^k q) N_k = \partial_j w_t^i N_j \qquad \text{on } \Gamma_c \times (0,T) \tag{4.35}$$

$$v_t^i = 0 \qquad \text{on } \Gamma_f \times (0,T) \tag{4.36}$$

for $i = 1, 2, 3$. In order to make the last term on the right-hand side of (4.33) superquadratic, we rewrite it as

$$\begin{aligned} \int_0^t \int_{\Omega_f} \partial_t (a_i^k q) \partial_k v_t^i \,\mathrm{d}x\,\mathrm{d}s &= \int_0^t \int_{\Omega_f} \partial_t a_i^k q \partial_k v_t^i \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_f} a_i^k \partial_t q \partial_k v_t^i \,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t \int_{\Omega_f} \partial_t a_i^k q \partial_k v_t^i \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_{\Omega_f} \partial_t a_i^k \partial_t q \partial_k v^i \,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{4.37}$$

where we used (4.29) in the last step. The lemma is thus established. □

Note that, by lemma 4.5, we have

$$E_1(t) + \int_s^t D_1(\tau)\mathrm{d}\tau \leqslant E_1(s) + \int_s^t (R_1(\tau), v_t(\tau))\,\mathrm{d}\tau \tag{4.38}$$

for any $0 \leqslant s \leqslant t$.

Proceeding in the same manner as for the lower level energy, we obtain the counterparts of lemmas 4.2 and 4.3. The following statement asserts the equipartition of the second level energy.

Lemma 4.6. *We have*

$$\begin{aligned} &\frac{\alpha}{2}\|w_t(t)\|_{L^2}^2 + \int_0^t \|\nabla w_t(s)\|_{L^2}^2\,\mathrm{d}s + \int_0^t \|w_t(s)\|_{L^2}^2\,\mathrm{d}s \\ &\quad\leqslant \int_0^t \|w_{tt}\|_{L^2}^2\,\mathrm{d}s + CE_1(t) + CE_1(0) + C\int_0^t \int_{\Gamma_c} w_t \cdot \frac{\partial w_t}{\partial N}\,\mathrm{d}\sigma(x)\,\mathrm{d}s \end{aligned} \tag{4.39}$$

for all $t \in [0, T]$.

From lemmas 4.5 and 4.6, we conclude that

$$E_1(t) + \int_0^t E_1(s)\,\mathrm{d}s \leqslant CE_1(0) + C\int_0^t (R_1(s), v_t(s))\,\mathrm{d}s \tag{4.40}$$

for all $t \in [0, T]$, where the constant C denotes a generic constant which depends on the domains.

When $0 \leqslant \alpha \leqslant 1/C$, the flux multiplier argument applied to the differentiated wave equation (4.30) gives

$$\begin{aligned} &\int_0^t \left(\|\nabla w_t\|_{L^2}^2 + \|w_t\|_{L^2}^2 + \|w_{tt}\|_{L^2}^2\right)\mathrm{d}s \\ &\quad\leqslant C\int_0^t \int_{\Gamma_c} \left(|w_{tt}|^2 + \left|\frac{\partial w_t}{\partial N}\right|^2\right)\mathrm{d}\sigma(x)\,\mathrm{d}s + CE_1(t) + CE_1(0) \end{aligned} \tag{4.41}$$

as the analogue of (4.21). Using the boundary condition (4.34), we write

$$\|w_{tt}\|_{L^2(\Gamma_c)}^2 \leqslant C\|v_t\|_{L^2(\Gamma_c)}^2 + C\gamma^2 \left\|\frac{\partial w_t}{\partial N}\right\|_{L^2(\Gamma_c)}^2 \leqslant CD_1(t). \tag{4.42}$$

We now substitute (4.42) in (4.41), multiply the resulting inequality with a small constant and add it to (4.31). We obtain

$$E_1(t) + \int_0^t E_1(s)\,\mathrm{d}s \leqslant C E_1(t) + C E_1(0) + C\int_0^t D_1(s)\,\mathrm{d}s$$
$$\leqslant C E_1(0) + C\left(E_1(0) + \int_0^t (R_1(s), v_t(s))\,\mathrm{d}s\right) \qquad (4.43)$$

when $0 \leqslant \alpha \leqslant 1/C$ and the domain is star-shaped.

We summarize the estimates in the following statement.

Lemma 4.7. *For any $\alpha, \gamma > 0$, we have*

$$E_1(t) + \int_0^t E_1(s)\,\mathrm{d}s \leqslant C\left(E_1(0) + \int_0^t (R_1(s), v_t(s))\,\mathrm{d}s\right) \qquad (4.44)$$

where $C = C_{\alpha,\gamma}$. If Ω_e is star-shaped, then the above inequality holds for all $\alpha \geqslant 0$.

4.3. Third level estimates

Here we repeat the procedure applied to the second time derivatives of the system. We introduce the next level of energy

$$E_2(t) = \frac{1}{2}\left(\|v_{tt}(t)\|_{L^2}^2 + \|w_{tt}(t)\|_{L^2}^2 + \|w_{ttt}(t)\|_{L^2}^2 + \|\nabla w_{tt}(t)\|_{L^2}^2\right) \qquad (4.45)$$

with the corresponding dissipation

$$D_2(t) = \frac{1}{C}\|\nabla v_{tt}(t)\|_{L^2}^2 + \alpha\|w_{ttt}(t)\|_{L^2}^2 + \gamma\left\|\frac{\partial w_{tt}}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2. \qquad (4.46)$$

Differentiating the full system (2.4)–(2.6) twice in time, we obtain

$$v_{ttt}^i - \partial_{tt}\partial_j(a_l^j a_l^k \partial_k v^i) + \partial_{tt}\partial_k(a_i^k q) = 0 \qquad \text{in } \Omega_f \times (0, T) \qquad (4.47)$$
$$a_j^k \partial_k v_{tt}^j + 2\partial_t a_j^k \partial_k v_t^j + \partial_{tt} a_j^k \partial_k v^j = 0 \qquad \text{in } \Omega_f \times (0, T) \qquad (4.48)$$
$$w_{tttt}^i - \Delta w_{tt}^i + \alpha w_{ttt}^i + w_{tt}^i = 0 \qquad \text{in } \Omega_e \times (0, T) \qquad (4.49)$$

with the boundary conditions

$$w_{ttt}^i = v_{tt}^i - \gamma\partial_j w_{tt}^i N_j \qquad \text{on } \Gamma_c \times (0, T) \qquad (4.50)$$
$$\partial_j w_{tt}^i N_j = \partial_{tt}(a_l^j a_l^k \partial_k v^i) N_j - \partial_{tt}(a_i^k q) N_k \qquad \text{on } \Gamma_c \times (0, T) \qquad (4.51)$$
$$v_{tt}^i = 0 \qquad \text{on } \Gamma_f \times (0, T) \qquad (4.52)$$

for $i = 1, 2, 3$. □

Lemma 4.8. *The inequality*

$$E_2(t) + \int_0^t D_2(s)\,\mathrm{d}s \leqslant E_2(0) + \int_0^t (R_2(s), v_{tt}(s))\,\mathrm{d}s \qquad (4.53)$$

holds for all $t \in [0, T]$, where

$$\int_0^t (R_2(s), v_{tt}(s))\,\mathrm{d}s = 2\int_0^t\int_{\Omega_f} \partial_t(a_l^j a_l^k)\partial_k v_t^i \partial_j v_{tt}^i\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t\int_{\Omega_f} \partial_{tt}(a_l^j a_l^k)\partial_k v^i \partial_j v_{tt}^i\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_{\Omega_f} \partial_{tt}(a_i^k q)\partial_k v_{tt}^i\,\mathrm{d}x\,\mathrm{d}s \qquad (4.54)$$

and $E_2(0) = (1/2)(\|v_{tt}(0)\|_{L^2}^2 + \|w_{tt}(0)\|_{L^2}^2 + \|w_{ttt}(0)\|_{L^2}^2 + \|\nabla w_{tt}(0)\|_{L^2}^2)$.

Proof of lemma 4.8. Multiplying (4.47) by v^i_{tt}, integrating over Ω_f and summing for $i = 1, 2, 3$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{tt}\|^2_{L^2} + \int_{\Omega_f} \partial_{tt}(a^j_l a^k_l \partial_k v^i)\partial_j v^i_{tt}\,\mathrm{d}x + \int_{\Gamma_c} \partial_{tt}(a^j_l a^k_l \partial_k v^i) v^i_{tt} N_j\,\mathrm{d}\sigma(x)$$
$$- \int_{\Omega_f} \partial_{tt}(a^k_i q)\partial_k v^i_{tt}\,\mathrm{d}x - \int_{\Gamma_c} \partial_{tt}(a^k_i q) v^i_{tt} N_k\,\mathrm{d}\sigma(x) = 0, \tag{4.55}$$

after integrating by parts. Similarly, we multiply (4.49) by w^i_{ttt}, sum for $i = 1, 2, 3$ and integrate over Ω_e to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{ttt}\|^2_{L^2} + \alpha\|w_{ttt}(t)\|^2_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{tt}\|^2_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w_{tt}\|^2_{L^2} - \int_{\Gamma_c} \partial_k w^i_{tt} w^i_{ttt} N_k\,\mathrm{d}\sigma(x) = 0. \tag{4.56}$$

Adding (4.55) and (4.56) and applying the boundary conditions (4.50) and (4.51) leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v_{tt}\|^2_{L^2} + \|w_{ttt}\|^2_{L^2} + \|\nabla w_{tt}\|^2_{L^2} + \|w_{tt}\|^2_{L^2}\right) + \alpha\|w_{ttt}(t)\|^2_{L^2} + \gamma\left\|\frac{\partial w_{tt}}{\partial N}\right\|^2_{L^2(\Gamma_c)}$$
$$+ \int_{\Omega_f} a^j_l a^k_l \partial_k v^i_{tt}\partial_j v^i_{tt}\,\mathrm{d}x + 2\int_{\Omega_f} \partial_t(a^j_l a^k_l)\partial_k v^i_t \partial_j v^i_{tt}\,\mathrm{d}x$$
$$+ \int_{\Omega_f} \partial_{tt}(a^j_l a^k_l)\partial_k v^i \partial_j v^i_{tt}\,\mathrm{d}x - \int_{\Omega_f} \partial_{tt}(a^k_i q)\partial_k v^i_{tt}\,\mathrm{d}x = 0. \tag{4.57}$$

The proof is concluded using the ellipticity of a and integrating in time. □

Lemma 4.9. *For any $\alpha, \gamma > 0$ there exists $C > 0$ such that*

$$\int_0^t E_2(s)\,\mathrm{d}s + E_2(t) \leqslant C\left(E_2(0) + \int_0^t (R_2(s), v_{tt}(s))\,\mathrm{d}s\right) \tag{4.58}$$

where $C = C_{\alpha,\gamma}$. If Ω_e is star-shaped, then the above inequality holds for all $\alpha \geqslant 0$.

4.4. Superlinear estimates

The goal in this subsection is to provide estimates on the perturbation terms

$$\int_0^t (R_1(s), v_t(s))\,\mathrm{d}s \tag{4.59}$$

and

$$\int_0^t (R_2(s), v_{tt}(s))\,\mathrm{d}s \tag{4.60}$$

from (4.32) and (4.54), respectively. The first of these two quantities is estimated in the following way.

Lemma 4.10. *We have*

$$|(R_1(t), v_t)| \leqslant C\|v\|^{1/2}_{H^1}\|v\|^{1/2}_{H^2}\|v_t\|_{H^1}\left(\|v\|_{H^2} + \|q\|_{H^1}\right) + C\|v\|^{3/2}_{H^1}\|v\|^{1/2}_{H^2}\|q_t\|_{H^1} \tag{4.61}$$

for all $t \in [0, T]$.

Proof of lemma 4.10. First, we have

$$|(R_1, v_t)| \leqslant C \int_{\Omega_f} |a||a_t||\nabla v||\nabla v_t| \, dx + C \int_{\Omega_f} |a_t||q||\nabla v_t| \, dx + C \int_{\Omega_f} |a_t||q_t||\nabla v| \, dx$$
$$= R_{11} + R_{12} + R_{13}. \tag{4.62}$$

Using Hölder and Gagliardo–Nirenberg inequalities, we have

$$R_{11} \leqslant C \|a_t\|_{L^3} \|\nabla v\|_{L^6} \|\nabla v_t\|_{L^2} \leqslant C \|v\|_{H^1}^{1/2} \|v\|_{H^2}^{3/2} \|v_t\|_{H^1} \tag{4.63}$$

where we also used $\|a_t\|_{L^3} \leqslant C\|\nabla v\|_{L^3} \leqslant C\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}$ resulting from lemma 3.1(iii) in the last inequality. Similarly,

$$R_{12} \leqslant C \|a_t\|_{L^3} \|q\|_{L^6} \|v_t\|_{H^1} \leqslant C \|v\|_{H^1}^{1/2} \|v\|_{H^2}^{1/2} \|q\|_{H^1} \|v_t\|_{H^1} \tag{4.64}$$

and

$$R_{13} \leqslant C \|a_t\|_{L^3} \|q_t\|_{L^6} \|\nabla v\|_{L^2} \leqslant C \|v\|_{H^1}^{3/2} \|v\|_{H^2}^{1/2} \|q_t\|_{H^1}. \tag{4.65}$$

The proof is then concluded by summing the last three inequalities. □

Lemma 4.11. *For* $\epsilon_0 \in (0, 1/C]$, *we have*

$$\int_0^t (R_2(s), v_{tt}(s)) \, ds$$
$$\leqslant \epsilon_0 \int_0^t \|\nabla v_{tt}\|_{L^2}^2 \, ds + C_{\epsilon_0} \int_0^t \left(\|v\|_{H^3}^2 + \|q\|_{H^2}^2\right) \left(\|v\|_{H^1}^{5/2} \|v\|_{H^3}^{3/2} + \|v_t\|_{H^1}^2\right) ds$$
$$+ C_{\epsilon_0} \int_0^t \|v\|_{H^1}^{3/2} \|v\|_{H^3}^{1/2} \|q_t\|_{H^1}^2 \, ds + \epsilon_0 \|q_t(t)\|_{H^1}^2 + \epsilon_0 \|v_t(t)\|_{H^2}^2 + \epsilon_0 \|v(t)\|_{H^3}^2$$
$$+ C_{\epsilon_0} \|v(t)\|_{H^1}^6 \|v(t)\|_{H^2}^4 + C_{\epsilon_0} \|v(t)\|_{H^1}^2 \|v(t)\|_{H^2}^2 \|v_t(t)\|_{L^2}^2$$
$$+ C \int_0^t \left(\|v\|_{H^2}^2 + \|v_t\|_{H^1}^{1/2} \|v_t\|_{H^2}^{1/2}\right) \|q_t\|_{H^1} \|v_t\|_{H^1} \, ds$$
$$+ C \int_0^t \left(\|v\|_{H^2}^3 + \|v_t\|_{H^1} \|v\|_{H^1}^{1/4} \|v\|_{H^3}^{3/4}\right) \|q_t\|_{H^1} \|v\|_{H^1}^{3/4} \|v\|_{H^3}^{1/4} \, ds$$
$$+ C \|v(0)\|_{H^3}^6 + C \|v_t(0)\|_{H^1}^4 + C \|q_t(0)\|_{H^1}^2 \tag{4.66}$$

for all $t \in [0, T]$.

Proof. From (4.54), we have

$$\int_0^t (R_2(s), v_{tt}(s)) \, ds$$
$$\leqslant C \left| \int_0^t \int_{\Omega_f} \partial_t (a_l^j a_l^k) \partial_k v_t^i \partial_j v_{tt}^i \, dx \, ds \right| + C \left| \int_0^t \int_{\Omega_f} \partial_{tt} (a_l^j a_l^k) \partial_k v^i \partial_j v_{tt}^i \, dx \, ds \right|$$
$$+ C \left| \int_0^t \int_{\Omega_f} \partial_{tt} a_i^k q \partial_k v_{tt}^i \, dx \, ds \right| + C \left| \int_0^t \int_{\Omega_f} \partial_t a_i^k q_t \partial_k v_{tt}^i \, dx \, ds \right|$$
$$+ C \left| \int_0^t \int_{\Omega_f} a_i^k q_{tt} \partial_k v_{tt}^i \, dx \, ds \right|$$
$$= R_{21} + R_{22} + R_{23} + R_{24} + R_{25}. \tag{4.67}$$

Using Hölder's inequality and lemma 3.1, we obtain

$$R_{21}+R_{22}+R_{23}\leqslant C\int_0^t (\|\nabla v\|_{L^\infty}+\|q\|_{L^\infty})\,(\|\nabla v\|_{L^2}\|\nabla v\|_{L^\infty}+\|\nabla v_t\|_{L^2})\,\|\nabla v_{tt}\|_{L^2}\,\mathrm{d}s$$
$$\leqslant C\int_0^t (\|v\|_{H^3}+\|q\|_{H^2})\,(\|\nabla v\|_{L^2}\|\nabla v\|_{L^\infty}+\|\nabla v_t\|_{L^2})\,\|\nabla v_{tt}\|_{L^2}\,\mathrm{d}s$$
$$\leqslant C\int_0^t (\|v\|_{H^3}+\|q\|_{H^2})\left(\|v\|_{H^1}^{5/4}\|v\|_{H^3}^{3/4}+\|v_t\|_{H^1}\right)\|\nabla v_{tt}\|_{L^2}\,\mathrm{d}s \qquad (4.68)$$

and

$$R_{24}\leqslant C\int_0^t \|\nabla v\|_{L^3}\|q_t\|_{L^6}\|\nabla v_{tt}\|_{L^2}\,\mathrm{d}s\leqslant C\int_0^t \|v\|_{H^1}^{3/4}\|v\|_{H^3}^{1/4}\|q_t\|_{H^1}\|\nabla v_{tt}\|_{L^2}\,\mathrm{d}s, \qquad (4.69)$$

where we also utilized the Sobolev and the interpolation inequalities. In order to treat the term R_{25}, denote $I=\int_0^t\int_{\Omega_f} a_i^k q_{tt}\partial_k v_{tt}^i\,\mathrm{d}x\,\mathrm{d}s$. By the time differentiated divergence-free condition (4.48)

$$I=-2\int_0^t\int_{\Omega_f}\partial_t a_i^k q_{tt}\partial_k v_t^i\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\Omega_f}\partial_{tt}a_i^k q_{tt}\partial_k v^i\,\mathrm{d}x\,\mathrm{d}s, \qquad (4.70)$$

whence, integrating by parts in t in both integrals gives

$$\begin{aligned}I=&\,2\int_{\Omega_f}\partial_t a_i^k(t)q_t(t)\partial_k v_t^i(t)\,\mathrm{d}x+\int_{\Omega_f}\partial_{tt}a_i^k(t)q_t(t)\partial_k v^i(t)\,\mathrm{d}x\\&-2\int_{\Omega_f}\partial_t a_i^k(0)q_t(0)\partial_k v_t^i(0)\,\mathrm{d}x-\int_{\Omega_f}\partial_{tt}a_i^k(0)q_t(0)\partial_k v^i(0)\,\mathrm{d}x\\&+3\int_0^t\int_{\Omega_f}\partial_{tt}a_i^k q_t\partial_k v_t^i\,\mathrm{d}x\,\mathrm{d}s+2\int_0^t\int_{\Omega_f}\partial_t a_i^k q_t\partial_k v_{tt}^i\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\Omega_f}\partial_{ttt}a_i^k q_t\partial_k v^i\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \qquad (4.71)$$

Applying lemma 3.1 along with Hölder's and the Gagliardo–Nirenberg inequalities, we obtain

$$\begin{aligned}R_{25}\leqslant&\, C\|\nabla v(t)\|_{L^3}\|q_t(t)\|_{L^6}\|\nabla v_t(t)\|_{L^2}+C\|a_{tt}(t)\|_{L^2}\|q_t(t)\|_{L^6}\|\nabla v(t)\|_{L^3}\\&+C\|\nabla v(0)\|_{L^3}\|q_t(0)\|_{L^6}\|\nabla v_t(0)\|_{L^2}+C\|a_{tt}(0)\|_{L^2}\|q_t(0)\|_{L^6}\|\nabla v(0)\|_{L^3}\\&+C\int_0^t\|a_{tt}\|_{L^3}\|q_t\|_{L^6}\|\nabla v_t\|_{L^2}\,\mathrm{d}s+C\int_0^t\|\nabla v\|_{L^3}\|q_t\|_{L^6}\|\nabla v_{tt}\|_{L^2}\,\mathrm{d}s\\&+C\int_0^t\|a_{ttt}\|_{L^2}\|q_t\|_{L^6}\|\nabla v\|_{L^3}\,\mathrm{d}s.\end{aligned}$$

The sum of the first two terms on the right-hand side is bounded by

$$\begin{aligned}&C\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|q_t\|_{H^1}\|v_t\|_{L^2}^{1/2}\|v_t\|_{H^2}^{1/2}+\left(\|\nabla v\|_{L^2}\|\nabla v\|_{L^\infty}+\|\nabla v_t\|_{L^2}\right)\|q_t\|_{H^1}\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\\&\quad\leqslant C\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|q_t\|_{H^1}\|v_t\|_{L^2}^{1/2}\|v_t\|_{H^2}^{1/2}+C\|v\|_{H^1}^{3/2}\|v\|_{H^2}\|v\|_{H^3}^{1/2}\|q_t\|_{H^1}\\&\qquad+C\|v_t\|_{L^2}^{1/2}\|v_t\|_{H^2}^{1/2}\|q_t\|_{H^1}\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\\&\quad\leqslant\epsilon_0\|q_t\|_{H^1}^2+\epsilon_0\|v_t\|_{H^2}^2+\epsilon_0\|v\|_{H^3}^2+C_{\epsilon_0}\|v\|_{H^1}^2\|v\|_{H^2}^2\|v_t\|_{L^2}^2+C_{\epsilon_0}\|v\|_{H^1}^6\|v\|_{H^2}^4\end{aligned} \qquad (4.72)$$

using parts (vi) and (vii) of lemma 3.1. Therefore, using in particular Agmon's inequality $\|u\|_{L^\infty}\leqslant C\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}$, we obtain

$$\begin{aligned}R_{25}\leqslant&\,\epsilon_0\|q_t(t)\|_{H^1}^2+\epsilon_0\|v_t\|_{H^2}^2+\epsilon_0\|v\|_{H^3}^2+C_{\epsilon_0}\|v\|_{H^1}^2\|v\|_{H^2}^2\|v_t\|_{L^2}^2+C_{\epsilon_0}\|v\|_{H^1}^6\|v\|_{H^2}^4\\&+C\|v(0)\|_{H^3}^6+C\|v_t(0)\|_{H^1}^4+C\|q_t(0)\|_{H^1}^2\end{aligned}$$

$$+C\int_0^t \left(\|v\|_{H^2}^2 + \|\nabla v_t\|_{L^2}^{1/2}\|\nabla v_t\|_{H^1}^{1/2}\right)\|q_t\|_{H^1}\|\nabla v_t\|_{L^2}\,\mathrm{d}s$$
$$+C\int_0^t \|\nabla v\|_{L^2}^{3/4}\|\nabla v\|_{H^2}^{1/4}\|q_t\|_{H^1}\|\nabla v_{tt}\|_{L^2}\,\mathrm{d}s$$
$$+C\int_0^t \left(\|v\|_{H^2}^3 + \|\nabla v_t\|_{L^2}\|\nabla v\|_{L^\infty} + \|\nabla v_{tt}\|_{L^2}\right)\|q_t\|_{H^1}\|\nabla v\|_{L^3}\,\mathrm{d}s.$$

Thus the proof of the lemma is complete. □

5. Proof of theorem 2.1

We introduce the norm

$$X(t) = E(t) + E_1(t) + E_2(t) + \epsilon_1\|\nabla v(t)\|_{L^2}^2 + \epsilon_1\|\nabla v_t(t)\|_{L^2}^2 \tag{5.1}$$

where $\epsilon_1 > 0$ is a small parameter which is to be determined. In order to control the terms $\|\nabla v(t)\|_{L^2}$ and $\|\nabla v_t(t)\|_{L^2}$, we use the estimates

$$\|\nabla v(t)\|_{L^2}^2 - \|\nabla v(0)\|_{L^2}^2 = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\|\nabla v(s)\|_{L^2}^2\,\mathrm{d}s \leqslant 2\int_0^t \|\nabla v\|_{L^2}\|\nabla v_t\|_{L^2}\,\mathrm{d}s, \tag{5.2}$$

whence

$$\|\nabla v(t)\|_{L^2}^2 \leqslant \|\nabla v(0)\|_{L^2}^2 + C\int_0^t \left(D(s) + D_1(s)\right)\mathrm{d}s, \tag{5.3}$$

and

$$\|\nabla v_t(t)\|_{L^2}^2 - \|\nabla v_t(0)\|_{L^2}^2 = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\|\nabla v_t(s)\|_{L^2}^2\,\mathrm{d}s \leqslant 2\int_0^t \|\nabla v_t\|_{L^2}\|\nabla v_{tt}\|_{L^2}\,\mathrm{d}s, \tag{5.4}$$

which implies

$$\|\nabla v_t(t)\|_{L^2}^2 \leqslant \|\nabla v_t(0)\|_{L^2}^2 + C\int_0^t \left(D_1(s) + D_2(s)\right)\mathrm{d}s. \tag{5.5}$$

From section 4.1, we have

$$E(t) + \int_0^t E(s)\,\mathrm{d}s + \int_0^t D(s)\,\mathrm{d}s \leqslant CE(0). \tag{5.6}$$

Section 4.2, combined with lemma 4.10, gives

$$E_1(t) + \int_0^t E_1(s)\,\mathrm{d}s + \int_0^t D_1(s)\,\mathrm{d}s \leqslant CE_1(0) + \int_0^t P_1(\|v\|_{H^2}, \|q\|_{H^1}, \|v_t\|_{H^1}, \|q_t\|_{H^1})\,\mathrm{d}s, \tag{5.7}$$

while from section 4.3, combined with lemma 4.11,

$$E_2(t) + \int_0^t E_2(s)\,\mathrm{d}s + \int_0^t D_2(s)\,\mathrm{d}s$$
$$\leqslant CE_2(0) + \epsilon_0\|v(t)\|_{H^3}^2 + \epsilon_0\|v_t(t)\|_{H^2}^2 + \epsilon_0\|q_t(t)\|_{H^1}^2 + \epsilon_0\int_0^t \|\nabla v_{tt}\|_{L^2}^2\,\mathrm{d}s$$
$$+P_2(\|v\|_{H^2}, \|v_t\|_{L^2}) + \int_0^t P_3(\|v\|_{H^3}, \|q\|_{H^2}, \|v_t\|_{H^2}, \|q_t\|_{H^1})\,\mathrm{d}s$$
$$+P_4(\|v(0)\|_{H^3}, \|v_t(0)\|_{H^1}, \|q_t(0)\|_{H^1}). \tag{5.8}$$

In (5.7), (5.8) and below, the symbols P_1, P_2, P_3 and P_4 denote the superlinear polynomials of their arguments, which are allowed to depend on ϵ_0 from lemma 4.11. Now, multiply (5.3) and (5.5) with ϵ_1 and add the resulting inequalities to the sum of (5.6), (5.7) and (5.8) while choosing (and fixing) ϵ_1 sufficiently small. We obtain

$$X(t) + \int_0^t X(s)\,\mathrm{d}s \leqslant CX(0) + \epsilon_0\|v(t)\|_{H^3}^2 + \epsilon_0\|v_t(t)\|_{H^2}^2 + \epsilon_0\|q_t(t)\|_{H^1}^2$$
$$+P_1(\|v\|_{H^2}, \|v_t\|_{L^2}) + \int_0^t P_2(\|v\|_{H^3}, \|q\|_{H^2}, \|v_t\|_{H^2}, \|q_t\|_{H^1})\,\mathrm{d}s$$
$$+P_3(\|v(0)\|_{H^3}, \|v_t(0)\|_{H^1}, \|q_t(0)\|_{H^1}) \tag{5.9}$$

where P_1, P_2 and P_3 are superlinear polynomials different from above.

Now, from (3.14), we obtain

$$\|v\|_{H^2}^2 + \|q\|_{H^1}^2 \leqslant CX(t) \tag{5.10}$$

and then, using (3.13) and (5.10),

$$\|v\|_{H^3}^2 + \|q\|_{H^2}^2 \leqslant CX(t). \tag{5.11}$$

From (3.15) and (5.11), we obtain

$$\|v_t\|_{H^2}^2 + \|q_t\|_{H^1}^2 \leqslant CX(t) + C\|v\|_{H^3}^{1/2}X(t)^{3/2} \leqslant CX(t) + CX(t)^2. \tag{5.12}$$

Using (5.11) and (5.12) and choosing $\epsilon_0 > 0$ sufficiently small, we obtain from (5.9)

$$X(t) + \int_0^t X(s)\,\mathrm{d}s \leqslant CX(0) + P(X(t)) + \int_0^t P(X(s))\,\mathrm{d}s + P(X(0)), \tag{5.13}$$

where P is a superlinear polynomial. We may rewrite this as

$$X(t) + \int_0^t X(s)\,\mathrm{d}s \leqslant C_0X(0) + C_0\sum_{j=1}^m \int_0^t X(s)^{\alpha_j}\,\mathrm{d}s + C_0\sum_{k=1}^n X(t)^{\beta_k} + C_0\sum_{k=1}^n X(0)^{\beta_k}, \tag{5.14}$$

for $C_0 \geqslant 1, \alpha_1, \ldots, \alpha_m > 1$ and $\beta_1, \ldots, \beta_n > 1$.

The proof of theorem 2.1 follows from the following auxiliary assertion.

Lemma 5.1. *Suppose that* $X\colon [0, \infty) \to [0, \infty]$ *is continuous for all* t *such that* $X(t)$ *is finite and assume that it satisfies*

$$X(t) + \int_\tau^t X(s)\,\mathrm{d}s \leqslant C_0\sum_{j=1}^m \int_\tau^t X(s)^{\alpha_j}\,\mathrm{d}s + C_0\sum_{k=1}^n X(t)^{\beta_k} + C_0\sum_{k=1}^n X(\tau)^{\beta_k} + C_0X(\tau), \tag{5.15}$$

where $\alpha_1, \ldots, \alpha_m > 1$ *and* $\beta_1, \ldots, \beta_n > 1$. *Also, assume that* $X(0) \leqslant \epsilon$. *If* $\epsilon \leqslant 1/C$, *where the constant* C *depends on* C_0, m, $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_n$, *we have* $X(t) \leqslant C\epsilon \mathrm{e}^{-t/C}$.

Remark 5.2. The global existence is based on the term $\int_\tau^t X(s)\,\mathrm{d}s$ on the left-hand side of (5.15) which provides a strong dissipative mechanism which controls the potentially exponential increase in the first two terms on the right-hand side of (5.15) (note the superlinear character of terms in the sums due to the assumptions $\alpha_1, \ldots, \alpha_m > 1$ and $\beta_1, \ldots, \beta_n > 1$) when observed on a long enough, but constant size, interval $[0, 8C_0]$. In the first part of the proof, we show that if $\epsilon > 0$ is sufficiently small, there exists a time $t_1 \in [4C_0, 8C_0]$ such that $X(t_1) \leqslant \epsilon/2$. Then one may repeat the argument with initial time t_1 and obtain a time $t_2 \in [t_1 + 4C_0, t_1 + 8C_0]$ such that $X(t_2) \leqslant \epsilon/4$, etc. On each interval $[t_j, t_{j+1}]$, where $j = 1, 2, \ldots$, the function X can only grow by a constant factor since the size of the interval is bounded by $8C_0$, and thus the exponential decay over $[0, \infty)$ is obtained.

Proof of lemma 5.1. First we show that the time of existence can be made arbitrarily large if $\epsilon > 0$ is sufficiently small. Let $X(0) \leqslant \epsilon$ for $\epsilon \in (0, 1/2]$. Also, let T be the time such that $X(t) < 2C_0\epsilon$ for $t \in [0, T)$ and $X(T) = 2C_0\epsilon$. Then, by (5.15), we obtain

$$X(t) \leqslant C_0 \sum_{j=1}^{m} \int_0^t X(s)^{\alpha_j} \, ds + C_0 \sum_{k=1}^{n} X(t)^{\beta_k} + C_0 \sum_{k=1}^{n} \epsilon^{\beta_k} + C_0\epsilon, \qquad t \in [0, T], \tag{5.16}$$

which at time $t = T$ gives

$$C_0\epsilon \leqslant C_0 \sum_{j=1}^{m} (2C_0\epsilon)^{\alpha_j} T + C_0 \sum_{k=1}^{n} (2C_0\epsilon)^{\beta_k} + C_0 \sum_{k=1}^{n} \epsilon^{\beta_k}. \tag{5.17}$$

Using $\alpha_1, \ldots, \alpha_m > 1$ and $\beta_1, \ldots, \beta_n > 1$, we obtain

$$T \geqslant \frac{1}{C\epsilon^k} \tag{5.18}$$

with positive constants C and k depending on C_0, m, $\alpha_1, \ldots, \alpha_m$, and $\beta_1, \ldots, \beta_n$. Thus, if $\epsilon \to 0$, we have $T \to \infty$.

Next, we show that if $\epsilon > 0$ is sufficiently small, $X(t)$ eventually equals $\epsilon/2$ and we also estimate from above the time t when this happens. First, let $T = 8C_0$; we claim that there exists $t \in [T/2, T]$ such that $X(t) \leqslant \epsilon/2$ provided $\epsilon > 0$ is sufficiently small (specified below). For the sake of obtaining a contradiction, assume that

$$X(t) > \epsilon/2, \qquad t \in \left[\frac{T}{2}, T\right], \tag{5.19}$$

where $T = 8C_0$. By the first part of this proof, we may choose $\epsilon > 0$ so small that

$$X(t) \leqslant 2C_0\epsilon, \qquad t \in [0, T]. \tag{5.20}$$

Then the inequality (5.15) used with $t = T$ and $\tau = 0$ combined with (5.19) and (5.20) gives

$$\frac{T}{2}\frac{\epsilon}{2} \leqslant C_0 T \sum_{j=1}^{m} (2C_0\epsilon)^{\alpha_j} + C_0 \sum_{k=1}^{n} (2C_0\epsilon)^{\beta_k} + C_0 \sum_{k=1}^{n} \epsilon^{\beta_k} + C_0\epsilon \tag{5.21}$$

which, dividing the equation by ϵ and using $T = 8C_0$, may be rewritten as

$$C_0 \leqslant 8C_0^2 \sum_{j=1}^{m} (2C_0)^{\alpha_j} \epsilon^{\alpha_j - 1} + C_0 \sum_{k=1}^{n} (2C_0)^{\beta_k} \epsilon^{\beta_k - 1} + C_0 \sum_{k=1}^{n} \epsilon^{\beta_k - 1}. \tag{5.22}$$

This leads to a contradiction if $\epsilon > 0$ is sufficiently small. Clearly, the upper bound for $\epsilon > 0$ when this happens can be easily obtained. This contradiction shows that $X(t_1) \leqslant \epsilon/2$ for some $t_1 \in [4C_0, 8C_0]$. Repeating this argument, we obtain the existence of $t_2 \in [t_1 + 4C_0, t_1 + 8C_0]$ such that $X(t_2) \leqslant \epsilon/4$. Continuing the procedure by mathematical induction yields finiteness and an exponential decay for $X(t)$. □

Proof of theorem 2.1. We fix $T = 8C_0$ as in the proof of lemma 5.1. Then $X(t) \leqslant 2C_0\epsilon$ for all $t \in [0, T]$ and by lemma 5.1 there exists $t_1 \in [T/2, T]$ such that $X(t_1) \leqslant \epsilon/2$. By (3.13), we have

$$\|v(t)\|_{H^3}^2 \leqslant CX(t) \leqslant C\epsilon \tag{5.23}$$

for all $t \in [0, T]$. In particular, the two assertions in part (viii) of lemma 3.1 hold for all $t \in [0, T]$. Indeed, for the first estimate in part (viii), we obtain

$$\|\delta_{jk} - a_k^j(t)\|_{H^2}^2 \leqslant T \int_0^T \|\partial_t a_k^j(s)\|_{H^2}^2 \, ds \leqslant CT \int_0^T \|v(s)\|_{H^3}^2 \, ds \leqslant C\epsilon \tag{5.24}$$

for all $t \in [0, T]$. Thus, we may establish the validity of (5.15) and by the inductive argument from the end of the proof of lemma 5.1, we conclude that $\|v(t)\|_{H^3}^2 \leqslant C\epsilon e^{-t/C}$ and $\|\delta_{jk} - a_k^j(t)\|_{H^2}^2 \leqslant C\epsilon$ for all $t > 0$. □

6. Construction of solutions

We first construct solutions in lemma 6.1 to the linear problem for given matrix a with coefficients $a_j^i = \delta_{ij}$ for $i, j = 1, 2, 3$, and for given nonzero forcing f, nonzero divergence condition g and nonzero difference of stresses h on the common boundary Γ_c. Then, in the general case of given smooth elliptic matrix $a(x, t)$, we apply a fixed point technique to the perturbed linear system (6.4)–(6.6) where

$$f^i = -\partial_j \left((\delta_{jk} - a_l^j a_l^k)\partial_k v^i\right) + \partial_k \left((\delta_{ik} - a_i^k)q\right), \tag{6.1}$$

$$g = (\delta_{jk} - a_j^k)\partial_k v^j, \tag{6.2}$$

$$h^i = (\delta_{jk} - a_l^j a_l^k)\partial_k v^i N_j + (\delta_{ik} - a_i^k)q N_k \tag{6.3}$$

for $i = 1, 2, 3$.

Lemma 6.1. *Let $\alpha \geqslant 0$, $\beta > 0$ and $\gamma > 0$. Consider the linear coupled Stokes-wave system*

$$v_t - \Delta v + \nabla q = f \qquad \text{in } \Omega_f \times (0, T) \tag{6.4}$$

$$\nabla \cdot v = g \qquad \text{in } \Omega_f \times (0, T) \tag{6.5}$$

$$w_{tt} - \Delta w + \alpha w_t + \beta w = 0 \qquad \text{in } \Omega_e \times (0, T) \tag{6.6}$$

with the boundary conditions

$$\nabla w \cdot N = \gamma^{-1}(v - w_t) \qquad \text{on } \Gamma_c \times (0, T) \tag{6.7}$$

$$v = 0 \qquad \text{on } \Gamma_f \times (0, T) \tag{6.8}$$

$$\nabla v \cdot N - qN = \nabla w \cdot N + h \qquad \text{on } \Gamma_c \times (0, T). \tag{6.9}$$

Assume that $(v_0, w_0, w_1) \in (V \cap H^4(\Omega_f)) \times H^3(\Omega_e) \times H^2(\Omega_e)$ is subject to the compatibility conditions

$$w_1 = v_0 - \gamma \nabla w_0 \cdot N \qquad \text{on } \Gamma_c \tag{6.10}$$

$$\frac{\partial w_0}{\partial N} \cdot \tau = \frac{\partial v_0}{\partial N} \cdot \tau - h(0) \cdot \tau \qquad \text{on } \Gamma_c \tag{6.11}$$

$$v_0 = 0 \qquad \text{on } \Gamma_f, \tag{6.12}$$

with

$$w_{tt}(0) = v_t(0) - \gamma \nabla w_1 \cdot N \qquad \text{on } \Gamma_c \tag{6.13}$$

$$\frac{\partial w_1}{\partial N} \cdot \tau = \frac{\partial v_t(0)}{\partial N} \cdot \tau - h_t(0) \cdot \tau \qquad \text{on } \Gamma_c \tag{6.14}$$

$$v_t(0) = 0 \qquad \text{on } \Gamma_f, \tag{6.15}$$

and

$$w_{ttt}(0) = v_{tt}(0) - \gamma \nabla w_{tt}(0) \cdot N \qquad \text{on } \Gamma_c \tag{6.16}$$

$$\frac{\partial w_{tt}(0)}{\partial N} \cdot \tau = \frac{\partial v_{tt}(0)}{\partial N} \cdot \tau - h_{tt}(0) \cdot \tau \qquad \text{on } \Gamma_c \tag{6.17}$$

$$v_{tt}(0) = 0 \qquad \text{on } \Gamma_f, \tag{6.18}$$

and $f \in L^\infty([0,T];H^1(\Omega_f))$, $f_t \in L^\infty([0,T];L^2(\Omega_f))$, $f_{tt} \in L^2([0,T];H^{-1}(\Omega_f))$, $g \in C([0,T];H^2(\Omega_f))$, $g_t \in L^\infty([0,T];H^1(\Omega_f))$, $g_{tt} \in L^2([0,T];L^2(\Omega_f)) \cap L^\infty([0,T];H^{-1}(\Omega_f))$, $h \in L^\infty([0,T];H^{3/2}(\Gamma_c))$, $h_t \in L^\infty([0,T];H^{1/2}(\Gamma_c))$, $h_{tt} \in L^2([0,T];H^{-1/2}(\Gamma_c))$ *for some time* $T > 0$ *with* $g(0) = 0$*. Then there exists a unique solution* (v, w, q) *satisfying*

$$\begin{aligned}
&v \in L^\infty([0,T];H^3(\Omega_f)), && v_t \in L^\infty([0,T];H^2(\Omega_f)),\\
&v_{tt} \in L^\infty([0,T];L^2(\Omega_f)), && \nabla v_{tt} \in L^2([0,T];L^2(\Omega_f))\\
&\partial_t^j w \in C([0,T];H^{3-j}(\Omega_e)), && j = 0,1,2,3\\
&q \in L^\infty([0,T];H^2(\Omega_f)), && q_t \in L^\infty([0,T];H^1(\Omega_f)).
\end{aligned}$$

Note that the pressure q_0 solves the elliptic problem

$$\begin{aligned}
&\Delta q_0 = -\partial_i v_0^k \partial_k v_0^i + \operatorname{div} f(0) && \text{in } \Omega_f\\
&\nabla q_0 \cdot N = \Delta v_0 \cdot N + f(0)\cdot N && \text{on } \Gamma_f\\
&q_0 = \partial_j v_0^i N_j N_i - \partial_j w_0^i N_j N_i - h^i(0) N_i && \text{on } \Gamma_c.
\end{aligned}$$

Proof of lemma 6.1. We change variables $u = v - z$, where $z = z_1 + z_2 + z_3$. Let $E\colon H^s(\Omega_f) \to H^s(\mathbb{R}^3)$ be the extension operator which is continuous for $s = -1, 0, 1, 2$ and satisfies $Eg = g$ in Ω_f. We define the variable z_1 on the whole space $\mathbb{R}^3$ as the unique solution to the stationary Stokes problem with nonzero divergence

$$-\Delta z_1 + \nabla q_1 = 0 \qquad \text{in } \mathbb{R}^3 \times (0,T) \tag{6.19}$$

$$\nabla \cdot z_1 = Eg \qquad \text{in } \mathbb{R}^3 \times (0,T). \tag{6.20}$$

The existence and uniqueness of the solution (z_1, q_1) to (6.19) and (6.20) is classical. Also, the estimate

$$\|z_1\|_{L^\infty([0,T];H^3(\Omega_f))} + \|q_1\|_{L^\infty([0,T];H^2(\Omega_f))} \leqslant C\|Eg\|_{L^\infty([0,T];H^2(\mathbb{R}^3))} \tag{6.21}$$

is valid for $Eg \in L^\infty([0,T];H^2(\mathbb{R}^3))$. Differentiating the system (6.19) and (6.20) in time, we have

$$\|(z_1)_t\|_{L^\infty([0,T];H^2(\Omega_f))} + \|(q_1)_t\|_{L^\infty([0,T];H^1(\Omega_f))} \leqslant C\|Eg_t\|_{L^\infty([0,T];H^1(\mathbb{R}^3))}. \tag{6.22}$$

Also, by differentiating twice in time,

$$\|(z_1)_{tt}\|_{L^2([0,T];H^1(\Omega_f))} + \|(q_1)_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \leqslant C\|Eg_{tt}\|_{L^2([0,T];L^2(\mathbb{R}^3))} \tag{6.23}$$

and

$$\|(z_1)_{tt}\|_{L^\infty([0,T];L^2(\Omega_f))} \leqslant C\|Eg_{tt}\|_{L^\infty([0,T];H^{-1}(\mathbb{R}^3))}. \tag{6.24}$$

Using the continuity of E, we have

$$\begin{aligned}
&\|Eg\|_{H^2(\mathbb{R}^3)} \leqslant C\|g\|_{H^2(\Omega_f)}, && \|Eg_t\|_{H^1(\mathbb{R}^3)} \leqslant C\|g_t\|_{H^1(\Omega_f)},\\
&\|Eg_{tt}\|_{L^2(\mathbb{R}^3)} \leqslant C\|g_{tt}\|_{L^2(\Omega_f)}, && \|Eg_{tt}\|_{H^{-1}(\mathbb{R}^3)} \leqslant C\|g_{tt}\|_{H^{-1}(\Omega_f)}.
\end{aligned}$$

Thus we conclude that $z_1 \in L^\infty([0,T];H^3(\Omega_f))$, $(z_1)_t \in L^\infty([0,T];H^2(\Omega_f))$, $(z_1)_{tt} \in L^2([0,T];H^1(\Omega_f)) \cap L^\infty([0,T];L^2(\Omega_f))$. In particular, the normal trace of $(z_1)_{tt}$ is well defined and

$$(z_1)_{tt} \cdot N \in L^2([0,T];H^{1/2}(\Gamma_f \cup \Gamma_c)). \tag{6.25}$$

We define z_2 as the solution of the Stokes problem with Dirichlet boundary data

$$-\Delta z_2 + \nabla q_2 = 0 \qquad \text{in } \Omega_f \times (0,T) \tag{6.26}$$

$$\nabla \cdot z_2 = 0 \qquad \text{in } \Omega_f \times (0,T) \tag{6.27}$$

$$z_2 = -z_1 + \lambda\psi \qquad \text{on } \Gamma_c \times (0,T), \tag{6.28}$$

$$z_2 = -z_1 \qquad \text{on } \Gamma_f \times (0,T), \tag{6.29}$$

where ψ is a smooth compactly supported function on Γ_c such that $\psi \geqslant 0$ with $\int_{\Gamma_c} \psi \cdot N \, d\sigma = 1$ and $\lambda(t) = \int_{\Gamma_c} z_1(\cdot, t) \cdot N \, d\sigma - \int_{\Gamma_f} z_1(\cdot, t) \cdot N \, d\sigma$ for $t \in (0, T)$. Observe that $z_1 + z_2$ solves the problem

$$-\Delta(z_1 + z_2) + \nabla(q_1 + q_2) = 0 \qquad \text{in } \Omega_f \times (0, T) \tag{6.30}$$

$$\nabla \cdot (z_1 + z_2) = g \qquad \text{in } \Omega_f \times (0, T) \tag{6.31}$$

$$z_1 + z_2 = \lambda\psi \qquad \text{on } \Gamma_c \times (0, T), \tag{6.32}$$

$$z_1 + z_2 = 0 \qquad \text{on } \Gamma_f \times (0, T), \tag{6.33}$$

with the smooth boundary data $\lambda\psi$ on $\Gamma_c \times (0, T)$. The variable z_3 is defined as the solution of the Stokes system with zero divergence

$$(z_3)_t - \Delta z_3 + \nabla q_3 = -(z_1)_t - (z_2)_t \qquad \text{in } \Omega_f \times (0, T) \tag{6.34}$$

$$\nabla \cdot z_3 = 0 \qquad \text{in } \Omega_f \times (0, T) \tag{6.35}$$

$$z_3 = 0 \qquad \text{on } \Gamma_c \cup \Gamma_f \times (0, T) \tag{6.36}$$

with the initial data $z_3(\cdot, 0) = 0$.

The existence and uniqueness of the solution to (6.30)–(6.33) is well-known (see [Te, theorem I.2.4]). Thus, we may conclude $z_1 + z_2 \in L^\infty([0, T]; H^3(\Omega_f))$, $(z_1 + z_2)_t \in L^\infty([0, T]; H^2(\Omega_f))$, $(z_1 + z_2)_{tt} \in L^2([0, T]; H^1(\Omega_f))$.

Next, denote by $A = -P\Delta$ the Stokes operator, where P is the Leray projection on the space of divergence-free functions. Then, we may rewrite the system (6.34) and (6.35) for z_3 in the equivalent form

$$(z_3)_t + Az_3 = -P((z_1)_t + (z_2)_t). \tag{6.37}$$

Note that

$$e^{tA} \colon D(A^\theta) \to L^2((0, T); D(A^{\theta+1/2})) \cap C([0, T], D(A^\theta))$$

for $\theta \in [0, 1]$. The solution of (6.37) is given by

$$z_3(t) = -\int_0^t e^{(s-t)A} P((z_1)_t(s) + (z_2)_t(s)) \, ds + e^{-tA} z_3(0) \tag{6.38}$$

$$= -\int_0^t e^{(s-t)A} P((z_1)_t(s) + (z_2)_t(s)) \, ds, \tag{6.39}$$

as $z_3(0) = 0$. Similarly, by differentiating (6.34) in time and using that the Stokes operator commutes with time derivatives, we obtain

$$(z_3)_t(t) = -\int_0^t e^{(s-t)A} P((z_1)_{tt}(s) + (z_2)_{tt}(s)) \, ds + e^{-tA} (z_3)_t(0) \tag{6.40}$$

and

$$(z_3)_{tt}(t) = -\int_0^t e^{(s-t)A} P((z_1)_{ttt}(s) + (z_2)_{ttt}(s)) \, ds + e^{-tA} (z_3)_{tt}(0). \tag{6.41}$$

We note that $(z_3)_t(0) \in D(A)$. Indeed, from (6.34) and $z_3(0) = 0$ we obtain $(z_3)_t(0) + \nabla q_3(0) = -(z_1 + z_2)_t(0) \in H^2(\Omega_f)$ supplemented by $\nabla \cdot (z_3)_t(0) = 0$ in Ω_f and $(z_3)_t(0) = 0$ on $\Gamma_f \cup \Gamma_c$. Now, we observe that $q_3(0)$ solves $\Delta q_3(0) = -\nabla \cdot (z_1 + z_2)_t(0) \in H^1(\Omega_f)$ with Neumann data of $\nabla q_3(0) \cdot N = -(z_1 + z_2)_t(0) \cdot N \in H^{3/2}(\Gamma_f \cup \Gamma_c)$. By standard elliptic regularity, we also have $q_3(0) \in H^3(\Omega_f)$. Thus, we conclude $(z_3)_t(0) \in D(A)$.

By the maximal regularity of the Stokes semigroup, we have that the singular integral on the right side of (6.40) is a mapping from the space $L^2([0, T]; D(A^{1/2}))$ to $L^2([0, T]; D(A^{3/2})) \cap C([0, T]; D(A))$. Thus, by the embeddings of $D(A^{3/2})$ in H^3 and $D(A)$ in H^2, it follows that $(z_3)_t \in L^2([0,T];H^3(\Omega_f)) \cap C([0,T];H^2(\Omega_f))$. In particular, we have $z_3 \in L^\infty([0,T];H^3(\Omega_f))$ since $z_3 \in H^1([0,T];H^3(\Omega_f))$. Now, integrating (6.41) by parts in time, we obtain

$$-\int_0^t e^{(s-t)A} P((z_1)_{ttt}(s) + (z_2)_{ttt})\,ds \tag{6.42}$$
$$= -\int_0^t Ae^{(s-t)A} P((z_1)_{tt}(s) + (z_2)_{tt}(s))\,ds - P((z_1)_{tt}(t) + (z_2)_{tt}(t))$$
$$+e^{-tA} P((z_1)_{tt}(0) + (z_2)_{tt}(0)),$$

whence

$$(z_3)_{tt}(t) = -\int_0^t Ae^{(s-t)A} P((z_1)_{tt}(s) + (z_2)_{tt}(s))\,ds - P((z_1)_{tt}(t) + (z_2)_{tt}(t)) + e^{-tA}(z_{tt}(0)) \tag{6.43}$$

and, by the maximal regularity of the Stokes semigroup, $(z_3)_{tt} \in L^2([0,T];H^1(\Omega_f))$.

Here we utilized $z_{tt}(0) \in H$. We note that from (6.40) it follows $(z_3)_{tt}(0) + A(z_3)_t(0) = -(z_1+z_2)_{tt}(0)$. Since we already have that $(z_3)_t(0) \in D(A)$, we may conclude $(z_3)_{tt}(0) + (z_1+z_2)_{tt}(0) \in H$ which is equivalent to $z_{tt}(0) \in H$.

Observe that $(z,\bar{q})$ satisfies the Stokes system

$$z_t - \Delta z + \nabla\bar{q} = 0 \qquad \text{in } \Omega_f \times (0,T) \tag{6.44}$$
$$\nabla \cdot z = g \qquad \text{in } \Omega_f \times (0,T) \tag{6.45}$$
$$z = \lambda\psi \qquad \text{on } \Gamma_c \times (0,T) \tag{6.46}$$
$$z = 0 \qquad \text{on } \Gamma_f \times (0,T) \tag{6.47}$$

with $z(\cdot,0) = 0$, and we have

$$z \in L^\infty([0,T];H^3(\Omega_f)), \quad z_t \in L^\infty([0,T];H^2(\Omega_f)), \quad z_{tt} \in L^2([0,T];H^1(\Omega_f)), \tag{6.48}$$

since $z = z_1 + z_2 + z_3$ and $\bar{q} = q_1 + q_2 + q_3$.

In terms of the new variable u we obtain the divergence-free linear Stokes-wave system

$$u_t - \Delta u + \nabla q = \tilde{f} \qquad \text{in } \Omega_f \times (0,T) \tag{6.49}$$
$$\nabla \cdot u = 0 \qquad \text{in } \Omega_f \times (0,T) \tag{6.50}$$
$$w_{tt} - \Delta w + \alpha w_t + \beta w = 0 \qquad \text{in } \Omega_e \times (0,T) \tag{6.51}$$

with boundary conditions

$$\nabla w \cdot N = \gamma^{-1}(u + z - w_t) \qquad \text{on } \Gamma_c \times (0,T) \tag{6.52}$$
$$u = 0 \qquad \text{on } \Gamma_f \times (0,T) \tag{6.53}$$
$$\nabla u \cdot N - qN = \nabla w \cdot N + \tilde{h} \qquad \text{on } \Gamma_c \times (0,T), \tag{6.54}$$

where $\tilde{f} = f + \nabla\bar{q}$ and $\tilde{h} = h - \nabla z \cdot N$.

Now, we employ Galerkin's method to a suitable variational form. Namely, we assume that $\phi_s = \phi_s(x)$ and $\psi_s = \psi_s(x)$ are smooth functions such that $\{\phi_s\}_{s=1}^\infty$ is an orthogonal basis of V and $\{\psi\}_{s=1}^\infty$ is an orthogonal basis of $H^1(\Omega_e)$, respectively. For any $n = 1, 2, \ldots$, we define the approximate solutions

$$u_n(t) = \sum_{s=1}^n g_{sn}(t)\phi_s \tag{6.55}$$

and

$$w_n(t) = \sum_{s=1}^{n} d_{sn}(t)\psi_s, \tag{6.56}$$

where the coefficients $g_{sn}(t)$ and $d_{sn}(t)$ for $s = 1, \dots, n$ and $t \in [0, T]$ can be determined uniquely (by the standard ODE theory) such that

$$\begin{aligned}(u_n'(t), \phi_s) + (\nabla u_n(t), \nabla\phi_s) + \gamma^{-1}(u_n(t) - w_n'(t), \phi_s)_{\Gamma_c} \\ = (\tilde{f}_n(t), \phi_s) - (\tilde{h}_n(t), \phi_s)_{\Gamma_c} - \gamma^{-1}(z_n(t), \phi_s)_{\Gamma_c}\end{aligned} \tag{6.57}$$

and

$$\begin{aligned}(w_n''(t), \psi_s) + (\nabla w_n(t), \nabla\psi_s) + \alpha(w_n'(t), \psi_s) + \beta(w_n(t), \psi_s) - \gamma^{-1}(u_n(t) - w_n'(t), \psi_s)_{\Gamma_c} \\ = \gamma^{-1}(z_n(t), \psi_s)_{\Gamma_c}\end{aligned} \tag{6.58}$$

for $t \in (0, T)$, and $g_{sn}(0) = (u_0, \phi_s)$, $d_{sn}(0) = (w_0, \psi_s)$, and $d_{sn}'(0) = (w_1, \psi_s)$ for all $s = 1, \dots, n$.

Multiply equation (6.57) by $g_{sn}(t)$ and sum for $s = 1, \dots, n$. Similarly, multiply equation (6.58) by $d_{sn}'(t)$ and sum for $s = 1, \dots, n$. Adding the resulting equations and integrating in time gives the first level energy estimate

$$\begin{aligned}&\frac{1}{2}\left(\|u_n(t)\|_{L^2}^2 + \beta\|w_n(t)\|_{L^2}^2 + \|w_n'(t)\|_{L^2}^2 + \|\nabla w_n(t)\|_{L^2}^2\right) \\ &\quad + \int_0^t \left(\|\nabla u_n\|_{L^2}^2 + \alpha\|w_n'\|_{L^2}^2 + \gamma^{-1}\|u_n - w_n'\|_{L^2(\Gamma_c)}^2\right) \mathrm{d}s \\ &\leqslant \int_0^t (\tilde{f}_n(s), u_n(s))\,\mathrm{d}s - \int_0^t (\tilde{h}_n(s), u_n(s))_{\Gamma_c}\,\mathrm{d}s - \gamma^{-1}\int_0^t (z_n(s), u_n(s) - w_n'(s))_{\Gamma_c}\,\mathrm{d}s + E(0).\end{aligned} \tag{6.59}$$

Thus, we obtain that the sequence $\{u_n\}$ remains in a bounded set of $L^\infty(0, T; H) \cap L^2(0, T; V)$, the sequence $\{w_n\}$ remains in a bounded set of $L^\infty(0, T; H^1(\Omega_e))$, and the sequence $\{w_n'\}$ remains in a bounded set of $L^\infty(0, T; L^2(\Omega_e)) \cap L^2(0, T; L^2(\Omega_e))$. In particular, (6.59) implies an upper bound on $\int_0^t \|w_n'\|_{L^2(\Gamma_c)}^2\,\mathrm{d}s$. Passing to the limit in the variational form (6.57), (6.58), we may conclude

$$\begin{aligned}(u_t(t), \phi) + (\nabla u(t), \nabla\phi) + \gamma^{-1}(u(t) - w_t(t), \phi)_{\Gamma_c} \\ = (\tilde{f}(t), \phi) - (\tilde{h}(t), \phi)_{\Gamma_c} - \gamma^{-1}(z(t), \phi)_{\Gamma_c}\end{aligned} \tag{6.60}$$

$$\begin{aligned}(w_{tt}(t), \psi) + (\nabla w(t), \nabla\psi) + \alpha(w_t(t), \psi) + \beta(w(t), \psi) - \gamma^{-1}(u(t) - w_t(t), \psi)_{\Gamma_c} \\ = \gamma^{-1}(z(t), \psi)_{\Gamma_c}\end{aligned} \tag{6.61}$$

for all $\phi \in V$ and $\psi \in H^1(\Omega_e)$.

Next, we obtain the regularity on the time derivatives $u_t(0)$ and $w_{tt}(0)$. We integrate by parts in (6.60) and (6.61) and take the limit as $t \to 0^+$ to obtain

$$(u_t(0), \phi) - (\Delta u_0, \phi) - (\nabla u_0 \cdot N, \phi)_{\Gamma_c} + \gamma^{-1}(u_0 - w_1, \phi)_{\Gamma_c} = (\tilde{f}(0), \phi) - (\tilde{h}(0), \phi)_{\Gamma_c} \tag{6.62}$$

$$(w_{tt}(0), \psi) - (\Delta w_0, \psi) + (\nabla w_0 \cdot N, \psi)_{\Gamma_c} + \alpha(w_1, \psi) + \beta(w_0, \psi) - \gamma^{-1}(u_0 - w_1, \psi)_{\Gamma_c} = 0. \tag{6.63}$$

Using the compatibility conditions (6.10) and (6.11), all the terms on the common interface Γ_c vanish. Indeed, by (6.10) and (6.11), we have $(\nabla w_0 \cdot N, \phi)_{\Gamma_c} = \gamma^{-1}(u_0 - w_1, \phi)_{\Gamma_c}$ and $(\nabla w_0 \cdot N - \nabla u_0 \cdot N + \tilde{h}(0), \phi)_{\Gamma_c} = 0$ for all $\phi \in V$, from where $(\nabla u_0 \cdot N, \phi)_{\Gamma_c} = (\gamma^{-1}(u_0 -$

$w_1) + \tilde{h}(0), \phi)_{\Gamma_c}$. Here we utilized that $g(0) = 0$, so that $z_0 = 0$, $u_0 = v_0$, $\tilde{h}(0) = h(0)$ and $\tilde{f}(0) = f(0)$. Similarly, by (6.10), we have $(\nabla w_0 \cdot N, \psi)_{\Gamma_c} = \gamma^{-1}(u_0 - w_1, \psi)_{\Gamma_c}$ for all $\psi \in H^1(\Omega_e)$. From (6.62) and (6.63), we deduce

$$(u_t(0), \phi) = (\Delta u_0, \phi) + (\tilde{f}(0), \phi)$$
$$(w_{tt}(0), \psi) = (\Delta w_0, \psi) - \alpha(w_1, \psi) - \beta(w_0, \psi)$$

for all $\phi \in V$ and $\psi \in H^1(\Omega_e)$. By density of V in H, this leads to

$$\|u_t(0)\|_{L^2(\Omega_f)} \leqslant C\|u_0\|_{H^2(\Omega_f)} + C\|\tilde{f}(0)\|_{L^2(\Omega_f)}$$
$$\|w_{tt}(0)\|_{L^2(\Omega_e)} \leqslant C\|w_0\|_{H^2(\Omega_e)} + C\|w_1\|_{L^2(\Omega_e)}.$$

Therefore, we conclude $u_t(0) \in H$ and $w_{tt}(0) \in L^2(\Omega_e)$.

Our next step is to reconstruct the system (6.49)–(6.51). Taking test functions $\phi \in V$ and $\psi \in H^1(\Omega_e)$ vanishing on the common boundary Γ_c, we obtain

$$(u_t, \phi) - (\Delta u, \phi) = (\tilde{f}, \phi) \tag{6.64}$$
$$(w_{tt}, \psi) - (\Delta w, \psi) + \alpha(w_t, \psi) + \beta(w, \psi) = 0. \tag{6.65}$$

By (6.64), we obtain

$$u_t = \Delta u + \tilde{f} \text{ in } H^{\perp}(\Omega_f), \tag{6.66}$$

where we denoted $H^{\perp}(\Omega_f) = \{u \in L^2(\Omega_f) : \ u = \nabla q, q \in H^1(\Omega_f), q|_{\Gamma_c} = \text{const}\}$. This leads to

$$u_t = \Delta u - \nabla q + \tilde{f} \tag{6.67}$$

for some $q \in H^1(\Omega_f)$. From (6.65), we have

$$w_{tt} = \Delta w - \alpha w_t - \beta w. \tag{6.68}$$

Now, in order to recover the boundary conditions, we integrate by parts in (6.60) and use relation (6.67) to obtain

$$(-\nabla u \cdot N + qN + \gamma^{-1}(u + z - w_t) + \tilde{h}, g)_{\Gamma_c} = 0 \tag{6.69}$$

for all $g = \phi_{|\Gamma_c}$ with $\phi \in V$. Similarly, we obtain

$$(\nabla w \cdot N - \gamma^{-1}(u + z - w_t), h)_{\Gamma_c} = 0 \tag{6.70}$$

for all $h = \psi_{|\Gamma_c}$ with $\psi \in H^1(\Omega_e)$. Using the last two equalities we can reconstruct the boundary conditions

$$\nabla w \cdot N = \gamma^{-1}(u + z - w_t) \tag{6.71}$$
$$\nabla u \cdot N - qN = \nabla w \cdot N + \tilde{h} \tag{6.72}$$

on $\Gamma_c \times (0, T)$.

Next, we show that the limit solutions (u, w, q) belong to the functional spaces stated in lemma 6.1. Indeed, from the first level energy estimate (6.59), we have

$$u \in L^\infty([0, T]; L^2(\Omega_f)) \cap L^2([0, T]; H^1(\Omega_f)),$$
$$w \in L^\infty([0, T]; H^1(\Omega_e)),$$
$$w_t \in L^\infty([0, T]; L^2(\Omega_e)).$$

Using the same arguments on the time differentiated linear systems together with the compatibility conditions (6.13)–(6.15) and (6.16)–(6.18), we obtain the higher level energy estimates for the approximate solutions $u_n(t)$ and $w_n(t)$ in line with (4.38) and (4.53). Thus, we may conclude

$$\begin{aligned} u_t &\in L^\infty([0,T]; L^2(\Omega_f)) \cap L^2([0,T]; H^1(\Omega_f)),\\ w_t &\in L^\infty([0,T]; H^1(\Omega_e)),\\ w_{tt} &\in L^\infty([0,T]; L^2(\Omega_e)) \end{aligned}$$

and

$$\begin{aligned} u_{tt} &\in L^\infty([0,T]; L^2(\Omega_f)) \cap L^2([0,T]; H^1(\Omega_f)),\\ w_{tt} &\in L^\infty([0,T]; H^1(\Omega_e)),\\ w_{ttt} &\in L^\infty([0,T]; L^2(\Omega_e)). \end{aligned}$$

We also use the pointwise Stokes estimates

$$\|u\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)} \leqslant C\|u_t\|_{H^s(\Omega_f)} + C\|\tilde{f}\|_{H^s(\Omega_f)} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)} + C\|\tilde{h}\|_{H^{s+1/2}(\Gamma_c)} \tag{6.73}$$

for $s = 0, 1$ and

$$\|u_t\|_{H^2(\Omega_f)} + \|q_t\|_{H^1(\Omega_f)} \leqslant C\|u_{tt}\|_{L^2(\Omega_f)} + C\|\tilde{f}_t\|_{L^2(\Omega_f)} + C\left\|\frac{\partial w_t}{\partial N}\right\|_{H^{1/2}(\Gamma_c)} + C\|\tilde{h}_t\|_{H^{1/2}(\Gamma_c)}, \tag{6.74}$$

which are obtained as in lemma 3.2. Observe that $\tilde{f} \in L^\infty([0,T]; H^1(\Omega_f))$, $\tilde{h} \in L^\infty([0,T]; H^{3/2}(\Gamma_c))$, $\tilde{f}_t \in L^\infty([0,T]; L^2(\Omega_f))$, $\tilde{h}_t \in L^\infty([0,T]; H^{1/2}(\Gamma_c))$, which follows by the assumptions on f and h and the regularity of the Stokes problem (6.44)–(6.46) for the variable z. Thus, we obtain

$$\begin{aligned} u &\in L^\infty([0,T]; H^3(\Omega_f)), \qquad & u_t &\in L^\infty([0,T]; H^2(\Omega_f))\\ q &\in L^\infty([0,T]; H^2(\Omega_f)), & q_t &\in L^\infty([0,T]; H^1(\Omega_f)). \end{aligned}$$

Finally, the elliptic estimates for the wave equation are given by (3.11) and (3.12), leading to

$$w \in L^\infty([0,T]; H^3(\Omega_e)), \qquad w_t \in L^\infty([0,T]; H^2(\Omega_e)).$$

Therefore, the limit solution (u, w, q) is regular and lies in the space given in the statement of lemma 6.1.

We would like to point out that, by the construction of the solution (v, w, q), the fluid velocity $v = u + z$ belongs to the functional spaces

$$v \in L^\infty([0,T]; H^3(\Omega_f)), \quad v_t \in L^\infty([0,T]; H^2(\Omega_f)), \quad \nabla v_{tt} \in L^2([0,T]; L^2(\Omega_f)).$$

In addition, we obtain $v_{tt} \in L^\infty([0,T]; L^2(\Omega_f))$ from the third level *a priori* energy estimate (4.53). Therefore, the proof of the lemma is established. □

Now, we consider the case of given time-dependent matrix $a(x,t)$ with smooth coefficients $a_i^k(x,t) \in C^\infty(\Omega_f \times [0,T])$. We assume that $a(x,t)$ is a small perturbation of the identity matrix satisfying

$$\begin{gathered} \|\delta_{jk} - a_j^k\|_{H^2}^2 \leqslant \epsilon, \qquad \|\partial_t(\delta_{jk} - a_j^k)\|_{H^2}^2 \leqslant \epsilon, \qquad \|\partial_{tt}(\delta_{jk} - a_j^k)\|_{L^3}^2 \leqslant \epsilon,\\ \|\partial_{ttt}(\delta_{jk} - a_j^k)\|_{L^2}^2 \leqslant \epsilon \end{gathered} \tag{6.75}$$

and

$$\begin{gathered} \|\delta_{jk} - a_l^j a_l^k\|_{H^2}^2 \leqslant \epsilon, \qquad \|\partial_t(\delta_{jk} - a_l^j a_l^k)\|_{H^2}^2 \leqslant \epsilon, \qquad \|\partial_{tt}(\delta_{jk} - a_l^j a_l^k)\|_{L^3}^2 \leqslant \epsilon,\\ \|\partial_{ttt}(\delta_{jk} - a_l^j a_l^k)\|_{L^2}^2 \leqslant \epsilon \end{gathered}$$

for all $t \in [0, T]$ with T sufficiently small. In particular, the ellipticity condition $a_l^j a_l^k \xi_j^i \xi_k^i \geqslant (1/C)|\xi|^2$ holds for $\xi \in \mathbb{R}^3 \times \mathbb{R}^3$ and $t \in [0, T]$.

We use a fixed point argument for the perturbed system

$$v_t^{(n+1)} - \Delta v^{(n+1)} + \nabla q^{(n+1)} = f^{(n)} \qquad \text{in } \Omega_f \times (0, T) \tag{6.76}$$

$$\nabla \cdot v^{(n+1)} = g^{(n)} \qquad \text{in } \Omega_f \times (0, T) \tag{6.77}$$

$$w_{tt}^{(n+1)} - \Delta w^{(n+1)} + \alpha w_t^{(n+1)} + \beta w^{(n+1)} = 0 \qquad \text{in } \Omega_e \times (0, T) \tag{6.78}$$

with the boundary conditions

$$\frac{\partial w^{(n+1)}}{\partial N} = \gamma^{-1}(v^{(n+1)} - w_t^{(n+1)}) \qquad \text{on } \Gamma_c \times (0, T) \tag{6.79}$$

$$v^{(n+1)} = 0 \qquad \text{on } \Gamma_f \times (0, T) \tag{6.80}$$

$$\frac{\partial v^{(n+1)}}{\partial N} - q^{(n+1)} \cdot N = \frac{\partial w^{(n+1)}}{\partial N} + h^{(n)} \qquad \text{on } \Gamma_c \times (0, T), \tag{6.81}$$

where

$$f^{i(n)} = -\partial_j \left((\delta_{jk} - a_l^j a_l^k) \partial_k v^{i(n)} \right) + \partial_k \left((\delta_{ik} - a_i^k) q^{(n)} \right), \tag{6.82}$$

$$g^{(n)} = (\delta_{jk} - a_j^k) \partial_k v^{j(n)}, \tag{6.83}$$

$$h^{i(n)} = (\delta_{jk} - a_l^j a_l^k) \partial_k v^{i(n)} N_j + (\delta_{ik} - a_i^k) q^{(n)} N_k \tag{6.84}$$

for $i = 1, 2, 3$. As in the proof of lemma 6.1, we change variables $u^{(n+1)} = v^{(n+1)} - z^{(n+1)}$, where $(z^{(n+1)}, \bar{q}^{(n+1)})$ satisfies the Stokes system

$$z_t^{(n+1)} - \Delta z^{(n+1)} + \nabla \bar{q}^{(n+1)} = 0 \qquad \text{in } \Omega_f \times (0, T) \tag{6.85}$$

$$\nabla \cdot z^{(n+1)} = g^{(n)} \qquad \text{in } \Omega_f \times (0, T) \tag{6.86}$$

$$z^{(n+1)} = \lambda^{(n)} \psi^{(n)} \qquad \text{on } \Gamma_c \times (0, T), \tag{6.87}$$

$$z^{(n+1)} = 0 \qquad \text{on } \Gamma_f \times (0, T), \tag{6.88}$$

with the initial data $z^{(n+1)}(\cdot, 0) = 0$, where $z^{(n+1)} = z_1^{(n+1)} + z_2^{(n+1)} + z_3^{(n+1)}$, $\lambda^{(n)}$ and $\psi^{(n)}$ are defined as in the proof of lemma 6.1. Then $u^{(n+1)}$ satisfies the divergence-free Stokes-wave system

$$u_t^{(n+1)} - \Delta u^{(n+1)} + \nabla q^{(n+1)} = f^{(n)} + \nabla \bar{q}^{(n+1)} \qquad \text{in } \Omega_f \times (0, T), \quad i = 1, 2, 3 \tag{6.89}$$

$$\nabla \cdot u^{(n+1)} = 0 \qquad \text{in } \Omega_f \times (0, T) \tag{6.90}$$

$$w_{tt}^{(n+1)} - \Delta w^{(n+1)} + \alpha w_t^{(n+1)} + \beta w^{(n+1)} = 0 \qquad \text{in } \Omega_e \times (0, T) \tag{6.91}$$

with the boundary conditions

$$\frac{\partial w^{(n+1)}}{\partial N} = \gamma^{-1}(u^{(n+1)} + z^{(n+1)} - w_t^{(n+1)}) \qquad \text{on } \Gamma_c \times (0, T) \tag{6.92}$$

$$u^{(n+1)} = 0 \qquad \text{on } \Gamma_f \times (0, T) \tag{6.93}$$

$$\frac{\partial u^{(n+1)}}{\partial N} - q^{(n+1)} \cdot N = \frac{\partial w^{(n+1)}}{\partial N} + h^{(n)} + \nabla z^{(n+1)} \cdot N \qquad \text{on } \Gamma_c \times (0, T), \tag{6.94}$$

where

$$f^{i(n)} = -\partial_j \left((\delta_{jk} - a_l^j a_l^k) \partial_k v^{i(n)} \right) + \partial_k \left((\delta_{ik} - a_i^k) q^{(n)} \right), \tag{6.95}$$

$$h^{i(n)} = (\delta_{jk} - a_l^j a_l^k) \partial_k v^{i(n)} N_j + (\delta_{ik} - a_i^k) q^{(n)} N_k \tag{6.96}$$

for $i = 1, 2, 3$.

Recall that the proof of lemma 6.1 gives the existence and uniqueness of $z^{(n+1)}$ satisfying (6.85)–(6.88) with $z^{(n+1)}(0) = 0$ and such that

$$z^{(n+1)} \in L^\infty([0,T]; H^3(\Omega_f)),$$
$$z_t^{(n+1)} \in L^\infty([0,T]; H^2(\Omega_f)),$$
$$z_{tt}^{(n+1)} \in L^2([0,T]; H^1(\Omega_f)).$$

We conclude the construction of solutions by the following auxiliary assertion.

Lemma 6.2. *Assume the initial data* (u_0, w_0, w_1) *is small, that is*

$$\|u_0\|^2_{H^3(\Omega_f)}, \|u_t(0)\|^2_{H^1(\Omega_f)}, \|u_{tt}(0)\|^2_{L^2(\Omega_f)}, \|w_0\|^2_{H^3(\Omega_e)}, \|w_1\|^2_{H^2(\Omega_e)} \leqslant \epsilon, \tag{6.97}$$

where $\epsilon > 0$ *is a small parameter. Then, the map*

$$\Lambda : (u^{(n)}, w^{(n)}, q^{(n)}) \to (u^{(n+1)}, w^{(n+1)}, q^{(n+1)})$$

is a contraction in the norms

$$\begin{aligned} &u \in L^\infty([0,T]; H^3(\Omega_f)), && u_t \in L^\infty([0,T]; H^2(\Omega_f)),\\ &u_{tt} \in L^\infty([0,T]; L^2(\Omega_f)), && \nabla u_{tt} \in L^2([0,T]; L^2(\Omega_f))\\ &\partial_t^j w \in C([0,T]; H^{3-j}(\Omega_e)), && j = 0,1,2,3\\ &q \in L^\infty([0,T]; H^2(\Omega_f)), && q_t \in L^\infty([0,T]; H^1(\Omega_f)) \end{aligned} \tag{6.98}$$

for time $T > 0$ *which depends on the given Lagrangian matrix* $a(x,t)$.

Proof of lemma 6.2. We consider the three energy level estimates for the system (6.89)–(6.94). For the first level energy $E^{(n+1)} = (1/2)(\|u^{(n+1)}\|^2_{L^2} + \beta\|w^{(n+1)}\|^2_{L^2} + \|\nabla w^{(n+1)}\|^2_{L^2} + \|w_t^{(n+1)}\|^2_{L^2})$, we have

$$\begin{aligned} E^{(n+1)}(t) &+ \int_0^t (\|\nabla u^{(n+1)}\|^2_{L^2} + \alpha\|w_t^{(n+1)}\|^2_{L^2})\,\mathrm{d}s\\ &= E^{(n+1)}(0) + \int_0^t (f^{(n)}(s) + \nabla\bar{q}^{(n+1)}, u^{(n+1)}(s))\,\mathrm{d}s\\ &\quad - \int_0^t \left(\nabla w^{(n+1)}\cdot N + h^{(n)} + \nabla z^{(n+1)}\cdot N, u^{(n+1)}\right)_{\Gamma_c}\mathrm{d}s\\ &\quad + \int_0^t \left(\nabla w^{(n+1)}\cdot N, u^{(n+1)} + z^{(n+1)} - \gamma\nabla w^{(n+1)}\cdot N\right)_{\Gamma_c}\mathrm{d}s, \end{aligned} \tag{6.99}$$

using (6.92)–(6.94), which gives

$$\begin{aligned} E^{(n+1)}(t) + \int_0^t D^{(n+1)}(s)\,\mathrm{d}s &\leqslant E^{(n+1)}(0) + C\sum_j \int_0^t \|(\delta_{jk} - a_l^j a_l^k(s))\partial_k v^{(n)}(s)\|^2_{L^2}\,\mathrm{d}s\\ &+ C\sum_{i,k}\int_0^t \|(\delta_{ik} - a_i^k(s))q^{(n)}\|^2_{L^2}\,\mathrm{d}s + C\int_0^t \|(\delta_{jk} - a_j^k)\partial_k v^{j(n)}\|^2_{L^2}\,\mathrm{d}s\\ &+ \epsilon_0\int_0^t \|\nabla u^{(n+1)}(s)\|^2_{L^2}\,\mathrm{d}s, \end{aligned} \tag{6.100}$$

where we denoted by $D^{(n+1)} = \|\nabla u^{(n+1)}\|^2_{L^2} + \alpha\|w_t^{(n+1)}\|^2_{L^2} + \gamma\|\nabla w^{(n+1)}\cdot N\|^2_{L^2(\Gamma_c)}$ the dissipation terms. Absorbing the last term on the right, we obtain

$$E^{(n+1)}(t) + \int_0^t D^{(n+1)}(s)\,\mathrm{d}s \leqslant E^{(n+1)}(0) + C\epsilon\int_0^t \left(\|\nabla v^{(n)}(s)\|^2_{L^2} + \|q^{(n)}(s)\|^2_{L^2}\right)\mathrm{d}s, \tag{6.101}$$

as $\|\delta_{jk} - a_l^j a_l^k\|_{H^2}^2 \leqslant \epsilon$. Similarly, we obtain for the second and the third energy levels

$$E_1^{(n+1)}(t) + \int_0^t D_1^{(n+1)}(s)\,\mathrm{d}s \leqslant E_1^{(n+1)}(0) + C\sum_j \int_0^t \|\partial_t \left((\delta_{jk} - a_l^j a_l^k)\partial_k v^{(n)}(s)\right)\|_{L^2}^2\,\mathrm{d}s$$
$$+C\sum_{i,k}\int_0^t \|\partial_t\left((\delta_{ik} - a_i^k(s))q^{(n)}\right)\|_{L^2}^2\,\mathrm{d}s + C\int_0^t \|\partial_t\left((\delta_{jk} - a_j^k)\partial_k v^{j(n)}\right)\|_{L^2}^2\,\mathrm{d}s$$
$$+\epsilon_0 \int_0^t \|\nabla u_t^{(n+1)}(s)\|_{L^2}^2\,\mathrm{d}s \tag{6.102}$$

and

$$E_2^{(n+1)}(t) + \int_0^t D_2^{(n+1)}(s)\,\mathrm{d}s$$
$$\leqslant E_2^{(n+1)}(0) + \int_0^t\int_{\Omega_f} \partial_{tt}\left((\delta_{jk} - a_l^j(s)a_l^k(s))\partial_k v^{i(n)}\right)\partial_j u_{tt}^{(n+1)}(s)\,\mathrm{d}x\,\mathrm{d}s \leqslant E_2^{(n+1)}(0)$$
$$-\int_0^t\int_{\Omega_f} \partial_{tt}\left((\delta_{ik} - a_i^k)q^{(n)}\right)\partial_k u_{tt}^{i(n+1)}\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t \|\partial_{tt}\left((\delta_{jk} - a_j^k)\partial_k v^{j(n)}\right)\|_{L^2}^2\,\mathrm{d}s$$
$$+\,\epsilon_0 \int_0^t \|\nabla u_{tt}^{(n+1)}(s)\|_{L^2}^2\,\mathrm{d}s, \tag{6.103}$$

respectively. We treat the pressure term on the right side as we did in the proof of lemma 4.11. Namely, when the two time derivatives fall on q, we use the divergence-free condition to write $(\delta_{ik} - a_i^k)\partial_k u_{tt}^{i(n+1)} = -2\partial_t(\delta_{ik} - a_i^k)\partial_k u_t^{i(n+1)} - \partial_{tt}(\delta_{ik} - a_i^k)\partial_k u^{i(n+1)}$. From (6.102) and (6.103), we have

$$E_1^{(n+1)}(t) + \int_0^t D_1^{(n+1)}(s)\,\mathrm{d}s$$
$$\leqslant E_1^{(n+1)}(0) + C\epsilon\int_0^t \left(\|\nabla v^{(n)}\|_{L^2}^2 + \|\nabla v_t^{(n)}\|_{L^2}^2 + \|q^{(n)}\|_{L^2}^2 + \|q_t^{(n)}\|_{L^2}^2\right)\mathrm{d}s \tag{6.104}$$

and

$$E_2^{(n+1)}(t) + \int_0^t D_2^{(n+1)}(s)\,\mathrm{d}s$$
$$\leqslant CE_2^{(n+1)}(0) + C\epsilon\|q_t^{(n)}(t)\|_{L^6}^2 + \epsilon_0\left(\|\nabla u_t^{(n+1)}(t)\|_{L^2}^2 + \|\nabla u^{(n+1)}(t)\|_{L^2}^2\right)$$
$$+C\epsilon\int_0^t\left(\|\nabla v^{(n)}\|_{L^6}^2 + \|\nabla v_t^{(n)}\|_{L^2}^2 + \|\nabla v_{tt}^{(n)}\|_{L^2}^2 + \|q^{(n)}\|_{L^6}^2 + \|q_t^{(n)}\|_{L^6}^2\right)\mathrm{d}s$$
$$+\epsilon_0\int_0^t \|\nabla u^{(n+1)}\|_{L^3}^2\,\mathrm{d}s + \epsilon_0\int_0^t\left(\|\nabla u_t^{(n+1)}\|_{L^2}^2 + \|\nabla u_{tt}^{(n+1)}\|_{L^2}^2\right)\mathrm{d}s, \tag{6.105}$$

respectively. The last term on the right side of (6.105) can be absorbed in the dissipation terms. We need the two pointwise estimates

$$\|\nabla u^{(n+1)}(t)\|_{L^2}^2 \leqslant \|\nabla u^{(n+1)}(0)\|_{L^2}^2 + C\int_0^t\left(D^{(n+1)}(s) + D_1^{(n+1)}(s)\right)\mathrm{d}s \tag{6.106}$$

and

$$\|\nabla u_t^{(n+1)}(t)\|_{L^2}^2 \leqslant \|\nabla u_t^{(n+1)}(0)\|_{L^2}^2 + C\int_0^t\left(D_1^{(n+1)}(s) + D_2^{(n+1)}(s)\right)\mathrm{d}s, \tag{6.107}$$

which are obtained as in (5.3) and (5.5). Next, using the Stokes estimates (6.73) and (6.74), we obtain

$$\|u^{(n+1)}\|_{H^{s+2}(\Omega_f)} + \|q^{(n+1)}\|_{H^{s+1}(\Omega_f)} \leqslant C\|u_t^{(n+1)}\|_{H^s(\Omega_f)} + C\|f^{(n)}\|_{H^s(\Omega_f)} + C\left\|\frac{\partial w^{(n+1)}}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)} + C\|h^{(n)}\|_{H^{s+1/2}(\Gamma_c)} + C\|g^{(n)}\|_{H^{s+1}(\Omega_f)} \tag{6.108}$$

for $s = 0, 1$ and

$$\|u_t^{(n+1)}\|_{H^2(\Omega_f)} + \|q_t^{(n+1)}\|_{H^1(\Omega_f)} \leqslant C\|u_{tt}^{(n+1)}\|_{L^2(\Omega_f)} + C\|f_t^{(n)}\|_{L^2(\Omega_f)} + C\left\|\frac{\partial w_t^{(n+1)}}{\partial N}\right\|_{H^{1/2}(\Gamma_c)} + C\|h_t^{(n)}\|_{H^{1/2}(\Gamma_c)} + C\|g_t^{(n)}\|_{H^1(\Omega_f)}. \tag{6.109}$$

We have

$$\|f^{(n)}\|_{H^s} + \|h^{(n)}\|_{H^{s+1/2}(\Gamma_c)} + \|g^{(n)}\|_{H^{s+1}} \leqslant \epsilon\left(\|v^{(n)}\|_{H^{s+2}} + \|q^{(n)}\|_{H^{s+1}}\right) \tag{6.110}$$

for $s = 0, 1$, as $\|\delta_{jk} - a_j^k\|_{H^2}^2 \leqslant \epsilon$ and $\|\delta_{jk} - a_l^j a_l^k\|_{H^2}^2 \leqslant \epsilon$, and

$$\|f_t^{(n)}\|_{L^2} + \|h_t^{(n)}\|_{H^{1/2}(\Gamma_c)} + \|g_t^{(n)}\|_{H^1} \leqslant \epsilon\left(\|v_t^{(n)}\|_{H^2} + \|q_t^{(n)}\|_{H^1} + \|v^{(n)}\|_{H^2} + \|q^{(n)}\|_{H^1}\right), \tag{6.111}$$

since also $\|\partial_t(\delta_{jk} - a_j^k)\|_{H^2}^2 \leqslant \epsilon$ and $\|\partial_t(\delta_{jk} - a_l^j a_l^k)\|_{H^2}^2 \leqslant \epsilon$. Using the boundary condition (6.92), we may write

$$\left\|\frac{\partial w^{(n+1)}}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)} \leqslant C\left(\|u^{(n+1)}\|_{H^{s+1}} + \|z^{(n+1)}\|_{H^{s+1}} + \|w_t^{(n+1)}\|_{H^{s+1}}\right) \tag{6.112}$$

for $s = 0, 1$. Thus, we have

$$\|u^{(n+1)}\|_{H^3(\Omega_f)} + \|q^{(n+1)}\|_{H^2(\Omega_f)} \leqslant C\|u_t^{(n+1)}\|_{H^1(\Omega_f)} + C\epsilon\left(\|v^{(n)}\|_{H^3} + \|q^{(n)}\|_{H^2}\right) + C\left(\|u^{(n+1)}\|_{H^2} + \|w_t^{(n+1)}\|_{H^2}\right), \tag{6.113}$$

as well as

$$\|u^{(n+1)}\|_{H^2(\Omega_f)} + \|q^{(n+1)}\|_{H^1(\Omega_f)} \leqslant C\|u_t^{(n+1)}\|_{L^2(\Omega_f)} + C\epsilon\left(\|v^{(n)}\|_{H^2} + \|q^{(n)}\|_{H^1}\right) + C\left(\|u_t^{(n+1)}\|_{H^1} + \|w_{tt}^{(n+1)}\|_{H^1}\right). \tag{6.114}$$

Similarly, we bound the right side of the estimate (6.109) for the time derivatives u_t and q_t to obtain

$$\|u_t^{(n+1)}\|_{H^2(\Omega_f)} + \|q_t^{(n+1)}\|_{H^1(\Omega_f)} \leqslant C\|u_{tt}^{(n+1)}\|_{L^2(\Omega_f)} + C\epsilon\left(\|v_t^{(n)}\|_{H^2} + \|q_t^{(n)}\|_{H^1} + \|v^{(n)}\|_{H^2} + \|q^{(n)}\|_{H^1}\right) + C\left\|w_t^{(n+1)}\right\|_{H^2}. \tag{6.115}$$

Here we also employ the elliptic estimates (3.11) and (3.12) for the wave equation:

$$\|w^{(n+1)}\|_{H^3} \leqslant C\|w_{tt}^{(n+1)}\|_{H^1} + C\|w_t^{(n+1)}\|_{H^1} + C\|w^{(n+1)}\|_{H^1} + C\left(\|v^{(n+1)}\|_{H^2} + \|w_t^{(n+1)}\|_{H^2}\right) \tag{6.116}$$

and

$$\|w_t^{(n+1)}\|_{H^2} \leqslant C\|w_{ttt}^{(n+1)}\|_{L^2} + C\|w_{tt}^{(n+1)}\|_{L^2} + C\|w_t^{(n+1)}\|_{L^2} + C\left(\|v_t^{(n+1)}\|_{H^1} + \|w_{tt}^{(n+1)}\|_{H^1}\right). \tag{6.117}$$

From the above estimates one can see that the map

$$\Lambda : (u^{(n)}, w^{(n)}, q^{(n)}) \to (u^{(n+1)}, w^{(n+1)}, q^{(n+1)})$$

is a contraction in the norms (6.98), which concludes the proof of the lemma. □

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