ON WELL-POSEDNESS, REGULARITY AND EXACT CONTROLLABILITY FOR PROBLEMS OF TRANSMISSION OF PLATE EQUATION WITH VARIABLE COEFFICIENTS

By

BAO-ZHU GUO (*Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100080, P.R. China and School of Computational and Applied Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa*)

AND

ZHI-CHAO SHAO (*School of Computational and Applied Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa*)

Abstract. A system of transmission of Euler-Bernoulli plate equation with variable coefficients under Neumann control and collocated observation is studied. Using the multiplier method on a Riemannian manifold, it is shown that the system is well-posed in the sense of D. Salamon. This establishes the equivalence between the exact controllability of an open-loop system and the exponential stability of a closed-loop system under the proportional output feedback. The regularity of the system in the sense of G. Weiss is also proved, and the feedthrough operator is found to be zero. These properties make this PDE system parallel in many ways to the finite-dimensional ones. Finally, the exact controllability of an open-loop system is developed under a uniqueness assumption by establishing the observability inequality for the dual system.

1. Introduction and main results. In distributed parameter control systems, many properties are closely related to each other. In the early 1970's, Russell found that for some time-invertible PDEs, the exponential stability of a closed-loop system can deduce the exact controllability of an open-loop system ([24, 25]). Conversely, it was also found that the exact controllability of an open-loop system implies the exponential stability of a closed-loop system provided that the open-loop system is well-posed in the sense of D. Salamon ([2, 11]). Lagnese [17] studied some LQ-optimal control problems

for beam equations to obtain the stability of the system. For the general result from optimizability to exponential stability, we refer to [30].

In the past two decades, it has been discovered that the well-posed and regular systems are a quite general class of linear infinite-dimensional systems, and that they cover many control systems described by partial differential equations with actuators and sensors supported on isolated points, sub-domains, or on parts of the boundary of the spatial region ([8]). This class of infinite-dimensional systems, although the input and output operators are allowed to be unbounded, may possess many properties that are parallel in many ways to finite-dimensional systems ([9]).

Although the abstract theory of well-posed and regular linear systems has been quite fruitful, only a few multi-dimensional PDEs have been verified to be well-posed and regular. For the systems with constant coefficients, the well-posedness and regularity of a multi-dimensional heat equation with both Dirichlet and Neumann type boundary controls were established in [4] and that for a wave equation with boundary Dirichlet input and collocated output was proved in [1] and [12]. The well-posedness and regularity for the multi-dimensional Schrödinger equation and the Euler-Bernoulli equation were reported in [19, 13, 14]. Very recently, these results were generalized to multi-dimensional wave and plate equations with variable coefficients under boundary controls and observations ([15, 16]).

In this paper, we study the following system of transmission of the Euler-Bernoulli plate equation with variable coefficients under Neumann control and collocated observation:

$$
\begin{cases}
\ddot{w}(x,t) + a(x)\mathcal{A}^2 w(x,t) = 0, \quad (x,t) \in Q, \\
w(x,0) = w^0(x),\ \dot{w}(x,0) = w^1(x), \quad x \in \Omega, \\
w_2(x,t) = 0,\ \dfrac{\partial w_2}{\partial \nu_{\mathcal{A}}}(x,t) = u(x,t), \quad (x,t) \in \Sigma, \\
w_1(x,t) = w_2(x,t),\ \dfrac{\partial w_1}{\partial \nu_{\mathcal{A}}}(x,t) = \dfrac{\partial w_2}{\partial \nu_{\mathcal{A}}}(x,t), \quad (x,t) \in \Sigma_1, \\
a_1 \mathcal{A} w_1(x,t) = a_2 \mathcal{A} w_2(x,t),\ a_1 \dfrac{\partial \mathcal{A} w_1}{\partial \nu_{\mathcal{A}}}(x,t) = a_2 \dfrac{\partial \mathcal{A} w_2}{\partial \nu_{\mathcal{A}}}(x,t), \quad (x,t) \in \Sigma_1, \\
y(x,t) = -\mathcal{A}(\mathscr{A}^{-1}\dot{w}(x,t)),\ (x,t) \in \Sigma,
\end{cases}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is an open bounded region with boundary $\partial \Omega = \Gamma$ of C^4-class, Ω_1 is a bounded region contained inside of Ω: $\overline{\Omega}_1 \subset \Omega$ and $\Gamma_1 = \partial \Omega_1$ is of C^4-class. $\Omega_2 = \Omega \setminus \Omega_1$, $Q = \Omega \times (0,T)$, $Q_1 = \Omega_1 \times (0,T)$, $Q_2 = \Omega_2 \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, and $\Sigma_1 = \Gamma_1 \times (0,T)$, $T > 0$ is a time moment. u is the input function (or control) and y is the output function (or observation).

$$
\mathcal{A}w(x,t) := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w(x,t)}{\partial x_j}\right),\ D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega), \tag{1.2}
$$

with $a_{ij} = a_{ji} \in C^{\infty}(\mathbb{R}^n)$ and for some constant $\alpha > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j} \geq \alpha \sum_{i=1}^{n} |\xi_i|^2, \ \forall \ x \in \Omega, \ \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{C}^n, \tag{1.3}$$

$$\mathscr{A}w = \mathcal{A}^2 w, D(\mathscr{A}) = H^4(\Omega) \cap H_0^2(\Omega), \tag{1.4}$$

$$\nu_{\mathcal{A}} := \left(\sum_{k=1}^{n} \nu_k a_{k1}, \sum_{k=1}^{n} \nu_k a_{k2}, \cdots, \sum_{k=1}^{n} \nu_k a_{kn} \right), \ \frac{\partial}{\partial \nu_{\mathcal{A}}} := \sum_{i,j=1}^{n} \nu_i a_{ij} \frac{\partial}{\partial x_j}, \tag{1.5}$$

with $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ the unit normal vector on Γ or Γ_1 pointing toward the exterior of Ω_2. $w_1 = w|_{\Omega_1}$, $w_2 = w|_{\Omega_2}$, $\dot{w} = w_t$, and $a(x)$ is given by

$$a(x) = \begin{cases} a_1, \ x \in \Omega_1, \\ a_2, \ x \in \Omega_2, \end{cases} \tag{1.6}$$

where a_1, a_2 are positive constants.

When $a_1 = a_2$ the system (1.1) is reduced to the ordinary plate equation with variable coefficients studied in [16, 34]. When $a_1 \neq a_2$, the exact controllability for the system (1.1) with constant spatial coefficients and additional lower-order terms on the right hand side of the governing equation (1.1) was established in [21] (see also [3]). [23] studied the uniform stability for the solutions of a transmission problem in non-homogeneous anisotropic elasticity. The stabilization of a thermoelastic plate with variable coefficients can be found in [6]. For the exact controllability and uniform stabilization of problems of transmission for wave and shell equations, we refer to [5, 7, 18, 22].

One of the aims of this paper is to establish the equivalence between the exact controllability of open-loop system (1.1) and the exponential stability of its closed-loop form under the proportional feedback $u = -ky, k > 0$. This is a consequence of the well-posedness for the system (1.1). The well-posedness together with the regularity property will put the system (1.1) into a general framework of well-posed and regular linear infinite-dimensional systems that has been studied extensively in the past two decades ([8]).

Let $\mathcal{H} = L^2(\Omega) \times H^{-2}(\Omega)$ and $U = L^2(\Gamma)$.

THEOREM 1.1. Let $T > 0$, $(w^0, w^1) \in \mathcal{H}$ and $u \in L^2(0, T; U)$. Then there exists a unique solution $(w, \dot{w}) \in C([0, T]; \mathcal{H})$ to Equation (1.1). Moreover, if $a_2 \leq a_1$, then there exists a constant $C_T > 0$, independent of (w^0, w^1, u), such that

$$\|(w(\cdot, T), \dot{w}(\cdot, T))\|_{\mathcal{H}}^2 + \|y\|_{L^2(0,T;U)}^2 \leq C_T \left[\|(w^0, w^1)\|_{\mathcal{H}}^2 + \|u\|_{L^2(0,T;U)}^2 \right].$$

Theorem 1.1 implies that the open-loop system (1.1) is well-posed provided that $a_2 \leq a_1$ in the sense of D. Salamon with the state space $\mathcal{H}$, input and output space U ([27]). From this result and [2, 11], we immediately have the following Corollary 1.1.

COROLLARY 1.1. Assume that $a_2 \leq a_1$. Then the system (1.1) is exactly controllable in $\mathcal{H}$ on $[0, T]$ for some $T > 0$ if and only if its closed-loop system under the proportional output feedback $u = -ky \, (k > 0)$ is exponentially stable.

THEOREM 1.2. Assume that $a_2 \leq a_1$. Then the system (1.1) is regular in the sense of G. Weiss with zero feedthrough operator. Precisely, if $w(\cdot, 0) = \dot{w}(\cdot, 0) = 0$ and $u(\cdot, t) \equiv u(\cdot) \in U$ is a step input, then the corresponding output y satisfies

$$\lim_{\sigma \to 0} \int_{\Gamma_0} \left| \frac{1}{\sigma} \int_0^\sigma y(x, t) dt \right|^2 dx = 0. \tag{1.7}$$

Theorems 1.1 and 1.2 ensure that the system (1.1) is a well-posed regular linear system. This makes the system (1.1) parallel in many ways to a linear finite-dimensional system.

It should be pointed out that due to variable coefficients, the classical multiplier method in Euclidean space seems inadequate to prove Theorems 1.1 and 1.2. Some computations on the Riemannian manifold are needed as in [34].

Because of the well-posedness claimed by Theorem 1.1, the exact controllability of system (1.1) can be obtained by establishing the observability inequality for the dual system (5.1). To do this, we need some additional assumptions (H1)-(H3) stated below.

Notice the condition (1.3) and let $A(x)$ and $G(x)$ be, respectively, the coefficient matrix and its inverse:

$$A(x) := (a_{ij}(x)), \ G(x) := (g_{ij}(x)) = A(x)^{-1}, \ \mathcal{G}(x) := \det(g_{ij}(x)).$$

For each $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, define the inner product and norm over the tangent space $\mathbb{R}_x^n$ of $\mathbb{R}^n$ by

$$g(X, Y) := \langle X, Y \rangle_g = \sum_{i,j=1}^n g_{ij} \alpha_i \beta_j,$$

$$|X|_g := \langle X, X \rangle_g^{1/2}, \ \forall \ X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \ Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}_x^n.$$

Then $(\mathbb{R}^n, g)$ is a Riemannian manifold with Riemannian metric g ([34]). Denote by D the Levi-Civita connection with respect to g. Let N be a vector field on $(\mathbb{R}^n, g)$. Then for each $x \in \mathbb{R}^n$, the covariant differential DN of N determines a bilinear form on $\mathbb{R}_x^n$:

$$DN(X, Y) = \langle D_X N, Y \rangle_g, \ \forall \ X, Y \in \mathbb{R}_x^n,$$

where $D_X N$ stands for the covariant derivative of the vector field N with respect to X.

ASSUMPTION (H1). There exists a vector field N on $(\mathbb{R}^n, g)$ such that

$$DN(X, X) = b(x)|X|_g^2, \ \forall \ X \in \mathbb{R}_x^n, \ x \in \Omega, \tag{1.8}$$

where $b(x)$ is a function defined on Ω so that

$$b_0 = \inf_{x \in \Omega} b(x) > 0. \tag{1.9}$$

ASSUMPTION (H2). There exists a vector field N on $(\mathbb{R}^n, g)$ such that

$$N(x) \cdot \nu \leq 0, \ x \in \Gamma_1. \tag{1.10}$$

ASSUMPTION (H3) [THE UNIQUENESS ASSUMPTION]. The following problem

$$
\begin{cases}
a(x)\mathcal{A}^2 u = \zeta u \text{ in } \Omega, \\[2mm]
u_2 = \dfrac{\partial u_2}{\partial \nu_{\mathcal{A}}} = 0 \text{ on } \Gamma, \\[2mm]
u_1 = u_2, \ \dfrac{\partial u_1}{\partial \nu_{\mathcal{A}}} = \dfrac{\partial u_2}{\partial \nu_{\mathcal{A}}} \text{ on } \Gamma_1, \\[2mm]
a_1 \mathcal{A} u_1 = a_2 \mathcal{A} u_2, \ a_1 \dfrac{\partial \mathcal{A} u_1}{\partial \nu_{\mathcal{A}}} = a_2 \dfrac{\partial \mathcal{A} u_2}{\partial \nu_{\mathcal{A}}} \text{ on } \Gamma_1, \\[2mm]
\mathcal{A} u_2 = 0 \text{ on } \Gamma_0,
\end{cases}
\tag{1.11}
$$

possesses a unique zero solution, where ζ is an arbitrary complex number and Γ_0 is relatively open in Γ defined by

$$
\Gamma_0 = \{x \in \Gamma \mid N(x) \cdot \nu > 0\}.
\tag{1.12}
$$

When $a_{ij}(x) = \delta_{ij}$, then for some fixed x_0, the radial field $N = x - x_0$ meets Assumption (H1) with $b(x) \equiv 1$ and Assumption (H3) is a valid fact (Theorem 4.2 of [21]). For the variable coefficients case, several corollaries were presented in [35] to show how to verify Assumption (H1) by means of the Riemannian geometry method, but for Assumption (H3), it is not verified even for Euler-Bernoulli plate equations without transmission. As was indicated in [34], the problem is not a Cauchy problem, and hence many uniqueness theorems cannot be applied. We propose it as an unsolved problem here.

THEOREM 1.3. Under Assumptions (H1)–(H3) and $a_2 \leq a_1$, the system (1.1) is exactly controllable on some $[0, T], T > 0$. That is, for any initial data $(w^0, w^1) \in \mathcal{H}$, there exists a control $u \in L^2(\Sigma)$ with $u = 0$ on $\Sigma \setminus \Sigma_0$ such that the solution of (1.1) satisfies $w(\cdot, T) = w_t(\cdot, T) = 0$.

The remainder of the paper is organized as follows. In Section 2, the system (1.1) is cast into an abstract setting studied in [2, 11] and some basic background on Riemannian geometry is introduced. The proof of Theorem 1.1 is given in Section 3. Section 4 is devoted to the proof of Theorem 1.2. Finally, in Section 5, we show the exact controllability under Assumptions (H1)–(H3).

2. Abstract formulation and preliminaries. Let $H = H^{-2}(\Omega)$. Let A be the positive self-adjoint operator in H induced by the bilinear form $\beta(\cdot, \cdot)$ defined by

$$
\langle Af, g \rangle_{H^{-2}(\Omega) \times H_0^2(\Omega)} = \beta(f, g) = \int_\Omega a(x) \mathcal{A} f(x) \cdot \overline{\mathcal{A} g(x)} dx, \ \forall f, g \in H_0^2(\Omega).
$$

By means of the Lax-Milgram theorem, A is a canonical isomorphism from $D(A) = H_0^2(\Omega)$ onto H. Set

$$
\begin{aligned}
H^4(\Omega, \Gamma_1) = \quad &\{w \in H_0^2(\Omega), w_i \in H^4(\Omega_i), i = 1, 2 \mid \\
&\quad a_1 \mathcal{A} w_1 = a_2 \mathcal{A} w_2 \text{ and } a_1 \dfrac{\partial \mathcal{A} w_1}{\partial \nu_{\mathcal{A}}} = a_2 \dfrac{\partial \mathcal{A} w_2}{\partial \nu_{\mathcal{A}}} \text{ on } \Gamma_1\}
\end{aligned}
\tag{2.1}
$$

with the norm

$$\|w\|_{H^4(\Omega,\Gamma_1)} = \left(\|w_1\|_{H^4(\Omega_1)}^2 + \|w_2\|_{H^4(\Omega_2)}^2\right)^{\frac{1}{2}}.$$

Similar to Lemmas 1.1, 1.2 and 1.3 of [21], it can be shown that

$$\left(\|w_1\|_{H^4(\Omega_1)}^2 + \|w_2\|_{H^4(\Omega_2)}^2\right)^{\frac{1}{2}} \sim \left(\|\mathscr{A}^2 w_1\|_{L^2(\Omega_1)}^2 + \|\mathscr{A}^2 w_2\|_{L^2(\Omega_2)}^2\right)^{\frac{1}{2}},$$

and $H^4(\Omega,\Gamma_1)$ is dense in $H_0^2(\Omega)$. Moreover, $Af = a(x)\mathscr{A}f$ whenever $f \in H^4(\Omega,\Gamma_1)$ and that $A^{-1}g = (a(x)\mathscr{A})^{-1}g$ for any $g \in L^2(\Omega)$. Hence A is an extension of $a(x)\mathscr{A}$ to the space $H_0^2(\Omega)$.

Just as in [15], it can be shown that $D(A^{1/2}) = L^2(\Omega)$ and $A^{1/2}$ is an isomorphism from $L^2(\Omega)$ onto H. Taking $s = \frac{3}{2}$ in (4.3) of Lemma 4.1 in Section 4, we can define the map $\Upsilon \in \mathcal{L}(L^2(\Gamma), H^{3/2}(\Omega))$ such that $\Upsilon u = v$ if and only if

$$\begin{cases} a(x)\mathscr{A}^2 v = 0 \text{ in } \Omega, \\[2mm] v_1 = v_2, \ \dfrac{\partial v_1}{\partial \nu_{\mathcal{A}}} = \dfrac{\partial v_2}{\partial \nu_{\mathcal{A}}}, \ a_1 \mathcal{A} v_1 = a_2 \mathcal{A} v_2, \ a_1 \dfrac{\partial \mathcal{A} v_1}{\partial \nu_{\mathcal{A}}} = a_2 \dfrac{\partial \mathcal{A} v_2}{\partial \nu_{\mathcal{A}}} \text{ on } \Gamma_1, \\[3mm] v_2 = 0, \ \dfrac{\partial v_2}{\partial \nu_{\mathcal{A}}} = u \text{ on } \Gamma. \end{cases} \qquad (2.2)$$

Since $D(A)$ is dense in H, so is $D(A^{1/2})$. We identify H with its dual H'. Then the following relations hold:

$$D(A^{1/2}) \hookrightarrow H = H' \hookrightarrow (D(A^{1/2}))'.$$

An extension $\tilde{A} \in \mathcal{L}(D(A)^{1/2}, (D(A^{1/2}))')$ of A is defined by

$$\langle \tilde{A}f, g \rangle_{(D(A^{1/2}))' \times D(A^{1/2})} = \langle A^{1/2}f, A^{1/2}g \rangle_H, \ \forall\, f,g \in D(A^{1/2}). \qquad (2.3)$$

So (1.1) can be written in $(D(A^{1/2}))'$ as

$$\ddot{w} + \tilde{A}w = Bu,$$

where $B \in \mathcal{L}(U, (D(A^{1/2}))')$ is given by

$$Bu = \tilde{A}\Upsilon u, \ \forall\, u \in U. \qquad (2.4)$$

Define $B^* \in \mathcal{L}(D(A^{1/2}), U)$ by

$$\langle B^*f, u \rangle_U = \langle f, Bu \rangle_{D(A^{1/2}) \times (D(A^{1/2}))'}, \ \forall\, f \in D(A^{1/2}), u \in U.$$

Then for any $f \in D(A^{1/2})$ and $u \in C_0^\infty(\Gamma_0)$, we have

$$\begin{aligned} \langle f, Bu \rangle_{D(A^{1/2}) \times (D(A^{1/2}))'} &= \langle f, \tilde{A}\tilde{A}^{-1}Bu \rangle_{D(A^{1/2}) \times (D(A^{1/2}))'} = \langle A^{1/2}f, A^{1/2}\tilde{A}^{-1}Bu \rangle_H \\ &= \langle A^{1/2}f, A^{1/2}\Upsilon u \rangle_H = \langle f, \Upsilon u \rangle_{L^2(\Omega)} \\ &= \langle AA^{-1}f, \Upsilon u \rangle_{L^2(\Omega)} = \langle -\mathcal{A}(\mathscr{A}^{-1}f), u \rangle_U. \end{aligned}$$

In the last step, we used Green's second formula twice and the connection boundary conditions on Γ_1. Since $C_0^\infty(\Gamma)$ is dense in $L^2(\Gamma)$, we finally obtain

$$B^*f = -\mathcal{A}(\mathscr{A}^{-1}f)\big|_\Gamma, \ \forall f \in D(A^{1/2}) = L^2(\Omega). \qquad (2.5)$$

Now we have formulated the open-loop system (1.1) into an abstract form of a second-order collocated system in $\mathcal{H}$:

$$\begin{cases} \ddot{w}(\cdot,t) + \tilde{A}w(\cdot,t) = Bu(\cdot,t), \\ y(\cdot,t) = B^*\dot{w}(\cdot,t), \end{cases} \tag{2.6}$$

where $\tilde{A}$, B and B^* are defined by (2.3), (2.4) and (2.5), respectively. The system (2.6) was well studied in [2, 11].

To end this section, we list some basic facts in Riemannian geometry that we need in subsequent sections (see also [16]).

Let $(\mathbb{R}^n, g)$ be the Riemannian manifold defined in Section 1. For any $\varphi \in C^2(\mathbb{R}^n)$ and $N = \sum_{i=1}^{n} \gamma_i(x)\frac{\partial}{\partial x_i}$, denote

$$\mathrm{div}_0(N) := \sum_{i=1}^{n} \frac{\partial \gamma_i(x)}{\partial x_i}, \ D\varphi := \nabla_g \varphi = \sum_{i,j=1}^{n} \frac{\partial \varphi}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j},$$

$$\mathrm{div}_g(N) := \sum_{i=1}^{n} \frac{1}{\sqrt{\mathcal{G}(x)}} \frac{\partial}{\partial x_i}(\sqrt{\mathcal{G}(x)}\gamma_i(x)),$$

$$\Delta_g \varphi := \sum_{i,j=1}^{n} \frac{1}{\sqrt{\mathcal{G}}} \frac{\partial}{\partial x_i}(\sqrt{\mathcal{G}} a_{ij} \frac{\partial \varphi}{\partial x_j}) = \mathcal{A}\varphi - (Df)\varphi, \ f(x) = \frac{1}{2}\log\det(a_{ij}(x)),$$

where div_0 is the divergence operator on Euclidean space $\mathbb{R}^n$ and ∇_g, div_g and Δ_g are the gradient operator, the divergence operator and the Beltrami-Laplace operator on $(\mathbb{R}^n, g)$ respectively.

Let $\mu = \frac{\nu_{\mathcal{A}}}{|\nu_{\mathcal{A}}|_g}$. μ is the unit outward-pointing normal to $\partial\Omega$ in terms of the Riemannian metric g. The following Lemma 2.1 provides some useful identities ([28], pp. 128, 138).

LEMMA 2.1. Let $\varphi, \psi \in C^1(\overline{\Omega})$ and let N be a vector field on $(\mathbb{R}^n, g)$. Then we have
  (1) divergence formula and theorem:

$$\mathrm{div}_0(\varphi N) = \varphi\mathrm{div}_0(N) + N(\varphi), \quad \mathrm{div}_g(\varphi N) = \varphi\mathrm{div}_g(N) + N(\varphi),$$

$$\int_\Omega \mathrm{div}_0(N)dx = \int_{\partial\Omega} N \cdot \nu d\Gamma, \quad \int_\Omega \mathrm{div}_g(N)dx = \int_{\partial\Omega} \langle N, \mu\rangle_g d\Gamma,$$

  (2) Green's identities:

$$\int_\Omega \psi\mathcal{A}\varphi dx = \int_{\partial\Omega} \psi\frac{\partial\varphi}{\partial\nu_{\mathcal{A}}}d\Gamma - \int_\Omega \langle\nabla_g\varphi, \nabla_g\psi\rangle_g dx,$$

$$\int_\Omega \psi\Delta_g\varphi d\Omega = \int_{\partial\Omega} \psi\frac{\partial\varphi}{\partial\mu}dS - \int_\Omega \langle\nabla_g\varphi, \nabla_g\psi\rangle_g d\Omega,$$

where $d\Omega$ and dS are the volume elements of Ω and $\partial\Omega$, respectively.

Denote by $T^2(\mathbb{R}_x^n)$ the set of all covariant tensors of order 2 on $\mathbb{R}_x^n$. Then $T^2(\mathbb{R}_x^n)$ is an inner product space of dimension n^2 with inner product of the following:

$$\langle F, G\rangle_{T^2(\mathbb{R}_x^n)} = \sum_{i,j=1}^{n} F(e_i, e_j)G(e_i, e_j), \ \forall \ F, G \in T^2(\mathbb{R}_x^n),$$

where $\{e_i\}_{i=1}^n$ is an arbitrarily chosen orthonormal basis of $(\mathbb{R}_x^n, g)$.

Let $\mathfrak{X}(\mathbb{R}^n)$ be the set of all tangential vector fields on $\mathbb{R}^n$. Denote by $\triangle : \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$ the Hodge-Laplace operator acting on the tangential fields; then for any orthonormal basis $\{e_i\}_{i=1}^n$ and its dual basis $\{w^i\}_{i=1}^n$ at $(\mathbb{R}_x^n, g)$, we have the following Weitzenböck formula (see [32]):

$$\triangle = -\sum_{k=1}^n D^2_{e_k e_k} + \sum_{j,k=1}^n e^k \wedge i(w^j) R(e_k, e_j), \tag{2.7}$$

where for any $X, Y \in \mathfrak{X}(\mathbb{R}^n)$ and $\eta \in A(\mathbb{R}^n)$, the set of all the differential forms on $\mathbb{R}^n$, $D^2_{XY} := D_X D_Y - D_{D_X Y}$ is a second order covariant differential operator, $R(X, Y) = -D_X D_Y + D_Y D_X + D_{[X,Y]}$ is the curvature operator with regard to the metric g, and $i(\eta)X := \eta(X)$ is the interior product operator. Furthermore, it has ([34]):

$$\Delta_g(N(\varphi)) = (-\triangle N)(\varphi) + 2\langle DN, D^2\varphi \rangle_{T^2(\mathbb{R}_x^n)} + N(\Delta_g \varphi) + \text{Ric}(N, D\varphi), \tag{2.8}$$

$$N(\Delta_g \varphi) = N(\mathcal{A}\varphi) - D^2 f(N, D\varphi) - D^2 \varphi(N, Df), \ \forall \ \varphi \in C^2(\mathbb{R}^n), \tag{2.9}$$

where $f(x) = \dfrac{1}{2}\log \det(a_{ij}(x))$ and $\text{Ric}(\cdot, \cdot)$ is the Ricci curvature tensor with respect to the Riemannian metric g, $D^2 f$ and $D^2 \varphi$ are the Hessians of f and φ in terms of g, respectively. The identity (2.9) is actually proved by (5.10) in Lemma 5.1 of Section 5, while the proof of (2.8) is presented in the Appendix.

The following Lemma 2.2 is straightforward.

LEMMA 2.2. Let $\varphi \in C^2(\overline{\Omega})$. Then there is a constant C depending on g, N and Ω only such that

(1)

$$\sup_{x \in \overline{\Omega}} |\mathcal{A}(\text{div}_g(N))| \leq C, \quad \sup_{x \in \overline{\Omega}} |Df(\text{div}_g(N))| \leq C, \quad \sup_{x \in \overline{\Omega}} |\text{div}_0(Df)| \leq C,$$

$$\sup_{x \in \overline{\Omega}} |\text{div}_0(N)| \leq C, \quad \sup_{x \in \overline{\Omega}} |\text{div}_g(N)| \leq C, \quad \sup_{x \in \partial\Omega} \left| \frac{1}{|\nu_{\mathcal{A}}|_g} \right| \leq C, \quad \sup_{x \in \partial\Omega} |Df \cdot \nu| \leq C.$$

(2)

$$|\langle D\varphi, D(\text{div}_g(N)) \rangle_g| \leq |D\varphi|_g \, |D(\text{div}_g(N))|_g \leq C \, |D\varphi|_g,$$

$$|(\triangle N)\varphi|_g \leq C \, |\triangle N|_g \, |D\varphi|_g \leq C \, |D\varphi|_g,$$

$$\left| \langle DN, D^2\varphi \rangle_{T^2(\mathbb{R}_x^n)} \right| \leq C \, |DN|_g \, |D^2\varphi|_g \leq C \, |D^2\varphi|_g,$$

$$|D^2 f(N, D\varphi)| \leq |D^2 f|_g \, |N|_g \, |D\varphi|_g \leq C \, |D\varphi|_g,$$

$$|D^2\varphi(N, Df)| \leq |D^2\varphi|_g \, |N|_g \, |Df|_g \leq C \, |D^2\varphi|_g,$$

$$|\text{Ric}(N, D\varphi)| \leq |\text{Ric}|_g \, |N|_g \, |D\varphi|_g \leq C \, |D\varphi|_g, \quad |Df(\varphi)| \leq C \, |D\varphi|_g,$$

$$|N(\varphi)| \leq C \, |D\varphi|_g, \quad |Df(N(\varphi))| \leq C \, |D^2\varphi|_g, \quad |\mathcal{A}\varphi| \leq C \, |D^2\varphi|_g.$$

(3)

$$\int_\Omega |\varphi|^2 \, dx \leq C\|\varphi\|^2_{H^2(\Omega)}, \quad \int_\Omega |D\varphi|^2_g \, dx \leq C\|\varphi\|^2_{H^2(\Omega)}, \quad \int_\Omega |D^2\varphi|^2_g \, dx \leq C\|\varphi\|^2_{H^2(\Omega)}.$$

3. The proof of Theorem 1.1. In this section, we use C_T to denote some positive constant that is independent of (y, u) although it may change values in different contexts. We rewrite (1.1) with zero initial values as the following equation (3.1):

$$\begin{cases} \ddot{w}(x,t) + a(x)\mathcal{A}^2 w(x,t) = 0, \ (x,t) \in Q, \\ w(x,0) = 0, \ \dot{w}(x,0) = 0, \ x \in \Omega, \\ w_2(x,t) = 0, \ \dfrac{\partial w_2}{\partial \nu_{\mathcal{A}}}(x,t) = u(x,t), \ (x,t) \in \Sigma, \\ w_1(x,t) = w_2(x,t), \ \dfrac{\partial w_1}{\partial \nu_{\mathcal{A}}}(x,t) = \dfrac{\partial w_2}{\partial \nu_{\mathcal{A}}}(x,t), \ (x,t) \in \Sigma_1, \\ a_1 \mathcal{A}w_1(x,t) = a_2 \mathcal{A}w_2(x,t), \ a_1 \dfrac{\partial \mathcal{A}w_1}{\partial \nu_{\mathcal{A}}}(x,t) = a_2 \dfrac{\partial \mathcal{A}w_2}{\partial \nu_{\mathcal{A}}}(x,t), \ (x,t) \in \Sigma_1, \\ y(x,t) = -\mathcal{A}(\mathscr{A}^{-1}\dot{w}(x,t)), \ (x,t) \in \Sigma. \end{cases} \tag{3.1}$$

By [31] (see also Proposition A1 of [19]), Theorem 1.1 is equivalent to saying that the solution of (3.1) satisfies

$$\|y\|^2_{L^2(0,T;U)} \leq C_T \|u\|^2_{L^2(0,T;U)}, \ \forall\, u \in L^2(0,T;U).$$

We return to the smoother space $H_0^2(\Omega)$ by the following transformation:

$$z = A^{-1}\dot{w}.$$

Then z satisfies

$$\begin{cases} \ddot{z}(x,t) + a(x)\mathcal{A}^2 z(x,t) = \Upsilon u_t(x,t), \ (x,t) \in Q, \\ z(x,0) = z^0(x), \ \dot{z}(x,0) = z^1(x), \ x \in \Omega, \\ z_2(x,t) = 0, \ \dfrac{\partial z_2}{\partial \nu_{\mathcal{A}}}(x,t) = 0, \ (x,t) \in \Sigma, \\ z_1(x,t) = z_2(x,t), \ \dfrac{\partial z_1}{\partial \nu_{\mathcal{A}}}(x,t) = \dfrac{\partial z_2}{\partial \nu_{\mathcal{A}}}(x,t), \ (x,t) \in \Sigma_1, \\ a_1 \mathcal{A}z_1(x,t) = a_2 \mathcal{A}z_2(x,t), \ a_1 \dfrac{\partial \mathcal{A}z_1}{\partial \nu_{\mathcal{A}}}(x,t) = a_2 \dfrac{\partial \mathcal{A}z_2}{\partial \nu_{\mathcal{A}}}(x,t), \ (x,t) \in \Sigma_1. \end{cases} \tag{3.2}$$

By (2.5), the output becomes

$$y(x,t) = -\mathcal{A}z_2(x,t)|_\Sigma.$$

Therefore, Theorem 1.1 is valid if and only if for some (and hence for all) $T > 0$, there exists a $C_T > 0$ such that the solution of (3.2) satisfies

$$\int_\Sigma |\mathcal{A}z(x,t)|^2 d\Sigma \leq C_T \int_\Sigma |u(x,t)|^2 d\Sigma. \tag{3.3}$$

Proof. We split the proof into eight steps.

STEP 1. Let N be a vector field of class C^2 on $\overline{\Omega}$ such that (Lemma 4.1, [15])

$$N(x) = \mu(x), \ x \in \Gamma; \ |N|_g \leq 1, \quad x \in \Omega. \tag{3.4}$$

Multiply both sides of the first equation of (3.2) by $N(\overline{z})$ and integrate over $[0, T] \times \Omega$, to give

$$\int_0^T \int_\Omega \ddot{z} N(\overline{z}) dx dt + \int_0^T \int_\Omega a(x) \mathcal{A}^2 z N(\overline{z}) dx dt - \int_0^T \int_\Omega \Upsilon u_t N(\overline{z}) dx dt = 0. \qquad (3.5)$$

Compute the first term on the left hand side of (3.5) to yield

$$\int_0^T \int_\Omega \ddot{z} N(\overline{z}) dx dt = \int_\Omega \dot{z} N(\overline{z}) dx \Big|_0^T - \int_0^T \int_\Omega \dot{z} N(\dot{\overline{z}}) dx dt$$

$$= \int_\Omega \dot{z} N(\overline{z}) dx \Big|_0^T - \int_\Omega z N(\dot{\overline{z}}) dx \Big|_0^T + \int_0^T \int_\Omega z N(\ddot{\overline{z}}) dx dt$$

$$= \int_\Omega \dot{z} N(\overline{z}) dx \Big|_0^T - \int_\Omega [\operatorname{div}_g(z \dot{\overline{z}} N) - \dot{\overline{z}} z \operatorname{div}_g(N) - \dot{\overline{z}} N(z)] dx \Big|_0^T$$

$$\quad + \int_0^T \int_\Omega [\operatorname{div}_g(z \ddot{\overline{z}} N) - \ddot{\overline{z}} z \operatorname{div}_g(N) - \ddot{\overline{z}} N(z)] dx dt$$

$$= 2\operatorname{Re} \int_\Omega \dot{z} N(\overline{z}) dx \Big|_0^T + \int_\Omega \dot{\overline{z}} z \operatorname{div}_g(N) dx \Big|_0^T$$

$$\quad + \int_0^T \int_\Omega [z a(x) \mathcal{A}^2 \overline{z} \operatorname{div}_g(N) - \Upsilon \overline{u_t} z \operatorname{div}_g(N) - \ddot{\overline{z}} N(z)] dx dt.$$

Hence

$$\operatorname{Re} \int_0^T \int_\Omega \ddot{z} N(\overline{z}) dx dt = \operatorname{Re} \int_\Omega \dot{z} N(\overline{z}) dx \Big|_0^T + \frac{1}{2} \int_\Omega \dot{\overline{z}} z \operatorname{div}_g(N) dx \Big|_0^T$$
$$- \frac{1}{2} \int_0^T \int_\Omega \Upsilon \overline{u_t} z \operatorname{div}_g(N) dx dt + \frac{1}{2} \int_0^T \int_\Omega z a(x) \mathcal{A}^2 \overline{z} \operatorname{div}_g(N) dx dt. \qquad (3.6)$$

By Green's second formula in the Riemannian manifold and the fact that

$$\frac{\partial z}{\partial \mu} = \langle \nabla_g z, \mu \rangle_g, \quad \frac{\partial}{\partial \mu} = \mu = \frac{1}{|\nu_\mathcal{A}|_g} \frac{\partial}{\partial \nu_\mathcal{A}} \text{ on } \Gamma \text{ and } \Gamma_1, \qquad (3.7)$$

the last term of (3.6), by taking the boundary conditions into account, is further expressed as

$$
\frac{1}{2}\int_0^T\int_\Omega za(x)\mathcal{A}^2\overline{z}\operatorname{div}_g(N)dxdt = \frac{1}{2}\int_0^T\int_\Omega za(x)[(\Delta_g+Df)(\mathcal{A}\overline{z})]\operatorname{div}_g(N)dxdt
$$

$$
= \frac{1}{2}\int_0^T\int_\Omega za(x)\Delta_g(\mathcal{A}\overline{z})\operatorname{div}_g(N)dxdt + \frac{1}{2}\int_0^T\int_\Omega za(x)Df(\mathcal{A}\overline{z})\operatorname{div}_g(N)dxdt
$$

$$
= \frac{1}{2}\int_0^T\int_\Omega a(x)\mathcal{A}\overline{z}\Delta_g(z\operatorname{div}_g(N))dxdt + \frac{1}{2}\int_0^T\int_\Gamma a_2z_2\operatorname{div}_g(N)\frac{\partial(\mathcal{A}\overline{z}_2)}{\partial\mu}d\Gamma dt
$$

$$
+ \frac{1}{2}\int_0^T\int_{\Gamma_1} a_2z_2\operatorname{div}_g(N)\frac{\partial(\mathcal{A}\overline{z}_2)}{\partial\mu}d\Gamma dt - \frac{1}{2}\int_0^T\int_{\Gamma_1} a_1z_1\operatorname{div}_g(N)\frac{\partial(\mathcal{A}\overline{z}_1)}{\partial\mu}d\Gamma dt
$$

$$
- \frac{1}{2}\int_0^T\int_\Gamma a_2\mathcal{A}\overline{z}_2\frac{\partial(z_2\operatorname{div}_g(N))}{\partial\mu}d\Gamma dt - \frac{1}{2}\int_0^T\int_{\Gamma_1} a_2\mathcal{A}\overline{z}_2\frac{\partial(z_2\operatorname{div}_g(N))}{\partial\mu}d\Gamma dt
$$

$$
+ \frac{1}{2}\int_0^T\int_{\Gamma_1} a_1\mathcal{A}\overline{z}_1\frac{\partial(z_1\operatorname{div}_g(N))}{\partial\mu}d\Gamma dt + \frac{1}{2}\int_0^T\int_\Omega za(x)Df(\mathcal{A}\overline{z})\operatorname{div}_g(N)dxdt
$$

$$
= \frac{1}{2}\int_0^T\int_\Omega a(x)\mathcal{A}\overline{z}[\mathcal{A}z\operatorname{div}_g(N) + 2\langle Dz, D(\operatorname{div}_g(N))\rangle_g + z\mathcal{A}(\operatorname{div}_g(N))]dxdt
$$

$$
- \frac{1}{2}\int_0^T\int_\Omega a(x)\mathcal{A}\overline{z}Df(z\operatorname{div}_g(N))dxdt + \frac{1}{2}\int_0^T\int_\Omega za(x)Df(\mathcal{A}\overline{z})\operatorname{div}_g(N)dxdt. \tag{3.8}
$$

Substitute (3.8) into (3.6) to obtain

$$
\operatorname{Re}\int_0^T\int_\Omega \ddot{z}N(\overline{z})dxdt = \operatorname{Re}\int_\Omega \dot{z}N(\dot{\overline{z}})dx\Big|_0^T + \frac{1}{2}\int_\Omega \overline{\dot{z}}z\operatorname{div}_g(N)dx\Big|_0^T
$$

$$
- \frac{1}{2}\int_0^T\int_\Omega \Upsilon\overline{u_t}z\operatorname{div}_g(N)dxdt + \frac{1}{2}\int_0^T\int_\Omega a(x)\left|\mathcal{A}z\right|^2\operatorname{div}_g(N)dxdt
$$

$$
+ \int_0^T\int_\Omega a(x)\mathcal{A}\overline{z}\langle Dz, D(\operatorname{div}_g(N))\rangle_g dxdt + \frac{1}{2}\int_0^T\int_\Omega a(x)z\mathcal{A}\overline{z}\mathcal{A}(\operatorname{div}_g(N))dxdt
$$

$$
- \frac{1}{2}\int_0^T\int_\Omega a(x)\mathcal{A}\overline{z}Df(z\operatorname{div}_g(N))dxdt + \frac{1}{2}\int_0^T\int_\Omega za(x)Df(\mathcal{A}\overline{z})\operatorname{div}_g(N)dxdt. \tag{3.9}
$$

Next, by Lemma 2.1 of [34] and (3.7),

$$
\begin{cases}
z_1 - z_2 = \dfrac{\partial(z_1-z_2)}{\partial\nu_\mathcal{A}} = 0 \Rightarrow H(z_1-z_2) = 0 \text{ for any vector field } H \text{ on } \Gamma_1, \\[2mm]
\dfrac{\partial^2(z_1-z_2)}{\partial\mu^2} = \dfrac{1}{|\nu_\mathcal{A}|_g^2}\dfrac{\partial^2(z_1-z_2)}{\partial\nu_\mathcal{A}^2} = \dfrac{1}{|\nu_\mathcal{A}|_g^2}(\nu_\mathcal{A}\cdot\nu)\mathcal{A}(z_1-z_2) = \Delta_g(z_1-z_2).
\end{cases} \tag{3.10}
$$

Applying Green's second formula in the Riemannian manifold again, and taking (2.8), (2.9), (3.7), (3.10) and boundary conditions into account, the second term on the left

hand side of (3.5) is computed as

$$
\begin{aligned}
\int_0^T \int_\Omega a(x)\mathcal{A}^2 z N(\overline{z}) dx dt &= \int_0^T \int_\Omega a(x)[(\Delta_g + Df)(\mathcal{A}z)]N(\overline{z}) dx dt \\
&= \int_0^T \int_\Omega a(x)\Delta_g(\mathcal{A}z)N(\overline{z}) dx dt + \int_0^T \int_\Omega a(x)Df(\mathcal{A}z)N(\overline{z}) dx dt \\
&= \int_0^T \int_\Omega a(x)\mathcal{A}z\Delta_g(N(\overline{z})) dx dt + \int_0^T \int_\Gamma N(\overline{z}_2) a_2 \frac{\partial(\mathcal{A}z_2)}{\partial \mu} d\Gamma dt \\
&\quad + \int_0^T \int_{\Gamma_1} N(\overline{z}_2) a_2 \frac{\partial(\mathcal{A}z_2)}{\partial \mu} d\Gamma dt - \int_0^T \int_{\Gamma_1} N(\overline{z}_1) a_1 \frac{\partial(\mathcal{A}z_1)}{\partial \mu} d\Gamma dt \\
&\quad - \int_0^T \int_\Gamma a_2 \mathcal{A}z_2 \frac{\partial(N(\overline{z}_2))}{\partial \mu} d\Gamma dt - \int_0^T \int_{\Gamma_1} a_2 \mathcal{A}z_2 \frac{\partial(N(\overline{z}_2))}{\partial \mu} d\Gamma dt \\
&\quad + \int_0^T \int_{\Gamma_1} a_1 \mathcal{A}z_1 \frac{\partial(N(\overline{z}_1))}{\partial \mu} d\Gamma dt + \int_0^T \int_\Omega a(x)Df(\mathcal{A}z)N(\overline{z}) dx dt \\
&= \int_0^T \int_\Omega a(x)\mathcal{A}z\Delta_g(N(\overline{z})) dx dt - \int_0^T \int_\Gamma a_2 \mathcal{A}z_2 \frac{\partial^2 \overline{z}_2}{\partial \mu^2} d\Gamma dt - \int_0^T \int_{\Gamma_1} a_2 \mathcal{A}z_2 \frac{\partial^2 \overline{z}_2}{\partial \mu^2} d\Gamma dt \\
&\quad + \int_0^T \int_{\Gamma_1} a_1 \mathcal{A}z_1 \frac{\partial^2 \overline{z}_1}{\partial \mu^2} d\Gamma dt + \int_0^T \int_\Omega a(x)Df(\mathcal{A}z)N(\overline{z}) dx dt \\
&= \int_0^T \int_\Omega a(x)\mathcal{A}z[(-\triangle N)(\overline{z}) + 2\langle DN, D^2\overline{z}\rangle_{T^2(\mathbb{R}_x^n)} + N(\Delta_g \overline{z}) \\
&\quad + \mathrm{Ric}(N, D\overline{z})] dx dt - \int_0^T \int_\Gamma a_2 |\mathcal{A}z_2|^2 d\Gamma dt \\
&\quad + \int_0^T \int_{\Gamma_1} (a_1 |\mathcal{A}z_1|^2 - a_2 |\mathcal{A}z_2|^2) d\Gamma dt + \int_0^T \int_\Omega a(x)Df(\mathcal{A}z)N(\overline{z}) dx dt \\
&= \int_0^T \int_\Omega a(x)\mathcal{A}z[(-\triangle N)(\overline{z}) + 2\langle DN, D^2\overline{z}\rangle_{T^2(\mathbb{R}_x^n)} + N(\mathcal{A}\overline{z}) - D^2 f(N, D\overline{z}) \\
&\quad - D^2\overline{z}(N, Df) + \mathrm{Ric}(N, D\overline{z})] dx dt + \int_0^T \int_\Omega a(x)Df(\mathcal{A}z)N(\overline{z}) dx dt \\
&\quad - \int_0^T \int_\Gamma a_2 |\mathcal{A}z_2|^2 d\Gamma dt + \int_0^T \int_{\Gamma_1} \frac{(a_2 - a_1)}{a_1 a_2} |a_1 \mathcal{A}z_1|^2 d\Gamma dt.
\end{aligned}
$$

$$(3.11)$$

In the last step of (3.11), the identity $\dfrac{\partial^2 \overline{z}_2}{\partial \mu^2} = \Delta_g \overline{z}_2 = \mathcal{A}\overline{z}_2$ on Γ that is similar to (3.10) was used.

Furthermore, by the divergence formula, one has

$$
\operatorname{Re} \int_0^T \int_\Omega a(x)\mathcal{A}zN(\mathcal{A}\overline{z})dxdt = \frac{1}{2}\int_0^T \int_\Omega N(a(x)\left|\mathcal{A}z\right|^2)dxdt
$$

$$
= \frac{1}{2}\int_0^T \int_\Gamma a_2\left|\mathcal{A}z_2\right|^2 d\Gamma dt + \frac{1}{2}\int_0^T \int_{\Gamma_1} a_2\left|\mathcal{A}z_2\right|^2 d\Gamma dt - \frac{1}{2}\int_0^T \int_{\Gamma_1} a_1\left|\mathcal{A}z_1\right|^2 d\Gamma dt
$$

$$
- \frac{1}{2}\int_0^T \int_\Omega a(x)\left|\mathcal{A}z\right|^2 \operatorname{div}_g(N)dxdt
$$

$$
= \frac{1}{2}\int_0^T \int_\Gamma a_2\left|\mathcal{A}z_2\right|^2 d\Gamma dt - \frac{1}{2}\int_0^T \int_{\Gamma_1} \frac{(a_2 - a_1)}{a_1 a_2}\left|a_1\mathcal{A}z_1\right|^2 d\Gamma dt
$$

$$
- \frac{1}{2}\int_0^T \int_\Omega a(x)\left|\mathcal{A}z\right|^2 \operatorname{div}_g(N)dxdt.
$$

So (3.11) can be further expressed as

$$
\operatorname{Re} \int_0^T \int_\Omega a(x)\mathcal{A}^2 zN(\overline{z})dxdt
$$

$$
= -\frac{1}{2}\int_0^T \int_\Omega a(x)\left|\mathcal{A}z\right|^2 \operatorname{div}_g(N)dxdt + \operatorname{Re} \int_0^T \int_\Omega a(x)\mathcal{A}z[(-\triangle N)(z)
$$

$$
+ 2\langle DN, D^2\overline{z}\rangle_{T^2(\mathbb{R}_x^n)} - D^2 f(N, D\overline{z}) - D^2\overline{z}(N, Df) + \operatorname{Ric}(N, D\overline{z})]dxdt \qquad (3.12)
$$

$$
- \frac{1}{2}\int_0^T \int_\Gamma a_2\left|\mathcal{A}z_2\right|^2 d\Gamma dt + \frac{1}{2}\int_0^T \int_{\Gamma_1} \frac{(a_2 - a_1)}{a_1 a_2}\left|a_1\mathcal{A}z_1\right|^2 d\Gamma dt
$$

$$
+ \operatorname{Re} \int_0^T \int_\Omega a(x)Df(\mathcal{A}z)N(\overline{z})dxdt.
$$

Finally, substitute (3.9) and (3.12) into (3.5) to obtain

$$
\frac{1}{2}\int_0^T \int_\Gamma a_2\left|\mathcal{A}z\right|^2 d\Gamma dt + \frac{1}{2}\int_0^T \int_{\Gamma_1} \frac{(a_1 - a_2)}{a_1 a_2}\left|a_1\mathcal{A}z_1\right|^2 d\Gamma dt
$$
$$
= \operatorname{RHS}_1 + \operatorname{RHS}_2 + \operatorname{RHS}_3 + \mathrm{b}_{0,\mathrm{T}},
$$

(3.13)

where

$$\text{RHS}_1 = \int_0^T \int_\Omega a(x)\mathcal{A}\overline{z}\langle Dz, D(\text{div}_g(N))\rangle_g dxdt + \frac{1}{2}\int_0^T \int_\Omega za(x)\mathcal{A}\overline{z}A(\text{div}_g(N))dxdt$$

$$+\text{Re}\int_0^T \int_\Omega a(x)\mathcal{A}z[(-\triangle N)(\overline{z}) + 2\langle DN, D^2\overline{z}\rangle_{T^2(\mathbb{R}_x^n)} - D^2f(N, D\overline{z})$$

$$-D^2\overline{z}(N, Df) + \text{Ric}(N, D\overline{z})]dxdt,$$

$$\text{RHS}_2 = -\frac{1}{2}\int_0^T \int_\Omega a(x)\mathcal{A}\overline{z}Df(z\text{div}_g(N))dxdt + \frac{1}{2}\int_0^T \int_\Omega za(x)\text{div}_g(N)Df(\mathcal{A}\overline{z})dxdt$$

$$+\text{Re}\int_0^T \int_\Omega a(x)Df(\mathcal{A}z)N(\overline{z})dxdt,$$

$$\text{RHS}_3 = -\frac{1}{2}\int_0^T \int_\Omega \Upsilon\overline{u_t}z\text{div}_g(N)dxdt - \text{Re}\int_0^T \int_\Omega \Upsilon u_t N(\overline{z})dxdt,$$

$$\text{b}_{0,\text{T}} = \text{Re}\int_\Omega \dot{z}N(\overline{z})dx\bigg|_0^T + \frac{1}{2}\text{Re}\int_\Omega \overline{\dot{z}}z\text{div}_g(N)dx\bigg|_0^T.$$

By the assumption on $a(x)$, we have

$$\frac{1}{2}\int_0^T \int_{\Gamma_1} \frac{(a_1 - a_2)}{a_1 a_2} |a_1 \mathcal{A}z_1|^2 d\Gamma dt \geq 0.$$

Therefore, we only need to estimate term by term for the right hand side of (3.13).

STEP 2 (ESTIMATE FOR RHS$_1$). Let $\Upsilon\overline{u_t} = 0$ in (3.13). We note that the transformation $z = A^{-1}\dot{w} \in H_0^2(\Omega)$ implies $\dot{z} = A^{-1}\ddot{w} = -w \in L^2(\Omega)$. Then Equation (3.2) associates with a C$_0$-group solution in the space $H_0^2(\Omega) \times L^2(\Omega)$; that is to say, for any $(z^0, z^1) \in H_0^2(\Omega) \times L^2(\Omega)$, the corresponding solution to (3.2) satisfies $(z, \dot{z}) \in H_0^2(\Omega) \times L^2(\Omega)$ and depends continuously on (z^0, z^1):

$$\frac{1}{2}\int_0^T \int_\Gamma a(x) |\mathcal{A}z_2|^2 d\Gamma dt \leq C_T \left\|(z^0, z^1)\right\|_{H_0^2(\Omega) \times L^2(\Omega)}^2.$$

This shows that B^* is admissible, and so is B ([8]). In other words,

$$u \mapsto \{w, \dot{w}\} \text{ is continuous from } L^2(\Sigma) \to C([0, T]; L^2(\Omega) \times H^{-2}(\Omega)). \qquad (3.14)$$

By (3.14), $z(t) \in C([0, T]; H_0^2(\Omega))$ that is continuous in $u \in L^2(\Sigma)$. Hence

$$\text{RHS}_1 \leq C_T\|u\|_{L^2(\Sigma)}^2, \ \forall\, u \in L^2(\Sigma), \qquad (3.15)$$

where we used Lemma 2.2.

STEP 3 (ESTIMATE FOR RHS$_2$). By formulae

$$\text{div}_0(a(x)z\text{div}_g(N)\mathcal{A}\overline{z}Df)$$

$$= za(x)\text{div}_g(N)Df(\mathcal{A}\overline{z}) + a(x)\mathcal{A}\overline{z}Df(z\text{div}_g(N)) + za(x)\text{div}_g(N)\mathcal{A}\overline{z}\text{div}_0(Df)$$

and

$$\text{div}_0(a(x)N(\overline{z})\mathcal{A}zDf) = a(x)Df(\mathcal{A}z)N(\overline{z}) + a(x)\mathcal{A}zDf(N(\overline{z})) + a(x)N(\overline{z})\mathcal{A}z\text{div}_0(Df),$$

we have

$$\frac{1}{2}\int_0^T \int_\Omega za(x)\mathrm{div}_g(N)Df(\mathcal{A}\overline{z})dxdt$$

$$= \frac{1}{2}\int_0^T \int_\Gamma a_2 z_2 \mathrm{div}_g(N)\mathcal{A}\overline{z}_2 Df\cdot\nu d\Gamma dt$$

$$+\frac{1}{2}\int_0^T\!\!\int_{\Gamma_1} a_2 z_2 \mathrm{div}_g(N)\mathcal{A}\overline{z}_2 Df\cdot\nu d\Gamma dt - \frac{1}{2}\int_0^T\!\!\int_{\Gamma_1} a_1 z_1 \mathrm{div}_g(N)\mathcal{A}\overline{z}_1 Df\cdot\nu d\Gamma dt$$

$$-\frac{1}{2}\int_0^T\!\!\int_\Omega a(x)\mathcal{A}\overline{z}Df(z\mathrm{div}_g(N))dxdt - \frac{1}{2}\int_0^T\!\!\int_\Omega za(x)\mathrm{div}_g(N)\mathcal{A}\overline{z}\mathrm{div}_0(Df)dxdt \tag{3.16}$$

$$= -\frac{1}{2}\int_0^T \int_\Omega a(x)\mathcal{A}\overline{z}Df(z\mathrm{div}_g(N))dxdt$$

$$= -\frac{1}{2}\int_0^T \int_\Omega za(x)\mathrm{div}_g(N)\mathcal{A}\overline{z}\mathrm{div}_0(Df)dxdt.$$

Similarly,

$$\int_0^T \int_\Omega a(x)Df(\mathcal{A}z)N(\overline{z})dxdt = \int_0^T \int_\Gamma N(\overline{z}_2)a_2\mathcal{A}z_2 Df\cdot\nu d\Gamma$$

$$-\int_0^T \int_\Omega a(x)\mathcal{A}zDf(N(\overline{z}))dxdt - \int_0^T \int_\Omega a(x)N(\overline{z})\mathcal{A}z\mathrm{div}_0(Df)dxdt. \tag{3.17}$$

By boundary conditions we see that the first term on the right hand side of (3.17) equals zero. Substitute (3.16) and (3.17) into RHS_2 to get

$$\mathrm{RHS}_2 = -\int_0^T \int_\Omega a(x)\mathcal{A}\overline{z}Df(z\mathrm{div}_g(N))dxdt$$

$$-\mathrm{Re}\int_0^T \int_\Omega a(x)\mathcal{A}zDf(N(\overline{z}))dxdt$$

$$-\frac{1}{2}\int_0^T \int_\Omega za(x)\mathrm{div}_g(N)\mathcal{A}\overline{z}\mathrm{div}_0(Df)dxdt \tag{3.18}$$

$$-\mathrm{Re}\int_0^T \int_\Omega a(x)N(\overline{z})\mathcal{A}z\mathrm{div}_0(Df)dxdt.$$

Thus, along the same lines as in Step 2, we get

$$\mathrm{RHS}_2 \le C_T\|u\|^2_{L^2(\Sigma)},\ \forall\, u\,\in L^2(\Sigma), \tag{3.19}$$

where we used Lemma 2.2 again.

The following Steps 4–6 are very similar to that of [16] for the variable coefficients case without transmission, but for the sake of completeness, we list the sketch of the proof here.

STEP 4 (REGULARITY OF $\dot{z}$). To handle RHS_3, we need the regularity of $\dot{z}$.

$$\dot{z} = A^{-1}\ddot{w} = A^{-1}(-Aw + \tilde{A}\Upsilon u) = -w + \Upsilon u \in L^2(\Sigma). \tag{3.20}$$

Since both $w \in C([0,T]; L^2(\Omega))$ and $\Upsilon u \in L^2(\Sigma)$ depend continuously on $u \in L^2(\Sigma)$, it follows that

$$\dot{z} \in L^2(\Sigma) \text{ continuously in } u \in L^2(\Sigma). \tag{3.21}$$

STEP 5 (ESTIMATES OF RHS$_3$ AND $b_{0,T}$ FOR SMOOTHER u). To estimate both RHS$_3$ and $b_{0,T}$, confine u within a smoother class that is dense in $L^2(\Sigma)$,

$$u \in C^1([0,T] \times \Gamma), \quad u(\cdot,0) = u(\cdot,T) = 0. \tag{3.22}$$

We will show the following two estimates

$$\text{RHS}_3 \le C_T \|u\|^2_{L^2(\Sigma)} \tag{3.23}$$

and

$$b_{0,T} \le C_T \|u\|^2_{L^2(\Sigma)} \tag{3.24}$$

for all u in the class of (3.22). From now on, we assume that $z_0 = z_1 = 0$ in (3.2).

STEP 6 (PROOF OF (3.24)). By the facts that $\dot{w} \in C([0,T]; H^{-2}(\Omega))$ continuously in $u \in L^2(\Sigma)$, $A^{-1} \in \mathcal{L}(H^{-2}(\Omega), H^2_0(\Omega))$ and $\dot{w}(\cdot,0) = 0$, we have

$$z(\cdot,0) = 0, \ z(\cdot,T) = A^{-1}\dot{w} \in H^2_0(\Omega) \text{ continuously in } u \in L^2(\Sigma). \tag{3.25}$$

Next by (3.20), (3.22) and $w(\cdot,0) = 0$,

$$\begin{cases} \dot{z}(\cdot,0) = -w(\cdot,0) + \Upsilon u(\cdot,0) = 0, \\ \dot{z}(\cdot,T) = -w(\cdot,T) \in L^2(\Omega) \text{ continuously in } u \in L^2(\Sigma), \end{cases} \tag{3.26}$$

where we used the regularity (3.14).

Using (3.14), (3.25) and (3.26), we readily obtain

$$b_{0,T} = \text{Re} \int_\Omega \dot{z} N(z) dx \Big|_0^T + \text{Re} \frac{1}{2} \int_\Omega \overline{z} z \, \text{div}_g(N) dx \Big|_0^T \le C_T \|u\|^2_{L^2(\Sigma)}. \tag{3.27}$$

STEP 7 (PROOF OF (3.23)). For the second term with u in the class (3.22), we integrate by parts with respect to t and make use of the divergence theorem again to obtain

$$-\text{Re} \int_0^T \int_\Omega \Upsilon u_t N(z) dx dt = -\text{Re} \int_\Omega \Upsilon u N(\overline{z}) dx \Big|_0^T + \text{Re} \int_0^T \int_\Omega \Upsilon u N(\dot{\overline{z}}) dx dt$$

$$= \text{Re} \int_0^T \int_\Omega \Upsilon u N(\dot{\overline{z}}) dx dt$$

$$= \text{Re} \int_0^T \int_\Omega \text{div}_0(\Upsilon u \dot{\overline{z}} N) dx dt - \text{Re} \int_0^T \int_\Omega \Upsilon u \dot{\overline{z}} \, \text{div}_0(N) dx dt$$

$$-\text{Re} \int_0^T \int_\Omega \dot{\overline{z}} N(\Upsilon u) dx dt \tag{3.28}$$

$$= \text{Re} \int_0^T \int_{\partial\Omega} \Upsilon u \dot{\overline{z}} N \cdot \nu d\Gamma dt - \text{Re} \int_0^T \int_\Omega \Upsilon u \dot{\overline{z}} \, \text{div}_0(N) dx dt$$

$$-\text{Re} \int_0^T \int_\Omega \dot{\overline{z}} N(\Upsilon u) dx dt$$

$$= -\text{Re} \int_0^T \int_\Omega \Upsilon u \dot{\overline{z}} \, \text{div}_0(N) dx dt - \text{Re} \int_0^T \int_\Omega \dot{\overline{z}} N(\Upsilon u) dx dt.$$

By the equalities (3.20), (3.28) and the fact that $\Upsilon u \in L^2(0,T;H^{3/2}(\Omega))$ implies $N(\Upsilon u) \in L^2(0,T;H^{1/2}(\Omega))$, all continuously in $u \in L^2(\Sigma)$, we have

$$-\mathrm{Re}\int_0^T\int_\Omega \Upsilon u_t N(\overline{z})dxdt \le C\|u\|_{L^2(\Sigma)}^2.$$

A similar estimate holds true for the first term of RHS$_3$ and we thereby obtain (3.23).

STEP 8. We can then extend estimate (3.23) of RHS$_3$ and (3.24) of $b_{0,T}$ to all $u \in L^2(\Sigma)$ by a density argument, which together with (3.19) and (3.15) gives (3.3). The proof is complete. $\qquad\square$

4. Proof of Theorem 1.2. In order to show Theorem 1.2, we need the following Lemma 4.1 which generalizes the regularity results for elliptic boundary problems to the transmission case.

LEMMA 4.1. Suppose ϕ solves the following equation:

$$\begin{cases} a(x)\mathcal{A}^2\phi(x) = h(x),\ x \in \Omega, \\[2mm] \phi_2(x) = 0,\ \dfrac{\partial\phi_2(x)}{\partial\nu_\mathcal{A}} = u(x),\ x \in \Gamma, \\[2mm] \phi_1(x) = \phi_2(x),\ \dfrac{\partial\phi_1(x)}{\partial\nu_\mathcal{A}} = \dfrac{\partial\phi_2(x)}{\partial\nu_\mathcal{A}},\ x \in \Gamma_1, \\[2mm] a_1\mathcal{A}\phi_1(x) = a_2\mathcal{A}\phi_2(x),\ a_1\dfrac{\partial\mathcal{A}\phi_1(x)}{\partial\nu_\mathcal{A}} = a_2\dfrac{\partial\mathcal{A}\phi_2(x)}{\partial\nu_\mathcal{A}},\ x \in \Gamma_1. \end{cases} \tag{4.1}$$

Then there exists a constant $C > 0$ independent of (ϕ,h,u) such that

$$\|\phi\|_{H^s(\Omega)} \le C(\|h\|_{H^{s-4}(\Omega)} + \|u\|_{H^{s-\frac{3}{2}}(\Gamma)})\quad \text{for } s \ge 4, \tag{4.2}$$

and

$$\|\phi\|_{H^s(\Omega)} \le C\|u\|_{H^{s-\frac{3}{2}}(\Gamma)}\quad \text{for } 0 < s < 4 \text{ and } h \equiv 0. \tag{4.3}$$

Proof. The solution of (4.1) can be written as

$$\phi(x) = \begin{cases} \phi_1(x),\ \ x \in \Omega_1, \\ \phi_2(x),\ \ x \in \Omega_2, \end{cases}$$

where ϕ_1 and ϕ_2 satisfy, respectively,

$$\begin{cases} a_2\mathcal{A}^2\phi_2(x) = h(x),\ \ x \in \Omega, \\[2mm] \phi_2(x) = 0,\ \dfrac{\partial\phi_2(x)}{\partial\nu_\mathcal{A}} = u(x),\ \ x \in \Gamma, \end{cases}$$

and

$$\begin{cases} a_1\mathcal{A}^2\phi_1(x) = h(x),\ \ x \in \Omega_1, \\[2mm] \phi_1(x) = \phi_2(x),\ \dfrac{\partial\phi_1(x)}{\partial\nu_\mathcal{A}} = \dfrac{\partial\phi_2(x)}{\partial\nu_\mathcal{A}},\ \ x \in \Gamma_1, \\[2mm] a_1\mathcal{A}\phi_1(x) = a_2\mathcal{A}\phi_2(x),\ a_1\dfrac{\partial\mathcal{A}\phi_1(x)}{\partial\nu_\mathcal{A}} = a_2\dfrac{\partial\mathcal{A}\phi_2(x)}{\partial\nu_\mathcal{A}},\ \ x \in \Gamma_1. \end{cases}$$

By virtue of the elliptic regularity (7.27) of [20] on p. 189, we have

$$\|\phi_2\|_{H^s(\Omega)} \leq C(\|h\|_{H^{s-4}(\Omega)} + \|u\|_{H^{s-\frac{3}{2}}(\Gamma)}) \text{ for } s \geq 4. \tag{4.4}$$

Similarly, by the elliptic regularity (7.28) of [20] on p. 189, we also have

$$\|\phi_2\|_{H^s(\Omega)} \leq C\|u\|_{H^{s-\frac{3}{2}}(\Gamma)} \text{ for } s < 4 \quad \text{and } h \equiv 0, \tag{4.5}$$

where $C > 0$ denotes, throughout the proof, some constants independent of functions involved although it may have different values in different cases.

For $s \geq 4$, it follows from the standard trace theorem in Sobolev space that $\phi_2|_{\Gamma_1} \in H^{s-\frac{1}{2}}(\Gamma_1)$, $\frac{\partial\phi_2}{\partial\nu_\mathcal{A}}|_{\Gamma_1} \in H^{s-\frac{3}{2}}(\Gamma_1)$ and

$$\left(\|\phi_2\|_{H^{s-\frac{1}{2}}(\Gamma_1)} + \|\frac{\partial\phi_2}{\partial\nu_\mathcal{A}}\|_{H^{s-\frac{3}{2}}(\Gamma_1)}\right) \leq C\|\phi_2\|_{H^s(\Omega)} \text{ for } s \geq 4. \tag{4.6}$$

As for $0 < s < 4$, applying the trace theorem 7.3 of [20] on p. 187, we have $\phi_2|_{\Gamma_1} \in H^{s-\frac{1}{2}}(\Gamma_1)$, $\frac{\partial\phi_2}{\partial\nu_\mathcal{A}}|_{\Gamma_1} \in H^{s-\frac{3}{2}}(\Gamma_1)$ and

$$\left(\|\phi_2\|_{H^{s-\frac{1}{2}}(\Gamma_1)} + \|\frac{\partial\phi_2}{\partial\nu_\mathcal{A}}\|_{H^{s-\frac{3}{2}}(\Gamma_1)}\right) \leq C\|\phi_2\|_{H^s(\Omega_1)} \text{ for } 0 < s < 4 \text{ and } h \equiv 0. \tag{4.7}$$

Applying the same elliptic regularity as above for both $s \geq 4$ and $0 < s < 4$, we get $\phi_1 \in H^s(\Omega_1)$ for any $s > 0$. At the same time, it follows from the connection boundary conditions on Γ_1 that

$$\|\phi_1\|_{H^s(\Omega_1)} \leq C\left(\|h\|_{H^{s-4}(\Omega_1)} + \|\phi_2\|_{H^{s-\frac{1}{2}}(\Gamma_1)} + \|\frac{\partial\phi_2}{\partial\nu_\mathcal{A}}\|_{H^{s-\frac{3}{2}}(\Gamma_1)}\right) \text{ for } s \geq 4, \tag{4.8}$$

and

$$\|\phi_1\|_{H^s(\Omega_1)} \leq C\left(\|\phi_2\|_{H^{s-\frac{1}{2}}(\Gamma_1)} + \|\frac{\partial\phi_2}{\partial\nu_\mathcal{A}}\|_{H^{s-\frac{3}{2}}(\Gamma_1)}\right) \text{ for } 0 < s < 4 \text{ and } h \equiv 0. \tag{4.9}$$

(4.2) and (4.3) are thus obtained by combining (4.4), (4.6), (4.8), (4.5), (4.7) and (4.9). □

It is known from the Appendix of [11] that the transfer function of the system (2.6) is

$$H(\lambda) = \lambda B^*(\lambda^2 + \tilde{A})^{-1}B, \tag{4.10}$$

where $\tilde{A}$, B and B^* are given by (2.3), (2.4) and (2.5), respectively. Moreover, from the well-posedness claimed by Theorem 1.1, it follows that there are constants $M, \beta > 0$ such that ([10])

$$\sup_{\text{Re}\lambda \geq \beta} \|H(\lambda)\|_{\mathcal{L}(U)} = M < \infty. \tag{4.11}$$

PROPOSITION 4.1. Theorem 1.2 is valid if for any $u \in C_0^\infty(\Gamma)$, the solution w to the following equation

$$
\begin{cases}
\lambda^2 w(x) + a(x)\mathcal{A}^2 w(x) = 0, \ x \in \Omega, \\[2mm]
w_2(x) = 0, \ \dfrac{\partial w_2(x)}{\partial \nu_{\mathcal{A}}} = u(x), \ x \in \Gamma, \\[2mm]
w_1(x) = w_2(x), \ \dfrac{\partial w_1(x)}{\partial \nu_{\mathcal{A}}} = \dfrac{\partial w_2(x)}{\partial \nu_{\mathcal{A}}}, \ \ x \in \Gamma_1, \\[2mm]
a_1 \mathcal{A} w_1(x) = a_2 \mathcal{A} w_2(x), \ a_1 \dfrac{\partial \mathcal{A} w_1(x)}{\partial \nu_{\mathcal{A}}} = a_2 \dfrac{\partial \mathcal{A} w_2(x)}{\partial \nu_{\mathcal{A}}}, \ x \in \Gamma_1
\end{cases}
\tag{4.12}
$$

satisfies

$$
\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} \int_\Gamma a_2 \left| \frac{1}{\lambda} \mathcal{A} w(x) \right|^2 dx = 0.
\tag{4.13}
$$

Proof. It was shown in [29] that in the frequency domain, (1.7) is equivalent to

$$
\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} H(\lambda)u = 0 \ \text{ in strong topology of } U \text{ for any } u \in U,
\tag{4.14}
$$

where $H(\lambda)$ is given by (4.10). Due to (4.11) and the density argument, it suffices to show that (4.14) is satisfied for all $u \in C_0^\infty(\Gamma)$.

Now assume that $u \in C_0^\infty(\Gamma)$, and put

$$
w(x) = ((\lambda^2 + \tilde{A})^{-1} Bu)(x).
$$

Then w satisfies (4.12) and

$$
(H(\lambda)u)(x) = -\lambda \mathcal{A}(\mathscr{A}^{-1} w)(x), \ x \in \Gamma.
\tag{4.15}
$$

Taking $s = 4$ in (4.2), we get a function $v \in H^4(\Omega, \Gamma_1)$ satisfying the following equation:

$$
\begin{cases}
a(x)\mathcal{A}^2 v(x) = 0, \ x \in \Omega, \\[2mm]
v_2(x) = 0, \ \dfrac{\partial v_2(x)}{\partial \nu_{\mathcal{A}}} = u(x), \ \ x \in \Gamma, \\[2mm]
v_1(x) = v_2(x), \ \dfrac{\partial v_1(x)}{\partial \nu_{\mathcal{A}}} = \dfrac{\partial v_2(x)}{\partial \nu_{\mathcal{A}}}, \ \ x \in \Gamma_1, \\[2mm]
a_1 \mathcal{A} v_1(x) = a_2 \mathcal{A} v_2(x), \ a_1 \dfrac{\partial \mathcal{A} v_1(x)}{\partial \nu_{\mathcal{A}}} = a_2 \dfrac{\partial \mathcal{A} v_2(x)}{\partial \nu_{\mathcal{A}}}, \ \ x \in \Gamma_1.
\end{cases}
$$

Thus (4.12) can be written as

$$
\begin{cases}
\lambda^2 w(x) + a(x)\mathcal{A}^2(w(x) - v(x)) = 0, \ x \in \Omega, \\[2mm]
(w_2 - v_2)(x) = \dfrac{\partial(w_2 - v_2)(x)}{\partial \nu_{\mathcal{A}}} = 0, \ x \in \Gamma, \\[2mm]
(w_1 - v_1)(x) = (w_2 - v_2)(x), \ \dfrac{\partial(w_1 - v_1)(x)}{\partial \nu_{\mathcal{A}}} = \dfrac{\partial(w_2 - v_2)(x)}{\partial \nu_{\mathcal{A}}}, \ x \in \Gamma_1, \\[2mm]
a_1 \mathcal{A}(w_1 - v_1)(x) = a_2 \mathcal{A}(w_2 - v_2)(x), \\[2mm]
a_1 \dfrac{\partial \mathcal{A}(w_1 - v_1)(x)}{\partial \nu_{\mathcal{A}}} = a_2 \dfrac{\partial \mathcal{A}(w_2 - v_2)(x)}{\partial \nu_{\mathcal{A}}}, \ x \in \Gamma_1
\end{cases}
$$

or

$$\lambda^2(\mathscr{A}^{-1}w)(x) = a(x)(-w(x) + v(x)).$$

So (4.15) becomes

$$(H(\lambda)u)(x) = \frac{1}{\lambda}a(x)\mathcal{A}w(x) - \frac{1}{\lambda}a(x)\mathcal{A}v(x). \tag{4.16}$$

Since $\mathcal{A}v(x)$ is independent of λ, the required result then follows from (4.16) and (4.14).
$\square$

In order to prove (4.13), we state the following Lemma 4.2 that comes from Lemma 4.1 of [16].

LEMMA 4.2. Let w be the solution of (4.12). Then there exists a function $\eta(x)$ independent of λ, which is continuous on Γ, such that

$$\Delta_g w(x) = \frac{\partial^2 w(x)}{\partial \mu^2} + \eta(x)\frac{\partial w(x)}{\partial \mu}, \ \forall \ x \in \Gamma. \tag{4.17}$$

Now we are in a position to prove Theorem 1.2.

Proof. Firstly, multiply both sides of the first equation of (4.12) by $\overline{w}$ and integrate by parts to give

$$
\begin{aligned}
0 \ &= \int_\Omega \lambda^2 |w|^2 + a(x)\mathcal{A}^2 w \cdot \overline{w}dx \\
&= \int_\Omega \lambda^2 |w|^2 \, dx + \int_\Omega a(x) |\mathcal{A}w|^2 \, dx - \int_\Gamma a_2 \mathcal{A}w_2 \frac{\partial \overline{w}_2}{\partial \nu_\mathcal{A}}d\Gamma \\
&= \int_\Omega \lambda^2 |w|^2 \, dx + \int_\Omega a(x) |\mathcal{A}w|^2 \, dx - \int_\Gamma a_2 \mathcal{A}w_2 \overline{u}d\Gamma,
\end{aligned}
$$

from which we get

$$\int_\Omega |w|^2 \, dx + \frac{1}{\lambda^2}\int_\Omega a(x) |\mathcal{A}w|^2 \, dx \leq \frac{1}{\lambda} \|u\|_{L^2(\Gamma)} \left\| \frac{1}{\lambda}a_2\mathcal{A}w_2 \right\|_{L^2(\Gamma)}. \tag{4.18}$$

Secondly, choose the vector field N on $\overline{\Omega}$ as in (3.4). As was done in Section 3, multiply both sides of the first equation of (4.12) by $N(\overline{w})$, integrate by parts and use (2.8), (2.9),

(3.7), (3.10), (3.17), (4.17) and the divergence formula to yield

$$
\begin{aligned}
0 \ &= \operatorname{Re} \int_\Omega [\lambda^2 w N(\overline{w}) + a(x)\mathcal{A}^2 w N(\overline{w})] dx \\
&= \frac{\lambda^2}{2} \int_\Omega [\operatorname{div}_0(|w|^2 N) - |w|^2 \operatorname{div}_0(N)] dx + \operatorname{Re} \int_\Omega a(x)\mathcal{A}w \Delta_g(N(\overline{w})) dx \\
&\quad + \operatorname{Re} \int_\Gamma N(\overline{w}_2) a_2 \frac{\partial(\mathcal{A}w_2)}{\partial \mu} d\Gamma + \operatorname{Re} \int_{\Gamma_1} N(\overline{w}_2) a_2 \frac{\partial(\mathcal{A}w_2)}{\partial \mu} d\Gamma \\
&\quad - \operatorname{Re} \int_{\Gamma_1} N(\overline{w}_1) a_1 \frac{\partial(\mathcal{A}w_1)}{\partial \mu} d\Gamma - \operatorname{Re} \int_\Gamma a_2 \mathcal{A}w_2 \frac{\partial(N(\overline{w}_2))}{\partial \mu} d\Gamma \\
&\quad - \operatorname{Re} \int_{\Gamma_1} a_2 \mathcal{A}w_2 \frac{\partial(N(\overline{w}_2))}{\partial \mu} d\Gamma + \operatorname{Re} \int_{\Gamma_1} a_1 \mathcal{A}w_1 \frac{\partial(N(\overline{w}_1))}{\partial \mu} d\Gamma \\
&\quad + \operatorname{Re} \int_\Omega a(x) Df(\mathcal{A}w) N(\overline{w}) dx \\
&= -\frac{\lambda^2}{2} \int_\Omega \operatorname{div}_0(N) |w|^2 dx + \operatorname{Re} \int_\Omega a(x)\mathcal{A}w \Delta_g(N(\overline{w})) dx \\
&\quad + \operatorname{Re} \int_\Gamma \frac{\overline{u}}{|\nu_\mathcal{A}|_g} a_2 \frac{\partial(\mathcal{A}w_2)}{\partial \mu} d\Gamma - \operatorname{Re} \int_\Gamma a_2 \mathcal{A}w_2 \frac{\partial^2 \overline{w}_2}{\partial \mu^2} d\Gamma \\
&\quad + \operatorname{Re} \int_{\Gamma_1} [a_1 |\mathcal{A}w_1|^2 - a_2 |\mathcal{A}w_2|^2] d\Gamma + \operatorname{Re} \int_\Gamma N(\overline{w}_2) a_2 \mathcal{A}w_2 Df \cdot \nu d\Gamma \\
&\quad - \operatorname{Re} \int_\Omega a(x)\mathcal{A}w Df(N(\overline{w})) dx - \operatorname{Re} \int_\Omega a(x) N(\overline{w})\mathcal{A}w \operatorname{div}_0(Df) dx.
\end{aligned}
$$

Hence

$$
\begin{aligned}
a_2 \left\| \frac{1}{\lambda}\mathcal{A}w \right\|^2_{L^2(\Gamma)} &= -\int_\Omega \operatorname{div}_0(N) |w|^2 dx + \frac{2}{\lambda^2} \operatorname{Re} \int_\Gamma \frac{\overline{u}}{|\nu_\mathcal{A}|_g} a_2 \frac{\partial(\mathcal{A}w_2)}{\partial \mu} d\Gamma \\
&\quad + \frac{2}{\lambda^2} \operatorname{Re} \int_\Gamma \frac{\overline{u}}{|\nu_\mathcal{A}|_g} \eta(x) a_2 \mathcal{A}w_2 d\Gamma + \frac{2}{\lambda^2} \operatorname{Re} \int_\Gamma \frac{\overline{u}}{|\nu_\mathcal{A}|_g} a_2 \mathcal{A}w_2 Df \cdot \nu d\Gamma \\
&\quad + \frac{2}{\lambda^2} \operatorname{Re} \int_\Gamma a_2 \mathcal{A}w_2 Df(\overline{w}_2) d\Gamma - \frac{1}{\lambda^2} \int_\Omega a(x) |\mathcal{A}w|^2 \operatorname{div}_g(N) dx \\
&\quad + \frac{2}{\lambda^2} \operatorname{Re} \int_\Omega a(x)\mathcal{A}w[(-\triangle N)(\overline{w}) + 2\langle DN, D^2\overline{w}\rangle_{T^2(\mathbb{R}^2_x)} - D^2 f(N, D\overline{w}) \\
&\quad - D^2\overline{w}(N, Df) + \operatorname{Ric}(N, D\overline{w})] dx - \frac{2}{\lambda^2} \operatorname{Re} \int_\Omega a(x)\mathcal{A}w Df(N(\overline{w})) dx \\
&\quad - \frac{2}{\lambda^2} \operatorname{Re} \int_\Omega a(x) N(\overline{w})\mathcal{A}w \operatorname{div}_0(Df) dx + \frac{1}{2} \int_{\Gamma_1} \frac{(a_2 - a_1)}{a_1 a_2} |a_1 \mathcal{A}w_1|^2 d\Gamma \\
&\leq C_1 \|w\|^2_{L^2(\Omega)} + \frac{C_2}{\lambda^2} \|u\|_{L^2(\Gamma)} \|w\|_{H^4(\Omega)} + \frac{C_3}{\lambda} \|u\|_{L^2(\Gamma)} \left\| \frac{1}{\lambda}\mathcal{A}w \right\|^2_{L^2(\Omega)} \\
&\quad + \frac{C_4}{\lambda^2} \|\mathcal{A}w\|^2_{L^2(\Omega)},
\end{aligned}
\tag{4.19}
$$

where $C_i > 0, i = 1, 2, 3, 4$ are constants independent of λ. Notice that in the last inequality above, we have used Lemma 2.2, the assumption $a_2 \leq a_1$ and the following facts:

$$\sup_{x \in \Gamma} |\eta(x)| \leq C, \quad \|Df(w)\|_{L^2(\Gamma)} \leq C \|u\|_{L^2(\Gamma)},$$

$$\|w\|_{H^2(\Omega)} \leq C \|\mathcal{A}w\|_{L^2(\Omega)}, \quad \left\|\frac{\partial(\mathcal{A}w)}{\partial \mu}\right\|_{L^2(\Gamma)} \leq C \|w\|_{H^4(\Omega)},$$

for some constant $C > 0$ independent of λ. The first two inequalities are apparent. The third estimate is also well known due to the vanishing condition of w on $\partial\Omega$, and the last estimate comes from the trace theorem in Sobolev space (see [20]).

Finally, setting $s = 4$ in (4.2) we deduce that the solution of (4.12) satisfies

$$\|w\|_{H^4(\Omega)} \leq C_5 \left[\left\|\frac{1}{\lambda^2}w\right\|_{L^2(\Omega)} + \|u\|_{H^{5/2}(\Gamma)}\right]$$

for some constant C_5 independent of λ. This together with (4.18) and (4.19) yields

$$a_2 \left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma)}^2 \leq (C_1 + C_3 + C_4)\frac{1}{\lambda}\|u\|_{L^2(\Gamma)}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma)} + C_2 C_5 \|u\|_{L^2(\Gamma)}\|w\|_{L^2(\Omega)}$$

$$+ C_2 C_5 \frac{1}{\lambda^2}\|u\|_{L^2(\Gamma)}\|u\|_{H^{5/2}(\Gamma)}$$

$$\leq (C_1 + C_3 + C_4)\frac{1}{\lambda}\|u\|_{L^2(\Gamma)}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma)}$$

$$+ C_2 C_5 \lambda^{-1/2}\|u\|_{L^2(\Gamma)}^{3/2}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Omega)}^{1/2} + C_2 C_5 \frac{1}{\lambda^2}\|u\|_{L^2(\Gamma)}\|u\|_{H^{5/2}(\Gamma)},$$

which implies that $\varlimsup_{\lambda \in \mathbb{R}, \lambda \to +\infty} a_2 \left\|\lambda^{-1}\mathcal{A}w\right\|_{L^2(\Gamma)} < +\infty$. Therefore

$$\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} a_2 \left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma)} = 0.$$

This is (4.13). The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Exact controllability. For the controllability problem, we confine the control to be real functions. We first consider the well-posedness and boundary regularity of the dual system of problem (1.1):

$$\begin{cases} \ddot{w}(x,t) + a(x)\mathcal{A}^2 w(x,t) = 0, & (x,t) \in Q, \\ w(x,0) = w^0, \ \dot{w}(x,0) = w^1, & x \in \Omega, \\ w_2(x,t) = 0, \ \dfrac{\partial w_2}{\partial \nu_{\mathcal{A}}}(x,t) = 0, & (x,t) \in \Sigma, \\ w_1(x,t) = w_2(x,t), \ \dfrac{\partial w_1}{\partial \nu_{\mathcal{A}}}(x,t) = \dfrac{\partial w_2}{\partial \nu_{\mathcal{A}}}(x,t), & (x,t) \in \Sigma_1, \\ a_1 \mathcal{A}w_1(x,t) = a_2 \mathcal{A}w_2(x,t), \ a_1 \dfrac{\partial \mathcal{A}w_1}{\partial \nu_{\mathcal{A}}}(x,t) = a_2 \dfrac{\partial \mathcal{A}w_2}{\partial \nu_{\mathcal{A}}}(x,t), & (x,t) \in \Sigma_1. \end{cases} \quad (5.1)$$

Define operator $\mathbb{A}$ by

$$\mathbb{A} = \begin{pmatrix} 0 & I \\ -a\mathscr{A} & 0 \end{pmatrix} \text{ with } D(\mathbb{A}) = H^4(\Omega, \Gamma_1) \times H_0^2(\Omega). \tag{5.2}$$

It is readily shown that $\mathbb{A}$ is skew-adjoint, i.e. $\mathbb{A}^* = -\mathbb{A}$. So $\mathbb{A}$ generates a C_0-group $e^{\mathbb{A}t}$ on $H_0^2(\Omega) \times L^2(\Omega)$.

PROPOSITION 5.1. For any given initial data $(w^0, w^1) \in H_0^2(\Omega) \times L^2(\Omega)$, Equation (5.1) admits a unique weak solution

$$w(t) \in C([0, T]; H_0^2(\Omega)) \cap C^1([0, T]; L^2(\Omega)). \tag{5.3}$$

Furthermore, if $(w^0, w^1) \in H^4(\Omega, \Gamma_1) \times H_0^2(\Omega)$, then

$$w(t) \in C([0, T]; H^4(\Omega, \Gamma_1)) \cap C^1([0, T]; H_0^2(\Omega)). \tag{5.4}$$

To discuss the boundary regularity of the solution of (5.1), we need the following Lemma 5.1 that is similar to Lemma 4.2.

LEMMA 5.1. Let w be a smooth function on $\overline{\Omega}$ such that $w|_\Gamma = 0$. Then there exists a continuous function $m(x)$ on Γ independent of w such that

$$\Delta_g w(x) = \frac{\partial^2 w(x)}{\partial \mu^2} + m(x)\frac{\partial w(w)}{\partial \mu}, \ \forall x \in \Gamma. \tag{5.5}$$

Furthermore, if w satisfies $\left.\dfrac{\partial w}{\partial \nu_{\mathcal{A}}}\right|_\Gamma = 0$, then

$$N(w)|_\Gamma = 0 \text{ for any vector field } N \text{ on } \overline{\Omega}. \tag{5.6}$$

By Lemma 5.1,

$$\mathcal{A}w = \Delta_g w + Df(w) = \Delta_g w = \frac{\partial^2 w}{\partial \mu^2} = \frac{1}{|\nu_{\mathcal{A}}|_g^2}\frac{\partial^2 w}{\partial \nu_{\mathcal{A}}^2} \text{ on } \Gamma. \tag{5.7}$$

Now, we establish the following multiplier identity that plays a key role in obtaining the boundary regularity and the observability inequality.

LEMMA 5.2. Let N be a vector field on $(\mathbb{R}^n, g)$. Assume that w solves problem (5.1). Let $f(x) = \dfrac{1}{2}\log\det(a_{ij}(x))$. Then

$$\frac{1}{2}\int_\Sigma a_2(\mathcal{A}w_2)^2 N \cdot \nu d\Sigma - \frac{1}{2}\int_{\Sigma_1}\left[a_1(\mathcal{A}w_1)^2 - a_2(\mathcal{A}w_2)^2\right] N \cdot \nu d\Sigma$$

$$= \int_\Omega \dot{w} N(w) dx\Big|_0^T + \frac{1}{2}\int_Q \left[|\dot{w}|^2 - a(x)(\mathcal{A}w)^2\right] \mathrm{div}_0 N dQ \tag{5.8}$$

$$+ \int_Q a(x)\mathcal{A}w \left[2\langle DN, D^2 w\rangle_{T^2(\mathbb{R}_x^n)} + \mathcal{F}N(w) + \mathrm{Ric}(N, Dw) - D^2 f(N, Dw)\right] dQ,$$

where $\mathcal{F}N \in \mathfrak{X}(\mathbb{R}^n)$ is defined by $\mathcal{F}N = D_{Df}N - \triangle N$ and $D^2 f$ is the Hessian of f in terms of g.

Proof. Multiply the first equation of (5.1) by $N(w)$ and integrate by parts to obtain

$$
\begin{aligned}
\int_Q \ddot{w} N(w) dQ &= \int_\Omega \dot{w} N(w) dx \Big|_0^T - \int_Q \dot{w} N(\dot{w}) dQ \\
&= \int_\Omega \dot{w} N(w) dx \Big|_0^T - \int_{Q_1} \dot{w}_1 N(\dot{w}_1) dQ - \int_{Q_2} \dot{w}_2 N(\dot{w}_2) dQ \\
&= \int_\Omega \dot{w} N(w) dx \Big|_0^T - \frac{1}{2} \int_{Q_1} [\operatorname{div}_0(|\dot{w}_1|^2 N) - |\dot{w}_1|^2 \operatorname{div}_0 N] dQ \\
&\quad - \frac{1}{2} \int_{Q_2} [\operatorname{div}_0(|\dot{w}_2|^2 N) - |\dot{w}_2|^2 \operatorname{div}_0 N] dQ \\
&= \int_\Omega \dot{w} N(w) dx \Big|_0^T - \frac{1}{2} \int_{\Sigma_1} |\dot{w}_1|^2 N \cdot (-\nu) d\Sigma + \frac{1}{2} \int_{Q_1} |\dot{w}_1|^2 \operatorname{div}_0 N dQ \\
&\quad - \frac{1}{2} \int_{\Sigma_1} |\dot{w}_2|^2 N \cdot \nu d\Sigma - \frac{1}{2} \int_\Sigma |\dot{w}_2|^2 N \cdot \nu d\Sigma + \frac{1}{2} \int_{Q_2} |\dot{w}_2|^2 \operatorname{div}_0 N dQ \\
&= \int_\Omega \dot{w} N(w) dx \Big|_0^T + \frac{1}{2} \int_Q |\dot{w}|^2 \operatorname{div}_0 N dQ.
\end{aligned}
\tag{5.9}
$$

Given $x \in \mathbb{R}^n$. Let $E_1, E_2, \cdots, E_n$ be a frame field normal at x on $(\mathbb{R}^n, g)$, which means that $\langle E_i, E_j \rangle_g = \delta_{ij}$ in some neighborhood of x and $(D_{E_i} E_j)(x) = 0$ for $1 \le i, j \le n$ (see [33]). Let $N = \sum_{i=1}^n \gamma_i E_i$. Then $N(w) = \sum_{i=1}^n \gamma_i E_i(w)$, where $E_i(w)$ is the covariant derivative of w with respect to E_i under the Riemannian metric g. Then at x

$$
\begin{aligned}
N(\mathcal{A}w) &= N(\Delta_g w + (Df)w) = N(\Delta_g w) + N(\langle Df, Dw \rangle_g) \\
&= N(\Delta_g w) + N(\langle E_i(f) E_i, E_j(w) E_j \rangle_g) \\
&= N(\Delta_g w) + \gamma_j E_j(E_i(f) E_i(w)) \\
&= N(\Delta_g w) + \gamma_j E_j E_i(f) E_i(w) + \gamma_j E_i(f) E_j E_i(w) \\
&= N(\Delta_g w) + D^2 f(N, Dw) + D^2 w(N, Df),
\end{aligned}
\tag{5.10}
$$

where $E_j E_i(f)$ and $E_j E_i(w)$ are, respectively, the second covariant derivatives of f and w at x, and

$$
\begin{aligned}
\langle Df, D(N(w)) \rangle_g &= E_i(f) E_i(N(w)) = E_i(f)[E_i(\gamma_j) E_j(w) + \gamma_j E_i E_j(w)] \\
&= DN(Dw, Df) + D^2 w(N, Df).
\end{aligned}
\tag{5.11}
$$

With the help of (2.8), (5.10) and (5.11), we obtain

$$
\begin{aligned}
\mathcal{A}(N(w)) &= (\Delta_g + Df)(N(w)) = \Delta_g(N(w)) + \langle D(N(w)), Df \rangle_g \\
&= (-\triangle N)(w) + 2\langle DN, D^2 w \rangle_{T^2(\mathbb{R}_x^n)} + N(\Delta_g w) + \operatorname{Ric}(N, Dw) \\
&\quad + DN(Dw, Df) + D^2 w(N, Df) \\
&= N(\mathcal{A}w) + 2\langle DN, D^2 w \rangle_{T^2(\mathbb{R}_x^n)} + \mathcal{F}N(w) + \operatorname{Ric}(N, Dw) - D^2 f(N, Dw),
\end{aligned}
\tag{5.12}
$$

where we denoted $\mathcal{F}N = D_{Df}N - \triangle N$. By virtue of the divergence theorem,

$$
\begin{aligned}
&\int_\Omega a(x)N((\mathcal{A}w)^2)dx \\
&= \int_\Gamma a_2(\mathcal{A}w_2)^2 N \cdot \nu d\Gamma + \int_{\Gamma_1} \left[a_2(\mathcal{A}w_2)^2 - a_1(\mathcal{A}w_1)^2 \right] N \cdot \nu d\Gamma \\
&\quad - \int_\Omega a(x)(\mathcal{A}w)^2 \mathrm{div}_0 N dx.
\end{aligned}
\tag{5.13}
$$

Since $w \in H_0^2(\Omega)$, by Green's identity, it follows from (5.12) and (5.13) that

$$
\begin{aligned}
&\int_Q a(x)\mathcal{A}^2 w N(w) dQ = \int_{Q_1} a_1 \mathcal{A}^2 w_1 N(w_1) dQ + \int_{Q_2} a_2 \mathcal{A}^2 w_2 N(w_2) dQ \\
&= -\int_{\Sigma_1} N(w_1) a_1 \frac{\partial \mathcal{A}w_1}{\partial \nu_\mathcal{A}} d\Sigma - \int_{Q_1} a_1 \langle D(\mathcal{A}w_1), D(N(w_1)) \rangle_g dQ \\
&\quad + \int_{\Sigma_1} N(w_2) a_2 \frac{\partial \mathcal{A}w_2}{\partial \nu_\mathcal{A}} d\Sigma + \int_\Sigma N(w_2) a_2 \frac{\partial \mathcal{A}w_2}{\partial \nu_\mathcal{A}} d\Sigma \\
&\quad - \int_{Q_2} a_2 \langle D(\mathcal{A}w_2), D(N(w_2)) \rangle_g dQ \\
&= \int_\Sigma N(w_2) a_2 \frac{\partial \mathcal{A}w_2}{\partial \nu_\mathcal{A}} d\Sigma + \int_{\Sigma_1} \left[N(w_2) a_2 \frac{\partial \mathcal{A}w_2}{\partial \nu_\mathcal{A}} - N(w_1) a_1 \frac{\partial \mathcal{A}w_1}{\partial \nu_\mathcal{A}} \right] d\Sigma \\
&\quad + \int_{Q_1} a_1 \mathcal{A}w \mathcal{A}(N(w_1)) dQ + \int_{\Sigma_1} a_1 \mathcal{A}w_1 \frac{\partial N(w_1)}{\partial \nu_\mathcal{A}} d\Sigma \\
&\quad + \int_{Q_2} a_2 \mathcal{A}w_2 \mathcal{A}(N(w_2)) dQ - \int_{\Sigma_1} a_2 \mathcal{A}w_2 \frac{\partial N(w_2)}{\partial \nu_\mathcal{A}} d\Sigma - \int_\Sigma a_2 \mathcal{A}w_2 \frac{\partial N(w_2)}{\partial \nu_\mathcal{A}} d\Sigma \\
&= -\int_\Sigma a_2 \mathcal{A}w_2 \frac{\partial N(w_2)}{\partial \nu_\mathcal{A}} d\Sigma + \int_{\Sigma_1} \left[a_1 \mathcal{A}w_1 \frac{\partial N(w_1)}{\partial \nu_\mathcal{A}} - a_2 \mathcal{A}w_2 \frac{\partial N(w_2)}{\partial \nu_\mathcal{A}} \right] d\Sigma \\
&\quad + \int_Q a(x)\mathcal{A}w[2\langle DN, D^2 w \rangle_{T^2(\mathbb{R}_x^n)} + \mathcal{A}N(w) + \mathrm{Ric}(N, Dw) \\
&\quad - D^2 f(N, Dw)] dQ + \frac{1}{2} \int_{\Sigma_1} \left[a_2(\mathcal{A}w_2)^2 - a_1(\mathcal{A}w_1)^2 \right] N \cdot \nu d\Sigma \\
&\quad + \frac{1}{2} \int_\Sigma a_2(\mathcal{A}w_2)^2 N \cdot \nu d\Sigma - \frac{1}{2} \int_Q a(x)(\mathcal{A}w)^2 \mathrm{div}_0 N dQ.
\end{aligned}
\tag{5.14}
$$

Notice that in the last step, we used the fact that $N(w_1) = N(w_2)$ on Γ_1.

Since $w_2|_\Sigma = \left. \frac{\partial w_2}{\partial \nu_\mathcal{A}} \right|_\Sigma = 0$ implies $N(w_2)|_\Sigma = 0$, it follows from identity (5.7) that

$$
\begin{aligned}
\frac{\partial N(w_2)}{\partial \nu_\mathcal{A}} &= N\left(\frac{\partial w_2}{\partial \nu_\mathcal{A}} \right) = \left\langle N, \frac{\nu_\mathcal{A}}{|\nu_\mathcal{A}|_g} \right\rangle_g \frac{\nu_\mathcal{A}}{|\nu_\mathcal{A}|_g} \left(\frac{\partial w_2}{\partial \nu_\mathcal{A}} \right) \\
&= N \cdot \nu \frac{1}{|\nu_\mathcal{A}|_g^2} \frac{\partial^2 w_2}{\partial \nu_\mathcal{A}^2} = \mathcal{A}w_2 N \cdot \nu \text{ on } \Sigma.
\end{aligned}
\tag{5.15}
$$

Now we treat the term $a_1 \mathcal{A}w_1 \dfrac{\partial N(w_1)}{\partial \nu_{\mathcal{A}}} - a_2 \mathcal{A}w_2 \dfrac{\partial N(w_2)}{\partial \nu_{\mathcal{A}}}$ on Σ_1. Since $a_1 \mathcal{A}w_1|_{\Sigma_1} = a_2 \mathcal{A}w_2|_{\Sigma_1}$, $(w_1 - w_2)|_{\Sigma_1} = \dfrac{\partial(w_1 - w_2)}{\partial \nu_{\mathcal{A}}}\bigg|_{\Sigma_1} = 0$, similar to (5.15), we have

$$a_1 \mathcal{A}w_1 \frac{\partial N(w_1)}{\partial \nu_{\mathcal{A}}} - a_2 \mathcal{A}w_2 \frac{\partial N(w_2)}{\partial \nu_{\mathcal{A}}} = [a_1(\mathcal{A}w_1)^2 - a_2(\mathcal{A}w_2)^2] N \cdot \nu \text{ on } \Sigma_1. \tag{5.16}$$

Combining (5.9), (5.14), (5.15) and (5.16), we arrive at (5.8). The proof is complete. $\square$

The following Lemma 5.3 claims the boundary regularity (hidden regularity) for problem (5.1).

LEMMA 5.3. There exists a constant $C_T > 0$ such that for any mild solution of (5.1), it has

$$\int_\Sigma (\mathcal{A}w_2)^2 d\Sigma \le C_T \left(\|w^0\|^2_{H^2_0(\Omega)} + \|w^1\|^2_{L^2(\Omega)} \right). \tag{5.17}$$

Proof. Choose the vector field N in (5.8) such that $N = \nu_{\mathcal{A}}$ on Γ and $N = 0$ in some neighborhood of $\overline{\Omega}_1$. Then $N \cdot \nu = |\nu_{\mathcal{A}}|^2_g \ge \lambda > 0$. This together with (5.8), Lemma 5.2 and Lemma 5.3 gives (5.17). $\square$

LEMMA 5.4. Let N be a vector field and w a solution to (5.1). If $\varphi \in C^2(\mathbb{R}^n)$, then it has

$$\int_Q [|\dot{w}|^2 - a(x)(\mathcal{A}w)^2]\varphi \, dQ = -\int_\Omega \dot{w}w\varphi \, dx \bigg|_0^T + \int_Q a(x)\mathcal{A}w[w\mathcal{A}\varphi + 2D\varphi(w)] \, dQ. \tag{5.18}$$

Proof. By virtue of Green's identity, we have

$$
\begin{aligned}
\int_\Omega \dot{w}w\varphi \, dx \bigg|_0^T &= \int_Q [(-a(x)\mathcal{A}^2 w)w\varphi + |\dot{w}|^2\varphi] \, dQ \\
&= -\int_Q a(x)\mathcal{A}w\mathcal{A}(w\varphi) \, dQ - \int_\Sigma a_2 \frac{\partial \mathcal{A}w_2}{\partial \nu_{\mathcal{A}}} w_2\varphi \, d\Sigma + \int_\Sigma a_2 \mathcal{A}w_2 \frac{\partial(w_2\varphi)}{\partial \nu_{\mathcal{A}}} \, d\Sigma \\
&\quad + \int_Q |\dot{w}|^2\varphi \, dQ \\
&= \int_Q [|\dot{w}|^2 - a(x)(\mathcal{A}w)^2]\varphi \, dQ - \int_Q a(x)\mathcal{A}w[w\mathcal{A}\varphi + 2D\varphi(w)] \, dQ \\
&\quad - \int_\Sigma a_2 \frac{\partial \mathcal{A}w_2}{\partial \nu_{\mathcal{A}}} w_2\varphi \, d\Sigma + \int_\Sigma a_2 \mathcal{A}w_2 \frac{\partial(w_2\varphi)}{\partial \nu_{\mathcal{A}}} \, d\Sigma \\
&= \int_Q [|\dot{w}|^2 - a(x)(\mathcal{A}w)^2]\varphi \, dQ - \int_Q a(x)\mathcal{A}w[w\mathcal{A}\varphi + 2D\varphi(w)] \, dQ.
\end{aligned}
\tag{5.19}
$$

In the last step above, we used the following formula:
$$\mathcal{A}(w\varphi) = (\mathcal{A}w)\varphi + 2D\varphi(w) + (\mathcal{A}\varphi)w. \qquad \square$$

Now we define the energy function for the system (5.1) as

$$E(t) \equiv E(w,t) = \frac{1}{2} \int_\Omega \left(|\dot{w}|^2 + a(x)|\mathcal{A}w|^2 \right) dx. \tag{5.20}$$

Then $E(t) \equiv E(0)$ for all $t > 0$. Let

$$L(t) = \int_\Omega (w^2 + |Dw|_g^2)dx \qquad (5.21)$$

be the lower-order term in terms of $E(t)$.

LEMMA 5.5. Suppose that Assumption (H3) holds. Let w solve the equation (5.1) with $\mathcal{A}w = 0$ on Σ_0. Then $w \equiv 0$ in Q.

   Proof. Set

$Y = \{w \in X \triangleq C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \ w \text{ solves (2.1) with } \mathcal{A}w = 0 \text{ on } \Sigma_0\}$.

We prove $Y = 0$. Actually, by (5.3) and (5.40) that will be shown later, we have

$$E(0) \le C \left(\|\mathcal{A}w_2\|_{L^2(\Sigma_0)}^2 + \|w\|_{L^\infty(0,T;H_0^1(\Omega))}^2 \right), \ \forall \ w \in X \text{ satisfying (5.1).} \qquad (5.22)$$

   Now, we show that there exists a constant $C > 0$ such that for any $w \in X$ satisfying (5.1)

$$\|w\|_{L^\infty(0,T;H_0^1(\Omega))}^2 \le C \left(\|\mathcal{A}w_2\|_{L^2(\Sigma_0)}^2 + \|w\|_{L^\infty(0,T;L^2(\Omega))}^2 \right). \qquad (5.23)$$

In fact, if (5.23) is not true, then there exists a solution sequence $\{w_n\} \in X$ to Equation (5.1) such that

$$\|\mathcal{A}w_{n2}\|_{L^2(\Sigma_0)}^2 + \|w_n\|_{L^\infty(0,T;L^2(\Omega))}^2 \to 0 \text{ as } n \to \infty, \qquad (5.24)$$

with

$$\|w_n\|_{L^\infty(0,T;H_0^1(\Omega))}^2 = 1. \qquad (5.25)$$

It then follows from (5.3) and (5.22) that $\{w_n\}$ is bounded in X and hence is relatively compact in $L^\infty(0,T;H_0^1(\Omega))$. By extracting a subsequence if necessary, we may assume without loss of generality that $\{w_n\}$ converges strongly to $w \in L^\infty(0,T;H_0^1(\Omega))$. By (5.25), it has

$$\|w\|_{L^\infty(0,T;H_0^1(\Omega))}^2 = 1. \qquad (5.26)$$

However, (5.24) implies $w = 0$ in Q. This contradicts (5.26).

   Next, from (5.22) and (5.23), we have

$$E(w,0) \le C \left(\|\mathcal{A}w_2\|_{L^2(\Sigma_0)}^2 + \|w\|_{L^\infty(0,T;L^2(\Omega))}^2 \right), \ \forall \ w \in X \text{ satisfying (5.1).} \qquad (5.27)$$

By the density argument, (5.27) still holds true for $w \in L^\infty(0,T;L^2(\Omega))$ satisfying (5.1). We thus have proved that $w \in Y$ implies that $v = \dot{w}$ solves (5.1) with $\mathcal{A}v|_{\Sigma_0} = 0$ and $v \in L^\infty(0,T;L^2(\Omega))$. This together with (5.27) gives

$$(v(0), \dot{v}(0)) \in H_0^2(\Omega) \times L^2(\Omega). \qquad (5.28)$$

   Finally, since by Proposition 5.1, $v \in X$, it follows from (5.27) that the map $\partial_t : w \to \dot{w}$ is continuous from Y to Y and the injection of $\{w \in Y; \ \dot{w} \in Y\}$ is compact. Therefore, Y is a finite dimensional space. There must exist an $\eta \in \mathbb{C}$ and $w \in Y \setminus \{0\}$ such that $\dot{w} = \eta w$, which implies

$$w(x,t) = e^{\eta t} w(x,0). \qquad (5.29)$$

Substitute (5.29) into (5.1) to obtain (1.11) with $u(x) = w(x,0)$ and $\zeta = -\eta^2$. By Assumption (H3) we have $w(x,t) \equiv 0$ and hence $Y = \{0\}$. $\qquad \square$

With these preparations, we can now show Theorem 1.3.

Proof. By the duality principle, the proof is accomplished if we can show that there is a constant $C_T > 0$ independent of the solution w of Equation (5.1) such that

$$E(0) \leq C_T \int_{\Sigma_0} (\mathcal{A}w_2)^2 d\Sigma, \tag{5.30}$$

for $\Sigma_0 = \Gamma_0 \times (0, T)$, where Γ_0 given by (1.12) is relatively open in Γ.

To this end, for any $x \in \overline{\Omega}$ we define $T \in T^2(\mathbb{R}_x^n)$ by

$$T(X, Y) = DN(X, Y) + DN(Y, X), \ \forall \ X, Y \in \mathbb{R}_x^n. \tag{5.31}$$

It is clear that $T(\cdot, \cdot)$ is symmetric, and from (1.8), we have

$$DN(X, Y) + DN(Y, X) = 2b(x)\langle X, Y \rangle_g \ \forall \ X, Y \in \mathbb{R}_x^n, \ x \in \overline{\Omega}. \tag{5.32}$$

Given $x \in \overline{\Omega}$, let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $(\mathbb{R}_x^n, g)$. By (5.32), we have

$$\langle DN, D^2 w \rangle_{T^2(\mathbb{R}_x^n)} = DN(e_i, e_j) D^2 w(e_i, e_j) = b(x) \Delta_g w = b(x) \mathcal{A}w + Df(w). \tag{5.33}$$

By Lemma 5.1, there exist constants $C > 0$, $\varepsilon > 0$ such that

$$\left| \int_\Omega \dot{w} N(w) dx \right| \leq \varepsilon \int_\Omega |\dot{w}|^2 dx + \frac{1}{4\varepsilon} \int_\Omega |N(w)|^2 dx \leq \varepsilon E(0) + \frac{C}{4\varepsilon} L(t), \tag{5.34}$$

$$\left| \int_\Omega \dot{w} w \varphi dx \right| \leq \varepsilon E(0) + \frac{C}{4\varepsilon} L(t), \tag{5.35}$$

$$\left| \int_Q a(x) \mathcal{A}w \left[\mathcal{F}N(w) + \mathrm{Ric}(N, Dw) - D^2 f(N, Dw) \right] dQ \right|$$
$$\leq \varepsilon T E(0) + \frac{C}{4\varepsilon} \int_0^T L(t) dt, \tag{5.36}$$

$$\left| \int_Q a(x) b(x) \mathcal{A}w Df(w) dQ \right| \leq \varepsilon T E(0) + \frac{C}{4\varepsilon} \int_0^T L(t) dt, \tag{5.37}$$

$$\left| \int_Q a(x) \mathcal{A}w [w \mathcal{A}\varphi + 2D\varphi(w)] dQ \right| \leq \varepsilon T E(0) + \frac{C}{4\varepsilon} \int_0^T L(t) dt. \tag{5.38}$$

Keeping Assumptions (H1)–(H3) and $a_2 \leq a_1$ in mind, by (5.8), (5.18) and (5.33)–(5.38), we have

$$\frac{1}{2} \int_{\Sigma_0} a_2(\mathcal{A}w_2)^2 N \cdot \nu d\Sigma$$

$$\geq \quad \frac{1}{2} \int_{\Sigma} a_2(\mathcal{A}w_2)^2 N \cdot \nu d\Sigma - \frac{1}{2} \int_{\Sigma_1} \left[a_1(\mathcal{A}w_1)^2 - a_2(\mathcal{A}w_2)^2 \right] N \cdot \nu d\Sigma$$

$$= \quad \int_{\Omega} \dot{w} N(w) dx \bigg|_0^T + \frac{1}{2} \int_Q \left[|\dot{w}|^2 - a(x)(\mathcal{A}w)^2 \right] \operatorname{div}_0 N dQ$$

$$\quad + \int_Q a(x) \mathcal{A}w \left[2\langle DN, D^2 w \rangle_{T^2(\mathbb{R}^n_x)} + \mathcal{F}N(w) + \operatorname{Ric}(N, Dw) - D^2 f(N, Dw) \right] dQ$$

$$\geq \quad 2b_0 \int_Q a(x)(\mathcal{A}w)^2 dQ + \frac{1}{2} \int_Q \left[|\dot{w}|^2 - a(x)(\mathcal{A}w)^2 \right] \operatorname{div}_0 N dQ$$

$$\quad - \left(2\varepsilon E(0) + \frac{C}{4\varepsilon}(L(T) + L(0)) \right) - 2 \left(\varepsilon T E(0) + \frac{C}{4\varepsilon} \int_0^T L(t) dt \right)$$

$$= \quad b_0 \int_Q \left(|\dot{w}|^2 + a(x)(\mathcal{A}w)^2 \right) dQ + \int_Q \left[|\dot{w}|^2 - a(x)(\mathcal{A}w)^2 \right] \left(\frac{1}{2} \operatorname{div}_0 N - b_0 \right) dQ$$

$$\quad - \left(2\varepsilon E(0) + \frac{C}{4\varepsilon}(L(T) + L(0)) \right) - 2 \left(\varepsilon T E(0) + \frac{C}{4\varepsilon} \int_0^T L(t) dt \right)$$

$$\geq \quad (b_0 - 4\varepsilon - 3T\varepsilon) E(0) - \left(\frac{3C}{4\varepsilon} \int_0^T L(t) dt + \frac{C}{2\varepsilon}(L(T) + L(0)) \right). \tag{5.39}$$

Setting $\varepsilon > 0$ small enough, we obtain

$$E(0) \leq C_T \int_{\Sigma_0} (\mathcal{A}w)^2 d\Sigma + C \left(\int_0^T L(t) dt + L(T) + L(0) \right). \tag{5.40}$$

Now we apply the standard compact uniqueness argument to absorb the lower-order terms in (5.40). In other words, we want to show that there exists a constant $C > 0$ such that

$$\|w\|^2_{L^\infty(0,T;H_0^1(\Omega))} \leq C \int_{\Sigma_0} (\mathcal{A}w)^2 d\Sigma \tag{5.41}$$

for solution w of (5.1). In fact, if (5.41) is not true, then there exists a solution sequence $\{w_n\}$ of Equation (5.1) such that

$$\int_{\Sigma_0} (\mathcal{A}w_n)^2 d\Sigma \to 0, \quad n \to \infty \tag{5.42}$$

with

$$\|w_n\|^2_{L^\infty(0,T;H_0^1(\Omega))} \equiv 1. \tag{5.43}$$

It then follows from (5.40) and (5.3) that $\{w_n\}$ is a bounded sequence in $C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ and hence relatively compact in $L^\infty(0,T; H_0^1(\Omega))$ because the injection

$$C([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \to L^\infty(0,T; H_0^1(\Omega))$$

is compact due to Simon's result [26]. By extracting a subsequence if necessary, we may assume without loss of generality that $\{w_n\}$ converges strongly to $w \in L^\infty(0, T; H_0^1(\Omega))$. By (5.43),

$$\|w\|_{L^\infty(0,T;H_0^1(\Omega))}^2 = 1. \tag{5.44}$$

Furthermore, $\{w_n\}$ and $\{\dot{w}_n\}$ converge to w and $\dot{w}$ in $L^\infty(0, T; H_0^2(\Omega))$ and $L^\infty(0, T; L^2(\Omega))$, respectively, in weak star topology. Therefore, w is a solution to (5.1) with

$$w \in C([0, T]; H_0^2(\Omega)) \cap C^1([0, T]; L^2(\Omega)). \tag{5.45}$$

By (5.40) and (5.17), we have

$$\mathcal{A}w = 0 \text{ on } \Sigma_0. \tag{5.46}$$

Finally, by virtue of Lemma 5.5, we have

$$w \equiv 0 \text{ in } Q, \tag{5.47}$$

which contradicts (5.44). The proof is complete. $\qquad\square$

Appendix. Since there is no explanation for the identity (2.8) in [34], here we give a brief proof.

Proof. Einstein summation is used throughout the proof . Let $x \in \mathbb{R}^n$ be fixed. Let $\{e_i\}_{i=1}^n$ be a frame field normal at the point x on Riemannian manifold $(\mathbb{R}^n, g)$. Then at x, it has

$$
\begin{aligned}
\langle DN, D^2\varphi \rangle_{T^2(\mathbb{R}_x^n)} &= DN(e_i, e_j)D^2\varphi(e_i, e_j) \\
&= \langle D_{e_i}N, e_j \rangle_g [e_i e_j(\varphi) - D_{e_i} e_j(\varphi)] \\
&= \langle D_{e_i}N, e_j \rangle_g e_i \langle \nabla_g \varphi, e_j \rangle_g \\
&= \langle D_{e_i}N, e_j \rangle_g (\langle D_{e_i} \nabla_g \varphi, e_j \rangle_g + \langle \nabla_g \varphi, D_{e_i} e_j \rangle_g) \\
&= \langle D_{e_i}N, e_j \rangle_g \langle D_{e_i} \nabla_g \varphi, e_j \rangle_g \\
&= \langle D_{e_i}N, D_{e_i} \nabla_g \varphi \rangle_g.
\end{aligned} \tag{A.1}
$$

By the formulae $\text{div}_g X = \langle D_{e_i}X, e_i \rangle_g$ for any $X \in \mathfrak{X}(\mathbb{R}^n)$, $\nabla_g \varphi = e_i(\varphi)e_i$ and $|\nabla_g \varphi|_g^2 = \sum_{i=1}^n (e_i(\varphi))^2$, we have, at x,

$$
\begin{aligned}
N(\Delta_g \varphi) &= N(\text{div}_g(\nabla_g \varphi)) = N\langle D_{e_i} \nabla_g \varphi, e_i \rangle_g \\
&= \langle \nabla_g \langle D_{e_i} \nabla_g \varphi, e_i \rangle_g, N \rangle_g \\
&= \langle (e_j \langle D_{e_i} \nabla_g \varphi, e_i \rangle_g)e_j, N \rangle_g \\
&= \langle \langle D_{e_j} D_{e_i}(\nabla_g \varphi), e_i \rangle_g e_j, N \rangle_g \\
&= \langle D_{e_i} D_{e_i}(\nabla_g \varphi), N \rangle_g.
\end{aligned} \tag{A.2}
$$

Computing the first term on the right hand side of (2.8) we obtain, at x,

$$
\begin{aligned}
(-\triangle N)(\varphi) \quad &= -(-D^2_{e_k e_k} N + e^k \wedge i(w^j) R(e_k, e_j) N)(\varphi)(\text{by the formula } (2.7)) \\
&= D_{e_k} D_{e_k} N(\varphi) - e^k \wedge i(w^j)\{\langle R(e_k, e_j)N, e_l\rangle_g e_l\}(\varphi) \\
&= \langle \nabla_g \varphi, D_{e_k} D_{e_k} N\rangle_g - \langle R(e_k, e_j)N, e_j\rangle_g e_k(\varphi) \quad\quad (\text{A}.3) \\
&= \langle \nabla_g \varphi, D_{e_k} D_{e_k} N\rangle_g - \langle R(\nabla_g \varphi, e_j)N, e_j\rangle_g \\
&= \langle \nabla_g \varphi, D_{e_k} D_{e_k} N\rangle_g - \mathrm{Ric}(\nabla_g \varphi, N).
\end{aligned}
$$

The left hand side of (2.8) at x can be expressed as

$$
\begin{aligned}
\Delta_g(N(\varphi)) \quad &= \mathrm{div}_g(\nabla_g(N(\varphi))) = \mathrm{div}_g(e_i(N(\varphi))e_i) \\
&= \langle D_{e_j} e_i(N(\varphi))e_i, e_j\rangle_g \\
&= e_i e_i \langle \nabla_g \varphi, N\rangle_g \quad\quad (\text{A}.4) \\
&= e_i\{\langle D_{e_i} \nabla_g \varphi, N\rangle_g + \langle \nabla_g \varphi, D_{e_i} N\rangle_g\} \\
&= \langle D_{e_i} D_{e_i}(\nabla_g \varphi), N\rangle_g + 2\langle D_{e_i} \nabla_g \varphi, D_{e_i} N\rangle_g + \langle \nabla_g \varphi, D_{e_i} D_{e_i} N\rangle_g.
\end{aligned}
$$

Combining identities (A.1)–(A.4), noting the arbitrariness of x and the fact that the Ricci curvature tensor $\mathrm{Ric}(\cdot, \cdot)$ is a symmetric second-order covariant tensor field, we finally obtain (2.8). $\qquad\square$

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