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On Weyl's Gauge Field

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The relationship between the scale transformation and Weyl's gauge transformation is investigated. It is shown that a scale invariant Lagrangian can be transformed into a scalar density which is invariant under x-dependent gauge transformations in Weyl's sense. The Lagrangian of the latter form gives a symmetric energy momentum tensor the trace of which can be shown to vanish provided that some equation for Weyl's gauge field is satisfied in addition to the equations of the original scale invariant fields. A simple example is investigated to show an extraordinary property of Weyl's gauge field.

§ 1. Introduction

Since the last few years, the so-called scale invariance has attracted our attention in connection with the high energy behaviour of some collision processes. The transformation considered in such cases differs from the famous gauge transformation proposed by Weyl[1)~3)] but is quite similar to the latter in some respects.

The aim of the present paper is to investigate the relationship between the transformations of both kinds and to give a prescription for the derivation of a symmetric energy-momentum tensor, the trace of which vanishes when the system is scale invariant. It will be shown that such an energy-momentum tensor not only depends on the original scale-invariant fields but also should have a contribution from the gauge field φ_μ which was first proposed by Weyl in order to unify the electromagnetic field with the gravitational field from the view point of the world geometry.

The field φ_μ was abandoned contrary to Weyl's intention owing to some defects. In fact this gauge field has an extraordinary property as will be shown in § 5, and it is a matter of course that the field φ_μ could not be identified with the electromagnetic field. In § 5 it will be shown that φ_μ has a Tachyon-like property with a Lagrangian density which has an opposite sign to that of the electromagnetic field. Such a "wrong" sign of the Lagrangian density forces us to have an expectation that this unusual field might play a role of a cohesion field which we have been looking for in order to explain the stable character of elementary particles. It may be worth while to investigate the behaviour of φ_μ from such a view point as mentioned just above. Section 5 is nothing but the introduction to such a new exploration.

§ 2. Review of Weyl's theory of gauge transformation

For the later discussion, let us begin with a brief review of Weyl's theory of gauge transformation.

Consider a vector field $V^\mu(x)$ in a Minkowski space. A derivative of $V^\mu(x)$,

$$V^\mu_{,\nu}(x) = \frac{\partial V^\mu}{\partial x^\nu},$$

has an invariant meaning, namely, it behaves as a component of a mixed tensor under any Lorentz transformation. However, if the group of transformations is replaced with general transformations of coordinates, the above derivative should be modified in the following way:

$$V^\mu_{;\nu} \stackrel{d}{\equiv} \partial_\nu V^\mu + \{{}_\nu{}^\mu{}_\lambda\} \cdot V^\lambda, \tag{2·1}$$

in order to retain the tensor character under general transformations. Here $\{{}_\nu{}^\mu{}_\lambda\}$ is the Christoffel three-index symbol which is written in terms of the metric tensor $g_{\mu\nu}$ as follows:

$$\{{}_\nu{}^\mu{}_\lambda\} \stackrel{d}{\equiv} \tfrac{1}{2} g^{\mu\rho} \{g_{\rho\nu,\lambda} + g_{\lambda\rho,\nu} - g_{\nu\lambda,\rho}\}.$$

Let us consider Weyl's gauge transformation:

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) \stackrel{d}{\equiv} \lambda^2 g_{\mu\nu}(x), \tag{2·2}$$

where λ is any real function of the x^μ's and the coordinate variable x^μ is supposed to be kept unchanged. Let $V^\mu(x)$ be assumed to be transformed as

$$V^\mu \to \bar{V}^\mu \stackrel{d}{\equiv} \lambda^n V^\mu$$

under the transformation (2·2). n is a constant which characterizes the transformation property of V.

If λ is assumed to be a constant, the covariant derivative $V^\mu_{;\nu}$ has the same transformation property as that of V^μ, because $g^{\mu\nu}$ should be transformed as

$$g^{\mu\nu} \to \bar{g}^{\mu\nu} \stackrel{d}{\equiv} \lambda^{-2} g^{\mu\nu}$$

owing to its definition and consequently $\{{}_\nu{}^\mu{}_\lambda\}$ becomes invariant. On the other hand, if λ depends on x, the well-known prescription for general gauge fields[4] leads to the gauge-invariant and generally covariant derivative which can be derived from (2·1) by a simple substitution of an ordinary derivative with a corresponding gauge-invariant derivative:

$$\partial_\tau g_{\rho\nu} \to D_\tau g_{\rho\nu} \stackrel{d}{\equiv} \partial_\tau g_{\rho\nu} + 2\varphi_\tau g_{\rho\nu},$$

$$\partial_\tau V^\mu \to D_\tau V^\mu \stackrel{d}{\equiv} \partial_\tau V^\mu + n\varphi_\tau V^\mu. \tag{2·3}$$

The additional terms in (2·3) are proportional to a new vector field φ_τ which is called Weyl's gauge field and obeys the following rule under the gauge transformation:

$$\varphi_\tau \to \bar{\varphi}_\tau \overset{d}{=} \varphi_\tau - \partial_\tau \lambda / \lambda .$$

The gauge-invariant and generally covariant derivative of a vector V^μ is defined by

$$\nabla_\nu V^\mu \overset{d}{=} D_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda \tag{2·4}$$

and the new affine connection Γ is given by

$$\Gamma^\mu_{\nu\lambda} \overset{d}{=} \tfrac{1}{2} g^{\mu\rho} (D_\lambda g_{\rho\nu} + D_\nu g_{\lambda\rho} - D_\rho g_{\nu\lambda}) . \tag{2·5}$$

For the sake of convenience, let us call $Q(x)$ a quantity with the gauge weight n when Q is transformed by the rule

$$Q \to \bar{Q} \overset{d}{=} \lambda^n Q .$$

Then, (2·2) shows that $g_{\mu\nu}$ has a gauge weight 2, and according to the definition (2·4) the gauge-invariant and generally covariant derivative of $g_{\mu\nu}$ is given by

$$\nabla_\lambda g_{\mu\nu} \overset{d}{=} D_\lambda g_{\mu\nu} - \Gamma^\tau_{\lambda\mu} g_{\tau\nu} - \Gamma^\tau_{\lambda\nu} g_{\mu\tau} .$$

The right-hand side of the above expression vanishes owing to the definition (2·5) regardless of any choice of $g_{\mu\nu}$ and φ_λ.

As an example, consider a vector V^μ with the weight 0. The square of the magnitude of this vector, i.e., $(V)^2 = g_{\mu\nu} V^\mu V^\nu$ has a gauge weight 2. Thus the gauge-invariant derivative of $(V)^2$ becomes

$$\nabla_\nu (V)^2 = D_\nu (V)^2 = \partial_\nu (V)^2 + 2\varphi_\nu (V)^2,$$

which shows that the scalar $(V)^2$ with a weight 2 undergoes a change when its constituent vectors are transferred from a point x to an infinitely closed point $x+dx$ by a parallel transport:

$$\{V(x)\}^2 \to \{V(x+dx)_{\parallel}\}^2 \overset{d}{=} \{1 - 2\varphi_\lambda dx^\lambda\} \cdot \{V(x)\}^2 .$$

Here $\{V(x+dx)_{\parallel}\}^2$ indicates a square of the magnitude of V^μ at $x+dx$ after the parallel transport. The fact that even a scalar changes its magnitude when it is transferred from point to point by an infinitesimal parallel transport is the most characteristic point of the geometry proposed by Weyl.

The φ_μ which had to be introduced in order to recover the same transformation property of $\nabla_\lambda V^\mu$ as that for the case of a constant λ, was regarded by Weyl as representing the electromagnetic potential. However, his idea was forced to be abandoned owing to some strong objections. In fact, it will be shown in the present paper that Weyl's gauge field is a kind of Tachyon field and is governed by a constraint of a completely new type which was never seen in the case of the electromagnetic field.

§ 3. Scale transformation and Weyl's gauge transformation

In connection with the behaviour of cross sections of some extremely high energy processes, many people are interested in the invariance of physical laws under a scale transformation

$$x^\mu \to x^{\mu'} \stackrel{d}{\equiv} \alpha x^\mu,$$
$$g_{\mu\nu} = \text{unchanged.} \qquad (\alpha = \text{any constant}) \tag{3·1}$$

It will be shown that the action integral of some system is invariant under the scale transformation (3·1) if it is Lorentz invariant and also gauge invariant in Weyl's sense with a constant λ.

In order to prove the above statement, let us consider a system of fields Ψ_A $(A=1, 2, \cdots N)$ with a Lagrangian

$$L_0(\Psi_A, \partial_\mu \Psi_A, \eta_{\mu\nu}), \tag{3·2}$$

where $\eta_{\mu\nu}$ is the metric tensor of the Minkowskian type. If (3·2) is Lorentz invariant, it is easy to make (3·2) be a scalar density under any linear transformation of coordinates by substituting $\eta_{\mu\nu}$ with a constant metric tensor $\zeta_{\mu\nu}$ and multiplying (3·2) with a factor $\sqrt{\zeta}$. Here ζ denotes $|\det(\zeta_{\mu\nu})|$ which is assumed to be $\neq 0$. After such a modification, the action integral

$$I_0 = \int L_0(\Psi_A, \partial_\mu \Psi_A, \zeta_{\mu\nu}) \sqrt{\zeta} d^4x \tag{3·3}$$

is invariant under any linear transformation of coordinates, especially under the transformation of the following type (dilatation):

$$x^\mu \to x^{\mu'} \stackrel{d}{\equiv} \alpha x^\mu. \qquad (\alpha = \text{any constant})$$

If Ψ_A is assumed to be a covariant tensor of rank r, it is transformed as follows:

$$\Psi_A \to \Psi_A'(x') \stackrel{d}{\equiv} \alpha^{-r} \Psi_A(x)$$

under the transformation stated above. Similarly, the metric tensor undergoes the change

$$\zeta_{\mu\nu} \to \zeta'_{\mu\nu} \stackrel{d}{\equiv} \alpha^{-2} \zeta_{\mu\nu}.$$

If Weyl's gauge transformation (with a constant λ) is performed subsequently to the above transformation, and the dilatation parameter α is particularly chosen to be equal to λ, the resultant transformation is nothing but the scale transformation (3·1), namely

$$x^\mu \to x^{\mu'} (= \lambda x^\mu) \to \bar{x}^{\mu'} = x^{\mu'} = \lambda x^\mu,$$

$$\zeta_{\mu\nu}\to\zeta'_{\mu\nu}(=\lambda^{-2}\zeta_{\mu\nu})\to\bar{\zeta}'_{\mu\nu}=\lambda^2\zeta'_{\mu\nu}=\zeta_{\mu\nu},$$
$$\Psi_A\to\Psi_A'(=\lambda^{-r}\Psi_A)\to\bar{\Psi}_A'=\lambda^n\Psi_A'=\lambda^{n-r}\Psi_A,$$

where Ψ_A has been assumed to have the gauge weight n and the Minkowskian $\eta_{\mu\nu}$ should be taken in place of $\zeta_{\mu\nu}$ after this transformation. Therefore if (3·3) is gauge invariant in Weyl's sense with a constant λ, it is necessarily invariant under the scale transformation (3·1). The inverse of this statement is also true as is easily shown.

In the above proof, we have considered only the case that Ψ_A is covariant tensors for the sake of simplicity, but it is not hard to extend the above consideration to the case of spinor fields.

§ 4. The vanishing trace of the energy-momentum tensor

It is believed that the trace of the energy-momentum tensor of a system vanishes if the action integral of that system is scale invariant, but any satisfactory proof for this fact has never been published so far.

To prove this, it is necessary to rewrite (3·3), which is assumed to be scale invariant, in a little more general form. We have to consider a gauge invariant Lagrangian for the case of x-dependent λ. As was already shown in § 2, we are able to rewrite (3·3) in an invariant form under any x-dependent gauge transformation by substituting $\partial_\mu\Psi_A$ with a gauge invariant derivative

$$D_\mu\Psi_A\overset{d}{=}\partial_\mu\Psi_A+n\varphi_\mu\Psi_A.$$

A gauge transformation with an x-dependent λ does not allow us to remain in the case of a constant metric tensor $\zeta_{\mu\nu}$. Therefore it is necessary to rewrite (3·3) in a generally covariant form by a further substitution of

$$\zeta_{\mu\nu}\quad\text{with}\quad g_{\mu\nu}(x)$$

and

$$D_\mu\Psi_A\quad\text{with}$$

$$\nabla_\mu\Psi_A\overset{d}{=}D_\mu\Psi_A-\sum\{\Gamma\Psi\}_A, \tag{4·1}$$

where the Γ in the second term of the right-hand side is the gauge-invariant affine coefficient defined by (2·5).*) Thus the action integral which we are going to discuss is

*) Strictly speaking, a metric tensor of the conformally flat world, namely, that of the following type

$$g_{\mu\nu}(x)=\{\xi(x)\}^2\eta_{\mu\nu}$$

is sufficient for our present discussion. Since, however, the case with such a particular metric is rather troublesome to deal with, we make use of a general non-degenerate metric tensor $g_{\mu\nu}(x)$ without any restriction.

$$I=\int \mathcal{L}d^4x \overset{d}{\equiv} \int (\mathcal{L}_0+\mathcal{L}_1)\,d^4x\,, \tag{4·2}$$

where

$$\mathcal{L}_0(\Psi_A,\,\partial_\mu\Psi_A,\,g_{\mu\nu},\,\partial_\lambda g_{\mu\nu},\,\varphi_\mu) \overset{d}{\equiv} \sqrt{-g}\cdot L_0(\Psi_A,\,\nabla_\mu\Psi_A,\,g_{\mu\nu}), \tag{4·2$'$}$$

and

$$\mathcal{L}_1(\varphi_\mu,\,\partial_\nu\varphi_\mu,\,g_{\mu\nu},\,\partial_\lambda g_{\mu\nu})$$

is some gauge-invariant scalar density which should be added to the given Lagrangian L_0 in order to determine the behaviour of the Weyl's gauge field φ_μ.

From the procedure stated so far it is evident that I is a gauge-invariant scalar. Consequently, for an infinitesimal gauge transformation,

$$\lambda(x)=1+\varepsilon(x), \qquad (\varepsilon=\text{an infinitesimal function})$$

$$\delta g_{\mu\nu}(x) \overset{d}{\equiv} \bar{g}_{\mu\nu}(x)-g_{\mu\nu}(x)=2\varepsilon(x)\cdot g_{\mu\nu}(x),$$

$$\delta\Psi_A(x) \overset{d}{\equiv} \bar{\Psi}_A(x)-\Psi_A(x)=n\varepsilon(x)\Psi_A\,,$$

$$\delta\varphi_\mu \overset{d}{\equiv} -\partial_\mu\varepsilon(x),$$

we have the following identity:

$$\delta I=\int \varepsilon(x)\left\{\frac{\delta\mathcal{L}}{\delta\Psi_A}n\Psi_A+\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\varphi_\mu}\right)+2\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}g_{\mu\nu}\right\}d^4x$$
$$+\int \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial\Psi_{A,\mu}}\delta\Psi_A+\frac{\partial\mathcal{L}}{\partial\varphi_{\rho,\mu}}\delta\varphi_\rho+\frac{\partial\mathcal{L}}{\partial g_{\rho\sigma,\mu}}\delta g_{\rho\sigma}-\varepsilon\frac{\delta\mathcal{L}}{\delta\varphi_\mu}\right\}d^4x\equiv 0\,.$$

Here the following abbreviation has been employed:

$$\frac{\delta\mathcal{L}}{\delta\Psi_A} \overset{d}{\equiv} \frac{\partial\mathcal{L}}{\partial\Psi_A}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\Psi_{A,\mu}}\right). \qquad (\text{similar notations for } \varphi_\mu \text{ and } g_{\mu\nu})$$

The well-known line of reasoning used in Noether's theorem decomposes the above integral to a couple of identities

$$-2\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}g_{\mu\nu}\equiv n\frac{\delta\mathcal{L}}{\delta\Psi_A}\Psi_A+\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\varphi_\mu}\right) \tag{4·3}$$

and

$$\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial\Psi_{A,\mu}}\delta\Psi_A+\,''\,+\,''\,-\,''\right\}\equiv 0\,. \tag{4·4}$$

Now, it is well known that the tensor density

$$\Theta^{\mu\nu} \overset{d}{\equiv} -2\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}$$

is a symmetric energy-momentum tensor density of the total system $\mathcal{L}$ and plays the role of source for the gravitational field in Einstein's theory of gravitation. Therefore the identity (4·3) leads to the vanishing trace of the energy-momentum tensor:

$$\Theta^{\mu\nu}g_{\mu\nu}=0\,,$$

provided that the field equations

$$\frac{\delta\mathcal{L}}{\delta\Psi_A}=0 \tag{4·5}$$

and

$$\frac{\delta\mathcal{L}}{\delta\varphi_\mu}=0 \tag{4·6}$$

have been employed.

The proof for $\Theta^\mu_\mu=0$ given above has two unsatisfactory points. The first point is that the $\Theta^{\mu\nu}$ depends not only on Ψ_A, but also on the gauge field φ_μ. Therefore if we want to have an energy-momentum tensor of the Ψ_A alone, the gauge field φ_μ needs to be expressed in terms of Ψ_A by solving Eq. (4·6). Thus the resultant $\Theta^{\mu\nu}$ with a property of vanishing trace has in general a non-local feature. The second weak point that is more serious is that the compatibility of Eq. (4·5) with (4·6) of which we are quite optimistic and do not customarily give any careful consideration.

§ 5. Example; Tachyon field

In order to investigate the compatibility of field equations pointed out in the end of § 4, let us consider a simple example.

Let ψ be a real scalar field the behaviour of which is determined by the following Lagrangian:*)

$$L_0=-\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\psi\partial_\nu\psi\,. \qquad (\eta_{00}=-1) \tag{5·1}$$

It is easily seen that the action integral of (5·1) is invariant under the scale transformation

$$x^\mu\to\lambda x^\mu,$$

$$\psi\to\lambda^{-1}\psi\,. \qquad (\lambda=\text{any constant})$$

The modified Lagrangian corresponding to (5·1) which is gauge-invariant scalar density under general transformations of coordinates is

$$\mathcal{L}_0=-\tfrac{1}{2}\sqrt{-g}\cdot g^{\mu\nu}\nabla_\mu\psi\cdot\nabla_\nu\psi\,, \tag{5·2}$$

*) The case in which the self-interaction $1/4\cdot f\psi^4$ exists is discussed in a report by the present author.[5)]

where

$$\nabla_\mu \psi = \partial_\mu \psi - \varphi_\mu \psi \, .$$

As to the Lagrangian describing the behaviour of the gauge field φ_μ, let us adopt

$$\mathcal{L}_1 = -\frac{a}{4}\sqrt{-g}\cdot g^{\mu\rho} g^{\nu\sigma} f_{\mu\nu} f_{\rho\sigma} \, , \tag{5·3}$$

where

$$f_{\mu\nu} \overset{d}{\equiv} \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu$$

and the parameter a is assumed to take a value $+1$ or -1. (5·3) is the only possible one among the various types of gauge invariant scalar densities which give a linear field equation of second order.

The total action integral

$$I = \int (\mathcal{L}_0 + \mathcal{L}_1) d^4x$$

is invariant under the gauge transformation

$$\psi \to \bar{\psi} = \lambda^{-1} \psi \, ,$$

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \lambda^2 g_{\mu\nu} \, ,$$

$$\varphi_\mu \to \bar{\varphi}_\mu = \varphi_\mu - \partial_\mu \lambda / \lambda \, . \tag{5·4}$$

The prescription stated in § 3 leads to the symmetric energy-momentum tensor density (with a gauge weight -2) of the following type:

$$\Theta^{\mu\nu} = -\sqrt{-g}(\nabla^\mu \psi \cdot \nabla^\nu \psi + a f^{\mu\rho} f^{\nu\sigma} g_{\rho\sigma}) - g^{\mu\nu}(\mathcal{L}_0 + \mathcal{L}_1) \, ,$$

the trace of which is

$$\Theta \overset{d}{\equiv} \Theta^{\mu\nu} g_{\mu\nu} = \sqrt{-g} \nabla_\mu \psi \cdot \nabla_\nu \psi \cdot g^{\mu\nu} . \tag{5·5}$$

The field equations for ψ and φ_μ are

$$\nabla_\mu (\sqrt{-g} \nabla^\mu \psi) \overset{d}{\equiv} \partial_\mu (\sqrt{-g} \nabla^\mu \psi) + \varphi_\mu (\sqrt{-g} \nabla^\mu \psi) = 0 \tag{5·6}$$

and

$$a \partial_\nu (\sqrt{-g} f^{\mu\nu}) = \sqrt{-g} \psi \cdot \nabla^\mu \psi \, . \tag{5·7}$$

Owing to the antisymmetry of $f^{\mu\nu}$ with respect to its suffices μ and ν, (5·7) gives a law of conservation

$$\partial_\mu (\sqrt{-g} \psi \cdot \nabla^\mu \psi) = 0 \, . \tag{5·8}$$

Now the right-hand side of (5·5) can be transformed into

$$\partial_\mu (\sqrt{-g} \psi \cdot \nabla^\mu \psi) - \psi \cdot \nabla_\mu (\sqrt{-g} \nabla^\mu \psi) \, ,$$

the second term of which vanishes with the aid of the field equation (5·6), while the first term is equal to zero owing to the law of conservation (5·8). Thus we have shown that indeed the trace Θ vanishes provided that the field equations are satisfied.

For the sake of simplicity, let us assume that the space time is conformally flat. In such a case it is possible to introduce a Cartesian system of coordinates by making use of some appropriate gauge transformation together with a general transformation of coordinates. The field equations in such a case become

$$\partial_\mu(\partial^\mu\psi-\varphi^\mu\psi)+\varphi_\mu(\partial^\mu\psi-\varphi^\mu\psi)=0 \tag{5·9}$$

and

$$a\partial_\nu f^{\mu\nu}=\psi(\partial^\mu\psi-\varphi^\mu\psi). \tag{5·10}$$

If the following notation

$$\bar{\varphi}^\mu \overset{d}{\equiv} \varphi^\mu-\partial^\mu\psi/\psi$$

is employed, the gauge invariant derivative $\nabla_\mu\psi$ takes a simple form

$$\nabla_\mu\psi=-\psi\cdot\bar{\varphi}_\mu ,$$

and Eqs. (5·9) and (5·10) turn out to be

$$\partial_\mu(\psi^2\bar{\varphi}^\mu)+\psi^2\bar{\varphi}_\mu\bar{\varphi}^\mu=0 \tag{5·9$'$}$$

and

$$a\partial_\nu\bar{f}^{\mu\nu}=-\psi^2\bar{\varphi}^\mu \tag{5·10$'$}$$

respectively, where

$$\bar{f}_{\mu\nu}=\partial_\mu\bar{\varphi}_\nu-\partial_\nu\bar{\varphi}_\mu .$$

The divergence of (5·10)$'$ leads to

$$\partial_\mu(\psi^2\bar{\varphi}^\mu)=0 , \tag{5·8$'$}$$

which gives a new constraint

$$\psi^2\bar{\varphi}_\mu\bar{\varphi}^\mu=0 \tag{5·9$''$}$$

when the first term of (5·9)$'$ is substituted by (5·8)$'$.

It is easily seen that in the case of $\psi\equiv0$, the gauge field $\bar{\varphi}_\mu$ satisfies the same equation as that of the electromagnetic field and in such a case, the parameter a should be put equal to $+1$ in order to obtain a positive definite energy density.

On the other hand, if ψ does not identically vanish, (5·10)$'$ has a feature which shows that the gauge field $\bar{\varphi}_\mu$ has a "mass" $\psi(x)$. Let us consider some region in the space time where $\psi(x)$ is almost constant (such a region can always exist if the extension of this region is chosen sufficiently small). In such a case the field equations and the constraints take the following form in a sufficiently good approximation:

$$\Box\bar{\varphi}_\mu - am^2\bar{\varphi}_\mu = 0\,, \tag{5·11}$$

$$\partial_\mu\bar{\varphi}^\mu = 0\,, \tag{5·12}$$

$$\bar{\varphi}_\mu\bar{\varphi}^\mu = 0\,, \tag{5·13}$$

where m^2 has been put in place of $\{\psi(x)\}^2$.

Now (5·11) has a plane wave solution:

$$\bar{\varphi}_\mu(x) = A_\mu \cos(k_\sigma x^\sigma + \alpha),$$

where

$$k^0 = \sqrt{(\boldsymbol{k})^2 + am^2}\,.$$

The constraint (5·13) gives a condition

$$|A^0| = |\boldsymbol{A}| = \sqrt{\sum_{k=1}^{3} (A_k)^2}$$

which transforms (5·12) into a simple form

$$A^0 k^0 = |\boldsymbol{A}| \cdot k \cdot \cos\theta\,,$$

where θ is an angle between $\boldsymbol{A}$ and $\boldsymbol{k}$. Therefore we obtain a relation

$$\sqrt{(\boldsymbol{k})^2 + am^2} = k\cos\theta < k$$

(A^0 is assumed to be positive), which shows that a should be equal to -1.*) Thus the field $\bar{\varphi}_\mu$ is not an ordinary field but describes a kind of Tachyon field with non-positive-definite energy density which is due to the opposite sign of $\mathcal{L}_1$.

This example shows that the postulate of gauge invariance and Lorentz invariance is not necessarily sufficient for the determination of the equation of the gauge field. In fact the parameter a in this example cannot be determined without a careful examination of solutions. It can occur that the Hamilton principle leads to an incompatible set of field equations or to those which have physically unacceptable solutions even though the field equations have an invariant appearance.

Before closing this article it should be noted that the unusual field φ_μ might play some role in establishing a model of a stable elementary particle. It can be expected that the "wrong" sign of the Lagrangian density of φ_μ gives rise to a destructive effect on the physical propagators of other particles and consequently suppresses the divergences being inherent in the quantum field theory. Many similar approaches have been tried so far but all these theories have a common weak point. That is, it is not easy to make such a cohesion field un-

*) If $a = +1$, the wave number vector k_μ should become imaginary. Such a solution does not have a simple correspondence with a particle of the ordinary type.

observable. According to the result stated in this section, however, Weyl's gauge field is a Tachyon-like field which describes particles with momenta larger than its rest mass when it is quantized. This peculiar property of φ_μ which is characteristic of Tachyons suggests that the field φ_μ makes its appearance only in the vicinity (of the order of the Compton wave length of φ_μ) of its source.

The third strong point of Weyl's gauge field is that the interaction of φ_μ with other fields is uniquely determined (especially in the case of extremely high energy where the masses of other fields can be ignored) by the principle of gauge invariance. The various theories of cohesion field so far proposed had no such a guiding principle for the determination of the type of their interactions.

The conjecture stated above seems to show that it is worth while to investigate in more detail the behaviour of Weyl's field both in the classical and quantum theory.

References

1) H. Weyl, Sitzber. Preuss. Akad. Wiss. (1918), 465.
2) T. Fulton, F. Rohrlich and L. Witten, Rev. Mod. Phys. **34** (1962), 442.
3) A. Bregman, Prog. Theor. Phys. **49** (1973), 667.
4) R. Utiyama, Phys. Rev. **101** (1956), 1597.
5) R. Utiyama, Nippon Butsuri-gakkai-shi **29** (1974), to be published.