

# On Yen's Path Logic for Petri Nets

Mohamed Faouzi Atig and Peter Habermehl

LIAFA, CNRS & Univ. of Paris 7, Case 7014, 75205 Paris 13, France.  
{atig,haberm}@liafa.jussieu.fr

**Abstract.** In [13], Yen defines a class of formulas for paths in Petri nets and claims that its satisfiability problem is EXPSpace-complete. In this paper, we show that in fact the satisfiability problem for this class of formulas is as hard as the reachability problem for Petri nets. Moreover, we salvage almost all of Yen's results by defining a fragment of this class of formulas for which the satisfiability problem is EXPSpace-complete by adapting his proof.

## 1 Introduction

Petri nets (or equivalently, vector addition systems) are one of the most popular mathematical model for the representation and analysis of parallel processes [2]. The reachability problem for Petri nets is one of the key problems in the area of automatic verification since many other problems (e.g. the liveness problem) were shown to be recursively equivalent to the reachability problem (see [4, 6]). It is well known that the reachability problem for Petri nets is decidable [11, 10, 7, 8]. However, the precise complexity of the reachability problem for Petri nets remains open (all known algorithms require non-primitive recursive space). The best known lower bound is exponential space given by Lipton in [9].

On the other hand, to obtain a uniform approach for deciding and studying the complexity of many Petri nets problems, Yen has defined in [13] a class of formulas for paths in Petri nets, each of them is of the form:

$$\exists \mu_1, \dots, \mu_n \exists \sigma_1, \dots, \sigma_n ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \dots \mu_{n-1} \xrightarrow{\sigma_n} \mu_n) \wedge \phi(\mu_1, \dots, \mu_n, \sigma_1, \dots, \sigma_n))$$

where  $\phi$  belongs to a certain set of predicates (constraining the markings and transitions sequence occurring in the formula) and  $\mu_0$  is the initial marking of the given Petri net. The above formula means that any marking  $\mu_i$  can be reached from  $\mu_{i-1}$  ( $1 \leq i \leq n$ ) in the Petri net through the firing sequence of transitions  $\sigma_i$  and such that the predicate  $\phi(\mu_1, \dots, \mu_n, \sigma_1, \dots, \sigma_n)$  holds. In [13], Yen claims that the satisfiability problem for such class of formulas (i.e., the problem of, given a Petri net and a formula, determining whether there exists a path in the Petri net satisfying the given formula) is complete for exponential space. This class of formulas is a useful and an interesting one since it is powerful enough to express many Petri nets properties. In particular, Petri nets problems such as *boundedness*, *coverability*, *fair-nontermination*, and *regularity detection* are reducible to the satisfiability problem for this class of formulas [13]. Moreover, Yen's result has been cited and used in several papers [1, 15, 14, 5, 3].

In this paper, we prove that the reachability problem for Petri nets is in fact as hard as the satisfiability problem for this class of formulas. However, we can salvage almost all of Yen's results by defining an interesting and useful fragment of this class of formulas of paths in Petri nets for which the satisfiability problem is EXPSPACE-complete. In proving the upper bound for this fragment, we correct an error in the proof given in [13]. Essentially, the fragment requires the marking  $\mu_n$  to be bigger than  $\mu_1$  allowing the path satisfying the formula to be repeated.

The *regularity detection* problem can not be expressed using our fragment and therefore to the best of our knowledge it's complexity (given as EXPSPACE in [13]) remains unclear.

## 2 Preliminaries

Let  $\mathbb{Z}$  (resp.  $\mathbb{N}$ ) denote the set of (resp. nonnegative) integers, and  $\mathbb{Z}^k$  (resp.  $\mathbb{N}^k$ ) the set of vectors of  $k$  (resp. nonnegative) integers.

Let  $\Sigma$  be a finite alphabet. We denote by  $\Sigma^*$  (resp.  $\Sigma^+$ ) the set of all finite (resp. non empty) words over  $\Sigma$  and by  $\varepsilon$  the empty word. We use  $|\Sigma|$  to denote the number of symbols in  $\Sigma$ . We denote by  $\mathbb{N}^\Sigma$  (resp.  $\mathbb{Z}^\Sigma$ ) the set of all mappings from  $\Sigma$  to  $\mathbb{N}$  (resp. to  $\mathbb{Z}$ ) and by  $\mathbf{0}$  the mapping that maps every symbol in  $\Sigma$  to 0. Notice that any mapping in  $\mathbb{N}^\Sigma$  can be considered as a mapping in  $\mathbb{Z}^\Sigma$ .

Let  $\Sigma$  and  $\Sigma'$  be two finite alphabets such that  $\Sigma \subseteq \Sigma'$ . Given a mapping  $\mu$  in  $\mathbb{Z}^{\Sigma'}$ , we write  $\mu|_\Sigma$  to denote the mapping that maps every  $a \in \Sigma$  to  $\mu(a)$ .

Let  $\Sigma$  be a finite alphabet and  $\mu_1$  and  $\mu_2$  two mappings from  $\Sigma$  to  $\mathbb{Z}$ , we denote by  $\mu_1 \odot \mu_2$  the inner product of  $\mu_1$  and  $\mu_2$  (i.e.,  $\mu_1 \odot \mu_2 = \sum_{a \in \Sigma} \mu_1(a) \mu_2(a)$ ).

The *Parikh image*  $\sharp : \Sigma^* \mapsto \mathbb{N}^\Sigma$  maps a word  $w$  to a mapping  $\sharp(w)$  from  $\Sigma$  to  $\mathbb{N}$  such that  $\sharp(w)(a)$  is the number of occurrences of  $a$  in  $w$ .

A Petri net  $\mathcal{N} = (P, T, F, \mu_0)$  consists of a finite set  $P$  of places, a finite set  $T$  of transitions disjoint from  $P$ , a weight function  $F : (P \times T) \cup (T \times P) \mapsto \mathbb{N}$ , and an initial marking  $\mu_0 \in \mathbb{N}^P$ . A marking is a map from  $P$  to  $\mathbb{N}$ . For a marking  $\mu$  of  $\mathcal{N}$  and a place  $p \in P$ , we say that, in  $\mu$ , the place  $p$  contains  $\mu(p)$  tokens. For markings  $\mu, \mu'$ , we write  $\mu + \mu'$  for the marking obtained by point wise addition of place contents. We write  $\mu \leq \mu'$  if  $\mu(p) \leq \mu'(p)$  for all  $p \in P$ , and we write  $\mu < \mu'$  if  $\mu \leq \mu'$  and  $\mu(p') \neq \mu'(p')$  for some place  $p' \in P$ . The marking  $\mathbf{0}$  maps every  $p \in P$  to 0.

A transition  $t \in T$  is enabled at a marking  $\mu$  if and only if  $F(p, t) \leq \mu(p)$  for all  $p \in P$ . If a transition  $t$  is enabled at a marking  $\mu$ , then  $t$  may be fired yielding to a new marking  $\mu'$  defined as follows:  $\mu'(p) = \mu(p) - F(p, t) + F(t, p)$  for all  $p \in P$ . We then write  $\mu \xrightarrow{t} \mu'$  to denote that the marking  $\mu'$  is reached from  $\mu$  by firing the transition  $t$ . A sequence of transitions  $\sigma = t_1 \cdots t_n$  is a firing sequence from  $\mu_0$  if and only if  $\mu_0 \xrightarrow{t_1} \mu_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} \mu_n$  for some sequence of markings  $\mu_1, \dots, \mu_n$ . Furthermore, we call  $\mu_0 \xrightarrow{\sigma} \mu_n$  a *computation* of  $\mathcal{N}$ .

A marking  $\mu$  is said to be reachable in  $\mathcal{N}$  if and only if  $\mu = \mu_0$  or there is some  $\sigma \in T^+$  such that  $\mu_0 \xrightarrow{\sigma} \mu$ . The reachability problem for a Petri net  $\mathcal{N}$  is, for a given marking  $\mu$ , to determine whether  $\mu$  is reachable in  $\mathcal{N}$ .

We define the size  $s(\mathcal{N})$  of a Petri Net  $\mathcal{N}$  as in [13], i.e. numbers are encoded in binary and the size of a Petri Net is then  $\lceil \log k \rceil + \lceil \log r \rceil$  (where  $k$  is the number of

places and  $r$  is the number of transitions) + the sum of the sizes of the elements of  $F$  + the size of  $\mu_0$ . The firing of a transition may result in removing (or adding)  $2^{s(\mathcal{N})}$  tokens from (to) a place.

Finally, we recall that the reachability problem for Petri nets is decidable.

**Theorem 1 ([11, 9]).** *The reachability problem for Petri nets is EXPSPACE-hard.*

### 3 Yen's Path Logic for Petri Nets

In this section we define the class of path formulas for Petri nets considered by Yen in [13]. We essentially follow his definitions. Let  $\mathcal{N} = (P, T, F, \mu_0)$  be a Petri net. Each path formula consists of the following elements:

1. *Variables:* There are two types of variables, namely, marking variables  $\mu_1, \mu_2, \dots$  and variables for transition sequences  $\sigma_1, \sigma_2, \dots$ , where each  $\mu_i$  denotes a marking of  $\mathcal{N}$  and each  $\sigma_i$  denotes a finite sequence of transition rules.
2. *Terms:* Terms are defined recursively as follows:
  - For every mapping  $\mathbf{c} \in \mathbb{N}^P$ ,  $\mathbf{c}$  is a term.
  - For all  $j > i$ ,  $\mu_j - \mu_i$  is a term, where  $\mu_i$  and  $\mu_j$  are marking variables.
  - $\mathcal{T}_1 + \mathcal{T}_2$  and  $\mathcal{T}_1 - \mathcal{T}_2$  are terms if  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are terms. (Consequently, every mapping  $\mathbf{c} \in \mathbb{Z}^P$  is also a term.)
3. *Atomic predicates:* There are two types of atomic predicates, namely, transition predicates and marking predicates.
  - (a) *Transition predicates:*
    - $\mathbf{z} \odot \#(\sigma_i) \geq c$  and  $\mathbf{z} \odot \#(\sigma_i) > c$  are predicates, where  $i > 1$ ,  $c \in \mathbb{N}$  is a constant, and  $\mathbf{z}$  is a mapping from  $T$  to  $\mathbb{Z}$ .
    - $\#(\sigma_1)(t) \geq c$  and  $\#(\sigma_1)(t) \leq c$  are predicates, where  $c \in \mathbb{N}$  is a constant and  $t \in T$  is a transition rule of  $\mathcal{N}$ .
  - (b) *Marking predicates:*
    - $\mu(p) \geq z$  and  $\mu(p) > z$  are predicates, where  $\mu$  is a marking variable,  $p \in P$  is a place of  $\mathcal{N}$ , and  $z \in \mathbb{Z}$  is an integer.
    - $\mathcal{T}_1(p_1) = \mathcal{T}_2(p_2)$ ,  $\mathcal{T}_1(p_1) < \mathcal{T}_2(p_2)$ , and  $\mathcal{T}_1(p_1) > \mathcal{T}_2(p_2)$  are predicates, where  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are terms and  $p_1, p_2 \in P$  are two places of  $\mathcal{N}$ .

A *predicate* is either a marking predicate, a transition predicate, or of the form  $\bigvee_{1 \leq i \leq k} \bigwedge_{1 \leq j \leq m_i} \phi_i^j$  (i.e., in the disjunctive normal form<sup>1</sup>) where each  $\phi_i^j$  is a marking or transition predicate. A *Path formula*  $f$  is a formula of the form:

$$\exists \mu_1, \dots, \mu_n \exists \sigma_1, \dots, \sigma_n ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \dots \mu_{n-1} \xrightarrow{\sigma_n} \mu_n) \wedge \phi(\mu_1, \dots, \mu_n, \sigma_1, \dots, \sigma_n))$$

where  $\phi$  is a predicate.

Given a Petri net  $\mathcal{N}$  and a path formula  $f$ , we use  $\mathcal{N} \models f$  to denote that  $f$  is true in  $\mathcal{N}$ . The *satisfiability problem* for such a path formula  $f$  asks if there exists an execution

<sup>1</sup> In [13], a predicate can be any positive boolean combination of predicates. In fact, we can show that our results (in particular Theorem 3) still hold even if we consider this general case.

of  $\mathcal{N}$  of the form  $\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \dots \mu_{n-1} \xrightarrow{\sigma_n} \mu_n$  such that  $\phi(\mu_1, \dots, \mu_n, \sigma_1, \dots, \sigma_n)$  holds. In this case, we say  $\mathcal{N}$  satisfies the path formula  $f$  (i.e.,  $\mathcal{N} \models f$ ).

The following result can be shown following [13].

**Lemma 1.** *Given a Petri net  $\mathcal{N} = (P, T, F, \mu_0)$  and a formula  $f$ , we can construct in polynomial time, a Petri net  $\mathcal{N}' = (P', T', F', \mu'_0)$  and a formula  $f'$  containing no transition predicates such that  $\mathcal{N} \models f$  if and only if  $\mathcal{N}' \models f'$ .*

Therefore, it is sufficient to consider formulas containing only marking predicates in order to decide satisfiability.

## 4 From the reachability problem to the satisfiability problem

In the following, we prove that the reachability problem for Petri nets is polynomially reducible to the satisfiability problem for path formulas.

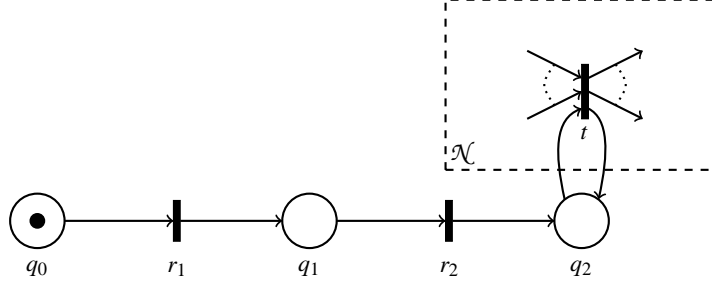
**Theorem 2.** *Given a Petri net  $\mathcal{N} = (P, T, F, \mu_0)$  and a marking  $\mu \in \mathbb{N}^P$ , we can construct, in polynomial time, a Petri net  $\mathcal{N}' = (P', T', F', \mu'_0)$  and a path formula  $f$  such that the marking  $\mu$  is reachable by  $\mathcal{N}$  if and only if  $\mathcal{N}' \models f$ .*

The rest of this section is devoted to the proof of Theorem 2. We first construct a Petri net  $\mathcal{N}' = (P', T', F', \mu'_0)$  with  $P \subseteq P'$  such that a marking  $\mu \in \mathbb{N}^P$  is reachable in  $\mathcal{N}$  if and only if there is a marking  $\mu' \in \mathbb{N}^{P'}$  such that  $\mu'|_P = \mu$  and  $\mu'$  is reachable in  $\mathcal{N}'$ . Then, we construct a path formula  $f$  for the Petri net  $\mathcal{N}'$  such that  $\mathcal{N}'$  satisfies the formula  $f$  if and only if there is a marking  $\mu' \in \mathbb{N}^{P'}$  such that  $\mu'|_P = \mu$  and  $\mu'$  is reachable in  $\mathcal{N}'$ . This implies that the marking  $\mu$  is reachable in  $\mathcal{N}$  if and only if  $\mathcal{N}'$  satisfies the formula  $f$ .

### 4.1 Constructing the Petri net $\mathcal{N}'$

The Petri net  $\mathcal{N}' = (P', T', F', \mu'_0)$  is built up from  $\mathcal{N}$  in a way described in Fig. 1. Formally,  $\mathcal{N}'$  contains all transitions and places of  $\mathcal{N}$ . In addition, three new places  $q_0, q_1, q_2$  and two new transitions  $r_1$  and  $r_2$  are added to  $\mathcal{N}'$ . Initially,  $\mathcal{N}'$  has just one token in the place  $q_0$  and  $\mu_0(p)$  tokens in each place  $p \in P$  (i.e.,  $\mu'_0(q_0) = 1, \mu'_0(q_1) = \mu'_0(q_2) = 0$ , and  $\mu'_0|_P = \mu_0$ ). The transition  $r_1$  (resp.  $r_2$ ) consumes exactly one token from the place  $q_0$  (resp.  $q_1$ ) and produces only one token in the place  $q_1$  (resp.  $q_2$ ), i.e.,  $F'(q_0, r_1) = F'(r_1, q_1) = 1$  (resp.  $F'(q_1, r_2) = F'(r_2, q_2) = 1$ ) and 0 otherwise. A transition  $t \in T$  of  $\mathcal{N}'$  consumes exactly one token from the place  $q_2$  and  $F(p, t)$  tokens from each place  $p \in P$ , and produces one token in the place  $q_2$  and  $F(t, p)$  token in each place  $p \in P$ . Formally, we have that for every  $i \in \{0, 1\}$ ,  $F'(q_i, t) = F'(t, q_i) = 0$ ,  $F'(q_2, t) = F(t, q_2) = 1$  and for every  $p \in P$ ,  $F'(p, t) = F(p, t)$  and  $F'(t, p) = F(t, p)$ .

Then, the relation between  $\mathcal{N}$  and  $\mathcal{N}'$  is giving by the following lemma.



**Fig. 1.** The Petri net  $\mathcal{N}'$

**Lemma 2.** Let  $\mu \in \mathbb{N}^P$  be a marking and  $\sigma \in T^+$  be a sequence of transitions of  $\mathcal{N}$ .  $\mu_0 \xrightarrow{\sigma} \mu$  is a computation of  $\mathcal{N}$  if and only if  $\mu'_0 \xrightarrow{r_1} \mu'_1 \xrightarrow{r_2 \sigma} \mu'$  is a computation of  $\mathcal{N}'$  where:

- $\mu'_1(q_0) = \mu'_1(q_2) = 0$ ,  $\mu'_1(q_1) = 1$ , and  $\mu'_1|_P = \mu_0$ .
- $\mu'(q_0) = \mu'(q_1) = 0$ ,  $\mu'(q_2) = 1$ , and  $\mu'|_P = \mu$ .

As an immediate consequence of Lemma 2, we get the following result.

**Corollary 1.** A marking  $\mu \in \mathbb{N}^P$  is reachable by  $\mathcal{N}$  if and only if there is a sequence of transitions  $\sigma \in T^*$  such that  $\mu'_0 \xrightarrow{r_1} \mu'_1 \xrightarrow{r_2 \sigma} \mu'$  is a computation of  $\mathcal{N}'$  with  $\mu'(q_0) = \mu'(q_1) = 0$ ,  $\mu'(q_2) = 1$ , and  $\mu'|_P = \mu$ .

#### 4.2 Constructing the path formula $f$ for the Petri net $\mathcal{N}'$

In the following, we construct a path formula  $f$  such that  $\mathcal{N}'$  satisfies  $f$  if and only if the marking  $\mu$  is reachable by  $\mathcal{N}$ . The path formula  $f$  is of the following form:

$$\exists \mu_1, \mu_2 \exists \sigma_1, \sigma_2 ((\mu'_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2) \wedge \phi_1(\mu_1) \wedge \phi_2(\mu_1, \mu_2))$$

where  $\phi_1$  and  $\phi_2$  are two predicates.

The predicate  $\phi_1(\mu_1) = \mu_1(q_1) \geq 1$  says that only the transition rule  $r_1$  is fired during the sequence of transitions  $\sigma_1$  (i.e.,  $\sigma_1 = r_1$ ). This implies that the marking  $\mu_1$  is defined as follows:  $\mu_1(q_0) = \mu_1(q_2) = 0$ ,  $\mu_1(q_1) = 1$ , and  $\mu_1|_P = \mu_0$ .

$$\phi_2(\mu_1, \mu_2) = (\mu_2(q_2) \geq 1) \wedge \bigwedge_{p \in P} (\mu_2(p) - \mu_1(p) = \mu(p) - \mu_0(p))$$

**Fig. 2.** The predicate  $\phi_2(\mu_1, \mu_2)$

The predicate  $\phi_2$  (given by Fig. 2) says that for each place  $p \in P$ , the difference between the number of tokens added to  $p$  and the number of tokens taken from  $p$ ,

during firing the sequence of transitions  $\sigma_2$ , is equal to  $\mu(p) - \mu_0(p)$ . This implies that  $\mu_2(q_0) = \mu_2(q_1) = 0$ ,  $\mu_2(q_2) = 1$ , and  $\mu_2|_P = \mu$ .

**Lemma 3.** *The Petri net  $\mathcal{N}'$  satisfies the path formula  $f$  if and only if  $\mu'_0 \xrightarrow{r_1} \mu_1 \xrightarrow{r_2 \sigma} \mu_2$  is a computation of  $\mathcal{N}'$  where  $\sigma \in T^*$ , and  $\mu_1$  and  $\mu_2$  are two markings defined as follows:*

- $\mu_1(q_0) = \mu_1(q_2) = 0$ ,  $\mu_1(q_1) = 1$ , and  $\mu_1|_P = \mu_0$ .
- $\mu_2(q_0) = \mu_2(q_1) = 0$ ,  $\mu_2(q_2) = 1$ , and  $\mu_2|_P = \mu$ .

As an immediate consequence of Lemma 3 and Corollary 1, we have that:

**Corollary 2.** *The marking  $\mu$  is reachable by  $\mathcal{N}$  if and only if  $\mathcal{N}' \models f$ .*

Hence, the reachability problem for Petri nets is polynomially reducible to the satisfiability problem for the class of path formulas.

*Remark 1.* It is also possible to reduce the reachability problem for Petri nets to the satisfiability problem for a path formula that contains only transition predicates.

## 5 From the satisfiability problem to the reachability problem

In this section, we show that the satisfiability problem for path formulas is polynomially reducible to the reachability problem for Petri nets.

**Theorem 3.** *Given a Petri net  $\mathcal{N} = (P, T, F, \mu_0)$  and a path formula  $f$ , we can construct, in polynomial time, a Petri net  $\mathcal{N}' = (P', T', F', \mu'_0)$  such that  $\mathcal{N} \models f$  iff the empty marking  $\mathbf{0}$  is reachable by  $\mathcal{N}'$ .*

The rest of this section is devoted to the proof of Theorem 3. Let us suppose that the path formula  $f$  is of the form:

$$\exists \mu_1, \dots, \mu_n \exists \sigma_1, \dots, \sigma_n ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2 \cdots \mu_{n-1} \xrightarrow{\sigma_n} \mu_n) \wedge \phi(\mu_1, \dots, \mu_n))$$

We assume, without loss of generality, that  $\phi$  contains only marking predicates (see Lemma 1). Furthermore, because  $\phi_1 \vee \phi_2$  is satisfiable if and only if  $\phi_1$  is satisfiable or  $\phi_2$  is satisfiable, we can assume that  $\phi$  is of the form  $\phi = \phi_1 \wedge \cdots \wedge \phi_m$  where for every  $i \in \{1, \dots, m\}$ ,  $\phi_i$  is a marking predicate of the form<sup>2</sup>:

$$y_0^i + \sum_{j=1}^n (\mathbf{y}_j^i \odot \mu_j) \leq z_0^i + \sum_{j=1}^n (\mathbf{z}_j^i \odot \mu_j)$$

where  $\mathbf{y}_j^i$  and  $\mathbf{z}_j^i$  are two mappings from  $P$  to  $\mathbb{N}$  and  $y_0^i$  and  $z_0^i$  are two nonnegative integers.

<sup>2</sup> According to [13] (Lemma 3.4, page 130) any marking predicate can be represented as a predicate of this form. Moreover, it is easy to see that the set of predicates of this form is slightly more general than the set of marking predicates defined in section 3.

For every  $i \in \{1, \dots, m\}$ , let  $\rho_i^-$  and  $\rho_i^+$  be two mappings from  $(\mathbb{N}^P)^n$  to  $\mathbb{N}$  such that: for every given sequence of markings  $\mu_1, \dots, \mu_n$  of  $\mathcal{N}$ , we have that  $\rho_i^-(\mu_1, \dots, \mu_n) = y_0^i + \sum_{j=1}^n (y_j^i \odot \mu_j)$  and  $\rho_i^+(\mu_1, \dots, \mu_n) = z_0^i + \sum_{j=1}^n (z_j^i \odot \mu_j)$ .

In the following, we compute a Petri net  $\mathcal{N}' = (P', T', F', \mu_0')$  such that  $\mathcal{N} \models f$  if and only if the empty marking  $\mathbf{0}$  is reachable by  $\mathcal{N}'$ . A computation of  $\mathcal{N}'$  can be divided in two phases: First,  $\mathcal{N}'$  guesses a sequence of markings  $\mu_1, \dots, \mu_n$  of  $\mathcal{N}$  such that:  $\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2 \cdots \mu_{n-1} \xrightarrow{\sigma_n} \mu_n$  is a computation of  $\mathcal{N}$  for some  $\sigma_1, \dots, \sigma_n \in T^+$ . Then, in the second phase,  $\mathcal{N}'$  checks for every  $i \in \{1, \dots, m\}$ , if  $\rho_i^-(\mu_1, \dots, \mu_n) \leq \rho_i^+(\mu_1, \dots, \mu_n)$  (i.e., the predicate  $\phi(\mu_1, \dots, \mu_n)$  is true).

The Petri net  $\mathcal{N}'$  contains all places of  $\mathcal{N}$ . In addition, the new places  $q_1, \dots, q_n$  and  $\bar{q}$  are added to  $\mathcal{N}'$  such that the total number of token in all these places is always less or equal to one. The sequence of places  $q_1, \dots, q_n$  is used during the first phase to guess the sequence of markings  $\mu_1, \dots, \mu_n$  of  $\mathcal{N}$ , while, the place  $\bar{q}$  is used during the second phase to check if the predicate  $\phi(\mu_1, \dots, \mu_n)$  is true for the guessed sequence of markings. Moreover, for every  $i \in \{1, \dots, m\}$ , the Petri net  $\mathcal{N}'$  has two places  $s_i^-$  and  $s_i^+$  to keep track (in some increasing way with respect to the sequence of guessed markings) of the value of  $\rho_i^-$  and  $\rho_i^+$ , respectively, such that a marking  $\mu \in \mathbb{N}^{P'}$  is reachable by  $\mathcal{N}'$ , if and only if one of the two following cases holds:

- *During the first phase:* If  $\mu(q_j) = 1$  for some  $j \in \{1, \dots, n\}$  (only one token in the place  $q_j$  and, consequently, the places  $q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_n$ , and  $\bar{q}$  are empty), then there is a sequence of markings  $\mu_1, \dots, \mu_{j-1} \in \mathbb{N}^P$  of  $\mathcal{N}$  such that:
  1.  $\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2 \cdots \mu_{j-1} \xrightarrow{\sigma_j} \mu|_P$  is a computation of  $\mathcal{N}$ , and
  2. for every  $i \in \{1, \dots, m\}$ , the number of tokens in the places  $s_i^-$  and  $s_i^+$  is  $\rho_i^-(\mu_1, \dots, \mu_{j-1}, \mu|_P, \dots, \mu|_P)$  and  $\rho_i^+(\mu_1, \dots, \mu_{j-1}, \mu|_P, \dots, \mu|_P)$ , respectively.
- *During the second phase:* If  $\mu(\bar{q}) = 1$  (only one token in the place  $\bar{q}$  and, consequently, the places  $q_1, \dots, q_n$  are empty), then there is a sequence of markings  $\mu_1, \dots, \mu_n \in \mathbb{N}^P$  of  $\mathcal{N}$  such that:
  1.  $\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2 \cdots \mu_{n-1} \xrightarrow{\sigma_n} \mu_n$  is a computation of  $\mathcal{N}$ ,
  2. for every place  $p \in P$ , the number of tokens in the place  $p$  is less or equal to  $\mu_n(p)$ , and
  3. for every  $i \in \{1, \dots, m\}$ , there is a nonnegative number  $c_i$  such that the number of tokens in  $s_i^-$  (resp.  $s_i^+$ ) is equal to  $\rho_i^-(\mu_1, \dots, \mu_n) - c_i$  (resp. less or equal to  $\rho_i^+(\mu_1, \dots, \mu_n) - c_i$ ).

Initially, the Petri net  $\mathcal{N}'$  has  $\mu_0(p)$  tokens in each place  $p \in P$ , one token in the place  $q_1$ , 0 token in the set of places  $q_2, \dots, q_n, \bar{q}$ , and for every  $i \in \{1, \dots, m\}$ , the places  $s_i^-$  and  $s_i^+$  have  $y_0^i$  and  $z_0^i$  tokens, respectively.

The set of transitions of  $\mathcal{N}'$  is defined in such a way that the above invariant is always preserved. Formally, the set of transitions of  $\mathcal{N}'$  is defined as the smallest set satisfying the following conditions:

- **The simulation of the first phase:**

- **Simulation of a computation of  $\mathcal{N}$  from  $\mu_{j-1}$  to  $\mu_j$ :** For every natural number  $j \in \{1, \dots, n\}$  and for every transition  $t \in T$ ,  $\mathcal{N}'$  has a transition  $t_j$  such that:
  1.  $F'(q_j, t_j) = F'(t_j, q_j) = 1$ ,  $F'(\bar{q}, t_j) = F'(t_j, \bar{q}) = 0$ , and  $F'(q_l, t_j) = F'(t_j, q_l) = 0$  for all  $l \in \{1, \dots, n\}$  and  $l \neq j$ . This means that in order to fire the transition  $t_j$ , the place  $q_j$  must contain one token.
  2. For every place  $p \in P$ ,  $F'(p, t_j) = F(p, t)$  and  $F'(t_j, p) = F(t, p)$ . This means that the transition  $t_j$  of  $\mathcal{N}'$  has the same effect over the set of places  $P$  as the transition  $t$  of  $\mathcal{N}$ .
  3. For every  $i \in \{1, \dots, m\}$ ,  $F'(s_i^-, t_j) = \sum_{p \in P} F(p, t) \sum_{k \geq j} (\mathbf{y}_k^i(p))$  and  $F'(t_j, s_i^-) = \sum_{p \in P} F(t, p) \sum_{k \geq j} (\mathbf{y}_k^i(p))$ . Hence, the invariant between the place  $s_i^-$  and the mapping  $\rho_i^-$  is preserved.
  4. For every  $i \in \{1, \dots, m\}$ ,  $F'(s_i^+, t_j) = \sum_{p \in P} F(p, t) \sum_{k \geq j} (\mathbf{z}_k^i(p))$  and  $F'(t_j, s_i^+) = \sum_{p \in P} F(t, p) \sum_{k \geq j} (\mathbf{z}_k^i(p))$ . Hence, the invariant between the place  $s_i^+$  and the mapping  $\rho_i^+$  is preserved.
- **Guessing the marking  $\mu_j$ :** For every natural number  $j \in \{1, \dots, n-1\}$  and for every transition  $t \in T$ ,  $\mathcal{N}'$  has a transition  $t_j^{j+1}$  such that:
  1.  $F'(q_j, t_j^{j+1}) = F'(t_j^{j+1}, q_{j+1}) = 1$ ,  $F'(\bar{q}, t_j^{j+1}) = F'(t_j^{j+1}, \bar{q}) = 0$ , and  $F'(q_l, t_j^{j+1}) = F'(t_j^{j+1}, q_{l'}) = 0$  for any  $l, l' \in \{1, \dots, n\}$ ,  $l \neq j$  and  $l' \neq j+1$ . This corresponds to moving the token from the place  $q_j$  to the place  $q_{j+1}$ .
  2. For every place  $p \in P$ ,  $F'(p, t_j^{j+1}) = F(p, t)$  and  $F'(t_j^{j+1}, p) = F(t, p)$ . This means that the transition  $t_j^{j+1}$  of  $\mathcal{N}'$  has the same effect over the set of places  $P$  as the transition  $t$  of  $\mathcal{N}$ .
  3. For every  $i \in \{1, \dots, m\}$ ,  $F'(s_i^-, t_j^{j+1}) = \sum_{p \in P} F(p, t) \sum_{k \geq j} (\mathbf{y}_k^i(p))$  and  $F'(t_j^{j+1}, s_i^-) = \sum_{p \in P} F(t, p) \sum_{k \geq j} (\mathbf{y}_k^i(p))$ . Hence, the invariant between the place  $s_i^-$  and the mapping  $\rho_i^-$  is preserved.
  4. For every  $i \in \{1, \dots, m\}$ ,  $F'(s_i^+, t_j^{j+1}) = \sum_{p \in P} F(p, t) \sum_{k \geq j} (\mathbf{z}_k^i(p))$  and  $F'(t_j^{j+1}, s_i^+) = \sum_{p \in P} F(t, p) \sum_{k \geq j} (\mathbf{z}_k^i(p))$ . Hence, the invariant between the place  $s_i^+$  and the mapping  $\rho_i^+$  is preserved.

Notice that firing the transition rule  $t_j^{j+1}$  in  $\mathcal{N}'$  simulates the firing of the transition rule  $t$  in  $\mathcal{N}$  over the set of places  $P$ . This guarantees that the guessed sequence of transitions  $\sigma_j$  contains at least one transition.

- **Guessing the marking  $\mu_n$ :** For every transition  $t \in T$ ,  $\mathcal{N}'$  has a transition  $t_n^{n+1}$  such that:
  1.  $F'(q_n, t_n^{n+1}) = F'(t_n^{n+1}, \bar{q}) = 1$ ,  $F'(t_n^{n+1}, q_n) = 0$ , and  $F'(q_l, t_n^{n+1}) = F'(t_n^{n+1}, q_l) = 0$  for all  $1 \leq l < n$ . This corresponds to moving the token from the place  $q_n$  to the place  $\bar{q}$ .
  2. For every place  $p \in P$ ,  $F'(p, t_n^{n+1}) = F(p, t)$  and  $F'(t_n^{n+1}, p) = F(t, p)$ . This means that the transition  $t_n^{n+1}$  of  $\mathcal{N}'$  has the same effect over the set of places  $P$  as the transition  $t$  of  $\mathcal{N}$ .

3. For every  $i \in \{1, \dots, m\}$ ,  $F'(s_i^-, t_n^{n+1}) = \sum_{p \in P} F(p, t) \sum_{k \geq j} (\mathbf{y}_k^i(p))$  and  $F'(t_n^{n+1}, s_i^-) = \sum_{p \in P} F(t, p) \sum_{k \geq j} (\mathbf{y}_k^i(p))$ . Hence, the invariant between the place  $s_i^-$  and the mapping  $\rho_i^-$  is preserved.
4. For every  $i \in \{1, \dots, m\}$ ,  $F'(s_i^+, t_n^{n+1}) = \sum_{p \in P} F(p, t) \sum_{k \geq j} (\mathbf{z}_k^i(p))$  and  $F'(t_n^{n+1}, s_i^+) = \sum_{p \in P} F(t, p) \sum_{k \geq j} (\mathbf{z}_k^i(p))$ . Hence, the invariant between the place  $s_i^+$  and the mapping  $\rho_i^+$  is preserved.

– **Simulation of the second phase:**

- **Decreasing the number of tokens in each place of  $P$ :** For every  $p \in P$ ,  $\mathcal{N}'$  has a transition  $t_p$  such that  $F'(\bar{q}, t_p) = F'(t_p, \bar{q}) = F'(p, t_p) = 1$  and 0 otherwise.
- **Decreasing the number of tokens in each place  $s_i^+$ :** For every  $i \in \{1, \dots, m\}$ ,  $\mathcal{N}'$  has a transition  $t_i^+$  such that  $F'(\bar{q}, t_i^+) = F'(t_i^+, \bar{q}) = 1$ ,  $F'(s_i^+, t_i^+) = 1$ , and 0 otherwise.
- **Decreasing the number of tokens in each place  $s_i^-$ :** For every  $i \in \{1, \dots, m\}$ ,  $\mathcal{N}'$  has a special transition  $t_i^-$  such that  $F'(\bar{q}, t_i^-) = F'(t_i^-, \bar{q}) = 1$ ,  $F'(s_i^-, t_i^-) = F'(s_i^+, t_i^-) = 1$ , and 0 otherwise. Notice that, while decrementing the number of tokens in  $s_i^-$ , we decrease also the number of tokens in  $s_i^+$  by one.
- **The end of the second phase:**  $\mathcal{N}'$  has a transition  $t_{\text{end}}$  such that  $F'(\bar{q}, t_{\text{end}}) = 1$  and 0 otherwise.

Then, Theorem 3 is an immediate consequence of the following lemma:

**Lemma 4.** *The marking  $\mathbf{0}$  is reachable in  $\mathcal{N}'$  if and only if  $\mathcal{N}$  satisfies  $f$ .*

Hence, the satisfiability problem for the class of path formulas is polynomially reducible to the reachability problem for Petri nets. As an immediate consequence of Theorem 2 and 3, we get the following result:

**Corollary 3.** *The satisfiability problem for the class of path formulas is as hard as the reachability problem for Petri nets.*

## 6 An EXPSPACE-complete fragment

In this section we consider a fragment of Yen's path logic for which we can show that its satisfiability problem is EXPSPACE-complete. The proof follows very closely Yen's proof [13] which is a generalization of Rackoff's proof [12] for the complexity of the boundedness problem. The basic idea is to show that if a path satisfying a formula exists, then there is a short one. This is done by induction on the number of places of the Petri Net. However we have to modify one crucial lemma whose proof in the paper of Yen [13] contains an error. To correct the lemma, Yen's logic has to be restricted. The restriction makes sure that if there is a path showing that a formula is satisfiable, then there is also a path starting at each intermediate marking of the path which satisfies the formula. This is achieved by requiring the last designated marking of the path to be bigger than the first designated marking. Formally,

**Definition 1.** A path formula  $f$  of the form

$$\exists \mu_1, \dots, \mu_n \exists \sigma_1, \dots, \sigma_n ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \dots \mu_{n-1} \xrightarrow{\sigma_n} \mu_n) \wedge \phi(\mu_1, \dots, \mu_n, \sigma_1, \dots, \sigma_n))$$

is called increasing if  $\phi(\mu_1, \dots, \mu_n, \sigma_1, \dots, \sigma_n)$  does not contain transition predicates and implies  $\mu_n \geq \mu_1$ .

Notice that for  $n = 1$ ,  $\mu_n \geq \mu_1$  is always true and that an increasing path formula can also be written as  $\exists \mu_1, \dots, \mu_n \exists \sigma_1, \dots, \sigma_n ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \dots \mu_{n-1} \xrightarrow{\sigma_n} \mu_n) \wedge \phi(\mu_1, \dots, \mu_n))$ .

For the rest of the section we consider increasing path formulas. We can suppose furthermore that the formulas  $\phi$  are conjunctions of marking predicates, since disjunctions can be considered separately. We first give some additional definitions. Given a predicate  $\phi$  and a set of positive integers  $D$  we define  $\phi^{[D]}$  to be the predicate resulting from removing all marking predicates of the form  $\mu_i(p) \geq c$  and  $\mu_i(p) > c$  from  $\phi$  for all  $i \notin D$ . Let  $(P, T, F, \mu_0)$  be a Petri Net with  $k$  places. We suppose an ordering on  $P$  and  $T$  and can then suppose that markings are vectors of  $\mathbb{N}^k$ .

The *transition vector* of a transition  $t$ , denoted by  $\hat{t}$  is a  $k$ -dimensional vector with  $\hat{t}(i) = F(t, p_i) - F(p_i, t)$  for all  $i$  with  $1 \leq i \leq k$ . The set of transition vectors, denoted by  $\hat{T}$  is  $\{\hat{t} \mid t \in T\}$ . A *generalized marking* is a mapping from  $P$  to  $\mathbb{Z}$  (i.e. a vector of  $\mathbb{Z}^k$ ). A *generalized firing sequence* is any sequence of transitions of  $T$ . A finite sequence of vectors  $w_1, \dots, w_m \in \mathbb{Z}^k$  is said to be a *path* (of length  $m - 1$ ) if  $w_1 = \mu_0$  and  $w_{i+1} - w_i \in \hat{T}$  for all  $i$  with  $1 \leq i < m$ . A path  $w_1, \dots, w_m$  corresponds to at least one generalized firing sequence  $t_1, \dots, t_{m-1}$  such that  $w_{i+1} - w_i = \hat{t}_i$  for all  $i$  with  $1 \leq i < m$ . Let  $w \in \mathbb{Z}^k$ . The vector  $w$  is  *$i$  bounded* if  $w(j) \geq 0$  for  $1 \leq j \leq i$ . If  $r \in \mathbb{N}^+$  is such that  $0 \leq w(j) < r$  for  $1 \leq j \leq i$ , then  $w$  is called  *$i$ - $r$  bounded*. A path  $p = w_1, \dots, w_m \in \mathbb{Z}^k$  is called  *$i$  bounded* ( *$i$ - $r$  bounded*) if each  $w_j$  in  $p$  is  $i$  bounded ( $i$ - $r$  bounded). Given a predicate  $\phi(\mu_1, \dots, \mu_n)$ , an  $i$  bounded ( $i$ - $r$  bounded) path  $w_1, \dots, w_m$  is called an  *$i$  bounded  $\phi$  path* if  $\exists 1 \leq j_1 \leq j_2 \leq \dots \leq j_n = m$  such that  $\phi^{[1, \dots, i]}(w_{j_1}, w_{j_2}, \dots, w_{j_n})$  is true. Let  $m'(i, \mu, \phi)$  be either the length of the shortest  $i$  bounded  $\phi$  path whose initial generalized marking is  $\mu$ , or 0 if it does not exist. Let  $g(i, \phi) = \max\{m'(i, \mu, \phi) \mid \mu \in \mathbb{Z}^k\}$ . We have  $g(i, \phi) \in \mathbb{N}$  (see [13]).

The following two lemmas are from [13].

**Lemma 5.** If there is an  $i$ - $r$  bounded  $\phi$ -path in the Petri Net  $(P, T, F, \mu_0)$ , then there is an  $i$ - $r$  bounded  $\phi$ -path of length  $\leq r^{(s(\mathcal{N}))^c}$ , for some constant independent of  $r$  and  $s(\mathcal{N})$ .

We derive  $g(i, \phi)$  recursively.

**Lemma 6.**  $g(0, \phi) \leq 2^{(s(\mathcal{N}))^c}$ , for some constant  $c$  independent of  $s(\mathcal{N})$ .

**Lemma 7.**  $g(i+1, \phi) \leq (2^{(s(\mathcal{N}))} (g(i, \phi) + 1))^{(s(\mathcal{N}))^c}$  for all  $i < k$ , where  $c$  is a constant independent of  $s(\mathcal{N})$ .

**Proof:**

- Case 1. If there is an  $(i+1) \cdot 2^{(s(\mathcal{N}))} (g(i, \phi) + 1)$  bounded  $\phi$  path, then using Lemma 5, there exists a short one with length  $\leq (2^{(s(\mathcal{N}))} (g(i, \phi) + 1))^{(s(\mathcal{N}))^c}$ .

– Case 2. Otherwise, let  $v_1, \dots, v_{m_0}, v_{m_0+1}, \dots, v_m$  be an  $(i+1)$  bounded  $\phi$  path such that  $v_{m_0}$  is the first vector not  $(i+1) - 2^{(s(\mathcal{N}))}(g(i, \phi) + 1)$  bounded. Without loss of generality, we assume that  $v_{m_0}(i+1) > 2^{(s(\mathcal{N}))}(g(i, \phi) + 1)$ . Furthermore we assume that no two of  $v_1, \dots, v_{m_0}$  can agree on the first  $i+1$  positions, otherwise the path could be made shorter. Therefore  $m_0 \leq (2^{(s(\mathcal{N}))}(g(i, \phi) + 1))^{i+1}$ . Now we show that if we take as initial marking  $v_{m_0}$ , there is an  $i$  bounded  $\phi$  path in the Petri Net<sup>3</sup>. There are two cases depending on  $\phi$ .

1.  $\phi$  is of the form  $\phi(\mu_1)$ . In this case, since  $v_1, \dots, v_m$  is an  $i+1$  bounded  $\phi$  path and  $\phi$  is just a predicate on the marking  $\mu_1$ ,  $v_{m_0}, v_{m_0+1}, \dots, v_m$  is clearly an  $i$  bounded  $\phi$  path.
2.  $\phi$  is of the form  $\phi(\mu_1, \dots, \mu_n)$  and it implies  $\mu_n \geq \mu_1$ . Since  $v_1, \dots, v_{m_0}, v_{m_0+1}, \dots, v_m$  is an  $(i+1)$  bounded  $\phi$  path it is an  $i$  bounded  $\phi$  path as well. Therefore  $\exists 1 \leq j_1 \leq j_2 \leq \dots \leq j_n = m$  such that  $\phi^{\{1, \dots, i\}}(v_{j_1}, v_{j_2}, \dots, v_{j_n})$  is true. Furthermore  $v_{j_n} \geq v_{j_1}$ . Let  $s' = t'_1, \dots, t'_o$  be a sequence of transitions corresponding to the path  $v_{j_1}, v_{j_1+1}, \dots, v_{j_n}$ . Let  $s = t_{m_0}, \dots, t_{m-1}$  be a sequence of transitions corresponding to the path  $v_{m_0}, v_{m_0+1}, \dots, v_m$ . Then  $ss'$  is a sequence of transitions corresponding to a path  $v_{m_0}, \dots, v_m, v'_{j_1+1}, \dots, v'_{j_n}$  where  $v'_i = v_i + v_{j_n} - v_{j_1}$  for all  $i$  such that  $j_1 + 1 \leq i \leq j_n$ . Clearly the path is an  $i$  bounded  $\phi$  path starting from  $v_{m_0}$  (since  $\phi^{\{1, \dots, i\}}(v_m, v'_{j_2}, \dots, v'_{j_n})$  is true, because all predicates stay true when adding to all markings the same positive vector).

Now, we can take the shortest  $i$  bounded  $\phi$  path  $p$  in  $(P, T, F, v_{m_0})$ . It's length is  $\leq g(i, \phi)$ . As  $v_{m_0}(i+1) > 2^{(s(\mathcal{N}))}(g(i, \phi) + 1)$  and each place of each transition vector in the Petri Net is at most  $2^{(s(\mathcal{N}))}$  in absolute value,  $p$  is also  $i+1$  bounded and the  $(i+1)$  position will never fall below  $2^{(s(\mathcal{N}))}$  in  $p$  (so that marking predicates of the form  $\mu_i(p') \geq c$  and  $\mu_i(p') > c$  will still hold in  $p$ ). Therefore  $v_1, \dots, v_{m_0-1}, p$  is an  $(i+1)$  bounded  $\phi$  path of length  $(2^{(s(\mathcal{N}))}(g(i, \phi) + 1))^{i+1} + g(i, \phi) < (2^{(s(\mathcal{N}))}(g(i, \phi) + 1))^{(s(\mathcal{N}))^c}$ .

□

The following theorem now follows easily [13] from the bound on  $g$ .

**Theorem 4.** *The satisfiability problem for increasing path formulas can be decided in  $O(2^{d \cdot s(\mathcal{N})} \cdot \log(s(\mathcal{N})))$  space, for some constant  $d$  independent of  $s(\mathcal{N})$ .*

Since unboundedness can be expressed in the logic and boundedness is EXPSPACE-hard [9] we have the following:

**Theorem 5.** *The satisfiability problem for increasing path formulas is EXPSPACE-complete.*

<sup>3</sup> At this point, there is a mistake in the proof of [13] (Lemma 3.7, page 130), as it assumes that this path always exists and takes the shortest one. However *no*  $i$  bounded  $\phi$  path might exist starting from  $v_{m_0}$ .

## 6.1 Some applications

In the following we consider the applications given in [13] and discuss if they are in the increasing fragment. The following six problems are all in the fragment and therefore in EXPSPACE. They have already been shown to be in EXPSPACE before Yen's paper.

1. *Boundedness problem.* Unboundedness of a Petri Net can be formulated as  $\exists \mu_1, \mu_2 \exists \sigma_1, \sigma_2 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2) \wedge (\mu_2 > \mu_1))$  which is clearly an increasing path formula.
2. *Coverability.* It can be formulated as  $\exists \mu_1, \exists \sigma_1 ((\mu_0 \xrightarrow{\sigma_1} \mu_1) \wedge (\mu_1 \geq v))$  which is an increasing path formula.
3. *(Strict) Self-Coverability Problem.* It can be solved by considering formulas of the form  $\exists \mu_1, \mu_2 \exists \sigma_1, \sigma_2 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2) \wedge ((\bigwedge_{s \in I} \mu_2(s) \geq \mu_1(s)) \wedge (\bigwedge_{s' \notin I} \mu_2(s') = \mu_1(s'))))$  where  $I$  is a set of places (For strict self-coverability, replace  $\geq$  by  $>$ ). The formulas are clearly increasing path formulas.
4. *u-Self-Coverability Problem.* This can be solved by considering formulas of the form  $\exists \mu_1, \mu_2 \exists \sigma_1, \sigma_2 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2) \wedge (\mu_2 - \mu_1 = u))$  where  $u \in \mathbb{N}^k$ . These formulas are increasing.
5. *Final-State Self-Coverability Problem.* This can be solved by considering formulas of the form  $\exists \mu_1, \mu_2, \mu_3 \exists \sigma_1, \sigma_2, \sigma_3 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2 \xrightarrow{\sigma_3} \mu_3) \wedge (\mu_3 \geq \mu_1) \wedge (\bigvee_{s \in F} \mu_2(s) > 0))$  for some set  $F$  of places. This formula is increasing.
6. *Fair Nontermination Problems.* All the formulas considered for these problems are of the form  $\exists \mu_1, \mu_2 \exists \sigma_1, \sigma_2 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2) \wedge (\mu_2 \geq \mu_1) \wedge \varphi(\sigma_1, \sigma_2))$  where  $\varphi(\sigma_1, \sigma_2)$  is a formula containing only transition predicates. By carefully inspecting this transition predicates one can easily see that eliminating them with Lemma 1 yields increasing formulas.

The following problems were claimed to be in EXPSPACE in [13].

1. *Regularity Detection Problem.* Nonregularity of a Petri Net is equivalent to the satisfiability of the following path formula:

$$\exists \mu_1, \mu_2, \mu_3, \mu_4 \exists \sigma_1, \sigma_2, \sigma_3, \sigma_4 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2 \xrightarrow{\sigma_3} \mu_3 \xrightarrow{\sigma_4} \mu_4) \wedge \varphi(\mu_1, \mu_2, \mu_3, \mu_4))$$

where  $\varphi(\mu_1, \mu_2, \mu_3, \mu_4)$  is  $(\mu_2 \geq \mu_1) \wedge (\bigvee_{i=1}^k \mu_2(i) > \mu_1(i)) \wedge (\bigwedge_{i=1}^k (\mu_1(i) < \mu_2(i)) \vee (\mu_3(i) \leq \mu_4(i)) \wedge (\bigvee_{i=1}^k \mu_3(i) > \mu_4(i)))$ . Unfortunately, this formula is not increasing and we can not apply our complexity result. To the best of our knowledge the complexity of regularity is therefore still unknown.

2. *(Potential) Determinism Detection Problem.* Nondeterminism of a Petri Net can be expressed using the formula  $\exists \mu_1, \exists \sigma_1 ((\mu_0 \xrightarrow{\sigma_1} \mu_1) \wedge ((\bigvee_{t, t', t \neq t'} (\mu_1 \geq v_t) \wedge (\mu_1 \geq v_{t'})))$  where the  $v_t$  are the minimal vectors for which  $t$  is enabled. Clearly, the formula is increasing. Non potential determinism can then be expressed as  $\exists \mu_1, \mu_2, \exists \sigma_1, \sigma_2 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2) \wedge ((\bigvee_{t, t', t \neq t'} (\mu_1 \geq v_t) \wedge (\mu_1 \geq v_{t'})) \wedge (\mu_2 \geq \mu_1)))$ . This formula is increasing and therefore the problem is in EXPSPACE.
3. *Frozen Token Detection Problem.* To decide if a Petri Net has a frozen token it is sufficient to check the formula  $\exists \mu_1, \mu_2, \exists \sigma_1, \sigma_2 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2) \wedge (\mu_1(p) >$

- 0)  $\wedge (\mu_2 \geq \mu_1) \wedge (\sigma_2 \neq \Lambda)$ ) where  $p$  is a designated place and  $\sigma_2 \neq \Lambda$  denotes  $\bigvee_{t \in T} \#_{\sigma_2}(t) > 0$ . Eliminating with Lemma 1 the transition predicates yields an increasing path formula and therefore the problem is in EXPSPACE.
4. *(Strong) Promptness Detection.* A Petri Net is not (strongly) prompt if and only if  $\exists \mu_1, \mu_2, \exists \sigma_1, \sigma_2 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2) \wedge ((\bigwedge_{t \in T_1} \#_{\sigma_2}(t) \leq 0) \wedge (\mu_2 \geq \mu_1) \wedge (\sigma_2 \neq \Lambda)))$  is true. Again eliminating with Lemma 1 the transition predicates yields an increasing path formula and therefore the problem is in EXPSPACE.
5. *y-Synchronization Problem.* Given a map  $y$  from the transitions  $T$  to  $\mathbb{Z}$ , a Petri net is not  $y$ -synchronized iff  $\exists \mu_1, \mu_2, \exists \sigma_1, \sigma_2 ((\mu_0 \xrightarrow{\sigma_1} \mu_1 \xrightarrow{\sigma_2} \mu_2) \wedge ((\bigwedge_{t \in T_1} \#_{\sigma_1}(t) \leq 0) \wedge (((\sum_{t \in T} y(t) \#(\sigma_2)(t) > 0) \vee (\sum_{t \in T} y(t) \#(\sigma_2)(t) < 0)) \wedge (\mu_2 \geq \mu_1))))$  is true. While eliminating with Lemma 1 the transition predicates we notice that the newly added places are always increasing. Thus this yields an increasing path formula and therefore the problem is in EXPSPACE.

## 7 Conclusion

In this paper, we have shown that the satisfiability problem for the class of path formulas considered by Yen [13] is as hard as the reachability problem for Petri nets. However for an important fragment we have shown that its satisfiability problem is EXPSPACE-complete. By doing this, we have corrected the proof given in [13]. Furthermore we show that almost all applications considered by Yen can be solved using our fragment. However, the exact complexity of the regularity detection problem remains open. It would be interesting to obtain a bigger fragment which is in EXPSPACE allowing to show the EXPSPACE complexity of the regularity detection problem.

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