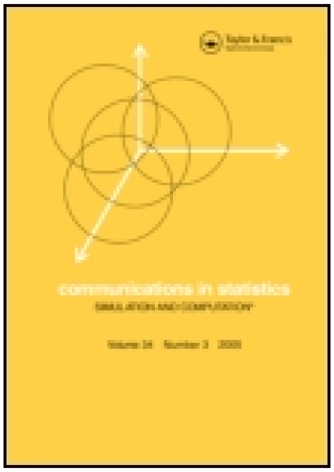

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One- and Two-Sample Bayesian Prediction Intervals Based on Type-I Hybrid Censored Data

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In this article, we consider a general form for the underlying distribution and a general conjugate prior, and describe a general procedure for determining the Bayesian prediction intervals for future lifetimes based on an observed Type-I hybrid censored data. For the illustration of the developed results, the Exponential(θ) and Pareto(α, β) distributions are used as examples. One-sample Bayesian predictive survival function can not be obtained in closed-form and so Gibbs sampling procedure is used to draw Markov Chain Monte Carlo (MCMC) samples, which are then used to compute the approximate predictive survival function. Finally, some numerical results are presented to illustrate all the inferential results developed here.

Keywords Bayesian prediction; Exponential distribution; Markov Chain Monte Carlo; Order statistics; Pareto distribution; Type-I hybrid censored sample.

Mathematics Subject Classification Primary 62G30; Secondary 62F15.

1. Introduction

In reliability analysis, experiments often terminate before all units on test have failed due to cost and time considerations. In such cases, failure information is available only on part of the sample, and on all units that had not failed, only partial information will be available. Such data are said to be censored. The two most common censoring schemes are Type-I and Type-II censoring schemes. They can be described as follows. Consider n identical units on a life-testing experiment. In the Type-I censoring scheme, the experiment is terminated when a pre-fixed censoring time T ia reached. On the other hand, in the Type-II censoring scheme, the experiment gets terminated when a pre-specified number $r \leq n$ of failures is observed. Under both censoring schemes, some information is lost since only a part of the sample is observed, but they do result in a saving in terms of time and cost. In the Type-I censoring scheme, the duration of the test is guaranteed but

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the level of efficiency may be too low or too high due to the uncertainty in the number of complete failures. In the Type-II censoring scheme, the level of efficiency is guaranteed (since the number of failures to be observed is fixed in advance), but the duration of the experiment may end up being too long since the exact time of the r-th failure is random. For these reasons, another censoring scheme becomes necessary if both efficiency level and guaranteed duration are sought.

A mixture of Type-I and Type-II censoring schemes, known as hybrid censoring scheme, has been discussed in the literature for this purpose. In the Type-I hybrid censoring scheme, the experiment is terminated as soon as a pre-specified number r out of n items has failed or a pre-fixed time T on test has been reached. In contrast, in the Type-II hybrid censoring scheme, the life-testing experiment gets terminated whenever the later of the two stopping rules is reached; see Childs et al. (2003). Type-I hybrid censoring has been discussed extensively in the reliability literature; see, for example, Epstein (1954), MIL-STD-781 C (1977), Chen and Bhattacharyya (1988), Gupta and Kundu (1998), Kundu (2007), Park et al. (2008), and Park and Balakrishnan (2009).

Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics (OS) from a random sample of size n from an absolutely continuous distribution function $F(x) \equiv F(x \mid \theta)$ with density function $f(x) \equiv f(x \mid \theta)$, where the parameter $\theta \in \Theta$ may be a real vector. Let K denote the number of $X_{i:n}$'s that are at most T. Then, K is a discrete random variable with support $\{0, 1, \ldots, n\}$ and probability density function (pdf) as

$$P(K = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, \ldots, n,$$

where $p = F(T)$ and $q = 1 - p = 1 - F(T)$.

Therefore, under the Type-I hybrid censoring scheme described above, we have one of the two following types of observations:

Case I. $X_{1:n} < \cdots < X_{r:n}$ if $X_{r:n} \leq T$ with $r \leq K \leq n$;

Case II. $X_{1:n} < \cdots < X_{K:n}$ if $T < X_{r:n}$ with $0 \leq K \leq r - 1$.

Thus, the likelihood function of such a Type-I hybrid censored sample is as follows:

Case I.

$$L_1(\theta; \mathbf{x}_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i)[1 - F(x_r)]^{n-r}, \tag{1.1}$$

where $\mathbf{x}_r = (x_1, \ldots, x_r)$ and $x_1 < \cdots < x_r \leq T$;

Case II.

$$L_2(\theta; \mathbf{x}_K) = \frac{n!}{(n-K)!} \prod_{i=1}^{K} f(x_i)[1 - F(T)]^{n-K}, \tag{1.2}$$

where $\mathbf{x}_K = (x_1, \ldots, x_K)$ and $x_1 < \cdots < x_K \leq T < x_{K+1}$.

We will use the following Lemma to develop the main results presented in the following sections.

Lemma 1.1.

1. *Conditional on* $K = k$, *the vectors* $(X_{1:n}, \dots, X_{k:n})$ *and* $(X_{k+1:n}, \dots, X_{n:n})$ *are mutually independent with*

$$(X_{1:n}, \dots, X_{k:n}) \stackrel{d}{=} (V_{1:k}, \dots, V_{k:k}),$$
$$(X_{k+1:n}, \dots, X_{n:n}) \stackrel{d}{=} (W_{1:n-k}, \dots, W_{n-k:n-k}),$$

   where $V_{1:k}, \dots, V_{k:k}$ *are OS from an iid sample of size* k *from* $F(x)$ *right-truncated at* T, *and* $W_{1:n-k}, \dots, W_{n-k:n-k}$ *are OS from an iid sample of size* $n-k$ *from* $F(x)$ *left-truncated at* T;
2. *Conditional on* $K = k$, *the conditional pdf of* $X_{s:n}$, *given* $X_{1:n} = x_1, \dots, X_{r:n} = x_r$ $(X_{r:n} < X_{s:n} \le T, r < s \le n)$, *is the same as the conditional pdf of* $X_{s:k}$, *given* $X_{r:k} = x_r$, *from a random sample of size* k *from* $F(x)$ *right-truncated at* T;
3. *Conditional on* $K = k$, *the conditional pdf of* $X_{s:n}$, *given* $X_{1:n} = x_1, \dots, X_{r:n} = x_r$ $(X_{r:n} \le T < X_{s:n}, r < s \le n)$, *is the same as the marginal pdf of the* (*s-k*)*-th order statistic from a random sample of size* $n-k$ *from* $F(x)$ *left-truncated at* T;
4. *Conditional on* $K = k$, *the conditional pdf of* $X_{s:n}$, *given* $X_{1:n} = x_1, \dots, X_{k:n} = x_k$ $(X_{s:n} > T, k+1 \le s \le n)$, *is the same as the marginal pdf of the* (*s-k*)*-th order statistic from a random sample of size* $n-k$ *from* $F(x)$ *left-truncated at* T.

For a proof of this result as well as some generalizations of this result, one may refer to Iliopoulos and Balakrishnan (2009).

When $r < s \le n$, by using the above Lemma, the conditional density function of $X_{s:n}$, given the Type-I hybrid censored data, is obtained readily as follows:

Case I.

$$f_1(x_s \mid \mathbf{x}_r) = \begin{cases} f_{11}(x_s \mid \mathbf{x}_r), & x_r < x_s \le T, \\ f_{12}(x_s \mid \mathbf{x}_r), & x_s > T, \end{cases} \tag{1.3}$$

where

$$\begin{aligned} f_{11}(x_s \mid \mathbf{x}_r) &= \frac{1}{P(r \le K \le n)} \sum_{k=s}^{n} f(x_s \mid \mathbf{x}_r, K = k) P(K = k) \\ &= \sum_{k=s}^{n} \frac{(k-r)!\phi_k(T)}{(s-r-1)!(k-s)!} \frac{[F(x_s) - F(x_r)]^{s-r-1}[F(T) - F(x_s)]^{k-s} f(x_s)}{[F(T) - F(x_r)]^{k-r}}, \quad x_r < x_s \le T, \end{aligned}$$

and

$$\begin{aligned} f_{12}(x_s \mid \mathbf{x}_r) &= \frac{1}{P(r \le K \le n)} \sum_{k=r}^{s-1} f(x_s \mid \mathbf{x}_r, K = k) P(K = k) \\ &= \sum_{k=r}^{s-1} \frac{(n-k)!\phi_k(T)}{(s-k-1)!(n-s)!} \frac{[F(x_s) - F(T)]^{s-k-1}[1 - F(x_s)]^{n-s} f(x_s)}{[1 - F(T)]^{n-k}}, \quad x_s > T, \end{aligned}$$

with

$$\phi_k(T) = \frac{P(K = k)}{\sum_{j=r}^{n} P(K = j)};$$

Case II.

$$f_2(x_s \mid \mathbf{x}_K) = \frac{1}{P(0 \leq K \leq r-1)} \sum_{k=0}^{r-1} f(x_s \mid \mathbf{x}_k, K = k) P(K = k)$$
$$= \sum_{k=0}^{r-1} \frac{(n-k)!\psi_k(T)}{(s-k-1)!(n-s)!} \frac{[F(x_s) - F(T)]^{s-k-1}[1 - F(x_s)]^{n-s} f(x_s)}{[1 - F(T)]^{n-k}}, \quad x_s > T, \tag{1.4}$$

where

$$\psi_k(T) = \frac{P(K = k)}{\sum_{j=0}^{r-1} P(K = j)}.$$

Prediction of future events on the basis of the past and present knowledge is a fundamental problem of statistics, arising in many contexts in a natural way. As in the case of estimation, a predictor can be either a point or an interval predictor. Several researchers have considered Bayesian prediction for future observations based on Type-I censored data; see AL-Hussaini (1999a) and AL-Hussaini et al. (2001). Bayesian prediction bounds for future observations based on Type-II censored data have been discussed by several authors, including Dunsmore (1974), Nigm and Hamdy (1987), Nigm (1988, 1989), AL-Hussaini and Jaheen (1995), AL-Hussaini (1999b), and Raqab and Madi (2005). Draper and Guttman (1987) discussed the two-sample Bayesian prediction of the future lifetime of an item based on a Type-I hybrid censored data from an exponential distribution. Ebrahimi (1992) developed the classical prediction intervals for future failures in the case of exponential distribution under Type-I hybrid censoring. Recently, Balakrishnan and Shafay (2011) considered a general form for the underlying distribution and a general conjugate prior and developed a general procedure for determining the one- and two-sample Bayesian prediction intervals for future lifetimes based on a Type-II hybrid censored data. In this paper, we discuss the same problem based on a Type-I hybrid censored data which involves some additional complications.

The rest of this article is organized as follows. In Sec. 2, we present the structure of the prior and posterior distributions. In Sec. 3, we derive the one-sample Bayesian predictive survival function and the one-sample Bayesian prediction bounds for the s-th $(r < s \leq n)$ ordered lifetime from Type-I hybrid censored sample. Next, we derive the two-sample Bayesian predictive survival function and the two-sample Bayesian prediction bounds for the s-th ordered lifetime from a future independent sample when the (observed) informative sample is a Type-I hybrid censored and the (unobserved) future sample is a complete sample from the same parent distribution. In Sec. 4, we present the results for the Exponential(θ) and Pareto(α, β) distributions as illustrative examples, wherein we adopt the Markov Chain Monte Carlo method to compute the approximate predictive survival function in the one-sample case. Finally, in Sec. 5, we present some numerical results for illustrating all the inferential methods developed here.

2. Prior and Posterior Distributions

Since the survival function (SF) $\overline{F}(x \mid \theta) = 1 - F(x \mid \theta)$ corresponding to any cumulative distribution function (CDF) $F(x \mid \theta)$, $\theta \in \Theta$, can be written in the form

$$\overline{F}(x \mid \theta) = \exp[-\lambda(x; \theta)], \tag{2.1}$$

where $\lambda(x; \theta) = -\ln \overline{F}(x \mid \theta)$, we shall consider the underling population SF to be given by (2.1). Of course, some conditions need to be imposed so that $\overline{F}(x \mid \theta)$ is a valid SF. These conditions are: $\lambda(x; \theta)$ is continuous, monotone increasing and differentiable function, with $\lambda(x; \theta) \to 0$ as $x \to -\infty$ and $\lambda(x; \theta) \to \infty$ as $x \to \infty$. The probability density function (pdf) corresponding to (2.1) is given by

$$f(x \mid \theta) = \lambda'(x; \theta) \exp[-\lambda(x; \theta)], \tag{2.2}$$

where $\lambda'(x; \theta)$ is the derivative of $\lambda(x; \theta)$ with respect to x.

With an appropriate choice of $\lambda(x; \theta)$ (notice that the derivative of $\lambda(x; \theta)$ with respect to x is the hazard rate function), several distributions that are used in reliability studies can be obtained as special cases. For example, if $\lambda(x; \theta) = \theta x$, we obtain the Exponential(θ) distribution. If $\lambda(x; \theta) = -\alpha \ln(\beta/x)$, we obtain the Pareto($\alpha, \beta$) distribution. If $\lambda(x; \theta) = \alpha x^{\beta}$, we obtain the Weibull($\alpha, \beta$) distribution. The Burr Type XII(α, β) distribution is obtained by taking $\lambda(x; \theta) = \alpha \ln(1 + x^{\beta})$. Appropriate conditions need to be imposed on $\lambda(x; \theta)$ to suit the domain on which $\overline{F}(x \mid \theta)$ in (2.1) is defined. For example, if $\overline{F}(x \mid \theta)$ is defined only on the positive half of the real line (as for the Exponential, Weibull and Burr Type XII distributions), then $\lambda(x; \theta) \to 0$ as $x \to 0^{+}$ and $\lambda(x; \theta) \to \infty$ as $x \to \infty$. If $\overline{F}(x \mid \theta)$ is defined on (β, ∞) (as in the Pareto distribution), then $\lambda(x; \theta) \to 0$ as $x \to \beta^{+}$ and $\lambda(x; \theta) \to \infty$ as $x \to \infty$. The exponential form of the SF in (2.1) provides some flexibility in developing general results, as carried out in the following sections.

Upon using (2.1) and (2.2) in (1.1) and (1.2), we obtain the likelihood function as follows:

Case I.

$$L_1(\theta; \mathbf{x}_r) = \frac{n!}{(n-r)!} \left(\prod_{i=1}^{r} \lambda'(x_i; \theta) \right) \exp\left[-\sum_{i=1}^{r} \lambda(x_i; \theta) - (n-r)\lambda(x_r; \theta) \right]; \tag{2.3}$$

Case II.

$$L_2(\theta; \mathbf{x}_K) = \frac{n!}{(n-K)!} \left(\prod_{i=1}^{K} \lambda'(x_i; \theta) \right) \exp\left[-\sum_{i=1}^{K} \lambda(x_i; \theta) - (n-K)\lambda(T; \theta) \right]. \tag{2.4}$$

From the Bayesian viewpoint, the unknown parameter is regarded as a realization of a random variable, which has some prior distribution. We consider here a general conjugate prior, suggested by AL-Hussaini (1999b), that is given by

$$\pi(\theta; \delta) \propto C(\theta; \delta) \exp[-D(\theta; \delta)], \tag{2.5}$$

where $\theta \in \Theta$ is the vector of parameters of the distribution in (2.1) and δ is the vector of prior parameters. The prior family in (2.5) includes several priors used in the literature as special cases.

Then, from (2.3), (2.4), and (2.5), the posterior density function of θ, given the Type-I hybrid censored data, is readily obtained as follows:

Case I.

$$\pi_1^*(\theta \mid \mathbf{x}_r) = I_1^{-1}\eta_1(\theta; \mathbf{x}_r)\exp[-\zeta_1(\theta; \mathbf{x}_r)], \tag{2.6}$$

where

$$\eta_1(\theta; \mathbf{x}_r) = C(\theta; \delta)\prod_{i=1}^{r}\lambda'(x_i; \theta), \quad \zeta_1(\theta; \mathbf{x}_r) = \sum_{i=1}^{r}\lambda(x_i; \theta) + (n-r)\lambda(x_r; \theta) + D(\theta; \delta)$$

and

$$I_1 = \int_{\theta\in\Theta}\eta_1(\theta; \mathbf{x}_r)\exp[-\zeta_1(\theta; \mathbf{x}_r)]d\theta;$$

Case II.

$$\pi_2^*(\theta \mid \mathbf{x}_K) = I_2^{-1}\eta_2(\theta; \mathbf{x}_K)\exp[-\zeta_2(\theta; \mathbf{x}_K)], \tag{2.7}$$

where

$$\eta_2(\theta; \mathbf{x}_K) = C(\theta; \delta)\prod_{i=1}^{K}\lambda'(x_i; \theta), \quad \zeta_2(\theta; \mathbf{x}_K) = \sum_{i=1}^{K}\lambda(x_i; \theta) + (n-K)\lambda(T; \theta) + D(\theta; \delta)$$

and

$$I_2 = \int_{\theta\in\Theta}\eta_2(\theta; \mathbf{x}_K)\exp[-\zeta_2(\theta; \mathbf{x}_K)]d\theta.$$

3. Bayesian Prediction Intervals

3.1. *One-Sample Bayesian Prediction*

Upon substituting (2.1) and (2.2) in (1.3) and (1.4), we obtain the conditional density function of $X_{s:n}$, given the Type-I hybrid censored data, as follows:

Case I.

$$f_1(x_s \mid \mathbf{x}_r) = \begin{cases} f_{11}(x_s \mid \mathbf{x}_r), & x_r < x_s \le T, \\ f_{12}(x_s \mid \mathbf{x}_r), & x_s > T, \end{cases} \tag{3.1}$$

where

$$f_{11}(x_s \mid \mathbf{x}_r) = \sum_{k=s}^{n}\sum_{w=0}^{k-s}\sum_{q=0}^{s-r-1} C_1\phi_k(T; \theta)\lambda'(x_s; \theta)h_{k,w,q}(x_s, x_r, T; \theta), \quad x_r < x_s \le T,$$

and

$$f_{12}(x_s \mid \mathbf{x}_r) = \sum_{k=r}^{s-1}\sum_{w=0}^{s-k-1} C_2\phi_k(T; \theta)\lambda'(x_s; \theta)g_w(x_s, T; \theta), \quad x_s > T,$$

with $C_1 = \frac{(-1)^{w+q}\binom{k-s}{w}\binom{s-r-1}{q}(k-r)!}{(s-r-1)!(k-s)!}$, $C_2 = \frac{(-1)^{w}\binom{s-k-1}{w}(n-k)!}{(s-k-1)!(n-s)!}$,

$$\phi_k(T;\theta) = \frac{\binom{n}{k}\exp[-(n-k)\lambda(T;\theta)][1-\exp[-\lambda(T;\theta)]]^k}{\sum_{j=r}^{n}\binom{n}{j}\exp[-(n-j)\lambda(T;\theta)][1-\exp[-\lambda(T;\theta)]]^j},$$

$$h_{k,w,q}(x,y,z;\theta) = \frac{\exp[-(s-r-q-1)\lambda(y;\theta)-w\lambda(z;\theta)-(k-s-w+q+1)\lambda(x;\theta)]}{[\exp[-\lambda(y;\theta)]-\exp[-\lambda(z;\theta)]]^{k-r}},$$

and

$$g_w(x,y;\theta) = \exp[-(n-s+w+1)\{\lambda(x;\theta)-\lambda(y;\theta)\}];$$

Case II.

$$f_2(x_s \mid \mathbf{x}_K) = \sum_{k=0}^{r-1}\sum_{w=0}^{s-k-1} C_2\psi_k(T;\theta)\lambda'(x_s;\theta)g_w(x_s,T;\theta), \quad x_s > T, \tag{3.2}$$

where

$$\psi_k(T,\theta) = \frac{\binom{n}{k}\exp[-(n-k)\lambda(T;\theta)][1-\exp[-\lambda(T;\theta)]]^k}{\sum_{j=0}^{r-1}\binom{n}{j}\exp[-(n-j)\lambda(T;\theta)][1-\exp[-\lambda(T;\theta)]]^j}.$$

From (2.6), (2.7), (3.1), and (3.2), we simply obtain the predictive density function of $X_{s:n}$ as follows:

Case I.

$$f_1^*(x_s \mid \mathbf{x}_r) = \begin{cases} f_{11}^*(x_s \mid \mathbf{x}_r), & x_r < x_s \le T, \\ f_{12}^*(x_s \mid \mathbf{x}_r), & x_s > T, \end{cases} \tag{3.3}$$

where

$$f_{11}^*(x_s \mid \mathbf{x}_r) = \sum_{k=s}^{n}\sum_{w=0}^{k-s}\sum_{q=0}^{s-r-1} C_1 \int_{\theta\in\Theta} \lambda'(x_s;\theta)\phi_k(T;\theta)h_{k,w,q}(x_s,x_r,T;\theta)\pi_1^*(\theta \mid \mathbf{x}_r)d\theta, \quad x_r < x_s \le T,$$

and

$$f_{12}^*(x_s \mid \mathbf{x}_r) = \sum_{k=r}^{s-1}\sum_{w=0}^{s-k-1} C_2 \int_{\theta\in\Theta} \lambda'(x_s;\theta)\phi_k(T;\theta)g_w(x_s,T;\theta)\pi_1^*(\theta \mid \mathbf{x}_r)d\theta, \quad x_s > T;$$

Case II.

$$f_2^*(x_s \mid \mathbf{x}_K) = \sum_{k=0}^{r-1}\sum_{w=0}^{s-k-1} C_2 \int_{\theta\in\Theta} \lambda'(x_s;\theta)\psi_k(T,\theta)g_w(x_s,T;\theta)\pi_2^*(\theta \mid \mathbf{x}_K)d\theta, \quad x_s > T. \tag{3.4}$$

From (3.3) and (3.4), we simply obtain the predictive survival function of $X_{s:n}$ as follows:

Case I.

$$\overline{F}_1^*(t \mid \mathbf{x}_r) = \begin{cases} \overline{F}_{11}^*(t \mid \mathbf{x}_r), & x_r < t \leq T, \\ \overline{F}_{12}^*(t \mid \mathbf{x}_r), & t > T, \end{cases} \tag{3.5}$$

where

$$\begin{aligned}
\overline{F}_{11}^*(t \mid \mathbf{x}_r) &= \int_t^T f_{11}^*(x_s \mid \mathbf{x}_r) dx_s + \int_T^\infty f_{12}^*(x_s \mid \mathbf{x}_r) dx_s \\
&= \sum_{k=s}^{n} \sum_{w=0}^{k-s} \sum_{q=0}^{s-r-1} \frac{C_1}{k-s-w+q+1} \int_{\theta \in \Theta} \phi_k(T; \theta) \{h_{k,w,q}(t, x_r, T; \theta) \\
&\quad - h_{k,w,q}(T, x_r, T; \theta)\} \pi_1^*(\theta \mid \mathbf{x}_r) d\theta \\
&\quad + \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n-s+w+1} \int_{\theta \in \Theta} \phi_k(T; \theta) \pi_1^*(\theta \mid \mathbf{x}_r) d\theta
\end{aligned}$$

and

$$\begin{aligned}
\overline{F}_{12}^*(t \mid \mathbf{x}_r) &= \int_t^\infty f_{12}^*(x_s \mid \mathbf{x}_r) dx_s \\
&= \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n-s+w+1} \int_{\theta \in \Theta} \phi_k(T; \theta) g_w(t, T; \theta) \pi_1^*(\theta \mid \mathbf{x}_r) d\theta;
\end{aligned}$$

Case II.

$$\begin{aligned}
\overline{F}_2^*(t \mid \mathbf{x}_K) &= \int_t^\infty f_2^*(x_s \mid \mathbf{x}_K) dx_s \\
&= \sum_{k=0}^{r-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n-s+w+1} \int_{\theta \in \Theta} \psi_k(T; \theta) g_w(t, T; \theta) \pi_2^*(\theta \mid \mathbf{x}_K) d\theta.
\end{aligned} \tag{3.6}$$

Then, the Bayesian predictive bounds of a two-sided equi-tailed $100(1-\gamma)\%$ interval for $X_{s:n}$, $r < s \leq n$, can be obtained by solving the following two equations:

$$\overline{F}^*(L_{X_{s:n}} \mid \mathbf{x}) = 1 - \frac{\gamma}{2} \quad \text{and} \quad \overline{F}^*(U_{X_{s:n}} \mid \mathbf{x}) = \frac{\gamma}{2},$$

where

$$\overline{F}^*(t \mid \mathbf{x}) = \begin{cases} \overline{F}_1^*(t \mid \mathbf{x}_r), & \text{Case I}, \\ \overline{F}_2^*(t \mid \mathbf{x}_K), & \text{Case II}, \end{cases}$$

and $L_{X_{s:n}}$ and $U_{X_{s:n}}$ denote the lower and upper bounds, respectively.

3.2. *Two-Sample Bayesian Prediction*

Let $Y_{1:m} \leq Y_{2:m} \leq \cdots \leq Y_{m:m}$ be the OS from a future random sample of size m from the same population. It is well known that the marginal density function of the s-th order statistic from a sample of size m from a continuous distribution with cdf $F(x)$ and pdf $f(x)$ is given by

$$f_{Y_{s:m}}(y \mid \theta) = \frac{m!}{(s-1)!(m-s)!}[F(y)]^{s-1}[1-F(y)]^{m-s}f(y), \tag{3.7}$$

where $1 \leq s \leq m$; see Arnold et al. (1992).

Upon substituting (2.1) and (2.2) in (3.7), we obtain

$$f_{Y_{s:m}}(y \mid \theta) = \sum_{w=0}^{s-1} C_3 \lambda'(y;\theta)\exp[-(m-s+w+1)\lambda(y;\theta)], \tag{3.8}$$

where $1 \leq s \leq m$ and $C_3 = \frac{(-1)^w \binom{s-1}{w} m!}{(s-1)!(m-s)!}$.

From (2.6), (2.7) and (3.8), we simply obtain the Bayesian predictive density function of $Y_{s:m}$ as follows:

Case I.

$$f^*_{1Y_{s:m}}(y \mid \mathbf{x}_r) = \sum_{w=0}^{s-1} C_3 \int_{\theta\in\Theta} \lambda'(y;\theta)\exp[-(m-s+w+1)\lambda(y;\theta)]\pi_1^*(\theta \mid \mathbf{x}_r)d\theta; \tag{3.9}$$

Case II.

$$f^*_{2Y_{s:m}}(y \mid \mathbf{x}_K) = \sum_{w=0}^{s-1} C_3 \int_{\theta\in\Theta} \lambda'(y;\theta)\exp[-(m-s+w+1)\lambda(y;\theta)]\pi_2^*(\theta \mid \mathbf{x}_K)d\theta. \tag{3.10}$$

From (3.9) and (3.10), we simply obtain the predictive survival function of $Y_{s:m}$ as follows:

Case I.

$$\begin{aligned}\overline{F}^*_{1Y_{s:m}}(t \mid \mathbf{x}_r) &= \int_t^\infty f^*_{1Y_{s:m}}(y \mid \mathbf{x}_r)dy \\ &= \sum_{w=0}^{s-1} \frac{C_3}{m-s+w+1} \int_{\theta\in\Theta} \exp[-(m-s+w+1)\lambda(t;\theta)]\pi_1^*(\theta \mid \mathbf{x}_r)d\theta;\end{aligned} \tag{3.11}$$

Case II.

$$\begin{aligned}\overline{F}^*_{2Y_{s:m}}(t \mid \mathbf{x}_K) &= \int_t^\infty f^*_{2Y_{s:m}}(y \mid \mathbf{x}_K)dy \\ &= \sum_{w=0}^{s-1} \frac{C_3}{m-s+w+1} \int_{\theta\in\Theta} \exp[-(m-s+w+1)\lambda(t;\theta)]\pi_2^*(\theta \mid \mathbf{x}_K)d\theta.\end{aligned} \tag{3.12}$$

Consequently, the Bayesian predictive bounds of a two-sided equi-tailed $100(1-\gamma)\%$ interval for $Y_{s:m}$, $1 \leq s \leq m$, can be obtained by solving the following two equations:

$$\overline{F}^*_{Y_{s:m}}(L_{Y_{s:m}} \mid \mathbf{x}) = 1 - \frac{\gamma}{2} \quad \text{and} \quad \overline{F}^*_{Y_{s:m}}(U_{Y_{s:m}} \mid \mathbf{x}) = \frac{\gamma}{2},$$

where

$$\overline{F}^*_{Y_{s:m}}(t \mid \mathbf{x}) = \begin{cases} \overline{F}^*_{1Y_{s:m}}(t \mid \mathbf{x}_r), & \text{Case I,} \\ \overline{F}^*_{2Y_{s:m}}(t \mid \mathbf{x}_K), & \text{Case II,} \end{cases}$$

and $L_{Y_{s:m}}$ and $U_{Y_{s:m}}$ denote the lower and upper bounds, respectively.

4. Illustrative Examples

In this section, we discuss the Bayesian prediction problems for the Exponential(θ) distribution when θ is unknown, and the Pareto(α, β) distribution when both parameters α and β are unknown, as illustrative examples.

4.1. *Exponential (θ) Model*

The distribution function in this case is

$$F(x \mid \theta) = 1 - \exp[-\theta x], \quad x > 0, \tag{4.1}$$

where $\theta > 0$, and so we have

$$\lambda(x; \theta) = \theta x \quad \text{and} \quad \lambda'(x; \theta) = \theta. \tag{4.2}$$

For the case when θ is unknown, we use the conjugate gamma prior for θ with density

$$\pi(\theta; \delta) = \frac{d^c}{\Gamma(c)} \theta^{c-1} \exp[-\theta d], \quad \theta > 0, \tag{4.3}$$

where c and d are positive constants, and so we have

$$C(\theta; \delta) = \theta^{c-1} \quad \text{and} \quad D(\theta; \delta) = \theta d, \tag{4.4}$$

where $\delta = (c, d)$.

Hence, the posterior density function is obtained as follows:

Case I.

$$\pi_1^*(\theta \mid \mathbf{x}_r) = \frac{\left(\sum_{i=1}^r x_i + (n-r)x_r + d\right)^{r+c}}{\Gamma(r+c)} \, \theta^{r+c-1} \exp\left[-\theta\left\{\sum_{i=1}^r x_i + (n-r)x_r + d\right\}\right]; \tag{4.5}$$

Case II.

$$\pi_2^*(\theta \mid \mathbf{x}_K) = \frac{\left(\sum_{i=1}^K x_i + (n-K)T + d\right)^{K+c}}{\Gamma(K+c)} \, \theta^{K+c-1} \times \exp\left[-\theta\left\{\sum_{i=1}^K x_i + (n-K)T + d\right\}\right]. \tag{4.6}$$

4.1.1. One-Sample Bayesian Prediction. The predictive survival function of $X_{s:n}$ in this special case is obtained as follows:

Case I.

$$\overline{F}_1^*(t \mid \mathbf{x}_r) = \begin{cases} \overline{F}_{11}^*(t \mid \mathbf{x}_r), & x_r < t \leq T, \\ \overline{F}_{12}^*(t \mid \mathbf{x}_r), & t > T, \end{cases}$$

where

$$\begin{aligned} \overline{F}_{11}^*(t \mid \mathbf{x}_r) = & \sum_{k=s}^{n} \sum_{w=0}^{k-s} \sum_{q=0}^{s-r-1} \frac{C_1}{k-s-w+q+1} \\ & \times \int_0^\infty \phi_k(T;\theta)\{h_{k,w,q}(t, x_r, T; \theta) - h_{k,w,q}(T, x_r, T; \theta)\}\pi_1^*(\theta \mid \mathbf{x}_r) d\theta \\ & + \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n-s+w+1} \int_0^\infty \phi_k(T;\theta)\pi_1^*(\theta \mid \mathbf{x}_r) d\theta \end{aligned} \tag{4.7}$$

and

$$\overline{F}_{12}^*(t \mid \mathbf{x}_r) = \sum_{k=r}^{s-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n-s+w+1} \int_0^\infty \phi_k(T;\theta) g_w(t, T; \theta)\pi_1^*(\theta \mid \mathbf{x}_r) d\theta, \tag{4.8}$$

with

$$\phi_k(T, \theta) = \frac{\binom{n}{k} \exp[-(n-k)\theta T][1 - \exp[-\theta T]]^k}{\sum_{j=r}^{n} \binom{n}{j} \exp[-(n-j)\theta T][1 - \exp[-\theta T]]^j},$$

$$h_{k,w,q}(x, y, z; \theta) = \frac{\exp[-\theta\{(s-r-q-1)y + wz + (k-s-w+q+1)x\}]}{[\exp[-\theta y] - \exp[-\theta z]]^{k-r}}$$

and

$$g_w(t, T; \theta) = \exp[-\theta\{(n-s+w+1)(t-T)\}];$$

Case II.

$$\overline{F}_2^*(t \mid \mathbf{x}_K) = \sum_{k=0}^{r-1} \sum_{w=0}^{s-k-1} \frac{C_2}{n-s+w+1} \int_0^\infty \psi_k(T, \theta) g_w(t, T; \theta)\pi_2^*(\theta \mid \mathbf{x}_K) d\theta, \tag{4.9}$$

where

$$\psi_k(T, \theta) = \frac{\binom{n}{k} \exp[-(n-k)\theta T][1 - \exp[-\theta T]]^k}{\sum_{j=0}^{r-1} \binom{n}{j} \exp[-(n-j)\theta T][1 - \exp[-\theta T]]^j}.$$

It does not seem to be possible to compute the probabilities in (4.7)–(4.9) analytically. Hence, we use the Markov Chain Monte Carlo (MCMC) technique for constructing the Bayesian prediction interval.

To compute $\int_0^\infty f(\theta)\pi_1^*(\theta \mid \mathbf{x}_r)d\theta$ by using the MCMC technique, we use the following procedure:

Step 1. Generate $\theta_1 \sim \text{Gamma}(r+c, \sum_{i=1}^r x_i + (n-r)x_r + d)$;

Step 2. Repeat Step 1 and obtain $\theta_1, \theta_2, \ldots, \theta_N$;

Step 3. The approximate value of $\int_0^\infty f(\theta)\pi_1^*(\theta \mid \mathbf{x}_r)d\theta$ is then obtained as

$$\int_0^\infty f(\theta)\pi_1^*(\theta \mid \mathbf{x}_r)d\theta = \frac{\sum_{i=1}^N f(\theta_i)}{N}.$$

Similarly, we can use the above algorithm to compute $\int_0^\infty g(\theta)\pi_2^*(\theta \mid \mathbf{x}_K)d\theta$.

4.1.2. Two-Sample Bayesian Prediction. The predictive survival function of $Y_{s:m}$ in this special case is obtained as follows:

Case I.

$$\overline{F}^*_{1Y_{s:m}}(t \mid \mathbf{x}_r) = I_1^{-1}\sum_{w=0}^{s-1}\frac{C_3}{m-s+w+1}\left(\sum_{i=1}^r x_i + (n-r)x_r + (m-s+w+1)t + d\right)^{-(r+c)}, \tag{4.10}$$

where

$$I_1 = \left(\sum_{i=1}^r x_i + (n-r)x_r + d\right)^{-(r+c)};$$

Case II.

$$\overline{F}^*_{2Y_{s:m}}(t \mid \mathbf{x}_K) = I_2^{-1}\sum_{w=0}^{s-1}\frac{C_3}{m-s+w+1}\left(\sum_{i=1}^K x_i + (n-K)T + (m-s+w+1)t + d\right)^{-(K+c)}, \tag{4.11}$$

where

$$I_2 = \left(\sum_{i=1}^K x_i + (n-K)T + d\right)^{-(K+c)},$$

4.2. *Pareto*(α, β) *Model*

The distribution function in this case is

$$F(x \mid \alpha, \beta) = 1 - \left(\frac{\beta}{x}\right)^{\alpha}, \quad x > \beta, \tag{4.12}$$

where $\alpha > 0$ and $\beta > 0$, and so we have

$$\lambda(x; \alpha, \beta) = \alpha \ln\left(\frac{x}{\beta}\right) \quad \text{and} \quad \lambda'(x; \alpha, \beta) = \frac{\alpha}{x}. \tag{4.13}$$

Under the assumption that both parameters α and β are unknown, we may consider a natural joint conjugate prior for α and β which was suggested by Lwin (1972) and generalized by Arnold and Press (1989). The generalized Lwin prior or the power-gamma prior, denoted by PG(a, b, c, d), is given by

$$\pi(\alpha, \beta; \delta) \propto \alpha^c \beta^{-1} \exp\left[-\alpha\left(d + a\ln\left(\frac{b}{\beta}\right)\right)\right], \quad \alpha > 0, \quad 0 < \beta < b, \tag{4.14}$$

where a, b, c, d are positive constants. This general prior is obtained by first specifying the prior for the parameter α and then specifying the conditional prior for β, given knowledge on the parameter α. More specifically, we take $\pi(\alpha)$ as a gamma distribution with parameters c and d, and $\pi(\beta \mid \alpha)$ as a power function distribution with parameters $a\alpha$ and b of the form

$$\pi(\beta \mid \alpha) \propto \alpha \beta^{a\alpha - 1} b^{-a\alpha}, \quad 0 < \beta < b,$$

to arrive at the joint prior given in (4.14). Thus, we have

$$C(\alpha, \beta; \delta) = \alpha^c \beta^{-1} \quad \text{and} \quad D(\alpha, \beta; \delta) = \alpha\left\{d + a\ln\left(\frac{b}{\beta}\right)\right\}, \tag{4.15}$$

where $\delta = (a, b, c, d)$.

Hence, the posterior density function is obtained as follows:

Case I.

$$\pi_1^*(\alpha, \beta \mid \mathbf{x}_r) = \pi_{11}^*(\alpha \mid \mathbf{x}_r)\pi_{12}^*(\beta \mid \alpha, \mathbf{x}_r), \tag{4.16}$$

where

$$\pi_{11}^*(\alpha \mid \mathbf{x}_r) = \frac{[I_1(\mathbf{x}_r, L)]^{r+c}}{\Gamma(r+c)} \alpha^{r+c-1} \exp[-\alpha I_1(\mathbf{x}_r, L)], \tag{4.17}$$

$$\pi_{12}^*(\beta \mid \alpha, \mathbf{x}_r) = \alpha(n+a)\beta^{\alpha(n+a)-1} L^{-\alpha(n+a)}, \tag{4.18}$$

$I_1(\mathbf{x}_r, z) = \sum_{i=1}^{r} \ln\left(\frac{x_i}{z}\right) + (n-r)\ln\left(\frac{x_r}{z}\right) + a\ln\left(\frac{b}{z}\right) + d$ and $L = \min(x_1, b)$;

Case II.

$$\pi_2^*(\alpha, \beta \mid \mathbf{x}_K) = \pi_{21}^*(\alpha \mid \mathbf{x}_K)\pi_{22}^*(\beta \mid \alpha, \mathbf{x}_K), \tag{4.19}$$

where

$$\pi_{21}^*(\alpha \mid \mathbf{x}_K) = \frac{[I_2(\mathbf{x}_K, L)]^{K+c}}{\Gamma(K+c)} \alpha^{K+c-1} \exp[-\alpha I_2(\mathbf{x}_K, L)], \tag{4.20}$$

$$\pi_{22}^*(\beta \mid \alpha, \mathbf{x}_K) = \alpha(n+a)\beta^{\alpha(n+a)-1} L^{-\alpha(n+a)}, \tag{4.21}$$

and $I_2(\mathbf{x}_K, z) = \sum_{i=1}^{K} \ln\left(\frac{x_i}{z}\right) + (n-K)\ln\left(\frac{T}{z}\right) + a\ln\left(\frac{b}{z}\right) + d$.

4.2.1. One-Sample Bayesian Prediction. The predictive survival function of $X_{s:n}$ in this special case is obtained as follows:

Case I.

$$\overline{F}_1^*(t\,|\,\mathbf{x}_r) = \begin{cases} \overline{F}_{11}^*(t\,|\,\mathbf{x}_r), & x_r < t \le T, \\ \overline{F}_{12}^*(t\,|\,\mathbf{x}_r), & t > T, \end{cases}$$

where

$$\overline{F}_{11}^*(t\,|\,\mathbf{x}_r) = \sum_{k=s}^{n}\sum_{w=0}^{k-s}\sum_{q=0}^{s-r-1} \frac{C_1}{k-s-w+q+1}\int_0^L\int_0^\infty \phi_k(T;\theta) \times \{h_{k,w,q}(t, x_r, T;\theta) - h_{k,w,q}(T, x_r, T;\theta)\}\pi_1^*(\alpha,\beta\,|\,\mathbf{x}_r)d\alpha\, d\beta + \sum_{k=r}^{s-1}\sum_{w=0}^{s-k-1}\frac{C_2}{n-s+w+1}\int_0^L\int_0^\infty \phi_k(T,\theta)\pi_1^*(\alpha,\beta\,|\,\mathbf{x}_r)d\alpha\, d\beta \quad (4.22)$$

and

$$\overline{F}_{12}^*(t\,|\,\mathbf{x}_r) = \sum_{k=r}^{s-1}\sum_{w=0}^{s-k-1}\frac{C_2}{n-s+w+1}\int_0^L\int_0^\infty \phi_k(T,\theta)g_w(t,T;\theta)\pi_1^*(\alpha,\beta\,|\,\mathbf{x}_r)d\alpha\, d\beta, \quad (4.23)$$

with

$$\phi_k(T,\theta) = \frac{\binom{n}{k}\exp\left[-(n-k)\alpha\ln\left(\frac{T}{\beta}\right)\right][1-\exp[-\alpha\ln\left(\frac{T}{\beta}\right)]]^k}{\sum_{j=r}^{n}\binom{n}{j}\exp[-(n-j)\alpha\ln\left(\frac{T}{\beta}\right)]\left[1-\exp[-\alpha\ln\left(\frac{T}{\beta}\right)]\right]^j},$$

$$h_{k,w,q}(x,y,z;\theta) = \frac{\exp[-\alpha\{(s-r-q-1)\ln\left(\frac{y}{\beta}\right) + w\ln\left(\frac{z}{\beta}\right) + (k-s-w+q+1)\ln\left(\frac{x}{\beta}\right)\}]}{\left[\exp[-\alpha\ln\left(\frac{y}{\beta}\right)] - \exp[-\alpha\ln\left(\frac{z}{\beta}\right)]\right]^{k-r}}$$

and

$$g_w(t,T;\theta) = \exp\left[-(n-s+w+1)\alpha\ln\left(\frac{t}{T}\right)\right];$$

Case II.

$$\overline{F}_2^*(t\,|\,\mathbf{x}_K) = \sum_{k=0}^{r-1}\sum_{w=0}^{s-k-1}\frac{C_2}{n-s+w+1}\int_0^L\int_0^\infty \psi_k(T;\alpha,\beta)g_w(t,T;\alpha,\beta)\pi_2^*(\alpha,\beta\,|\,\mathbf{x}_K)d\alpha\, d\beta, \quad (4.24)$$

where

$$\psi_k(T;\alpha,\beta) = \frac{\binom{n}{k}\exp[-(n-k)\alpha\ln\left(\frac{T}{\beta}\right)][1-\exp[-\alpha\ln\left(\frac{T}{\beta}\right)]]^k}{\sum_{j=0}^{r-1}\binom{n}{j}\exp[-(n-j)\alpha\ln\left(\frac{T}{\beta}\right)]\left[1-\exp[-\alpha\ln\left(\frac{T}{\beta}\right)]\right]^j}.$$

It does not seem to be possible to compute the probabilities in (4.22)–(4.24) analytically. Hence, we use the Gibbs sampling technique to generate MCMC samples, and then use the MCMC technique for constructing the Bayesian prediction interval.

To compute $\int_0^L \int_0^\infty f(\alpha, \beta)\pi_1^*(\alpha, \beta \mid \mathbf{x}_r) d\alpha\, d\beta$ by using the MCMC technique, we use the following procedure:

Step 1. Generate $\alpha_1 \sim \text{Gamma}\left(r + c, \sum_{i=1}^r \ln\left(\frac{x_i}{L}\right) + (n - r)\ln\left(\frac{x_r}{L}\right) + a\ln\left(\frac{b}{L}\right) + d\right)$;

Step 2. Generate $\beta_1 \sim$ Power function$(\alpha_1(n + a), L)$;

Step 3. Repeat Steps 1 and 2 and obtain $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_N, \beta_N)$;

Step 4. The approximate value of $\int_0^L \int_0^\infty f(\alpha, \beta)\pi_1^*(\alpha, \beta \mid \mathbf{x}_r) d\alpha\, d\beta$ is then obtained as

$$\int_0^L \int_0^\infty f(\alpha, \beta)\pi_1^*(\alpha, \beta \mid \mathbf{x}_r) d\alpha\, d\beta = \frac{\sum_{i=1}^N f(\alpha_i, \beta_i)}{N}.$$

Similarly, we can use the above algorithm to compute $\int_0^L \int_0^\infty g(\alpha, \beta)\pi_2^*(\alpha, \beta \mid \mathbf{x}_K) d\alpha\, d\beta$.

4.2.2. Two-Sample Bayesian Prediction. The predictive density function of $Y_{s:m}$ in this special case is obtained as follows:

Case I.

$$f_{1Y_{s:m}}^*(y \mid \mathbf{x}_r) = \begin{cases} f_{11Y_{s:m}}^*(y \mid \mathbf{x}_r), & 0 < y \le L, \\ f_{12Y_{s:m}}^*(y \mid \mathbf{x}_r), & y > L, \end{cases} \tag{4.25}$$

where

$$\begin{aligned} f_{11Y_{s:m}}^*(y \mid \mathbf{x}_r) &= \int_0^y \int_0^\infty f_{Y_{s:m}}(y \mid \alpha, \beta)\pi_1^*(\alpha, \beta \mid \mathbf{x}_r) d\alpha\, d\beta \\ &= \frac{(r + c)(n + a)[I_1(\mathbf{x}_r, L)]^{r+c}}{y[I_1(\mathbf{x}_r, y)]^{r+c+1}} \sum_{w=0}^{s-1} \frac{C_3}{n + a + m - s + w + 1}, \quad 0 < y \le L; \end{aligned}$$

and

$$\begin{aligned} f_{12Y_{s:m}}^*(y \mid \mathbf{x}_r) &= \int_0^L \int_0^\infty f_{Y_{s:m}}(y \mid \alpha, \beta)\pi_1^*(\alpha, \beta \mid \mathbf{x}_r) d\alpha\, d\beta \\ &= \frac{(r + c)(n + a)}{I_1(\mathbf{x}_r, L)} \sum_{w=0}^{s-1} \frac{C_3}{n + a + m - s + w + 1} \\ &\quad \times \frac{1}{y}\left[1 + \frac{(m - s + w + 1)\ln\left(\frac{y}{L}\right)}{I_1(\mathbf{x}_r, L)}\right]^{-(r+c+1)}, \quad y > L. \end{aligned}$$

Case II.

$$f_{2Y_{s:m}}^*(y \mid \mathbf{x}_K) = \begin{cases} f_{21Y_{s:m}}^*(y \mid \mathbf{x}_K), & 0 < y \le L, \\ f_{22Y_{s:m}}^*(y \mid \mathbf{x}_K), & y > L, \end{cases} \tag{4.26}$$

where

$$
\begin{aligned}
&f^*_{21Y_{s:m}}(y \mid \mathbf{x}_K) \\
&\quad = \int_0^y \int_0^\infty f_{Y_{s:m}}(y \mid \alpha, \beta)\pi_2^*(\alpha, \beta \mid \mathbf{x}_K) d\alpha\, d\beta \\
&\quad = \frac{(K+c)(n+a)[I_2(\mathbf{x}_K, L)]^{K+c}}{y[I_2(\mathbf{x}_K, y)]^{K+c+1}} \sum_{w=0}^{s-1} \frac{C_3}{n+a+m-s+w+1}, \quad 0 < y \le L,
\end{aligned}
$$

and

$$
\begin{aligned}
f^*_{22Y_{s:m}}(y \mid \mathbf{x}_K) &= \int_0^L \int_0^\infty f_{Y_{s:m}}(y \mid \alpha, \beta)\pi_2^*(\alpha, \beta \mid \mathbf{x}_K) d\alpha\, d\beta \\
&= \frac{(K+c)(n+a)}{y\ I_2(\mathbf{x}_K, L)} \sum_{w=0}^{s-1} \frac{C_3}{n+a+m-s+w+1} \\
&\quad \times \left[1 + \frac{(m-s+w+1)\ln\left(\frac{y}{L}\right)}{I_2\left(\mathbf{x}_K, L\right)}\right]^{-(K+c+1)}, \quad y > L.
\end{aligned}
$$

From (4.25) and (4.26), we simply obtain the predictive survival function as follows:

Case I.

$$
\overline{F}^*_{1Y_{s:m}}(t \mid \mathbf{x}_r) = \begin{cases} \overline{F}^*_{11Y_{s:m}}(t \mid \mathbf{x}_r), & 0 < t \le L, \\ \overline{F}^*_{12Y_{s:m}}(t \mid \mathbf{x}_r), & t > L, \end{cases} \tag{4.27}
$$

where

$$
\begin{aligned}
\overline{F}^*_{11Y_{s:m}}(t \mid \mathbf{x}_r) &= \int_t^L f^*_{11Y_{s:m}}(y \mid \mathbf{x}_r) dy + \int_L^\infty f^*_{12Y_{s:m}}(y \mid \mathbf{x}_r) dy \\
&= \sum_{w=0}^{s-1} \frac{C_3}{n+a+m-s+w+1} \left(1 - \left[\frac{I_1(\mathbf{x}_r, t)}{I_1(\mathbf{x}_r, L)}\right]^{-(r+c)}\right) \\
&\quad + (n+a) \sum_{w=0}^{s-1} \frac{C_3}{(n+a+m-s+w+1)(m-s+w+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{F}^*_{12Y_{s:m}}(t \mid \mathbf{x}_r) &= \int_t^\infty f^*_{12Y_{s:m}}(y \mid \mathbf{x}_r) dy \\
&= (n+a) \sum_{w=0}^{s-1} \frac{C_3}{(n+a+m-s+w+1)(m-s+w+1)} \\
&\quad \times \left[1 + \frac{(m-s+w+1)\ln\left(\frac{t}{L}\right)}{I_1(\mathbf{x}_r, L)}\right]^{-(r+c)};
\end{aligned}
$$

Case II.

$$\overline{F}^*_{2Y_{s:m}}(t \mid \mathbf{x}_K) = \begin{cases} \overline{F}^*_{21Y_{s:m}}(t \mid \mathbf{x}_K), & 0 < t \le L, \\ \overline{F}^*_{22Y_{s:m}}(t \mid \mathbf{x}_K), & t > L, \end{cases} \tag{4.28}$$

where

$$\begin{aligned} \overline{F}^*_{21Y_{s:m}}(t \mid \mathbf{x}_K) &= \int_t^L f^*_{21Y_{s:m}}(y \mid \mathbf{x}_K)dy + \int_L^\infty f^*_{22Y_{s:m}}(y \mid \mathbf{x}_K)dy \\ &= \sum_{j=0}^{s-1} \frac{C_3}{n+a+m-s+w+1}\left(1 - \left[\frac{I_2(\mathbf{x}_K, t)}{I_2(\mathbf{x}_K, L)}\right]^{-(K+c)}\right) \\ &\quad + (n+a)\sum_{w=0}^{s-1} \frac{C_3}{(n+a+m-s+w+1)(m-s+w+1)}, \end{aligned}$$

and

$$\begin{aligned} \overline{F}^*_{22Y_{s:m}}(t \mid \mathbf{x}_K) &= \int_t^\infty f^*_{22Y_{s:m}}(y \mid \mathbf{x}_K)dy \\ &= (n+a)\sum_{j=0}^{s-1} \frac{C_3}{(n+a+m-s+w+1)(m-s+w+1)} \\ &\quad \times \left[1 + \frac{(m-s+w+1)\ln\left(\frac{t}{L}\right)}{I_2(\mathbf{x}_K, L)}\right]^{-(K+c)}. \end{aligned}$$

5. Numerical Results

To illustrate the inferential procedures developed in the preceding sections, we present here a numerical study for the Exponential(θ) distribution when θ is unknown and the Pareto(α, β) distribution when both parameters α and β are unknown.

Example 5.1. To illustrate the prediction results for the Exponential(θ) distribution when θ is unknown, let us consider the data given by Bartholomew (1963) consisting of lifetimes of 20 items on a life-test for a pre-fixed time of 150 h. During that period, 15 items failed with the following lifetimes, measured in hours:

3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, and 138.

We shall use these data to consider two different Type-I hybrid censoring schemes:

1. When $r = 15$ and $T = 130$. Since $T < x_{15:20}$, the life-test would have terminated in this case at T, and we would have obtained the following data: 3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, and 109;
2. When $r = 15$ and $T = 150$. Since $x_{15:20} < T$, the life-test would have terminated in this case at time $x_{15:20} = 138$, and we would have obtained the following data: 3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, and 138.

As done previously by Bartholomew (1963) and Childs et al. (2003), we assume these data to have come from the Exponential(θ) distribution, where θ is unknown. Based on the above two Type-I hybrid censoring schemes, we then used the results presented earlier in Sec. 4.1 to construct 95% one-sample Bayesian prediction intervals for OS $X_{s:n}$, $s = 16, \ldots, 20$, from the same sample as well as 95% two-sample Bayesian prediction intervals for OS $Y_{s:m}$, $s = 1, 5, 10, 15, 20$, from a future sample of size $m = 20$. To examine the sensitivity of the Bayesian prediction intervals with respect to the hyperparameters (c, d), Table 1 presents the lower and upper 95% one-sample Bayesian prediction bounds for $X_{s:n}$, $s = 16, \ldots, 20$, for the choices of $c = 0.9, 1, 1.1$ and $d = 50, 55, 60$. Similarly, the lower and upper 95% two-sample Bayesian prediction bounds for $Y_{s:m}$, $s = 1, 5, 10, 15, 20$, for the choices of $c = 0.9$, 1, 1.1 and $d = 50, 55, 60$, are presented in Table 2.

Example 5.2. To illustrate the prediction results for the Pareto(α, β) distribution when both parameters α and β are unknown, we generated OS from a sample of size $n = 20$ from the Pareto distribution. The generated OS from the Pareto distribution (with $\alpha = 3$ and $\beta = 6$) are as follows:

6.046, 6.229, 6.445, 6.493, 6.856, 7.061, 7.097, 7.100, 7.163, 7.226, 7.344, 8.910,

9.290, 9.360, 9.525, 9.836, 10.263, 11.113, 15.769, and 39.211.

We will use these data to consider two different Type-I hybrid censoring schemes:

1. When $r = 15$ and $T = 9.4$. Since $T < x_{15:20}$, the life-test would have terminated in this case at T, and we would have obtained the following data: 6.046, 6.229, 6.445, 6.493, 6.856, 7.061, 7.097, 7.100, 7.163, 7.226, 7.344, 8.910, 9.290, and 9.360;
2. When $r = 15$ and $T = 9.6$. Since $x_{15:20} < T$, the life-test would have terminated in this case at time $x_{15:20}$, and we would have obtained the following data: 6.046, 6.229, 6.445, 6.493, 6.856, 7.061, 7.097, 7.100, 7.163, 7.226, 7.344, 8.910, 9.290, 9.360, and 9.525.

We assume these data to have come from the Pareto(α, β) distribution, where both parameters α and β are unknown. Based on the above two Type-I hybrid censoring schemes, we then used the results presented earlier in Sec. 4.2 to construct 95% one-sample Bayesian prediction intervals for OS $X_{s:n}$, $s = 16, \ldots, 20$, from the same sample as well as 95% two-sample Bayesian prediction intervals for OS $Y_{s:m}$, $s = 1, 5, 10, 15, 20$, from a future sample of size $m = 20$. To examine the sensitivity of the Bayesian prediction intervals with respect to the hyperparameters (a, b, c, d), we used three different choices of the hyperparameters (a, b, c, d): $(1, 9, 3, 1)$, $(1, 9, 6, 2)$, $(1, 9, 9, 3)$. The corresponding results for the one-sample and two-sample predictions, for these three choices of the hyperparameters, are presented in Tables 3 and 4, respectively.

6. Concluding Remarks

1. From Tables 1–4, we notice that, when we use the same value of r but larger T, the Bayesian prediction bounds become tighter as expected since the duration of the life-testing experiment is longer in this case.

Table 1
95% one-sample Bayesian prediction bounds for $X_{s:n}$, $s = 16, \ldots, 20$, from the exponential distribution

		$r = 15$ and $T = 130$						$r = 15$ and $T = 150$					
		$d = 50$		$d = 55$		$d = 60$		$d = 50$		$d = 55$		$d = 60$	
c	s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$
	16	136.604	328.554	136.643	329.544	136.683	330.536	138.186	209.349	138.187	209.670	138.188	209.991
	17	145.707	383.155	145.785	384.313	145.862	385.473	139.669	270.042	139.677	270.564	139.686	271.086
0.9	18	158.676	458.486	158.802	459.879	158.930	461.272	142.406	350.914	142.421	351.740	142.300	349.450
	19	177.641	578.352	177.836	580.121	178.032	581.891	145.740	468.033	145.775	469.199	145.982	472.444
	20	210.129	850.751	210.439	853.382	210.750	856.015	152.309	736.407	152.744	738.439	153.177	740.469
	16	136.530	326.114	136.568	327.094	136.607	328.075	138.183	208.645	138.184	208.965	138.186	209.285
	17	145.563	380.303	145.640	381.449	145.716	382.597	139.653	268.875	139.661	269.395	139.669	269.915
1	18	158.442	455.065	158.568	456.443	158.693	457.823	142.223	346.147	142.243	346.919	142.262	347.690
	19	177.283	574.033	177.476	575.784	177.670	577.536	145.679	465.392	145.713	466.551	145.747	467.709
	20	209.561	844.428	209.868	847.036	210.177	849.645	151.557	731.828	151.991	733.846	152.424	735.864
	16	136.458	323.719	136.496	324.689	136.533	325.659	138.181	207.948	138.182	208.267	138.183	208.585
	17	145.422	377.502	145.497	378.637	145.573	379.772	139.637	267.720	139.646	268.238	139.654	268.756
1.1	18	158.212	451.705	158.336	453.069	158.461	454.435	142.187	344.416	142.208	345.184	142.226	345.952
	19	176.930	569.788	177.122	571.522	177.314	573.258	145.620	462.784	145.654	463.936	145.688	465.087
	20	209.001	838.211	209.306	840.795	209.612	843.381	150.811	727.308	151.244	729.314	151.676	731.319

Table 2
95% two-sample Bayesian prediction bounds for $Y_{s:m}$, $s = 1, 5, 10, 15, 20$, from the exponential distribution

		$r = 15$ and $T = 130$						$r = 15$ and $T = 150$					
		$d = 50$		$d = 55$		$d = 60$		$d = 50$		$d = 55$		$d = 60$	
c	s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$
	1	0.130	21.476	0.130	21.546	0.131	21.616	0.125	20.590	0.126	20.655	0.126	20.720
	5	8.640	72.241	8.668	72.477	8.697	72.714	8.386	68.945	8.413	69.164	8.440	69.382
0.9	10	28.961	151.869	29.046	152.365	29.138	152.862	28.181	144.622	28.272	145.081	28.333	145.539
	15	63.178	285.761	63.379	286.696	63.624	287.630	61.521	271.880	61.711	272.742	61.898	273.604
	20	159.387	851.435	159.908	854.219	160.429	857.003	155.164	811.830	155.656	814.404	156.148	816.978
	1	0.129	21.315	0.130	21.384	0.130	21.454	0.125	20.446	0.125	20.510	0.126	20.575
	5	8.586	71.663	8.614	71.898	8.643	72.132	8.337	68.433	8.364	68.650	8.390	68.867
1	10	28.761	150.620	28.855	151.112	28.949	151.605	27.988	143.519	28.077	143.974	28.166	144.429
	15	62.815	283.384	63.020	284.311	63.226	285.238	61.193	269.783	61.388	270.639	61.582	271.494
	20	158.444	844.543	158.962	847.304	159.480	850.066	154.298	805.735	154.788	808.290	155.277	810.845
	1	0.128	21.155	0.129	21.225	0.129	21.293	0.124	20.304	0.124	20.368	0.125	20.432
	5	8.533	71.095	8.561	71.327	8.589	71.560	8.289	67.929	8.315	68.144	8.341	68.360
1.1	10	28.606	149.390	28.678	149.879	28.784	150.367	27.839	142.433	27.922	142.884	28.035	143.336
	15	62.435	281.045	62.636	281.964	62.844	282.883	60.836	267.718	61.024	268.567	61.212	269.416
	20	157.512	837.759	158.028	840.499	158.543	843.238	153.443	799.730	153.929	802.266	154.416	804.801

Table 3
95% two-sample Bayesian prediction bounds for $X_{s:n}$, $s = 16, \ldots, 20$, from the Pareto distribution

	$r = 15$ and $T = 9.4$						$r = 15$ and $T = 9.6$					
(a, b, c, d)	(1, 9, 3, 1)		(1, 9, 3, 2)		(1, 9, 4, 1)		(1, 9, 3, 1)		(1, 9, 3, 2)		(1, 9, 4, 1)	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$
16	9.661	19.892	9.747	23.599	9.635	18.486	9.529	12.217	9.530	12.859	9.528	11.968
17	10.015	24.354	10.181	29.752	9.965	22.348	9.544	15.311	9.545	16.682	9.538	14.789
18	10.539	32.227	10.814	41.006	10.454	29.057	9.566	20.439	9.580	23.266	9.556	19.394
19	11.349	50.418	11.795	68.494	11.209	44.208	9.591	31.927	9.672	29.481	9.578	31.443
20	12.876	140.334	13.670	221.713	12.630	115.547	10.472	86.745	11.112	122.821	10.255	75.458

Table 4
95% two-sample Bayesian prediction bounds for $Y_{s:m}$, $s = 1, 5, 10, 15, 20$, from the Pareto distribution

	$r = 15$ and $T = 9.4$						$r = 15$ and $T = 9.6$					
(a, b, c, d)	(1, 9, 3, 1)		(1, 9, 3, 2)		(1, 9, 4, 1)		(1, 9, 3, 1)		(1, 9, 3, 2)		(1, 9, 4, 1)	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$
1	5.693	6.446	5.642	6.509	5.714	6.421	5.710	6.426	5.661	6.485	5.728	6.404
5	6.067	7.730	6.070	8.023	6.065	7.609	6.066	7.630	6.069	7.901	6.065	7.524
10	6.618	10.315	6.711	11.185	6.587	9.957	6.592	10.017	6.680	10.803	6.564	9.708
15	7.574	16.813	7.838	19.630	7.489	15.700	7.510	15.881	7.758	18.350	7.433	14.944
20	10.975	135.895	12.012	217.714	10.652	110.864	10.725	114.813	11.685	178.355	10.435	95.802

2. It is evident from Tables 1 and 2 that, in the case of the exponential distribution, the lower as well as upper bounds are relatively insensitive to the specification of the hyperparameters (c, d).
3. It is also evident from Tables 3 and 4 that, in the case of the Pareto distribution, the lower bounds are relatively insensitive to the specification of the hyperparameters (a, b, c, d) while the upper bounds are somewhat sensitive.
4. If the vector of prior parameters δ is unknown, the empirical Bayes approach could be used in estimating such prior parameters based on past samples; see, for example, Maritz and Lwin (1989). Alternatively, one could use the hierarchical Bayesian method in which some suitable prior for δ could be proposed; see, for example, Geisser (1990) and Bernardo and Smith (1994). Work in these directions are currently under progress and we hope to report these findings in a future article.

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