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ONE-DIMENSIONAL BROWNIAN PARTICLE SYSTEMS WITH RANK-DEPENDENT DRIFTS

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We study interacting systems of linear Brownian motions whose drift vector at every time point is determined by the relative ranks of the coordinate processes at that time. Our main objective has been to study the long-range behavior of the spacings between the Brownian motions arranged in increasing order. For finitely many Brownian motions interacting in this manner, we characterize drifts for which the family of laws of the vector of spacings is tight and show its convergence to a unique stationary joint distribution given by independent exponential distributions with varying means. We also study one particular countably infinite system, where only the minimum Brownian particle gets a constant upward drift, and prove that independent and identically distributed exponential spacings remain stationary under the dynamics of such a process. Some related conjectures in this direction are also discussed.

1. Introduction. In this paper, we consider systems of interacting one-dimensional Brownian motions $X = (X_i(t), i \in I, t \geq 0)$, where i ranges over an index I which is either the finite set $\{1, \dots, N\}$ or the countable set of positive integers $\mathbb{N}$.

For $I = \{1, 2, \dots, N\}$, if we define ordered coordinates of any vector $x \in \mathbb{R}^N$ by

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(N)},$$

then the locations $X_i(t)$ of the Brownian particles evolve according to the system of stochastic differential equations

$$dX_i(t) = \sum_{j \in I} \delta_j 1\{X_i(t) = X_{(j)}(t)\}\, dt + dB_i(t) \qquad (i \in I) \tag{1}$$

for some sequence of *drifts* $\delta_1, \delta_2, \ldots \in \mathbb{R}$. Here, the B_i's are assumed to be independent $(\mathcal{F}_t)$-Brownian motions for some suitable underlying filtration $(\mathcal{F}_t)$. Less formally, the Brownian particles evolve independently, except that the ith-ranked particle is given drift δ_i.

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For these finite systems, with arbitrary initial distribution of $(X_i(0), i \in I)$, and arbitrary drifts δ_i, the existence and uniqueness in law of such an N-particle system is guaranteed by the standard theory of SDE's; see Lemma 6 for more details. We would also like to define SDE (1) when I is the entire countably infinite collection $\mathbb{N}$. This is more problematic since the ordered process may no longer remain well defined and the existence of the solution of SDE (1) depends both on the initial distribution and the sequence of drifts. We consider one such system in Section 3, with a drift sequence $(\delta, 0, 0, \ldots)$ $(\delta > 0)$ for which the finite-dimensional arguments can be suitably extended. For this system to exist in the weak sense, starting from an initial vector $X_1(0) < X_2(0) < \cdots$, we show that it suffices to assume that

$$\liminf_{n\to\infty} \frac{(X_n(0) - X_1(0))^2}{n} > 0;$$

see Lemma 11 for the details of the proof.

Now, consider the *Brownian spacings system* derived from this ordered Brownian particle system with rank-dependent drifts, that is,

$$\Delta_k(t) := X_{(k+1)}(t) - X_{(k)}(t) \qquad \text{for } k, k+1 \in I.$$

The ordered particle system derived from independent Brownian motions with no drift (meaning $\delta_i \equiv 0$) was studied by Harris [15], Arratia [1] and Sznitman [29], [30], page 187. By Donsker's theorem, this system can be interpreted as a scaling limit of ordered particle systems derived from independent symmetric nearest-neighbor random walks on $\mathbb{Z}$. Harris [15] considers the spacings of an infinite ordered Brownian particle system defined by

$$\Delta_i^*(t) := B_{(i+1)}(t) - B_{(i)}(t), \qquad i \in \mathbb{Z},$$

where $\{B_i\}$ is a family of independent Brownian motions with no drifts and initial states $B_i(0) = B_{(i)}(0)$ such that $B_0(0) = 0$ and the $B_i(0)$ for $i \in \mathbb{Z}\backslash\{0\}$ are points of a Poisson process of rate λ on $\mathbb{R}$. That is, $(B_{(i)}(t), t \geq 0)_{i\in\mathbb{Z}}$ is the almost surely unique collection of processes with continuous paths such that $B_{(i)}(t) \leq B_{(i+1)}(t)$ for all $i \in \mathbb{Z}, t \geq 0$ and the union of the graphs of these processes is identical to the union of the graphs of the Brownian paths $(B_i(t), t \geq 0)_{i\in\mathbb{Z}}$. We call $(B_{(i)}(t), i \in \mathbb{Z}, t \geq 0)$ the *Harris system of ordered Brownian motions with rate* λ and their differences $(\Delta_i^*(t), i \in \mathbb{Z}, t \geq 0)$ the *Harris system of Brownian spacings with rate* λ.

As observed by Arratia [2], Section 4, the corresponding stationary system of spacings between particles of the exclusion process associated with a nearest-neighbor random walk on $\mathbb{Z}$ can be interpreted as a finite or infinite series of queues, also known as the *zero-range process with constant rate*; see [16–20, 24] for background on systems of Brownian queues. Such connections between systems of queues and one-dimensional interacting particle systems have been exploited by a number of authors, in particular Kipnis [22], Srinivasan [28], Ferrari and Fontes [13, 14] and Seppäläinen [27]. Ferrari [12] surveys old and new results on the limiting behavior of a tagged particle in various interacting particle systems. Also see articles by Baryshnikov [4] and O'Connell and Yor [24] for some recent studies of Brownian queues in tandem, connected to the directed percolation and the directed polymer models, and the GUE random matrix ensemble.

More recently rank-dependent SDEs have been considered by several authors as possible models for financial or economic data. Fernholz, in [10], introduces the so-called *Atlas model* which we study in this paper. It is a model of finitely many Brownian particles where, at every time point, the minimum Brownian motion gets a constant positive drift, while the rest get no drift. The general rank-dependent interacting Brownian models, whose drifts and volatilities depend on time-varying ranks, have been considered by Banner, Fernholz and Karatzas in [3], with whom our work in this paper bears close resemblance. For SDE (1), they work under a specific condition on the drift sequence required for stability of the solution process. We prove in Theorem 8 [condition (13)] that this condition is indeed necessary and sufficient. Although their method is mostly based on a beautiful analysis of the local times of intersections of different Brownian motions, they also note the connection with the Harrison–Williams theory of reflected Brownian motions which we use extensively in this paper; see Lemma 4 for a complete statement.

In that same article [3], the authors establish marginal convergence of spacings Δ_k to exponential distributions. They leave open the question of joint convergence, which we settle in this paper (in Theorem 8) by proving that the vector of spacings converges in law to a vector of independent exponentials with different means. They also study ergodic properties of such processes, including a demonstration of the exchangeability of the indices of the Brownian particles under rank-dependent drifts and volatilities. In their later papers, Fernholz and Karatzas also consider generalized versions of (1), where the drifts and the volatilities depend on both the index and the rank of a Brownian particle. As expected, in most such cases, explicit descriptions of their properties become very difficult to obtain. However, many interesting results can still be recovered. A good source for what has been done thus far can be found in the recent survey article by Fernholz and Karatzas [11]. Also, see the article by Chatterjee and Pal [5] for a follow-up in this direction, where the authors consider an increasing number of Brownian particles in a rank-dependent motion and establish a connection with the Poisson–Dirichlet family of point processes.

McKean and Shepp, in [23], also consider Brownian motions interacting via their ranks. They start with two Brownian motions, their objective being to find the optimal drift under constraints (as a control) such that the probability that both Brownian motions never hit zero is maximized. The solutions is, as they establish, the Atlas model for the two particle system.

An interesting related model, studied by Rost and Vares in [26], replaces the ordered particles in the Harris model by linear Brownian motions repelled by their nearest neighbors through a potential. The authors study stationary measures for the spacings of such processes and show that rescaled combinations of spacings converge to an Ornstein–Uhlenbeck process.

Our purpose here is to draw attention to the general class of Brownian particle systems with rank-dependent drifts, as considered in [3]. Many natural questions about these systems remain open. We are particularly interested in an infinite version of the Atlas model, with a drift sequence $(\delta, 0, 0, \ldots)$, with *Atlas drift* $\delta > 0$. One result we obtain for this system is the following.

THEOREM 1. *For each $\delta > 0$, a sequence of independent* Exponential(2δ) *variables provides an equilibrium distribution of spacings for the infinite Atlas model with the Atlas drift δ.*

Theorem 1, which essentially follows from Theorem 14, suggests a number of interesting conjectures and open problems. In particular, we can immediately formulate the following.

CONJECTURE 2. *For each $\delta > 0$, Theorem 1 describes the unique equilibrium distribution of spacings for the infinite Atlas particle system with Atlas drift δ.*

Let $(\Delta_1(t), \Delta_2(t), \ldots)_{t\geq 0}$ denote the equilibrium state of spacings of the infinite Atlas Brownian particle system described by Theorem 1. This process has some subtle features. For each $k = 1, 2, \ldots$ and each $t > 0$,

$$(\Delta_k(t), \Delta_{k+1}(t), \ldots) \stackrel{d}{=} (\Delta_1(t), \Delta_2(t), \ldots) \tag{2}$$

and the common distribution of these sequences is that of independent exponential (2δ) variables. But, while both sides of (2) define stationary sequence-valued processes as t varies, these processes do not have the same law for all k. In particular, the finite-dimensional distributions of nonnegative stationary process $(\Delta_k(t), t \geq 0)$ depend on k.

Harris [15], equation (7.1), gave an explicit formula for the law of $B_{(0)}(t)$, the location at time t of the particle initially at 0 in the Harris system of ordered Brownian motions, from which he deduced for $2\lambda = 1$ that

$$\frac{B_{(0)}(t)}{t^{1/4}} \stackrel{d}{\to} \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\lambda}} B(1) \qquad \text{as } t \to \infty, \tag{3}$$

where $B(1)$ is standard Gaussian. As remarked by Arratia [1], page 71, the conclusion for general $\lambda > 0$ follows from the case $2\lambda = 1$ by Brownian scaling; see also De Masi and Ferrari [6], Rost and Vares [26] and Arratia [1], where variants (or generalizations) of (3) are proved for a tagged particle in the exclusion process on $\mathbb{Z}$ associated with a simple symmetric random walk. Harris conjectured that the process $B_{(0)}$ is not Markov and left open the problem of describing the long-run behavior of paths of $B_{(0)}$. These questions were answered by Dürr, Goldstein and

Lebowitz in [7], Theorem 7.1, where they prove that the *tagged* process $B_{(0)}(t)$, suitably rescaled, converges to fractional Brownian motion with Hurst parameter $1/4$. In fact, they show a general theorem where convergence to fractional Brownian motion holds for systems with processes which have stationary increments and perform *elastic collisions* as in the Harris model; see also [8] where the same authors generalize their results in the case where an external potential is present.

In this connection, we have the following conjecture.

CONJECTURE 3. *For each $\delta > 0$, in the infinite Atlas model with Atlas drift $\delta > 0$ and initially ordered values $X_{(k)}(0)$, $k = 1, 2, \dots$, which are the points of a Poisson process with rate 2δ on $(0, \infty)$,*

$$\frac{X_{(k)}(t) - X_{(k)}(0)}{t^{1/4}} \xrightarrow{d} \left(\frac{2}{\pi}\right)^{1/4} \frac{c_k}{\sqrt{2\delta}} B(1) \qquad \text{as } t \to \infty$$

for some sequence of constants $c_k > 0$ with $c_k \to 1$ as $k \to \infty$.

The paper is organized as follows. In the next section, we analyze the finite Brownian particle system with rank-dependent drifts. The main result is Theorem 8, which describes the convergence in total variation of the laws of the spacings to that of independent exponential distributions. The precise condition needed on the drift sequence for such stability is also proved. In Section 3, we look at countably infinite Brownian particles with the dynamics of the Atlas model. The main result, Theorem 1, follows readily from Theorem 14.

2. The finite Brownian particle system. We first present some results regarding the asymptotic behavior of spacings for the N-particle system with arbitrary rank-dependent drifts δ_i, $1 \le i \le N$. We start by recording the following two lemmas, which clarify the issues of existence and uniqueness of the N-particle system with arbitrary drifts δ_i, and characterize the associated ordered particle system. See the book by Revuz and Yor [25] for a background and the definitions of concepts from the calculus of continuous semimartingales.

LEMMA 4. *Let X_i, $1 \le i \le N$, be a solution of the SDE (1) defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, for some arbitrary initial condition and arbitrary drifts $\{\delta_j\}$. Then, for each $1 \le j \le N$, the jth ordered process $X_{(j)}$ is a continuous semimartingale with decomposition*

$$dX_{(j)}(t) = d\beta_j(t) + \frac{1}{\sqrt{2}}\big(dL_{(j-1,j)}(t) - dL_{(j,j+1)}(t)\big), \tag{4}$$

where the β_j's for $1 \le j \le N$ are independent $(\mathcal{F}_t)$-Brownian motions with unit variance coefficient and drift coefficients δ_j, where $L_{(0,1)} = L_{(N,N+1)} = 0$ and, for $1 \le j \le N-1$,

$$L_{(j,j+1)}(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1\big(\big(X_{(j+1)}(s) - X_{(j)}(s)\big)/\sqrt{2} \le \varepsilon\big)\, ds, \qquad t \ge 0, \tag{5}$$

which is half of the continuous increasing local time process at 0 *of the semimartingale* $(X_{(j+1)} - X_{(j)})/\sqrt{2}$. *Moreover, the ordered system is a Brownian motion in the domain*

$$\{(x_{(i)}, 1 \le i \le N) \in \mathbb{R}^N : x_{(1)} \le x_{(2)} \le \cdots \le x_{(N)}\} \tag{6}$$

with constant drift vector $(\delta_j, 1 \le j \le N)$ *and normal reflection at each of the* $N-1$ *boundary hyperplanes* $\{x_{(i)} = x_{(i+1)}\}$ *for* $1 \le i \le N-1$.

PROOF. Sznitman [29], page 594, [30], Lemma 3.7, gave these results in the case of zero drifts. As he observed, they follow from Tanaka's formula [25], page 223, and the definition of Brownian motion in a polyhedron with normal reflection; see, for example, Varadhan and Williams [31]. Sznitman further proves that the Brownian motions $\{\beta_j\}$ are independent, which is essentially due to Knight's theorem [25], page 183. The factors of $\sqrt{2}$ are most easily checked in the case $N = 2$. That they are the same for all N follows by a localization argument. The results of the lemma for general drifts δ_j are deduced from the results with no drift by application of the next two lemmas. $\square$

LEMMA 5. *Let* $X_1, X_2, \ldots$ *be a sequence of independent Brownian motions indexed by a finite set* I. *Consider the ordered process* $X_{(1)}, X_{(2)}, \ldots$. *The following then holds almost surely*

$$\sum_{i=1}^{|I|} \int_0^t 1\{X_i(s) = X_{(j)}(s)\}\, ds = t \qquad \forall j < |I| + 1, \forall t \in [0, \infty). \tag{7}$$

Moreover, the points of increments of the finite variation processes $L_{(j,j+1)}$ *are almost surely disjoint.*

When I *is countable, equation* (7) *still holds true, as long as the assumptions on the initial values* $X_1(0) < X_2(0) < \cdots$ *are sufficient to guarantee the existence of the ordered processes for all finite times.*

PROOF. For any two indices $k < l$, the Lebesgue measure of the set $\{t \in [0, \infty) : X_k = X_l\}$ is zero. This follows because the zero set of Brownian motion is of Lebesgue measure zero almost surely.

Now, the event $\sum_{i=1}^{|I|} \int_0^t 1\{X_i(s) = X_{(j)}(s)\}\, ds > t$ for some j and some t implies that there exists some pair (k, l) such that the Lebesgue measure of the set $\{0 \le s \le t : X_k(s) = X_{(j)}(s) = X_l(s)\}$ is positive. This is of measure zero according to the previous paragraph, and by countable additivity.

The other possibility of $\sum_{i=1}^{|I|} \int_0^t 1\{X_i(s) = X_{(j)}(s)\}\, ds < t$ is trivially ruled out by our assumption that the ordered processes are always achieved.

For the second assertion, note that, according to the general theory of semimartingale local times [25], the process $L_{(j,j+1)}$ increases only on the random closed set of times t when $X_{(j)}(t) = X_{(j+1)}(t)$. These random sets are almost surely disjoint as j varies. This is because, with probability 1, there is *no triple collision*, meaning a time $t > 0$ at which $X_i(t) = X_j(t) = X_k(t)$. It follows from the fact that the bivariate process $(X_i - X_j, X_j - X_k)$ is a linear transformation of a standard planar Brownian motion which does not hit points. $\square$

Call $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ the *canonical set-up* if $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is the usual space of continuous paths in $\mathbb{R}^N$, with the usual right-continuous filtration, and X_i is the ith coordinate process.

LEMMA 6. *Let $\delta = (\delta_i, 1 \le i \le N) \in \mathbb{R}^N$ and let μ be a probability distribution on $\mathbb{R}^N$.*

(i) *In the canonical setup there is a unique probability measure $\mathbb{P}^{\delta,\mu}$ under which the coordinate processes $(X_i, 1 \le i \le N)$ solve the system of SDEs* (1) *with initial distribution μ. In particular, for $\delta = 0$, the law $\mathbb{P}^{0,\mu}$ is the Wiener measure governing standard Brownian motion in $\mathbb{R}^N$ with initial distribution μ.*

(ii) *In the canonical setup, for each $t > 0$, the law $\mathbb{P}^{\delta,\mu}$ is absolutely continuous with respect to $\mathbb{P}^{0,\mu}$ on $\mathcal{F}_t$, with density*

$$\exp\left(\sum_{j=1}^{N} \delta_j \beta_j(t) - \frac{1}{2} \sum_{j=1}^{N} \delta_j^2 t\right), \tag{8}$$

where β_j, which is the same as in (4), *can be defined by the expression*

$$\beta_j(t) = \sum_{i=1}^{N} \int_0^t 1\{X_i(s) = X_{(j)}(s)\}\, dX_i(s), \qquad 1 \le j \le N. \tag{9}$$

Under $\mathbb{P}^{\delta,\mu}$, the β_j's are independent Brownian motions on $\mathbb{R}$ with drift coefficients $\{\delta_j\}$ and unit diffusion coefficients.

(iii) *If $(X_i, 1 \le i \le N)$ is a realization of the N-particle system with drifts $\{\delta_i\}$ and initial distribution μ on an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the $\mathbb{P}$ joint distribution of the processes X_i is identical to the $\mathbb{P}^{\delta,\mu}$ distribution of the coordinate processes on the canonical space, as specified in* (ii).

PROOF. This is an instance of a well-known general construction of the solution of an SDE with drift terms from one with no drift terms [25], Chapter IX, Theorem (1.10).

Under $\mathbb{P}^{0,\mu}$, the fact that $\{\beta_j\}$ is a collection of independent Brownian motions also follows from expression (9) and equation (7). Under $\mathbb{P}^{\delta,\mu}$, the process β_j is a stochastic integral with respect to Brownian motions with drifts. By expanding the drift term, it is obvious that β_j is a Brownian motion itself with drift δ_j. $\square$

Note that the SDE defining the N-particle system with drifts is a typical example of an SDE for which there is uniqueness in law, but not pathwise uniqueness.

For a solution X of SDE (1), let

$$\bar{X}(t) := \frac{1}{N}\sum_{i=1}^{N} X_i(t) = \frac{1}{N}\sum_{j=1}^{N} X_{(j)}(t),$$

which is the *center of mass* of the particle system, where we regard each particle as having mass $1/N$. We call the $\mathbb{R}^N$-valued process

$$(X_i - \bar{X}, 1 \le i \le N)$$

the *centered system.* Note that the $N-1$ spacings defined by taking differences of order statistics of the original system are identical to the $N-1$ spacings defined by differences of the order statistics of the centered system. So, the N order statistics of the centered system, which are constrained to have average 0, are an invertible linear transformation of the $N-1$ spacings of the original system.

LEMMA 7. *For the N-particle system, with arbitrary drifts and initial distribution*:

(i) *the center of mass process is a Brownian motion with drift*

$$\bar{\delta}_N := N^{-1}\sum_{j=1}^{N} \delta_j \tag{10}$$

and diffusion coefficient $1/N$; *explicitly,*

$$\bar{X}(t) = \bar{X}(0) + \bar{\delta}_N t + \frac{B(t)}{\sqrt{N}}, \tag{11}$$

where $B(t) := N^{-1/2}\sum_{j=1}^{N} B_j(t)$ *is a standard BM on* $\mathbb{R}$. *Consequently,*

$$\frac{\bar{X}(t)}{t} \to \bar{\delta}_N \qquad \textit{almost surely as } t \to \infty.$$

(ii) *The shifted center of mass process* $\bar{X}(t) - \bar{X}(0)$ *is independent of the centered system* $(X_i(\cdot) - \bar{X}(\cdot))$, $1 \le i \le n$.

PROOF. Part (i) clearly follows from the SDE (1) by summing over i.

For part (ii), note that if σ_1 and σ_2 are two independent sub-σ-algebras of a probability space $(A, \mathcal{A}, P)$ and we change P to another probability Q via defining $dQ/dP = fg$, where $f \in \sigma_1$, $g \in \sigma_2$, then σ_1 and σ_2 remain independent under Q.

Now, as in Lemma 6, when (X_i) denotes N independent Brownian motions with initial distribution μ, the shifted average $\bar{X}(t) - \bar{X}(0)$ and the centered process $Y = X - \bar{X}\mathbf{1}$ are independent. This follows from the facts that, conditionally on X_0, the processes are Gaussian with zero covariance, and that $\bar{X}(t) - \bar{X}(0)$ is independent of X_0. Now, to get to $\mathbb{P}^{\delta,\mu}$, the Radon–Nikodym derivative is given by (8). Note that, from expression (9), we get

$$\beta_j(t) = \sum_{i=1}^{N} \int_0^t 1\{Y_i(s) = Y_{(j)}(s)\}\, dY_i(s) + \bar{X}(t) - \bar{X}(0).$$

Thus, from (8), it is clear that $d\mathbb{P}^{\delta,\mu}/d\mathbb{P}^{0,\mu}$ can be written as fg, where $f \in \sigma(Y)$ and $g \in \sigma(\bar{X}(\cdot) - \bar{X}(0))$. Now, by first localizing at finite time, and using the argument in the preceding paragraph, we establish the claim in part (ii). $\square$

THEOREM 8. *For $1 \le k \le N$, let*

$$\alpha_k := \sum_{i=1}^{k} (\delta_i - \bar{\delta}_N), \tag{12}$$

where $\bar{\delta}_N$ is the average drift, as in (10). *For each fixed initial distribution of the N-particle system with drifts $\{\delta_i, i = 1, 2, \dots, N\}$, the collection of laws of $X_{(N)}(t) - X_{(1)}(t)$ for $t \ge 0$ is tight if and only if*

$$\alpha_k > 0 \qquad \textit{for all } 1 \le k \le N-1, \tag{13}$$

in which case the following results all hold:

(i) *The distribution of the spacings system $(X_{(j+1)} - X_{(j)}, 1 \le j \le N-1)$ at time t converges in total variation norm as $t \to \infty$ to a unique stationary distribution for the spacings system, which is that of independent exponential variables Y_j with rates $2\alpha_j$, $1 \le j \le N-1$. Moreover, the spacings system is reversible at equilibrium.*

(ii) *The distribution of the centered system at time t converges in total variation norm as $t \to \infty$ to a unique stationary distribution for the centered system, which is the distribution of*

$$\big(S_{\pi(i)-1} - \bar{S}, 1 \le i \le N\big),$$

where $S_0 := 0$ and $S_i := Y_1 + \cdots + Y_i$ for $1 \le i \le N-1$, where π is a uniform random permutation of $\{1, \dots, N\}$ which is independent of the Y_i, and

$$\bar{S} := \frac{1}{N} \sum_{i=1}^{N} S_{\pi(i)-1} = \frac{1}{N} \sum_{i=1}^{N-1} S_i = \frac{1}{N} \sum_{i=1}^{N-1} (N-i) Y_i.$$

Moreover, the centered system is reversible at equilibrium.

(iii) *As $t \to \infty$,*

$$X_i(t)/t \to \bar{\delta}_N \qquad \textit{almost surely for each } 1 \le i \le N$$

and the same is true for $X_{(i)}(t)/t$ instead of $X_i(t)/t$.

REMARK. Regard the system as split into *left-hand particles* of rank 1 to k and *right-hand particles* of rank $k+1$ to N. If these two parts of the system are started at some strictly positive distance from each other, they evolve independently like copies of the k-particle system and the $(N-k)$-particle system, respectively, until the first time there is a collision between a left-hand particle and a right-hand particle. It follows by summing over the corresponding drifts that the centers of mass of the two subsystems left to themselves would have almost sure asymptotic speeds $\bar{\delta}_k$ and $\hat{\delta}_k$, respectively, where

$$\bar{\delta}_k := \frac{1}{k}\sum_{i=1}^{k}\delta_i \quad \text{and} \quad \hat{\delta}_k := (N-k)^{-1}\sum_{i=k+1}^{N}\delta_i.$$

Since

$$\alpha_k = k\bar{\delta}_k - k\bar{\delta}_N = \frac{k(N-k)}{N}(\bar{\delta}_k - \hat{\delta}_k),$$

we see that $\alpha_k > 0$ if and only if $\bar{\delta}_k > \hat{\delta}_k$, which ensures that the right-hand system cannot avoid an eventual collision with the left-hand system. According to (12), for arbitrary prescribed $\bar{\delta}_N \in \mathbb{R}$, and $\alpha_k > 0$, the unique drift vector determining an ergodic N-particle system whose average drift is $\bar{\delta}_N$ and whose asymptotic spacings are independent exponential variables with rates $2\alpha_k$ is given by

$$\delta_i = \bar{\delta}_N + \alpha_i - \alpha_{i-1}, \qquad 1 \le i \le N, \tag{14}$$

where $\alpha_0 := \alpha_N := 0$. Given an arbitrary cumulative probability distribution function F on the line, and arbitrary $\bar{\delta} \in \mathbb{R}$ and $\varepsilon > 0$, it is clear that by taking N suitably large, we can choose $(\alpha_k, 1 \le k \le N-1)$ and hence $(\delta_j, 1 \le j \le N)$ so that, for all sufficiently large t,

$$\mathbb{P}\left(\sup_x \left|\frac{1}{N}\sum_{i=1}^{N} 1\bigl(X_i(t) - \bar{X}(t) \le x\bigr) - F(x)\right| > \varepsilon\right) < \varepsilon$$

and $\bar{X}(t)/t \to \bar{\delta}$ almost surely as $t \to \infty$. Thus, no matter what its initial distribution, the N-particle system looks asymptotically like a cloud of particles with mass distribution close to F drifting along the line at speed $\bar{\delta}$.

We would also like to mention here that a special case of the above theorem has been considered in a recent article by Jourdain and Malrieu [21]. They consider SDE (1) with an increasing sequence of δ_i's and establish joint convergence of the spacing system to independent exponentials as t goes to infinity.

PROOF OF THEOREM 8. According to Lemma 4, the ordered N-particle system is a Brownian motion $(X_{(k)}(t), 1 \le k \le N)_{t \ge 0}$ in the domain (6) with identity covariance matrix, constant drift vector $(\delta_j, 1 \le j \le N)$ and normal reflections at each of the $N-1$ boundary hyperplanes $\{x_{(i)} = x_{(i+1)}\}$ for $1 \le i \le N-1$. Note that the vector $(1, 1, \ldots, 1)$ of all ones is in the intersection of these boundary hyperplanes.

Let β be an independent one-dimensional Brownian motion ($\beta_0 = 1$) with a negative drift $-\theta$ ($\theta > 0$) which is reflected at the origin. The process defined by

$$Y_i'(t) := X_i(t) - \bar{X}(t) + \frac{\beta}{\sqrt{N}}, \qquad i = 1, 2, \ldots, N,$$

has the same spacings as the original process X. Moreover, by the independence of the centered system and the center of mass process established in Lemma 7, and the fact that $\bar{Y}' = \beta/\sqrt{N}$ is the projection of Y' on the subspace generated by the vector $(1, 1, \dots, 1)$, it follows that the vector $(Y_1', Y_2', \dots, Y_N')$ is a Brownian motion with identity covariance matrix which is normally reflected in the wedge

$$\left\{ y \in \mathbb{R}^N : \sum_{i=1}^{N} y_i \geq 0, y_1 \leq y_2 \leq \cdots \leq y_N \right\}.$$

The drift vector in this wedge can be written as a linear combination of spacings by summation by parts:

$$\sum_{i=1}^{N}\left(\delta_i - \bar{\delta}_N - \frac{\theta}{\sqrt{N}}\right) y_i = -\sum_{k=1}^{N-1}\left(\sum_{i=1}^{k}(\delta_i - \bar{\delta}_N)\right)(y_{k+1} - y_k) - \frac{\theta}{\sqrt{N}}\sum_{i=1}^{N} y_i. \tag{15}$$

Noting that $Y'_{(k+1)} - Y'_{(k)} \equiv X_{(k+1)} - X_{(k)}$ for all k, condition (13) for stability and part (i) of the lemma are now read from the general result about equilibrium distributions of reflecting Brownian motions stated in the following lemma, and standard theory of Harris recurrent Markov processes (see, e.g., [9]).

Part (ii) is established by showing that the centered system $(X_i(t) - \bar{X}(t), 1 \leq i \leq N)$, $t \geq 0$, is a Harris positive recurrent diffusion with the indicated invariant measure. The recurrence property has been proven in detail in [3]. The invariance is evident because a uniform randomization of labels relative to the centered order statistics is clearly invariant for the centered motion. This convergence in distribution, combined with part (i) of Lemma 6, gives convergence in probability of $X_i(t)/t$ to $\bar{\delta}_N$ for each $1 \leq i \leq N$. Almost sure convergence can now be justified by appeal to an ergodic theorem for the Harris recurrent centered diffusion. $\square$

The proof of Theorem 8 is completed by the following lemma, which we deduce from the general theory of stationary distributions for reflecting Brownian motions in polyhedra due to Williams [32].

LEMMA 9. *Let $R := (R_t, t \geq 0)$ be a Brownian motion in the domain*

$$\{x \in \mathbb{R}^K : b_i(x) \geq 0 \text{ for } i = 1, \dots, K\}$$

for some collection of K linearly independent linear functionals b_i, with R having identity covariance matrix, normal reflection at the boundary and constant drift δ with

$$\sum_{i=1}^{K} \delta_i x_i = -\sum_{i=1}^{K} a_i b_i(x), \qquad (x_1, \dots, x_K) = x \in \mathbb{R}^K. \tag{16}$$

This process R has a stationary probability distribution if and only if $a_i > 0$ for all $i = 1, \dots, K$, in which case, in the stationary state, the $b_i(R_t)$ are independent exponential variables with rates $2a_i$ and the process in its stationary state is reversible.

PROOF. This is read from the particular case of [32], Theorem 1.2, when the matrix Q is identically 0. According to that theorem, R is in duality with itself relative to the measure ρ on the domain whose density function with respect to Lebesgue measure at x is $\exp(2\delta \cdot x)$. Using (16), the linear change of variables to $y_i = b_i(x)$ shows that the ρ distribution of the $b_i(x)$ has joint density at $(y_1, \ldots, y_K)$ equal to $c \prod_{i=1}^{K} \exp(-2a_i y_i)$ for some $c > 0$. It follows that

$$\rho \text{ has finite total mass if and only if } a_i > 0 \text{ for all } i, \tag{17}$$

in which case, when ρ is normalized to be a probability, the ρ distribution of the $b_i(x)$ is that of independent exponential variables with rates $2a_i$. The "if" part of the conclusion is now evident. For the "only if" part, we argue that if a stationary probability distribution ρ' existed, it would obviously have a strictly positive density on the domain. Then, R sampled at time $0, 1, 2, \ldots$ would be an irreducible Harris recurrent Markov process with respect to ρ', hence $\rho = c\rho'$ for some $c > 0$ by the uniqueness of the invariant measure of a Harris recurrent Markov chain, and then $a_i > 0$ for all i by (17). □

From Theorem 8 we immediately deduce the following.

COROLLARY 10. *For each $\delta > 0$, the N-particle Atlas system with drift vector $(\delta, 0, \ldots, 0)$ is ergodic with average speed δ/N. The stationary distribution of*

$$\bigl(X_{(j+1)} - X_{(j)}, 1 \le j \le N-1\bigr)$$

is that of independent exponentials $(\zeta_j, 1 \le j \le N-1)$, where the rate of ζ_j is $2\delta(1 - j/N)$.

3. The infinite Atlas model. The infinite Atlas model can be described loosely as a countable collection of linear Brownian motions such that at every time point, the minimum Brownian motion is given a positive drift of $\delta > 0$ and the rest are left untouched. This is an example of (1), where $I = \mathbb{N}$, $\delta_1 = \delta$ and all other $\delta_i = 0$. Throughout this section, we take $\delta = 1$, since for our purposes here, the general case follows from the case when $\delta = 1$ by scaling.

The infinite Atlas model for $\delta = 1$ can be constructed rigorously in the weak solution framework in the following way. Start with the canonical setup (as in Lemma 6) of Brownian path space,

$$(C[0,\infty], \{\mathcal{F}_t\}_{0 \le t < \infty}, \mathbb{W}^x), \tag{18}$$

where $\{\mathcal{F}_t\}$ is the right-continuous filtration generated by the coordinates which satisfy the usual conditions and $\mathbb{W}^x$ is the law of the Brownian motion starting from $x \in \mathbb{R}$. We now look at the countable product of a sequence of spaces like (18),

$$\Omega = C[0,\infty]^{\mathbb{N}}, \qquad \mathcal{G}_t = \bigotimes_1^\infty \mathcal{F}_t(i), \qquad P^x = \bigotimes_1^\infty \mathbb{W}^{x_i}, \tag{19}$$

where the natural coordinate mapping is a countable collection of independent Brownian motions under P^x starting from the sequence $x = (x_1, x_2, \dots)$.

Let x be a sequence such that $x_{(1)} > -\infty$ and let $X = (X_1, X_2, \dots)$ denote the sequence of infinite independent Brownian motions starting from x. We have the following lemma whose proof will follow later.

LEMMA 11. *Assume that the initial sequence x is arranged in increasing order, $x_1 \le x_2 \le \cdots$, and satisfies*

$$\liminf_{n\to\infty} \frac{(x_n - x_1)^2}{n} > 0. \tag{20}$$

Then, P^x-almost surely, $X_{(1)}(t) > -\infty$ for all $t \ge 0$ and the process

$$N_t := \sum_{i=1}^{\infty} \int_0^t 1_{\{X_i(s) = X_{(1)}(s)\}}\, dX_i(s) \tag{21}$$

is a $\{\mathcal{G}_t\}$-martingale whose quadratic variation $\langle N\rangle_t \equiv t$. The stochastic exponential of N, given by

$$D_t = \exp(N_t - t/2), \tag{22}$$

is hence, again, a nonnegative $\{\mathcal{G}_t\}$-martingale.

We change the measure P^x by using the martingale D, that is, define Q by

$$Q^x|_{\mathcal{G}_t} := D_t \cdot P^x|_{\mathcal{G}_t}, \qquad t \ge 0.$$

By Girsanov's theorem, the probability measure Q^x exists and is well defined, and the coordinate process under Q^x is a solution of the infinite Atlas model (1). Hence, Q^x is the *unique* law of the Atlas model starting at x.

Our aim is the following: suppose the initial points $X_1(0) < X_2(0) < X_3(0) < \cdots$ are spread according to the Poisson process with rate two on the positive half-line and we run the infinite Atlas model starting from these points. We shall prove that the product law of independent Exponential(2) is invariant under the dynamics of the vector process Δ of spacings given by

$$\Delta_i(t) = X_{(i+1)}(t) - X_{(i)}(t), \qquad i = 1, 2, \dots. \tag{23}$$

The proof is achieved through a series of lemmas, the main argument involving the comparison of the infinite Atlas model with the finite Atlas model and suitably passing to the limit. Throughout the proof, the probability space is given by (19).

We start with the following lemma, whose proof follows directly from Lemma 5. We will find it convenient to use the following notation for the operator which sorts a given finite vector. If $x \in \mathbb{R}^n$, $1 \le n < \infty$, define

$$\mathcal{S}(x) = \bigl(x_{(1)}, x_{(2)}, \dots, x_{(n)}\bigr). \tag{24}$$

LEMMA 12. *Every $\omega \in \Omega$ comprises a sequence processes $\omega(t) = (\omega_1(t), \omega_2(t), \ldots)$. For any $N \in \mathbb{N}$, let us denote the ordered values of the processes with the first N indices by*

$$Z^N(\omega) = (Z_1^N, Z_2^N, \ldots, Z_N^N)(\omega) = \mathcal{S}((\omega_1, \omega_2, \ldots, \omega_N)).$$

Then,

$$D_N(t) = \exp\left(\sum_{i=1}^N \int_0^t 1_{\{X_i(s) = Z_1^N(s)\}} \, dX_i(s) - t/2\right) \tag{25}$$

is a $\mathcal{G}_t$-martingale. We denote by Q_N^x the probability measure obtained by changing the measure P^x by the martingale D_N.

Since, under every P^x, each of the Brownian motions is independent, it follows, by applying Girsanov's theorem, that, under Q_N^x, the first N coordinates $(\omega_1, \ldots, \omega_N)$ evolve according to the finite Atlas model, while the rest of the coordinates are independent Brownian motions with the corresponding initial starting points.

Let μ denote the probability measure whereby the points $0 = X_1(0) < X_2(0) < \cdots$ are such that the spacings $X_{i+1}(0) - X_i(0)$ are i.i.d. Exponential(2).

LEMMA 13. *For μ-almost every x, the measure Q^x exists and we can define*

$$Q \cdot \mu = \int Q^x \, d\mu(x).$$

PROOF. The proof follows from Lemma 11 and the law of large numbers. □

We now have our main theorem in this section which proves that μ is an invariant measure for the spacings of the infinite Atlas model.

THEOREM 14. *For any $K \in \mathbb{N}$, any function $F : \mathbb{R}^K \to \mathbb{R}$ which is smooth and has compact support and any time t, we have*

$$\mathrm{E}^{Q\cdot\mu}[F(\Delta_1, \Delta_2, \ldots, \Delta_K)(t)] = \mathrm{E}^{\mu}[F(\Delta_1, \Delta_2, \ldots, \Delta_K)(0)]. \tag{26}$$

Here, as defined in (23), Δ_i is the ith spacing $X_{(i+1)} - X_{(i)}$.

PROOF OF THEOREM 1. Since the previous result holds for a class of functions which determines the marginal distributions of a sequence-valued process, Theorem 1 follows readily for $\delta = 1$. The theorem for the infinite Atlas model with a general $\delta > 0$ follows by scaling. □

PROOF OF LEMMA 11. Fix an arbitrary time $T > 0$ and consider x as in the assumption of the lemma. We have the following claim.

CLAIM 15.

$$P^x\Big(\omega : \exists M(\omega) \text{ s.t. } \forall n \geq M, \inf_{0\leq s\leq T}\big(X_n(s) - X_{(1)}(s)\big) > 0\Big) = 1.$$

We prove the above claim by establishing the following:

$$P^x\Big(\omega : \exists M(\omega) \text{ s.t. } \forall n \geq M, \inf_{0\leq s\leq T}\big(X_n(s) - X_1(s)\big) > 0\Big) = 1. \tag{27}$$

This follows by defining the events

$$A_i := \Big\{\inf_{0\leq s\leq T}\big(X_i(s) - X_1(s)\big) \leq 0\Big\}, \qquad i = 1, 2, \dots. \tag{28}$$

Since $X_i - X_1$ is a Brownian motion starting from $(x_i - x_1)$, which for sufficiently large i is strictly positive [by (20)], by Bernstein's inequality [25], page 153, one can easily estimate $P^x(A_i) \leq \exp(-(x_i - x_1)^2/2T)$ for all sufficiently large i. Again, by assumption (20), we get

$$\begin{aligned}\limsup_{n\to\infty}(P^x(A_n))^{1/n} &\leq \limsup_n \exp\big(-(x_n - x_1)^2/2Tn\big)\\ &= \exp\Big(-\frac{1}{2T}\liminf_n \frac{(x_n - x_1)^2}{n}\Big) < 1.\end{aligned}$$

Thus, by Cauchy's root test, it follows that the series sum $\sum_{i=1}^{\infty} P^x(A_i) < \infty$. One can now apply the Borel–Cantelli lemma to obtain that $P^x(\limsup_i A_i) = 0$, which proves (27) and hence the required claim.

We now prove that the process N exists and is a martingale in the time interval $[0, T]$. Since T is arbitrary, this proves Lemma 11. Define the finite approximations

$$B_k(t) = \sum_{i=1}^{k}\int_0^t 1_{\{X_i(s)=X_{(1)}(s)\}}\,dX_i(s), \qquad 0 \leq t \leq T, k = 1, 2, \dots.$$

Each B_k is a stochastic integral with bounded, progressively measurable integrands. It is clear that they are martingales with quadratic variations

$$\begin{aligned}\langle B_k\rangle_t &= \sum_{i=1}^{k}\int_0^t 1_{\{X_i(s)=X_{(1)}(s)\}}\,ds\\ &= \mathbb{L}\Big\{0 \leq s \leq t : \min_{1\leq i\leq k} X_i(s) = X_{(1)}(s)\Big\},\end{aligned}$$

where $\mathbb{L}$ refers to Lebesgue measure on the line. We shall show that the sequence of martingales $\{B_k\}$ is a Cauchy sequence in the $\mathbb{H}^2$-norm and hence has a limit which is denoted by N, as in (21).

To see this, observe that for any $n, k \in \mathbb{N}$, we get

$$\langle B_{n+k} - B_n\rangle_T = \mathbb{L}\Big\{0 \leq s \leq T : \min_{n+1\leq i\leq n+k} X_i(s) = X_{(1)}(s)\Big\}.$$

By Claim 15, we see that $P^x(\lim_{n,k\to\infty}\langle B_{n+k}-B_n\rangle_T=0)=1$. It is also clear that $\langle B_{n+k}-B_n\rangle_T\le T$, for all $n,k\in\mathbb{N}$. It follows from the dominated convergence theorem that

$$\lim_{n,k\to\infty} E^{P^x}(\langle B_{n+k}-B_n\rangle_T)=0.$$

This, by definition, shows that the sequence of martingales $\{B_n\}$ is a Cauchy sequence in the $\mathbb{H}^2$-norm. Since the space of continuous martingales under that norm is complete, the sequence $\{B_n\}$ converges to a limiting martingale, which we denote N. It also follows that $\langle N\rangle_t=\lim_{n\to\infty}\langle B_n\rangle_t=t$. Thus, N is actually a $\{\mathcal{G}_t\}$-Brownian motion. This completes the proof of Lemma 11. $\square$

REMARK. The condition in (20) is clearly loose. In fact, if we consider A_i as in (28), all we require is that, for every $T>0$, one should have

$$\sum_{i=1}^{\infty} P^x(A_i)\le\sum_{i=1}^{\infty}\exp\bigl(-(x_i-x_1)^2/2T\bigr)<\infty,$$

which is a much weaker condition than required by (20).

The proof of Theorem 14 relies on probability estimates proved in the next three lemmas (17, 18 and 19). The second one is actually a generalization of the first. But, we choose to treat the first separately since it is simpler and more transparent. But, first, we will perform some basic computations.

LEMMA 16. *If $Y\sim\mathrm{Gamma}(r,\lambda)$ for some $r\ge 1$, then, for any $t>0$, we have*

$$\begin{aligned}\mathrm{E}(e^{-Y^2/2t}) &\le e^{t\lambda^2/2}(2\lambda^2 t)^{r/2}\frac{\Gamma(r/2)}{2\Gamma(r)}\\ &=\sqrt{\pi}\left(\frac{\lambda^2 t}{2}\right)^{r/2}\frac{e^{t\lambda^2/2}}{\Gamma((r+1)/2)}.\end{aligned}$$

PROOF.

$$\begin{aligned}\mathrm{E}(e^{-Y^2/2t}) &= \int_0^\infty e^{-y^2/(2t)}\frac{\lambda^r}{\Gamma(r)}y^{r-1}e^{-\lambda y}\,dy\\ &=\frac{\lambda^r}{\Gamma(r)}e^{t\lambda^2/2}\int_0^\infty y^{r-1}e^{-(y+t\lambda)^2/(2t)}\,dy \qquad (29)\\ &=\frac{\lambda^r}{\Gamma(r)}e^{t\lambda^2/2}\int_{\sqrt{t}\lambda}^\infty(\sqrt{t}z-t\lambda)^{r-1}e^{-z^2/2}\sqrt{t}\,dz,\qquad z=\frac{y+t\lambda}{\sqrt{t}}.\end{aligned}$$

For $z\ge\sqrt{t}\lambda$, since $r\ge 1$, one has

$$(\sqrt{t}z-t\lambda)^{r-1}\le(\sqrt{t}z)^{r-1}.$$

Thus, one can bound the (29) by

$$\begin{aligned}
\mathrm{E}(e^{-Y^2/2t}) &\le \frac{\lambda^r}{\Gamma(r)} e^{t\lambda^2/2} \int_{\sqrt{t}\lambda}^{\infty} (\sqrt{t}z)^{r-1} e^{-z^2/2} \sqrt{t}\, dz \\
&= \frac{(\lambda^2 t)^{r/2}}{\Gamma(r)} e^{t\lambda^2/2} \int_{\sqrt{t}\lambda}^{\infty} z^{r-1} e^{-z^2/2}\, dz \\
&= \frac{(\lambda^2 t)^{r/2}}{\Gamma(r)} e^{t\lambda^2/2} \int_{t\lambda^2/2}^{\infty} (2w)^{(r-1)/2} e^{-w} (2w)^{-1/2}\, dw, \qquad w = z^2/2 \\
&= \frac{(\lambda^2 t)^{r/2}}{\Gamma(r)} e^{t\lambda^2/2} 2^{r/2-1} \int_{t\lambda^2/2}^{\infty} w^{r/2-1} e^{-w}\, dw \\
&\le \frac{(2\lambda^2 t)^{r/2}}{2\Gamma(r)} e^{t\lambda^2/2} \Gamma(r/2).
\end{aligned}$$

The final identity in the lemma is due to the *duplication formula*:

$$\frac{\Gamma(r/2)}{\Gamma(r)} = \sqrt{\pi} \frac{2^{1-r}}{\Gamma((r+1)/2)}. \qquad \square$$

LEMMA 17. *For any $t > 0$ and all positive integers N satisfying $N+1 \ge 16et$, under $P \cdot \mu$, we have the following estimate of the probability of the event that during time $[0,t]$, the globally lowest ranked process is in fact the lowest ranked process among the processes with the first N indices*:

$$P \cdot \mu\{X_{(1)}(s) = Z_1^N(s), 0 \le s \le t\} \ge 1 - C_1 e^{2t} \left(\sqrt{\frac{4et}{N+1}}\right)^N,$$

where C_1 is a positive constant. To remind the reader, the processes $(Z_k^N, 1 \le k \le N)$ have been defined in Lemma 12.

PROOF. Note that the complement of the event has the following upper bound:

$$\begin{aligned}
&1 - P \cdot \mu\{X_{(1)}(s) = Z_1^N(s), 0 \le s \le t\} \\
&\quad \le P \cdot \mu\left(\bigcup_{i \ge N+1} \{X_i(s) \le Z_1^N(s), \text{ for some } s \in [0,t]\}\right) \\
&\quad \le P \cdot \mu\left(\bigcup_{i \ge N+1} \{X_i(s) \le X_1(s), \text{ for some } s \in [0,t]\}\right) \qquad (30) \\
&\quad \le \sum_{i=N+1}^{\infty} P \cdot \mu\bigl(X_i(s) \le X_1(s), \text{ for some } s \in [0,t]\bigr).
\end{aligned}$$

The final bound above is the so-called *union bound*.

We will now use exponential bounds for Brownian suprema to estimate $P \cdot \mu(X_i(s) \le X_1(s), \text{for some } s \in [0,t])$ which is the same as $P \cdot \mu(\inf_{0 \le s \le t}(X_i(s) -$

$X_1(s)) \le 0)$. Note that under $P \cdot \mu$, the process $X_i - X_1$ is a Brownian motion whose initial distribution is the law of $X_i(0) - X_1(0)$. This law is Gamma$(i-1, 2)$ since it is the sum of $(i-1)$ i.i.d. Exp(2). Conditional on $X(0) = x$, such that $\{X_i(0) - X_1(0) = y\}$, it follows from Bernstein's inequality that

$$P^x\Bigl(\inf_{0\le s\le t}\bigl(X_i(s) - X_1(s)\bigr) \le 0\Bigr) \le \exp(-y^2/2t).$$

Thus, for $i \ge 2$, we define $Y = X_i(0) - X_1(0) \sim$ Gamma$(i-1, 2)$ and use Lemma 16 to get

$$\begin{aligned} P \cdot \mu\Bigl(\inf_{0\le s\le t}\bigl(X_i(s) - X_1(s)\bigr) \le 0\Bigr) &\le \mathrm{E}(\exp(-Y^2/2t)) \\ &\le \sqrt{\pi}(2t)^{(i-1)/2}\frac{e^{2t}}{\Gamma(i/2)}. \end{aligned} \tag{31}$$

Plugging the estimate into (30), we get

$$1 - P \cdot \mu\bigl\{X_{(1)} = Z_1^N(s), 0 \le s \le t\bigr\} \le \sqrt{\pi}e^{2t}\sum_{i=N+1}^{\infty}\frac{(2t)^{(i-1)/2}}{\Gamma(i/2)}. \tag{32}$$

By Stirling's approximation, there exists some C, a positive constant, such that

$$\Gamma(z) \ge C^{-1}e^{-z}z^{z-1/2} \qquad \forall z \in \mathbb{R}^+. \tag{33}$$

Thus, from (32), we get

$$\begin{aligned} &1 - P \cdot \mu\bigl\{X_{(1)} = Z_1^N(s), 0 \le s \le t\bigr\} \\ &\quad\le C\sqrt{\pi}e^{2t}\sum_{i=N+1}^{\infty}\frac{(2t)^{(i-1)/2}e^{i/2}}{(i/2)^{(i-1)/2}} \\ &\quad= C\sqrt{\pi}e^{2t}\sum_{i=N}^{\infty}\Bigl(\frac{4et}{i+1}\Bigr)^{i/2}e^{1/2} \le \sqrt{\pi}Ce^{2t+1/2}\sum_{i=N}^{\infty}\biggl(\sqrt{\frac{4et}{N+1}}\biggr)^{i} \\ &\quad\le 2C\sqrt{\pi}e^{2t+1/2}\biggl(\sqrt{\frac{4et}{N+1}}\biggr)^{N}, \qquad \text{when } \sqrt{4et/(N+1)} \le 1/2. \end{aligned}$$

This proves the estimate. $\square$

LEMMA 18. *For any $t > 0$ and all positive integers $1 \le k < N \in \mathbb{N}$ satisfying $N - k + 2 \ge 16et$, under $P \cdot \mu$, we have the following estimate of the probability of the event that in the time interval $[0, t]$, the globally lowest k ranked processes are, in fact, the lowest k ranked processes among the ones with the first N indices*:

$$\begin{aligned} &P \cdot \mu\bigl\{\bigl(X_{(1)}, \ldots, X_{(k)}\bigr)(s) = (Z_1^N, \ldots, Z_k^N)(s), 0 \le s \le t\bigr\} \\ &\qquad\ge 1 - C_2(k)e^{2t}(N-1)^{k-1}\biggl(\sqrt{\frac{4et}{N-k+2}}\biggr)^{N-k+1} \end{aligned} \tag{34}$$

for some positive constant C_2 depending on k. For $k = 1$, we recover the bound in Lemma 17.

PROOF. As in the previous lemma, we bound the probability of complement of the event:

$$
\begin{aligned}
1 - P \cdot \mu\{&(X_{(1)}, \dots, X_{(k)})(s) = (Z_1^N, \dots, Z_k^N)(s), 0 \le s \le t\} \\
&\le P \cdot \mu\left(\bigcup_{i \ge N+1} \{X_i(s) \le Z_k^N(s), \text{ for some } s \in [0, t]\}\right) \\
&\le \sum_{i=N+1}^{\infty} P \cdot \mu\{X_i(s) \le Z_k^N(s), \text{for some } s \in [0, t]\}.
\end{aligned} \tag{35}
$$

Now, if $[N]$ denotes the set $\{1, 2, \dots, N\}$, let us note that

$$Z_k^N = \max_{i_1 < i_2 < \dots < i_{k-1}} \min_{l \in [N] \setminus \{i_1, i_2, \dots, i_{k-1}\}} X_l.$$

Thus, to get an upper bound on $P \cdot \mu\{X_i(s) \le Z_k^N(s), \text{for some } s \in [0, t]\}$, one can once more apply the union bound to obtain

$$
\begin{aligned}
P \cdot \mu\{&X_i(s) \le Z_k^N(s), \text{for some } s \in [0, t]\} \\
&\le \sum_{i_1 < \dots < i_{k-1}} P \cdot \mu\left(X_i(s) \le \min_{l \in [N] \setminus \{i_1, \dots, i_{k-1}\}} X_l(s), \text{for some } s \in [0, t]\right) \\
&\le \sum_{i_1 < \dots < i_{k-1}} P \cdot \mu\big(X_i(s) \le X_{i^*}, \text{for some } s \in [0, t], \\
&\qquad\qquad i^* = \min\{[N] \setminus \{i_1, \dots, i_{k-1}\}\}\big).
\end{aligned} \tag{36}
$$

Now, we can count the frequency with which i^* takes its possible values as the choice $i_1 < i_2 < \dots < i_{k-1}$ varies in $\{1, 2, \dots, N\}$. Let $g(i)$ be the number of ways to pick $i_1 < i_2 < \dots < i_{k-1}$ such that $i^* = i$. It is then straightforward to see that

$$
\begin{aligned}
g(1) &= \#\{\{i\} : i_1 > 1\} = \binom{N-1}{k-1}, \\
g(2) &= \#\{\{i\} : i_1 = 1, i_2 > 2\} = \binom{N-2}{k-2}, \\
&\vdots \\
g(l) &= \#\{\{i\} : i_1 = 1, i_2 = 2, \dots, i_{l-1} = l-1, i_l > l\} = \binom{N-l}{k-l}, \qquad l \le k, \\
g(l) &= 0 \qquad \forall l > k.
\end{aligned}
$$

Thus, by (36), we get

$$\sum_{i_1<i_2<\cdots<i_{k-1}} P\cdot\mu\Big(X_i(s) \le \min_{l\in[N]\setminus\{i_1,\dots,i_{k-1}\}} X_l(s), \text{ for some } s\in[0,t]\Big) \\ \le \sum_{l=1}^{k}\binom{N-l}{k-l} P\cdot\mu\big(X_i(s)\le X_l(s), \text{ for some } s\in[0,t]\big). \tag{37}$$

Now, again as we did when deriving the last key estimate, $X_i - X_l$ for $i > l$ is a Brownian motion under $P\cdot\mu$ with the initial distribution Gamma$(i-l, 2)$. Thus, by Lemma 16, we can bound

$$P\cdot\mu\big(X_i(s)\le X_l(s), \text{ for some } s\in[0,t]\big) \le \sqrt{\pi}\frac{(2t)^{(i-l)/2}e^{2t}}{\Gamma((i-l+1)/2)}.$$

Combining the above inequality with estimates (35), (36) and (37), we get

$$1 - P\cdot\mu\big\{\big(X_{(1)},\dots,X_{(k)}\big)(s) = (Z_1^N,\dots,Z_k^N)(s), 0\le s\le t\big\} \\ \le \sum_{i=N+1}^{\infty}\sum_{l=1}^{k}\binom{N-l}{k-l}\sqrt{\pi}e^{2t}\frac{(2t)^{(i-l)/2}}{\Gamma((i-l+1)/2)}. \tag{38}$$

We will again give a loose, but good enough, upper bound for the infinite sum. But, first, we need to note that, for any $n\ge k$, we have

$$\binom{n}{k}\Big/\binom{n-1}{k-1} = \frac{n}{k}\ge 1.$$

Thus, it follows that

$$\binom{N-l}{k-l}\le\binom{N-1}{k-1}\qquad \forall 1\le l\le k.$$

We can use this to simplify (38):

$$1 - P\cdot\mu\big\{\big(X_{(1)},\dots,X_{(k)}\big)(s) = (Z_1^N,\dots,Z_k^N)(s), 0\le s\le t\big\} \\ \le \sum_{i=N+1}^{\infty}\sum_{l=1}^{k}\binom{N-1}{k-1}\sqrt{\pi}e^{2t}\frac{(2t)^{(i-l)/2}}{\Gamma((i-l+1)/2)}.$$

We again use Stirling's approximation (33) to get

$$\sum_{i=N+1}^{\infty}\sum_{l=1}^{k}\binom{N-1}{k-1}\sqrt{\pi}e^{2t}\frac{(2t)^{(i-l)/2}}{\Gamma((i-l+1)/2)} \\ \le C\binom{N-1}{k-1}\sqrt{\pi}e^{2t}\sum_{i=N+1}^{\infty}\sum_{l=1}^{k}\frac{(2t)^{(i-l)/2}e^{(i-l+1)/2}}{((i-l+1)/2)^{(i-l)/2}} \\ = C\binom{N-1}{k-1}\sqrt{\pi}e^{2t+1/2}\sum_{i=N+1}^{\infty}\sum_{l=1}^{k}\bigg(\sqrt{\frac{4et}{i-l+1}}\bigg)^{i-l}. \tag{39}$$

Since $i-l \geq N+1-k$ for $l \leq k < N+1 \leq i$, it is clear that $4et/(i-l+1) \leq 4et/(N+2-k)$. Hence,

$$\sum_{l=1}^{k}\left(\sqrt{\frac{4et}{i-l+1}}\right)^{i-l} \leq \sum_{l=1}^{k}\left(\sqrt{\frac{4et}{N-k+2}}\right)^{i-l}.$$

By our assumption,

$$\sqrt{\frac{4et}{N-k+2}} \leq \frac{1}{2}. \tag{40}$$

Thus,

$$\left(\sqrt{\frac{4et}{N-k+2}}\right)^{i-l} \leq \left(\sqrt{\frac{4et}{N-k+2}}\right)^{i-k}, \qquad l \leq k < i,$$

and consequently

$$\sum_{l=1}^{k}\left(\sqrt{\frac{4et}{i-l+1}}\right)^{i-l} \leq k\left(\sqrt{\frac{4et}{N-k+2}}\right)^{i-k}.$$

Plugging this bound into (39), we get

$$\begin{aligned}
&\sum_{i=N+1}^{\infty}\sum_{l=1}^{k}\binom{N-1}{k-1}\sqrt{\pi}e^{2t}\frac{(2t)^{(i-l)/2}}{\Gamma((i-l+1)/2)} \\
&\quad \leq C\binom{N-1}{k-1}\sqrt{\pi}e^{2t+1/2}\sum_{i=N+1}^{\infty}k\left(\sqrt{\frac{4et}{N-k+2}}\right)^{i-k} \\
&\quad \leq Ck\binom{N-1}{k-1}\sqrt{\pi}e^{2t+1/2}\left(\sqrt{\frac{4et}{N-k+2}}\right)^{N+1-k} \qquad (41)\\
&\qquad \times \sum_{j=0}^{\infty}\left(\sqrt{\frac{4et}{N-k+2}}\right)^{j} \\
&\quad \leq C_2(k)e^{2t}(N-1)^{k-1}\left(\sqrt{\frac{4et}{N-k+2}}\right)^{N-k+1}, \qquad \text{by (40)},
\end{aligned}$$

for some positive constant C_2 depending on k. This proves the lemma. □

LEMMA 19. *For any $N \in \mathbb{N}$, define μ_N to be the law under which*

$$\begin{aligned}
&X_1(0) = 0, \\
&X_{i+1}(0) - X_i(0) \sim \operatorname{Exp}\bigl(2(1-i/N)\bigr), \qquad i = 1, 2, \ldots, N-1, \qquad (42)\\
\textit{and}\quad &X_{i+1}(0) - X_i(0) \sim \operatorname{Exp}(2), \qquad i = N, N+1, \ldots,
\end{aligned}$$

and all of these spacings are independent.

For any $t > 0$ and any three integers $k < J < N$ satisfying

$$J - k + 2 \geq 16et, \tag{43}$$

we have the following bound on the probability that under $P \cdot \mu_N$, during the time interval $[0, t]$, the lowest k ranked processes among the processes with the first N indices are, in fact, the lowest k among those with the first J indices:

$$\begin{aligned}
&P \cdot \mu_N\{(Z_1^J, \ldots, Z_k^J)(s) = (Z_1^N, \ldots, Z_k^N)(s), 0 \leq s \leq t\} \\
&\quad \geq 1 - C_2(k) e^{2t} (J-1)^{k-1} \left(\sqrt{\frac{4et}{J-k+2}}\right)^{J-k+1} \qquad (44) \\
&\qquad \times \left[1 - \left(\sqrt{\frac{4et}{J-k+2}}\right)^{N-J}\right].
\end{aligned}$$

PROOF. We follow the same line of argument as in the last two estimates. Thus,

$$\begin{aligned}
&1 - P \cdot \mu_N\{(Z_1^J, \ldots, Z_k^J)(s) = (Z_1^N, \ldots, Z_k^N)(s), 0 \leq s \leq t\} \\
&\quad \leq \sum_{i=J+1}^{N} P \cdot \mu_N\bigl(X_i(s) \leq Z_k^J(s), \text{for some } s \in [0, t]\bigr).
\end{aligned}$$

Following similar counting arguments as in (36) and (37), we can bound

$$\begin{aligned}
&1 - P \cdot \mu_N\{(Z_1^J, \ldots, Z_k^J)(s) = (Z_1^N, \ldots, Z_k^N)(s), 0 \leq s \leq t\} \\
&\quad \leq \sum_{i=J+1}^{N} \sum_{l=1}^{k} \binom{J-l}{k-l} P \cdot \mu_N\bigl(X_i(s) \leq X_l(s), \text{for some } s \in [0, t]\bigr).
\end{aligned} \tag{45}$$

Now, under μ_N, the gap $X_i(0) - X_l(0) = \sum_{j=l}^{i-1} Y_j$, where the Y_j's are independent and Y_j is distributed as $\mathrm{Exp}(2(1 - j/N))$. Thus, by the exponential bound used before, we get

$$P \cdot \mu_N\bigl(X_i(s) \leq X_l(s), \text{for some } s \in [0, t]\bigr) \leq \mathrm{E}\left[\exp\left(-\frac{1}{2t}\left(\sum_{j=l}^{i-1} Y_j\right)^2\right)\right]. \tag{46}$$

Now, since each Y_j is Exponential$(2(1 - j/N))$, the random variables

$$Y_j^* = (1 - j/N) Y_j$$

are i.i.d. Exp(2), and each $Y_j^* \leq Y_j$ since $j \leq N$. Thus, $\sum_{j=l}^{i-1} Y_j^* \leq \sum_{j=l}^{i-1} Y_j$ and, hence,

$$\mathrm{E}\left[\exp\left(-\frac{1}{2t}\left(\sum_{j=l}^{i-1} Y_j\right)^2\right)\right] \leq \mathrm{E}\left[\exp\left(-\frac{1}{2t}\left(\sum_{j=l}^{i-1} Y_j^*\right)^2\right)\right].$$

But, each Y_j^* is Exp(2) and, hence,

$$\mathrm{E}\left[\exp\left(-\frac{1}{2t}\left(\sum_{j=l}^{i-1} Y_j^*\right)^2\right)\right] = \mathrm{E}[e^{-Y^2/2t}],$$

where Y is a Gamma$(i-l, 2)$ random variable. Thus, plugging the bound from Lemma 16 into (46), we derive

$$P \cdot \mu_N\big(X_i(s) \le X_l(s), \text{for some } s \in [0,t]\big) \le \sqrt{\pi}\frac{e^{2t}(2t)^{(i-l)/2}}{\Gamma((i-l+1)/2)}.$$

We now follow approximations similar to (39) and (41) for the right-hand side of (45), to obtain

$$\begin{aligned}
1 - P \cdot \mu_N\{&(Z_1^J, \dots, Z_k^J)(s) = (Z_1^N, \dots, Z_k^N)(s), 0 \le s \le t\} \\
&\le \sum_{i=J+1}^{N} \sum_{l=1}^{k} \binom{J-1}{k-1} \sqrt{\pi} \frac{e^{2t}(2t)^{(i-l)/2}}{\Gamma((i-l+1)/2)} \\
&= C \binom{J-1}{k-1} \sqrt{\pi} e^{2t+1/2} \sum_{i=J+1}^{N} \sum_{l=1}^{k} \left(\sqrt{\frac{4et}{(i-l+1)}}\right)^{i-l} \\
&\le C \binom{J-1}{k-1} \sqrt{\pi} e^{2t+1/2} \sum_{i=J+1}^{N} k \left(\sqrt{\frac{4et}{(J-k+2)}}\right)^{i-k}, \qquad \text{by (43)}, \\
&\le C_2(k) e^{2t} (J-1)^{k-1} \left(\sqrt{\frac{4et}{J-k+2}}\right)^{J-k+1} \sum_{j=0}^{N-J-1} \left(\sqrt{\frac{4et}{J-k+2}}\right)^{j}.
\end{aligned}$$

Now, if we call $r = \sqrt{4et/(J-k+2)}$, then $r \le 1/2$, by assumption (43). The finite geometric sum can be easily bounded as

$$\sum_{j=0}^{N-J-1} r^j = \frac{1 - r^{N-J}}{1-r} \le 2(1 - r^{N-J}).$$

Thus, suitably altering the constant C_2, we get our desired bound,

$$\begin{aligned}
1 - P \cdot \mu_N\{&(Z_1^J, \dots, Z_k^J)(s) = (Z_1^N, \dots, Z_k^N)(s), 0 \le s \le t\} \\
&\le C_2(k) e^{2t} (J-1)^{k-1} \left(\sqrt{\frac{4et}{J-k+2}}\right)^{J-k+1} \left[1 - \left(\sqrt{\frac{4et}{J-k+2}}\right)^{N-J}\right].
\end{aligned}$$

This proves the lemma. $\square$

PROOF OF THEOREM 14. For every $K \le N \in \mathbb{N}$, let

$$Y_i^N = Z_{i+1}^N - Z_i^N, \qquad 1 \le i \le N.$$

From Corollary 10, we know that the law of the first $(N-1)$ spacings under μ_N (defined in Lemma 19) is exactly the stationary distribution of spacings for the finite Atlas model with N particles. Since, under Q_N (see Lemma 12), the dynamics of the processes $\{X_1(t), \dots, X_N(t)\}$ is that of a finite Atlas model independent of the rest of the Brownian motions, it follows by stationarity that for any $t > 0$, we have

$$\mathrm{E}^{Q_N \cdot \mu_N}[F(Y_1^N, \dots, Y_K^N)(t)] = \mathrm{E}^{\mu_N}[F(Y_1^N, \dots, Y_K^N)(0)]. \tag{47}$$

We will show that for fixed t and K, as N tends to infinity, the two sides of the above equation converge to the corresponding sides of (26). This will prove the theorem.

However, to do this, we will need an intermediary stage where, for an integer $J < N$, the dynamics of the process is according to Q_J, while the initial distribution of the spacings is either μ or μ_N. Define, for $K < J < N$, the following quantities:

$$\begin{aligned}
a &= \mathrm{E}^{Q_N \cdot \mu_N}[F(Y_1^N, \dots, Y_K^N)(t)] - \mathrm{E}^{Q_J \cdot \mu_N}[F(Y_1^J, \dots, Y_K^J)(t)], \\
b &= \mathrm{E}^{Q_J \cdot \mu_N}[F(Y_1^J, \dots, Y_K^J)(t)] - \mathrm{E}^{Q_J \cdot \mu}[F(Y_1^J, \dots, Y_K^J)(t)], \\
c &= \mathrm{E}^{Q_J \cdot \mu}[F(Y_1^J, \dots, Y_K^J)(t)] - \mathrm{E}^{Q \cdot \mu}[F(\Delta_1, \dots, \Delta_K)(t)]
\end{aligned}$$

and

$$d = \mathrm{E}^{\mu}[F(\Delta_1, \dots, \Delta_K)(0)] - \mathrm{E}^{\mu_N}[F(Y_1^N, \dots, Y_K^N)(0)].$$

It is clear that a, b, c and d all depend on t, K, J and N, although we choose to suppress this dependence in the notation. Also, it clearly follows from their definitions, combined with equality (47), that

$$|\mathrm{E}^{Q \cdot \mu}[F(\Delta_1, \dots, \Delta_K)(t)] - \mathrm{E}^{\mu}[F(\Delta_1, \dots, \Delta_K)(0)]| \le |a| + |b| + |c| + |d|.$$

We will now show that a, b, c, d all go to zero if we select a sequence of J and N such that J, N and N/J^2 go to infinity. This will prove the theorem.

Step 1 (Estimate of a). For $x_1 \le x_2 \le \cdots \le x_{K+1}$, define

$$G(x_1, x_2, \dots, x_{K+1}) := F(x_2 - x_1, x_3 - x_2, \dots, x_{K+1} - x_K).$$

Then, clearly, G is also a continuous bounded function. Assume that $\sup_x |G(x)| \le \alpha$. Now, by changing the measures from Q_N and Q_J to P, we get

$$\begin{aligned}
a &= \mathrm{E}^{Q_N \cdot \mu_N}[G(Z_1^N, \dots, Z_{K+1}^N)(t)] - \mathrm{E}^{Q_J \cdot \mu_N}[G(Z_1^J, \dots, Z_{K+1}^J)(t)] \\
&= \mathrm{E}^{P \cdot \mu_N}[D_N(t) G(Z_1^N, \dots, Z_{K+1}^N)(t) - D_J(t) G(Z_1^J, \dots, Z_{K+1}^J)(t)],
\end{aligned} \tag{48}$$

where D_N and D_J are the Radon–Nikodym derivative processes defined in (25). Define the event

$$\Gamma := \{(Z_1^N, \dots, Z_{K+1}^N)(s) = (Z_1^J, \dots, Z_{K+1}^J)(s), \forall s \in [0, t]\}.$$

If $\omega \in \Gamma$, then, by definition, $P \cdot \mu_N$-almost surely $D_N(t, \omega) = D_J(t, \omega)$. Thus, (48) can be written as

$$a = \mathrm{E}^{P\cdot\mu_N}\big[\big(D_N(t)G(Z_1^N, \dots, Z_{K+1}^N) - D_J(t)G(Z_1^J, \dots, Z_{K+1}^J)\big)1_{\Gamma^c}\big].$$

Thus, $|a|$ is bounded above by

$$\begin{aligned}
&\mathrm{E}^{P\cdot\mu_N}|D_N(t)G(\cdot)1_{\Gamma^c}| + \mathrm{E}^{P\cdot\mu_N}|D_J(t)G(\cdot)1_{\Gamma^c}| \\
&\quad\le \alpha\big(\mathrm{E}^{P\cdot\mu_N}|D_N(t)1_{\Gamma^c}| + \mathrm{E}^{P\cdot\mu_N}|D_J(t)1_{\Gamma^c}|\big) \\
&\quad\le \alpha\big(\|D_N(t)\|_N + \|D_J(t)\|_N\big)\sqrt{P\cdot\mu_N(\Gamma^c)}.
\end{aligned}\tag{49}$$

The final inequality is due to the Cauchy–Schwarz inequality, where the norm $\|\cdot\|_N$ refers to the $\mathbf{L}^2$-norm under the measure $P \cdot \mu_N$.

Now, under $P \cdot \mu_N$, both of the Radon–Nikodym derivatives $D_N(t)$ and $D_J(t)$ are equal in law to $\exp(B_t - t/2)$, where B is a standard Brownian motion. Thus, it is straightforward to see that

$$\|D_N(t)\|_N = \|D_J(t)\|_N = \exp(t/2).$$

Also, by Lemma 19 (put $k = K + 1$ in the statement), for large enough J and N, we have that $P \cdot \mu_N(\Gamma^c)$ is less than

$$C_2(K+1)e^{2t}(J-1)^K\left(\sqrt{\frac{4et}{J-K+1}}\right)^{J-K}\left[1-\left(\frac{4et}{J-K+1}\right)^{N-J}\right].$$

If we plug everything back into (49), we see that $|a|$ goes to zero as J, N and N/J^2 go to infinity, while keeping t and K fixed.

Step 2 (Estimate of b). Under Q_J^x, the vector $(Y_1^J, \dots, Y_K^J)$ depends only on the first J processes $X_1, X_2, \dots, X_J$ and is independent of X_i, $i > J$. Thus,

$$\mathrm{E}^{Q_J^x}[F(Y_1^J, \dots, Y_K^J)] = H(x_1, x_2, \dots, x_J),$$

where H is a bounded function since F is bounded. Thus, we have the following equality:

$$|b| = |\mathrm{E}^{\mu_N}(H(X_1(0), \dots, X_J(0))) - \mathrm{E}^{\mu}(H(X_1(0), \dots, X_J(0)))|.$$

If ν_N and ν denote the law of the vector $(X_1(0), \dots, X_J(0))$ under μ_N and μ, respectively, then, since we have assumed G (or, equivalently, F) to be bounded in absolute value by α, we have $|b| \le \alpha\|\nu_N - \nu\|_{\mathrm{TV}}$. Here, $\|\cdot\|_{\mathrm{TV}}$ refers to the total variation norm. We will now show that $\|\nu_N - \nu\|_{\mathrm{TV}}$ goes to zero as N, J and N/J^2 go to infinity.

Under μ_N, $X_1(0) = 0$ and each initial spacing $X_{i+1}(0) - X_i(0)$, $1 \le i \le N-1$, follows independent $\mathrm{Exp}(2(1 - i/N))$. While, under μ, $X_1(0) = 0$ and the spacings follow i.i.d. $\mathrm{Exp}(2)$. Now, the law of the vector $(X_i(0), 1 \le i \le J)$ is determined by the first J spacings, $(X_{i+1}(0) - X_i(0), 1 \le i \le J)$, which gives us the following inequality:

$$\begin{aligned}\|\nu_N - \nu\|_{\mathrm{TV}} &\le \int_{\mathbb{R}^J} \left| \prod_{i=1}^J 2(1-i/N)e^{-2(1-i/N)x_i} - 2^J e^{-2\sum_{i=1}^J x_i} \right| dx \\ &= \int_{\mathbb{R}^J} \left| \prod_{i=1}^J (1-i/N) e^{2ix_i/N} - 1 \right| 2^J e^{-2\sum_{i=1}^J x_i}\, dx.\end{aligned}$$

By an application of Cauchy–Schwarz inequality, we get

$$\begin{aligned}\|\nu_N - \nu\|_{\mathrm{TV}}^2 &\le \int_{\mathbb{R}^J} \left| \prod_{i=1}^J (1-i/N) e^{2ix_i/N} - 1 \right|^2 2^J e^{-2\sum_1^J x_i}\, dx, \\ &= 2^J \prod_{i=1}^J (1-i/N)^2 \int_{\mathbb{R}^J} \exp\left\{ -2\sum_{i=1}^J (1-2i/N)x_i \right\} dx \qquad (50) \\ &\quad - 2^{J+1} \prod_{i=1}^J (1-i/N) \int_{\mathbb{R}^J} \exp\left\{ -2\sum_{i=1}^J (1-i/N)x_i \right\} dx + 1.\end{aligned}$$

By the standard identity $\int e^{-\lambda x}\, dx = \lambda^{-1}$ for $\lambda > 0$, we get

$$\begin{aligned}\|\nu_N - \nu\|_{\mathrm{TV}}^2 &\le 2^J \prod_{i=1}^J (1-i/N)^2 \prod_{i=1}^J \bigl(2(1-2i/N)\bigr)^{-1} \\ &\quad - 2^{J+1} \prod_{i=1}^J (1-i/N) \prod_{i=1}^J \bigl(2(1-i/N)\bigr)^{-1} + 1 \qquad (51) \\ &= \frac{(1-1/N)^2(1-2/N)^2\cdots(1-J/N)^2}{(1-2/N)(1-4/N)\cdots(1-2J/N)} - 1.\end{aligned}$$

The following inequality is straightforward to prove:

$$e^{-2x} \le 1 - x \le e^{-x} \qquad \text{for all } 0 \le x \le \tfrac{1}{2}\log 2.$$

By our assumption that J, N and N/J^2 are going to infinity, we can assume, for all sufficiently large values of J and N, that $2J/N \le \log 2/2$. By the previous inequality, we get

$$e^{-2i/N} \le (1-i/N) \le e^{-i/N}, \qquad 1 \le i \le 2J/N,$$

and, consequently,

$$\begin{aligned}&\frac{(1-1/N)^2(1-2/N)^2\cdots(1-J/N)^2}{(1-2/N)(1-4/N)\cdots(1-2J/N)} \le \frac{\exp(-2\sum_{i=1}^J i/N)}{\exp(-2\sum_{i=1}^J 2i/N)} \\ &\text{and} \quad \frac{\exp(-4\sum_{i=1}^J i/N)}{\exp(-\sum_{i=1}^J 2i/N)} \le \frac{(1-1/N)^2(1-2/N)^2\cdots(1-J/N)^2}{(1-2/N)(1-4/N)\cdots(1-2J/N)}.\end{aligned} \qquad (52)$$

Thus, we can give upper and lower bounds:

$$e^{-J(J+1)/N} \le \frac{(1-1/N)^2(1-2/N)^2\cdots(1-J/N)^2}{(1-2/N)(1-4/N)\cdots(1-2J/N)} \le e^{J(J+1)/N}.$$

Combining this with (51), we see that as N, J and N/J^2 go to infinity, we clearly get that $|b|$ converges to zero.

Step 3 (Estimate of c). This is similar to the methods we used to estimate a. As in Step 1, we first employ a change of measure to obtain

$$c = \mathrm{E}^{P\cdot\mu}[D_J(t)G(Z_1^J,\dots,Z_{K+1}^J)(t)] - \mathrm{E}^{P\cdot\mu}[D(t)G(X_{(1)},\dots,X_{(K+1)})(t)].$$

Consider the event

$$\Gamma(t,J,K) = \{(X_{(1)},\dots,X_{(K+1)})(s) = (Z_1^J,\dots,Z_{K+1}^J)(s), 0\le s\le t\}. \tag{53}$$

On $\Gamma(t,J,K)$, the random processes $G(Z_1^J,\dots,Z_{K+1}^J)(s)$ are identical to $G(X_{(1)},\dots,X_{(K+1)})(s)$ in time $s\in[0,t]$. Also, the processes Z_1^J and $X_{(1)}$ are the same in the time interval $[0,t]$. Thus, the processes $D(t)$ and $D_J(t)$ are also the same. Since, by our assumption, $|G|$ is bounded by α, the following upper bound on $|c|$ holds:

$$\begin{aligned} |c| &\le \alpha \mathrm{E}^{P\cdot\mu}[(|D(t)|+|D_J(t)|)1_{\Gamma^c(t,J,K)}] \\ &\le \alpha(\|D_t\|+\|D_J(t)\|)\sqrt{P\cdot\mu(\Gamma^c(t,J,K))}, \end{aligned} \tag{54}$$

where, we denote by $\|\cdot\|$ the $\mathbf{L}^2$-norm under the measure $P\cdot\mu$.

Now, by Lemma 18 (for $k=K+1$ and $N=J$ in the statement), for large enough J, we get

$$P\cdot\mu(\Gamma^c(t,J,K)) \le C_2(K+1)e^{2t}(J-1)^K\left(\sqrt{\frac{4et}{J-K+1}}\right)^{J-K}. \tag{55}$$

Now, since t and K are fixed, as J tends to infinity, $P\cdot\mu(\Gamma^c(t,J,K))$ goes to zero. Finally, as in the estimate of a in Step 1, note that, by (22) and (25), we can assert that

$$D(t) = \exp(X_t - t/2) \quad \text{and} \quad D_N(t) = \exp(Y_t - t/2),$$

where X and Y are Brownian motions. Thus, $\|D(t)\| = e^{t/2} = \|D_N(t)\|$. If we plug these values in (54), we get that $|c|$ goes to zero as J tends to infinity.

Step 4 (Estimate of d). At time zero the indices are arranged in increasing order and, hence, $(\Delta_1,\Delta_2,\dots,\Delta_K)$ is obviously equal to $(Y_1^N,Y_2^N,\dots,Y_K^N)$. Since $|F|\le\alpha$, it follows that $|d|$ is bounded by α times the total variation distance between the law of $(\Delta_1,\Delta_2,\dots,\Delta_K)$ under μ and μ_N. A computation exactly like (50) gives us

$$\left(\frac{|d|}{\alpha}\right)^2 \le \frac{(1-1/N)^2(1-2/N)^2\cdots(1-K/N)^2}{(1-2/N)(1-4/N)\cdots(1-2K/N)} - 1.$$

But, since K is fixed and N grows to infinity, the right-hand side above goes to zero by a logic similar to (52). This proves the estimate.

Thus, combining Steps 1, 2, 3 and 4, we have proven the theorem. □

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