

One-Dimensional Disordered Supersymmetric Quantum Mechanics: A Brief Survey

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Abstract. We consider a one-dimensional model of localization based on the Witten Hamiltonian of supersymmetric quantum mechanics. The low energy spectral properties are reviewed and compared with those of other models with off-diagonal disorder. Using recent results on exponential functionals of a Brownian motion we discuss the statistical properties of the ground state wave function and their multifractal behaviour.

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1 Introduction

Many physical systems can be described using the concept of a random Hamiltonian. Such a formulation is useful when the Hamiltonian depends on a set of quenched variables. In most cases it is convenient to describe them by a set of random variables distributed according to some probability law. Consider, for instance, the quantum dynamics of a particle interacting with randomly distributed scatterers. If the potential is a sum of two-body potentials the Hamiltonian may be taken in the form

$$H = \frac{\mathbf{p}^2}{2m} + \sum_{k=1}^N V(\mathbf{r} - \mathbf{r}_k).$$

Here the quenched variables are the positions of the impurities \mathbf{r}_k and the number N of scatterers. They can, for instance, be modelled by independently random variables distributed according to a Poisson distribution.

In this context one of the most elementary quantities of physical interest is the average density of states $\rho(E)$. If the potential is repulsive and short range then $\rho(E)$ vanishes exponentially at the bottom of the spectrum

$$\rho(E) \underset{E \rightarrow 0}{\sim} e^{-C E^{-d/2}}$$

where d is the dimension of the space. This non-analytic behaviour was first discussed by Lifshits (Lifshits 1965) and then studied by a number of authors (see for example Luttinger and Waxler 1987). The physical mechanism which leads to this behaviour is the occurrence of large regions of space that are free from impurities and where the particle can move freely. Although these are exponentially rare events, they nevertheless contribute in the thermodynamic limit. This singular behaviour may also be derived using instanton techniques (Neuberger 1982).

However there do exist some systems for which the density of states has a very different behaviour:

1. The vibrations of a chain consisting of harmonic strings and random masses gives a spectral density with an accumulation of states at low frequency. This model, first introduced by Dyson (Dyson 1953), is in fact equivalent to the one-dimensional Anderson model with off-diagonal disorder.
2. In particle physics, the investigation of the random Dirac operator has stimulated interesting conjectures related to chiral symmetry breaking (Floratos et al. 1980). A particular model in dimension $2+1$ is to take as a Hamiltonian the square of the Euclidean Dirac operator coupled to a random magnetic field $H = -\not{D}^2 = -(\partial_\mu + iA_\mu)^2 + \frac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}$. In this case the low energy density of states must fulfil the inequality (Casher et al. 1984) $\rho(E) > \rho_o(E)$, where $\rho_o(E)$ is the free density of states. This means that there is an accumulation of low energy states which contributes to the chiral condensate.
3. The one-dimensional Schrödinger Hamiltonian $H = -\frac{d^2}{dx^2} + \phi^2(x) + \sigma_3\phi'(x)$ which was first introduced by Witten (Witten 1981) as a toy model of supersymmetric quantum mechanics (for a review see for example Cooper et al. 1995) provides a localization model with very unusual low energy spectral properties. In certain cases it gives rise to an accumulation of levels at low energy. The density of states displays either a power law behaviour $\rho(E) \underset{E \rightarrow 0}{\sim} E^{\mu-1}$ or a logarithmic singularity $\rho(E) \underset{E \rightarrow 0}{\sim} \frac{1}{E|\ln E|^3}$ of the same form as in the Dyson model (Dyson 1953).

A common feature of these three models is the fact that the zero energy wave function is exactly known for any realization of the disorder. Using this property we have presented, for the supersymmetric model (Comtet et al. 1995), a physical picture that accounts for the different behaviours of the density of states at the bottom of the spectrum. We believe that this model is probably generic, by which we mean representative of a whole class of systems in which the disorder is encoded in the ground state. Since this model is easier to handle, because one can use a wealth of techniques specific to one-dimensional systems, we will concentrate on this case. The recent literature shows a revival of interest for these problems, mainly in the context of condensed matter physics. We will briefly comment on this work and also draw attention to earlier work which is scattered in the literature and has so far remained unnoticed.

In part 2 we will review the basic mechanism which leads to these singularities and underline the differences with the usual Lifshits singularities. We will also mention some recent applications of the supersymmetric model to quantum spin chains. Applications to classical diffusion in a random medium (Bouchaud et al. 1990, Bouchaud and Georges 1990, Oshanin et al. 1993b) will not be discussed here.

In part 3, following (Broderix and Kree 1995) and (Shelton and Tsvelik 1997) we compute the correlation function of the zero energy states.

In part 4 we discuss the fluctuation properties of the ground state wave function using two different approaches. When $\phi(x)$ is white noise the wave function is an exponential functional of the Brownian motion. Such functionals were studied extensively both in the mathematical (Yor 1992) and physical literature (Monthus and Comtet 1994, Comtet et al. 1996). We will use our previous work to compute their statistical properties.

2 Spectral properties

The one-dimensional Schrödinger Hamiltonians

$$H_{\pm} = -\frac{d^2}{dx^2} + \phi^2(x) \pm \phi'(x) \quad (1)$$

may be rewritten in the factorized forms $H_+ = Q^\dagger Q$ and $H_- = QQ^\dagger$, where $Q \equiv -\frac{d}{dx} + \phi(x)$. This implies that H_+ and H_- have the same spectrum for $E > 0$. When $\phi(x)$ is random, they are characterized by the same localization length and density of states. These quantities have been computed exactly in two cases for which we now recall the main results.

2.1 White noise potential

$$\begin{cases} \langle \phi(x) \rangle = F_0 \\ \langle \phi(x)\phi(x') \rangle - \langle \phi(x) \rangle \langle \phi(x') \rangle = \sigma \delta(x - x') \end{cases}$$

The integrated density of states $N(E)$ and the localization length $\lambda(E)$ are respectively

$$\begin{aligned} N(E) &= \frac{2\sigma}{\pi^2} \frac{1}{J_\mu^2(z) + N_\mu^2(z)} \\ \lambda^{-1}(E) &= -\frac{\sigma z}{2} \frac{d}{dz} \ln (J_\mu^2(z) + N_\mu^2(z)) \end{aligned} \quad (2)$$

where $z \equiv \frac{\sqrt{E}}{\sigma}$ and $\mu \equiv \frac{F_0}{\sigma}$. $J_\mu(z)$ and $N_\mu(z)$ are Bessel functions.

Equation (2) was first obtained by Ovchinnikov and Erikmann (Ovchinnikov and Erikmann 1977) and then rediscovered independently in (Bouchaud et al. 1987). These results can be derived either by the node counting method (Luck

1992) or by the replica trick (Bouchaud et al. 1990). By this latter approach one can also compute the Green's function at non-coinciding points.

The low energy behaviour of the density of states and the localization length are given for $\mu = 0$ by

$$\begin{aligned} N(E) &\underset{E \rightarrow 0}{\sim} \frac{1}{\ln^2 E} \\ \lambda(E) &\underset{E \rightarrow 0}{\sim} -\ln E \end{aligned}$$

and by

$$\begin{aligned} N(E) &\underset{E \rightarrow 0}{\sim} E^\mu \\ \lambda(E) &\underset{E \rightarrow 0}{\sim} E^{\mu-1} \end{aligned}$$

when $\mu \neq 0$.

2.2 Random telegraph process

The function $\phi(x)$ is described by an ensemble of rectangular barriers (Bender-skii and Pastur 1974) with alternating heights ϕ_0 and ϕ_1 of random length l distributed according to an exponential law $p_{0,1}(l) = n_{0,1}e^{-n_{0,1}l}$.

In the case $\phi_0 = -\phi_1$ this model yields the same low energy behaviour as above (Comtet et al. 1995). The parameter μ is now given by $\mu = \frac{n_1 - n_0}{2\phi_0}$. The main interest of this model is to provide a physical picture of the low energy behaviour.

For $\mu \neq 0$ the potential $V(x) = \phi^2 + \phi'$ is constant everywhere except at the positions where $\phi(x)$ has a discontinuity. One thus obtains a sequence of δ functions with alternating signs. The attractive δ potentials can support bound states which would have exactly zero energy if one would ignore the couplings to the other peaks. By taking carefully into account these couplings, one can recover the low energy power law behaviour (Comtet et al. 1995). The physical picture that emerges from this analysis is that the *low energy states are localized at the positions of the impurities*. Therefore this is just the opposite mechanism from that in the Lifshits case. It would be interesting to generalize this approach to higher dimensions where similar behaviour can also occur.

For $\mu = 0$, since the positive and negative δ functions play a symmetric role this argument doesn't apply anymore. A study of the low energy states on a finite interval shows that the existence of quasi zero-modes can account for the logarithmic behaviour of the density of states.

Imposing Dirichlet boundary conditions on a finite interval $[-R, R]$ one finds that the ground state energy is (Comtet et al. 1995, Monthus et al. 1996)

$$E_0(R, \{\phi\}) \simeq \frac{1}{\int_{-R}^R dx' \psi_0^2(x')} \left(\frac{1}{\int_{-R}^0 \frac{dx}{\psi_0^2(x)}} + \frac{1}{\int_0^R \frac{dx}{\psi_0^2(x)}} \right).$$

This result is obtained by approximating the true ground state wave function near the boundaries by a suitable linear combination of the two linearly independent solutions of $H_+\psi = 0$:

$$\psi_0(x) = e^{\int^x dx' \phi(x')} \quad (3)$$

and

$$\psi_1(x) = \psi_0(x) \int^x \frac{dx'}{\psi_0^2(x')}. \quad (4)$$

If $\phi(x)$ is a white noise or a random telegraph process with Poissonian lengths, the typical behaviour of $\psi_0(x)$, given by the central limit theorem, is $\psi_0(x) \sim e^{\pm\sqrt{x}}$. Replacing $\psi_0(x)$ by its typical behaviour one finds that the energy E_0 is exponentially small in the length of the system $E_0 \sim e^{-\sqrt{R}}$. Therefore a quasi zero mode of energy E has a typical spatial extension $2R$ such that $R \sim \ln^2 E$. Coming back to the whole line one finds that the number of such states per unit length is

$$N(E) = \frac{1}{2R} \sim \frac{1}{\ln^2 E}.$$

Obviously this argument may be generalized to any one-dimensional disordered system for which the zero energy wave functions can be expressed in terms of the potential. This is in particular the case of the Anderson model with off-diagonal disorder. The discrete Schrödinger equation may be written in the form

$$\beta_{n+1}\varphi_{n+1} + \beta_n\varphi_{n-1} = E\varphi_n \quad (5)$$

where β_n are random variables. The model of Dyson of an harmonic chain with random masses belongs to this class. For any configuration of the disorder one can write down two independent zero energy solutions. One of them is obtained by solving the recurrence relation (5) with $E = 0$. A zero energy state satisfying the boundary conditions $\varphi_0 = 1$ and $\varphi_1 = 0$ is

$$\varphi_{2n} = \prod_{k=1}^n \frac{\beta_{2k-1}}{\beta_{2k}}.$$

If the β_n are independent identically distributed random variables, the typical behaviour of φ_n , given by the central limit theorem, is again of the form $|\varphi_n| \sim e^{\pm\sqrt{n}}$. One will therefore get the same low energy behaviour as before. For earlier references see (Theodorou and Cohen 1976, Markos 1988, Bovier 1989); another derivation of the logarithmic singularity is given in (Eggarter and Riedinger 1978).

2.3 Remarks

1. All these arguments are based on typical realizations of disorder. There exist however certain quantities whose behaviour cannot be obtained by this type of reasoning. This is in particular the case of the average ground state energy. Its dependence on the size of the sample has been obtained in (Monthus et al. 1996). It is given by the stretched-exponential function $\langle E_0 \rangle = \exp(-R^\xi)$ where the exponent ξ depends only on the nature of the correlations in the potential (the Gaussian white noise corresponds to $\xi = 1/3$). In (Monthus et al. 1996) it is shown that this behaviour is indeed supported by atypical realizations of the random potential.
2. The existence of a singular behaviour in the density of states implies, by the Thouless formula, that there will be a corresponding singularity in the localization length $\lambda(E)$. This quantity indeed diverges as $\ln E$ which reflects the appearance of a critical state at $E = 0$. It was recently pointed out (Steiner et al. 1997, Balents and Fisher 1997) that there exists another length scale in the system - the correlation length which controls the decay of the two-point Green's function. It behaves like $\ln^2 E$ and thus diverges faster than the localization length. These results are in agreement with those presented in (Bouchaud et al. 1990); although the full correlations were not computed exactly, it is shown in this paper that the two-point Green's function is indeed given by

$$\overline{\langle x | \frac{1}{H - E} | y \rangle} \underset{E \rightarrow 0}{\simeq} \sum_n c_n e^{-\frac{\sigma \pi^2 (2m+1)^2}{2 \ln^2 E} |x-y|}$$

It would be interesting to compare this method with Berezinskii's diagrammatic technique (Berezinskii 1974) recently used in (Steiner et al. 1997).

3. The existence of a critical state at $E = 0$ is best understood when this model is reinterpreted in the context of classical diffusion in a random medium. A diffusive behaviour at large time requires the existence of an extended state at $E = 0$ (Tossatti 1990).
4. The thermodynamic properties of some one-dimensional spin systems with random exchange couplings can be reinterpreted by using a mapping of the spin system onto a model of free fermions. This approach, which can be traced back to the pioneering work of Lieb, Schultz and Mattis (Lieb et al. 1961), was used by Smith (Smith 1970) in the context of the X-Y model. Exact result for quantum phase transition in random X-Y spin chains with a comparison with the renormalization group approach (Fisher 1994) were obtained by McKenzie (McKenzie 1996). Quite recently, a similar approach with a different type of disorder was developed by Fabrizio and Mélin (Fabrizio and Mélin 1997). It is also worth mentioning the nice paper of Steiner, Fabrizio and Gogolin (Steiner et al. 1997) extending this analysis to the case of correlations and boundary effects.

3 Correlation functions of the ground state wave function

The localization properties of the wave function can be characterized by the density-density correlation function. Various techniques have been developed, mainly by the Russian school (Lifshits et al. 1988), to compute such quantities in the weak disorder limit. In the supersymmetric model, one may take advantage of the fact that the ground state wave function is known exactly as a functional of the disorder. If the disordered potential $\phi(x)$ is white noise, this allows one to compute the corresponding n -point function by using a mapping with Liouville quantum mechanics. Such a calculation was recently carried out by Shelton and Tsvelik (Shelton and Tsvelik 1997) for the case $n = 2$ and $n = 3$ in the context of spin Peierls systems. An extension of this result to arbitrary n is given below. We first consider periodic boundary conditions. For completeness we also give the corresponding formula due to Broderix and Kree in the case of free boundary conditions (Broderix and Kree 1995).

3.1 Periodic boundary conditions

We consider the supersymmetric Hamiltonian (1) in which we set $\frac{\beta U(x)}{2} \equiv \int_0^x dx' \phi(x')$ for consistency with the notation of previous work (Monthus and Comtet 1994, Comtet et al. 1996). We are interested in the following two sections in the statistical properties of the zero mode wave function

$$\psi_0(x) = \frac{e^{\frac{\beta U(x)}{2}}}{\left[\int_0^L dx' e^{\beta U(x')} \right]^{1/2}} \quad (6)$$

when the disordered potential $\phi(x)$ is white noise. We consider a system of length L and impose periodic boundary conditions. This is achieved if the disordered potential is a Brownian bridge ($U(0) = U(L)$). The average over the disorder is performed through the Wiener measure

$$\langle \dots \rangle = \mathcal{N} \int_{U(0)=0}^{U(L)=0} \mathcal{D}U(x) \dots e^{-\frac{1}{2\sigma} \int_0^L dx \left(\frac{dU(x)}{dx} \right)^2} \quad (7)$$

where \mathcal{N} is a normalization to be determined. Our aim is to compute the correlation functions of the square wave function $\psi_0^2(x)$:

$$C_n(x_1, \dots, x_n) \equiv \langle |\psi_0(x_1)|^2 \dots |\psi_0(x_n)|^2 \rangle. \quad (8)$$

In order to perform the average over $U(x)$ it is convenient to exponentiate the denominator coming from the normalization of the wave function. By using the integral representation of the Γ function, one gets

$$\begin{aligned} & C_n(x_1, \dots, x_n) \\ &= \frac{\mathcal{N}}{\Gamma(n)} \int_0^\infty dp p^{n-1} \int_{U(0)=0}^{U(L)=0} \mathcal{D}U(x) e^{-\int_0^L dx \left[\frac{1}{2\sigma} \left(\frac{dU(x)}{dx} \right)^2 + p e^{\beta U(x)} \right] + \beta U(x_1) + \dots + \beta U(x_n)}. \end{aligned}$$

This expression establishes a link with one-dimensional Liouville quantum mechanics. For a detailed discussion of this model we refer to (Monthus and Comtet 1994) and (Kolokolov 1993, Kolokolov 1994). Considerable simplification occurs if one makes use of the fact that, in this theory, a change in the coupling constant can be interpreted as a translation in U space. This suggests the change of variable $p = \alpha e^{\beta U}$, where $\alpha \equiv \frac{\sigma \beta^2}{2}$, which leads to

$$C_n(x_1, \dots, x_n) = \frac{2\sqrt{\pi L} \alpha^{n+\frac{1}{2}}}{\Gamma(n)} \int_{-\infty}^{+\infty} dU \int_{U(0)=U}^{U(L)=U} \mathcal{D}U(x) e^{-S_L} e^{\beta U(x_1) + \dots + \beta U(x_n)} \tag{9}$$

where the action

$$S_L = \int_0^L dx \left[\frac{1}{2\sigma} \left(\frac{dU(x)}{dx} \right)^2 + \alpha e^{\beta U(x)} \right] \tag{10}$$

is associated with the Liouville Hamiltonian

$$H_L = -\frac{\sigma}{2} \frac{d^2}{dU^2} + \alpha e^{\beta U}. \tag{11}$$

The role of the normalization constant \mathcal{N} is to insure the relation

$$\int dx_1 \dots dx_n C_n(x_1, \dots, x_n) = 1.$$

Since \mathcal{N} is independent of n this relation may be used in the case $n = 1$ to find \mathcal{N} .

One may express the correlation function in terms of the Liouville propagator $G_x(U, U') \equiv \langle U | e^{-xH_L} | U' \rangle$. Choosing $L \geq x_1 \geq x_2 \geq \dots \geq x_n \geq 0$ where $x_{ij} \equiv x_i - x_j$ one gets

$$C_n(x_1, \dots, x_n) = \frac{2\sqrt{\pi L} \alpha^{n+\frac{1}{2}}}{\Gamma(n)} \int_{-\infty}^{+\infty} dU_1 dU_2 \dots dU_n G_{L-x_{1n}}(U_n, U_1) e^{\beta U_1} G_{x_{12}}(U_1, U_2) e^{\beta U_2} \dots G_{x_{n-1n}}(U_{n-1}, U_n) e^{\beta U_n}.$$

Using the eigenstates $|k\rangle$ of the Liouville Hamiltonian¹ gives

$$C_n(x_1, \dots, x_n) = \frac{2\sqrt{\pi L} \alpha^{n+\frac{1}{2}}}{\Gamma(n)} \int_0^{+\infty} dk_1 \dots dk_n \langle k_n | e^{\beta U} | k_1 \rangle \dots \langle k_{n-1} | e^{\beta U} | k_n \rangle e^{-\frac{\alpha(L-x_{1n})}{4} k_1^2 - \dots - \frac{\alpha x_{n-1n}}{4} k_n^2}. \tag{12}$$

Knowledge of the wave functions gives the matrix elements

$$\langle k | e^{\beta U} | k' \rangle = \frac{1}{8} \sqrt{kk' \sinh \pi k \sinh \pi k'} \frac{k^2 - k'^2}{\cosh \pi k - \cosh \pi k'}. \tag{13}$$

¹ The Liouville Hamiltonian has a continuous spectrum $H_L \psi_k(U) = \frac{\alpha k^2}{4} \psi_k(U)$ where the wave function is $\psi_k(U) = \frac{\sqrt{\beta k \sinh \pi k}}{\pi} K_{ik} \left(2e^{\frac{\beta U}{2}} \right)$.

Expression (12) allows one to get the long range behaviour (when all the distances involved are large compared to $\frac{1}{\alpha}$). Using the Laplace method one eventually finds

$$C_n(x_1, \dots, x_n) \simeq \frac{1}{(4\pi\alpha)^{\frac{n-1}{2}} \Gamma(n)} \frac{\sqrt{L}}{[(L - x_1 + x_n)(x_1 - x_2) \cdots (x_{n-1} - x_n)]^{3/2}}. \quad (14)$$

Despite the coefficients being different to those found by Shelton and Tsvelik (Shelton and Tsvelik 1997) when $n = 2$ and $n = 3$, we get the same behaviour as a function of the distances. This expression shows the existence of long range correlations. A nice interpretation of the algebraic tail was recently given in (Laloux and Le Doussal 1997). It is suggested that the exponent $3/2$ is associated with configurations of the disorder where $U(x)$ returns to its starting point. It is interesting to point out that the same exponent appears in conventional localization theory. In this case wave functions with the same (Berezinskii 1974, Gogolin 1975) or nearly equal (Gor'kov et al. 1983) energy are weakly correlated at large distances. The correlation function decays exponentially with the power law preexponential factor $(\frac{L}{x})^{3/2}$.

3.2 Free boundary conditions

Computation of correlation functions of the type in (8) was first performed by Broderix and Kree (Broderix and Kree 1995) in a different context: $|\psi_0(x)|^2$ is interpreted as the equilibrium Gibbs measure for the potential $U(x)$. Setting $U(0) = 0$ and letting $U(L)$ be free they get

$$C_n^{\text{BK}}(x_1, \dots, x_n) \simeq \frac{1}{\pi(4\pi\alpha)^{\frac{n-1}{2}} \Gamma(n)} \frac{1}{\sqrt{x_n(L - x_1)}} \frac{1}{[(x_1 - x_2) \cdots (x_{n-1} - x_n)]^{3/2}} \quad (15)$$

which is very similar to (14). The different boundary prescription therefore induces a slight change in the x dependance near the boundary.

4 Multifractality

In a recent series of publications, de Chamon et al. (de Chamon et al. 1996) have considered a two-dimensional localization model which exhibits a localization transition at zero energy. They consider a two-dimensional Dirac Hamiltonian in a random magnetic field $B(\mathbf{r}) = \Delta\phi(\mathbf{r})$. In this case the zero energy solution (Aharonov and Casher 1979) can be constructed explicitly for any realization of $\phi(\mathbf{r})$. They consider the particular "ground state" solution $\psi(\mathbf{r}) \propto (e^{-\phi(\mathbf{r})}, 0)$. For a Gaussian disorder $P[\phi] = \exp -\frac{1}{2g} \int d\mathbf{r} (\nabla\phi)^2$ the successive moments of $\psi(\mathbf{r})$ are encoded in the partition function $\mathcal{Z}(q) \equiv \int d\mathbf{r} e^{-2q\phi(\mathbf{r})}$ since $\int d\mathbf{r} \psi^{2q}(\mathbf{r}) = \frac{\mathcal{Z}(q)}{\mathcal{Z}(1)^q}$. The multifractal exponents of the wave function can be obtained by using

a formal equivalence with the problem of a directed polymer on the Cayley tree (Derrida and Spohn 1988).

In the following we consider a one-dimensional version of this model with, however, a different type of disorder and take into account the wave function normalization as in (Kogan et al. 1996). The starting point is the two-dimensional Euclidean Dirac operator

$$i\mathcal{D} = i\sigma_1 (\partial_y + iA_y) + i\sigma_2 (\partial_x + iA_x)$$

σ_i are the Pauli matrices and the gauge field is given by

$$\begin{cases} A_y = f(x) \\ A_x = 0. \end{cases}$$

We may take eigenstates of the form $\psi(x, y) = (\chi(x)e^{i\omega y}, 0)$. The eigenvalue equation $-\mathcal{D}^2\chi = E^2\chi$ then becomes

$$[-\partial_x^2 + (\omega + f(x))^2 - \sigma_3 f'(x)] \chi(x) = E^2\chi(x).$$

We are thus led to a one-dimensional Schrödinger equation with a supersymmetric potential $V(x) = \varphi^2(x) - \sigma_3\varphi'(x)$ where $\varphi(x) = \omega + f(x)$.

In an earlier work by one of us (Comtet et al. 1988) this approach was used to study the density of states of the two-dimensional Dirac operator. If $f(x)$ is white noise we may use the density of states of the one-dimensional problem and integrate over the free motion on the y axis. The resulting expression displays an enhancement at low energy as compared to the free case.

Here our purpose is to characterize the fluctuation properties of the ground state wave function $\chi(x) = e^{-\int dy\varphi(y)}$. By using two different approaches for two boundary prescription we compute exactly the successive moments of the normalized ground state

$$\psi_0(x) = \frac{\chi(x)}{\left[\int_0^L dx' \chi^2(x')\right]^{1/2}}$$

In order to keep unified conventions we will parametrize $\psi_0(x)$ as in equation (6).

4.1 Moments of $\psi_0(x)$ for periodic boundary conditions

If the random potential obeys periodic boundary conditions, the situation is the one described in section 3. The moment of order $2n$ is then related to the n -point correlation function at coinciding points

$$\langle |\psi_0(x)|^{2n} \rangle = C_n(x, \dots, x).$$

In terms of the Liouville propagator this is expressed as

$$\begin{aligned} \langle |\psi_0(x)|^{2n} \rangle &= \frac{2\sqrt{\pi L} \alpha^{n+\frac{1}{2}}}{\Gamma(n)} \int_{-\infty}^{+\infty} dU_1 G_L(U_1, U_1) e^{n\beta U_1} \\ &= \frac{2\sqrt{\pi L} \alpha^{n+\frac{1}{2}}}{\Gamma(n)} \int_0^\infty dk \langle k | e^{n\beta U} | k \rangle e^{-\frac{\alpha L}{4} k^2} \end{aligned}$$

where the matrix element is

$$\langle k | e^{n\beta U} | k \rangle = \frac{\Gamma(n)^2}{4\pi\Gamma(2n)} \prod_{m=0}^{n-1} (m^2 + k^2).$$

The moments are eventually given by

$$\langle |\psi_0(x)|^{2n} \rangle = \frac{\alpha^n}{2^n(2n-1)!!} \sum_{m=1}^n a_m^n 2^m (2m-1)!! \frac{1}{(\alpha L)^m} \tag{16}$$

where the coefficients a_m^n are defined by the equation

$$\prod_{m=0}^{n-1} (m^2 + X) = \sum_{m=1}^n a_m^n X^m.$$

For example $a_n^n = 1$ and $a_1^n = \Gamma(n)^2$.

One may extract from this expression the asymptotic dependance of the moments (when L is large compared to $\frac{1}{\alpha}$)

$$\langle |\psi_0(x)|^{2n} \rangle \simeq \left(\frac{\alpha}{2}\right)^{n-1} \frac{\Gamma(n)^2}{(2n-1)!!} \frac{1}{L}. \tag{17}$$

4.2 Moments of $\psi_0(x)$ for free boundary conditions

If one leaves $U(L)$ free instead of imposing periodic boundary, then the Brownian that enters in the wave function (6) is no longer a Brownian bridge but a free Brownian motion starting from $U(0) = 0$.

In the average over the disordered potential one must now take into account all the Brownian paths starting from 0 without any restriction on the final point. Except for this slight modification the formalism is the same as in previous sections. One is led to

$$\begin{aligned} \langle \psi_0^{2n}(x_1) \rangle &= \frac{\beta\alpha^n}{\Gamma(n)} \int_{-\infty}^{+\infty} dU \int_{U(0)=U} \mathcal{D}U(x) e^{-S_L} e^{n\beta U(x_1)} \\ &= \frac{\beta\alpha^n}{\Gamma(n)} \int_{-\infty}^{+\infty} dU dU_1 dU_2 G_{L-x}(U_2, U_1) e^{n\beta U_1} G_x(U_1, U). \end{aligned}$$

It is interesting to present another derivation of the moments using the language of previous works (Oshanin et al. 1993a, Monthus and Comtet 1994,

Comtet et al. 1996) devoted to the study of the statistical properties of the exponential functional $Z_L^{(\mu)} \equiv \int_0^L dx e^{-(\mu\alpha x + \sqrt{2\alpha}W(x))}$. Here μ is the drift for the Brownian motion $\beta U(x) = -(\alpha\mu x + \sqrt{2\alpha}W(x))$ and $W(x)$ is a Brownian motion of average 0 and variance 1 defined on $[0, L]$.

We will prove that the moments of $\psi_0(x)$ can be expressed in terms of the characteristic function $\phi^{(\mu)}(p, L) \equiv \langle e^{-pZ_L^{(\mu)}} \rangle$ (Laplace transform of the distribution law of $Z_L^{(\mu)}$). For this purpose we may note that $\psi_0(x)$ involves two exponential functionals. We may write

$$\psi_0^2(x) = \frac{1}{\int_0^L dy e^{\beta(U(y)-U(x))}}$$

and separate the denominator into two parts

$$\int_0^L dy e^{\beta(U(y)-U(x))} = \int_0^x dy e^{-[\mu\alpha y + \sqrt{2\alpha}B(y)]} + \int_0^{L-x} dy e^{-[\mu\alpha y + \sqrt{2\alpha}\tilde{B}(y)]}$$

where $B(y) \equiv W(x-y) - W(x)$ and $\tilde{B}(y) \equiv W(x+y) - W(x)$, respectively defined on $y \in [0, x]$ and $y \in [0, L-x]$, are two independent Brownian motions starting from zero $B(0) = \tilde{B}(0) = 0$. The denominator is then a sum of two statistically independent exponential functionals $Z_x^{(-\mu)}$ and $\tilde{Z}_{L-x}^{(\mu)}$.

The moments may then be rewritten as

$$\begin{aligned} \langle \psi_0^{2n}(x) \rangle &= \left\langle \frac{1}{\left(Z_x^{(-\mu)} + \tilde{Z}_{L-x}^{(\mu)} \right)^n} \right\rangle \\ &= \frac{1}{\Gamma(n)} \int_0^\infty dp p^{n-1} \langle e^{-pZ_x^{(-\mu)}} \rangle \langle e^{-p\tilde{Z}_{L-x}^{(\mu)}} \rangle \\ &= \frac{1}{\Gamma(n)} \int_0^\infty dp p^{n-1} \phi^{(-\mu)}(p, x) \phi^{(\mu)}(p, L-x). \end{aligned}$$

Where the characteristic functions are given in (Monthus and Comtet 1994, Comtet et al. 1996). In the case $\mu = 0$ which is of interest for us

$$\phi^{(0)}(p, L) = \frac{2}{\pi} \int_0^\infty ds \cosh \frac{\pi s}{2} K_{is} \left(2\sqrt{\frac{p}{\alpha}} \right) e^{-\frac{\alpha L}{4}s^2}. \tag{18}$$

One may extract from this expression the dominant behaviour when $\alpha x \gg 1$ and $\alpha(L-x) \gg 1$

$$\langle \psi_0^{2n}(x) \rangle \simeq \left(\frac{\alpha}{2} \right)^{n-1} \frac{\Gamma(n)^2}{\pi(2n-1)!!} \frac{1}{\sqrt{x(L-x)}}. \tag{19}$$

One may check that this result agrees with the one of Broderix and Kree (15) in the case $n = 1$. Whenever x belongs to the interval $[0, L]$, but is not on the edges, the moments still behave as $\langle \psi_0^{2n}(x) \rangle \sim \frac{1}{L}$.

We may also explore the behaviour of the moments of the wave function on the edges $\langle \psi_0^{2n}(0) \rangle = \langle \psi_0^{2n}(L) \rangle$. This quantity is equal to the moments of the partition function $Z_L^{(0)}$ for negative orders

$$\langle \psi_0^{2n}(0) \rangle = \left\langle \frac{1}{(Z_L^{(0)})^n} \right\rangle = \frac{1}{\Gamma(n)} \int_0^\infty dp p^{n-1} \phi^{(0)}(p, L).$$

We may easily extract the dominant behaviour of this quantity with the help of (18) and eventually find

$$\langle \psi_0^{2n}(0) \rangle \simeq \frac{\alpha^n \Gamma(n)}{\sqrt{\pi \alpha L}}. \quad (20)$$

This shows that the wave function fluctuates more on the edges of the interval (when $x \ll \frac{1}{\alpha}$ or $L - x \ll \frac{1}{\alpha}$) than in the bulk (when $x \gg \frac{1}{\alpha}$ and $L - x \gg \frac{1}{\alpha}$).

The square of the wave function on the edge is interpreted in the problem of classical diffusion in the random potential $U(x)$, as the steady current density when a constant density of particle is imposed at $x = 0$ in the presence of a trap at $x = L$. The distribution of the steady current was found in (Oshanin et al. 1993b, Monthus and Comtet 1994). Setting $J \equiv \frac{1}{\alpha} \psi_0^2(0)$ the distribution of J is a log-normal law for small values of J

$$P(J) \underset{J \rightarrow 0}{\simeq} \frac{1}{2\sqrt{\pi \alpha L}} e^{-\frac{1}{4\alpha L} \ln^2 J}$$

and possesses an exponential tail for large values of J

$$P(J) \underset{J \rightarrow \infty}{\simeq} \frac{1}{\sqrt{\pi \alpha L}} \frac{1}{J} e^{\frac{\pi^2}{4\alpha L} - J}$$

which is responsible of the behaviour of the moments given in (20).

4.3 Discussion of the results

We have just seen that the one-dimensional model is closely related to the model studied in (de Chamon et al. 1996). Since the one-dimensional model is exactly solvable, it is interesting to study its multifractal properties, although one cannot expect it to give the same behaviour. In fact, in both models the potential is long-range correlated; though in $d = 2$ the correlations are logarithmic whereas in $d = 1$ they are linear.

We define the scaling exponent $\tilde{\tau}(q)$ which characterizes the behaviour of the critical wave function at $E = 0$ as

$$\langle |\psi(\mathbf{r})|^{2q} \rangle \underset{L \rightarrow \infty}{\sim} L^{-d - \tilde{\tau}(q)}.$$

Using equation (17) we get $\tilde{\tau}(q) = 0$. This exponent is the scaling exponent of the average moments of the wave function. We may introduce a slightly different scaling exponent

$$\tau(q) = \lim_{L \rightarrow \infty} \frac{\langle \ln \int d\mathbf{r} |\psi_0(\mathbf{r})|^{2q} \rangle}{\ln(1/L)},$$

which is the mean exponent of the critical wave function. In terms of the partition function $Z_L^{(0)}(\beta) = \int_0^L dx e^{\beta U(x)}$ introduced before, it may be rewritten

$$\tau(q) = \lim_{L \rightarrow \infty} \frac{\langle \ln Z_L^{(0)}(q\beta) \rangle - q \langle \ln Z_L^{(0)}(\beta) \rangle}{\ln(1/L)}.$$

The mean free energy that appears has been calculated in (Comtet et al. 1996) to be

$$\langle \ln Z_L^{(0)}(\beta) \rangle = 2\sqrt{\frac{\alpha L}{\pi}} + C - \ln \alpha - \frac{\pi}{3\sqrt{\alpha L}} + O\left(\frac{1}{(\alpha L)^{3/2}}\right),$$

where $C = -\Gamma'(1)$ is the Euler-Mascheroni constant. Again, one eventually gets a scaling exponent $\tau(q) = 0$ which agrees with the behaviour found in (de Chamon et al. 1996) in the strong disorder regime. By using the mapping with the random directed polymer model (Derrida and Spohn 1988), de Chamon et al. have shown that this regime corresponds in fact to the low temperature phase of this model. The existence of such a link also appears in the one-dimensional case. A comparison of the moments of $Z_L^{(0)}(\beta)$ with those of the random energy model (REM) (Derrida 1981) reveals some striking similarities. For the one-dimensional case the expression of the moments given in (Monthus and Comtet 1994) read

$$\langle (Z_L^0)^n \rangle = \frac{1}{\alpha^n} \left(\frac{\Gamma(n)}{\Gamma(2n)} \sum_{k=1}^n (-1)^{n-k} e^{\alpha L k^2} C_{2n}^{k+n} + \frac{(-1)^n}{n!} \right).$$

In particular, for large L one obtains

$$\langle (Z_L^0)^n \rangle \underset{\alpha L \gg 1}{\simeq} \frac{1}{\alpha^n} \frac{\Gamma(n)}{\Gamma(2n)} e^{\alpha L n^2}.$$

These moments grow in the same manner as in the REM. The main difference is that, unlike in the REM, there is here no transition above which the behaviour would change from $e^{\alpha L n^2}$ to $e^{\alpha L n}$. In some sense one can consider that the transition temperature is sent to infinity. This explains why, in this one-dimensional case, one only probes the low temperature phase. It would be extremely interesting to explore intermediate cases with weaker correlations, as recently suggested in (Bouchaud and Mézard 1997).

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