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One-Dimensional Electron Gas with Delta-Function Interaction at Finite Temperature

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Integral equations which describe the thermodynamic properties of a one-dimensional electron gas with repulsive and attractive delta-function interactions are obtained. From these equations one can calculate the energy, entropy, magnetization, particle density and pressure at given temperature, magnetic field and chemical potential.

§ 1. Introduction

In recent papers Gaudin[1)] and Yang[2)] gave the ground state energy of a one-dimensional electron gas with a delta-function interaction[3)~5)] as a solution of a set of coupled integral equations. We try to treat the thermodynamic properties of this system as a one-dimensional Bose gas and a one-dimensional Heisenberg model.[8)] For this purpose it is necessary to obtain all of the energy eigenvalues of the Hamiltonian. In § 2 we review the work of Gaudin and Yang on the wave function. There appear two kinds of parameters k and Λ. In § 3 we make conjectures on the distributions of k's and Λ's in the complex plane. In § 4 the energy spectrum of the Hamiltonian for repulsive interaction is obtained and the integral equations which describe the thermodynamic properties are derived. In § 5 these integral equations are solved for some special cases. In §§ 6 and 7 we treat the electron gas with an attractive delta-function interaction.

§ 2. Wave function

We consider the eigenvalue problem of the Hamiltonian

$$\mathcal{H}=-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i{}^2}+4c\sum_{i<j}\delta(x_i-x_j)+\mu_0 H(2M-N), \tag{2·1}$$

where N is the number of electrons and M is the number of down-spin electrons. The wave function has the following form:

$$\Psi(x_1s_1, x_2s_2, \cdots, x_Ns_N)=\sum_j {}_M\Phi_j(x_1, x_2, \cdots, x_N)G_j{}^M. \tag{2·2}$$

Here x_i and s_i are the coordinate and spin-coordinate of the i-th electron, respectively. For a spin-$\frac{1}{2}$ electron, s_i is $+$ or $-$. $G_j{}^M$ is a spin function of which a typical one is

$$G_1{}^M=(\underbrace{++\cdots++}_{N-M}\underbrace{--\cdots-}_{M}).$$

${}_M\Phi_1$ is an eigenfunction of (2·1) which is antisymmetric to the permutation of $x_1, x_2, \cdots, x_{N-M}$ and to the permutation of $x_{N-M+1}, \cdots, x_N$, satisfying the condition

$$(1-\sum_{j=N-M+1}^{N} P_{N-M,j})\,{}_M\Phi_1=0\,. \tag{2·3}$$

Here $P_{i,j}$ is an operator which changes x_i and x_j. We can construct a full wave function Ψ using the fact that Ψ is totally antisymmetric. Gaudin and Yang gave the solution for this problem as follows:

$${}_M\Phi_1=\sum_P [Q,P]\exp(i\sum_{j=1}^{M} k_{Pj}x_{Qj}), \tag{2·4}$$

in the region $x_{Q1}<x_{Q2}<\cdots<x_{QN}$. Here Q and P are permutations of $1, 2, \cdots, N$ and $[Q,P]$ are $N!\times N!$ coefficients which are given by

$$[Q,P]=\varepsilon(Q_1)\varepsilon(Q_2)\sum_R A_R\prod_{j=1}^{M} F_P(\Lambda_{Rj}, y_j), \tag{2·5}$$

$$F_P(\Lambda, y)=\prod_{j=1}^{y-1}\frac{k_{Pj}-\Lambda+ic}{k_{P(j+1)}-\Lambda-ic}, \tag{2·6}$$

$$A_R=\prod_{\substack{i<j\\ Ri>Rj}} e\left(\frac{\Lambda_{Rj}-\Lambda_{Ri}}{2c}\right), \tag{2·7}$$

$$e(x)\equiv(x+i)/(x-i),$$

where $y_1<y_2, \cdots<y_M$ are coordinates of $x_{N-M+1}, x_{N-M+2}, \cdots x_N$ along the chain, Q_1 and Q_2 signify the orders of $1, 2, \cdots, N-M$ and $N-M+1, \cdots, N$ in the permutation Q. The parameters $\Lambda_1, \Lambda_2, \cdots, \Lambda_M$ are newly introduced. The periodic boundary condition

$${}_M\Phi_1(x_1, x_2, \cdots, x_i, \cdots, x_N)={}_M\Phi_1(x_1, \cdots, x_i+L, \cdots, x_N), \qquad i=1, 2, \cdots, N,$$

gives an equation for k's and Λ's as follows:

$$e^{ik_jL}=\prod_{\alpha=1}^{M} e\left(\frac{k_j-\Lambda_\alpha}{c}\right), \tag{2·8}$$

$$\prod_{j=1}^{N} e\left(\frac{\Lambda_\alpha-k_j}{c}\right)=\prod_{\beta\neq\alpha} e\left(\frac{\Lambda_\alpha-\Lambda_\beta}{2c}\right). \tag{2·9}$$

§ 3. Conjectures on the distribution of k's and Λ's in the complex plane

In this section we make three conjectures which are essential in the later sections of this paper.

Conjecture 1. If a set of solutions $(k_1, k_2, \cdots, k_N; \Lambda_1, \Lambda_2, \cdots, \Lambda_M)$ of (2·8) and (2·9) contain a complex k (or Λ), $\bar{k}$ (or $\bar{\Lambda}$) is also contained in the set of k's (or Λ's).

Corollary 1. At $c>0$, k's are real.

Proof: The conjecture 1 demands that the distributions of k's and Λ's are symmetric with respect to real axis. So we see that if $\mathrm{Im}\, k_j>0$, the absolute value of the right-hand side of (2·8) is larger than unity. On the other hand the left-hand side is smaller than unity because $\mathrm{Im}\, k_j>0$. So $\mathrm{Im}\, k_j>0$ is impossible. In the same way we can prove that $\mathrm{Im}\, k_j<0$ is also impossible. [Q.E.D.]

Conjecture 2. Complex Λ always forms a bound state with several other Λ's. In this set of Λ's the real parts of these Λ's are the same and the imaginary parts are $(n-1)ci$, $(n-3)ci, \cdots, -(n-1)ci$ for the bound state of $n-\Lambda$'s within the accuracy of $O(\exp(-\delta N))$, where δ is a positive number.

Conjecture 3. In the case $c<0$, complex k_α makes a pair with its complex conjugate $\bar{k}_\alpha$ and a real Λ, which we write as Λ_α'. The real parts of k_α, $\bar{k}_\alpha$ and Λ_α' are the same and the imaginary parts of k_α and $\bar{k}_\alpha$ are c and $-c$ within the accuracy of $O(\exp(-\delta L))$.

§ 4. Derivation of integral equations for the case of a repulsive interaction

In this case all k's are real by the corollary 1 in § 3. But Λ's are not necessarily real. We write Λ's as $\Lambda_\alpha^{n,j}$. Here n means that this belongs to a bound state of $n-\Lambda$'s, j specifies the imaginary part and α is the number of this bound state in the bound states of $n-\Lambda$'s. We write the real part of $\Lambda_\alpha^{n,j}$ by Λ_α^n. By the conjecture 2 we have

$$\Lambda_\alpha^{n,j}=\Lambda_\alpha^n+(n+1-2j)ci+O(\exp(-\delta N)),\ j=1,2,\cdots,n. \tag{4·1}$$

In the case of M_n bound states of n spins and N electrons we derive equations for Λ_α^n's and k_j's from Eqs. (2·8) and (2·9). Equations (2·8) can be rewritten as

$$e^{ik_jL}=\prod_{n=1}^{\infty}\prod_{\alpha=1}^{M_n} e\left(\frac{k_j-\Lambda_\alpha^n}{nc}\right), \qquad j=1,2,\cdots,N. \tag{4·2a}$$

Let us consider a product

$$\prod_{j=1}^{N} e\left(\frac{\Lambda_\alpha^n-k_j}{nc}\right).$$

By (4·1) this is transformed as

$$\prod_{j=1}^{N}\prod_{l=1}^{n} e\left(\frac{\Lambda_\alpha^{n,l}-k_j}{c}\right),$$

and by (2·9)

$$=\prod_{l=1}^{n}\left\{-\prod_{(m,\beta)}\prod_{h=1}^{m} e\left(\frac{\Lambda_\alpha^{n,l}-\Lambda_\beta^{m,h}}{2c}\right)\right\}=\prod_{l=1}^{n}\left\{\prod_{(m,\beta)\neq(n,\alpha)}\prod_{h=1}^{m} e\left(\frac{\Lambda_\alpha^{n,l}-\Lambda_\beta^{m,h}}{2c}\right)\right\}.$$

Substituting (4·1) we have finally

$$\prod_{j=1}^{N} e\left(\frac{\Lambda_\alpha{}^n - k_j}{nc}\right) = -\prod_{m=1}^{\infty}\prod_{\beta=1}^{M_m} E_{nm}\left(\frac{\Lambda_\alpha{}^n - \Lambda_\beta{}^m}{c}\right), \qquad \begin{array}{l} n=1,2,\cdots, \\ \alpha=1,2,\cdots,M_n, \end{array} \tag{4·2b}$$

where

$$E_{nm}(x) \equiv \begin{cases} e\left(\frac{x}{|n-m|}\right)e^2\left(\frac{x}{|n-m|+2}\right)e^2\left(\frac{x}{|n-m|+4}\right)\cdots e^2\left(\frac{x}{n+m-2}\right)e\left(\frac{x}{n+m}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } n\neq m, \\ e^2\left(\frac{x}{2}\right)e^2\left(\frac{x}{4}\right)\cdots e^2\left(\frac{x}{2n-2}\right)e\left(\frac{x}{2n}\right) \qquad \text{for } n=m. \end{cases}$$

The logarithms of these equations give

$$k_jL = 2\pi I_j - \sum_{n=1}^{\infty}\sum_{\alpha=1}^{M_n}\theta\left(\frac{k_j - \Lambda_\alpha{}^n}{nc}\right), \qquad j=1,2,\cdots,N, \tag{4·3a}$$

$$\sum_{j=1}^{N}\theta\left(\frac{\Lambda_\alpha{}^n - k_j}{nc}\right) = 2\pi J_\alpha{}^n + \sum_{m=1}^{\infty}\sum_{\beta=1}^{M_m}\Theta_{nm}\left(\frac{\Lambda_\alpha{}^n - \Lambda_\beta{}^m}{c}\right), \qquad \begin{array}{l} n=1,\cdots, \\ \alpha=1,2,\cdots,M_n \end{array} \tag{4·3b}$$

where $\theta(x) \equiv 2\tan^{-1}x$, $-\pi < \theta \leqq \pi$ and

$$\Theta_{nm}(x) \equiv \begin{cases} \theta\left(\frac{x}{|n-m|}\right) + 2\theta\left(\frac{x}{|n-m|+2}\right) + 2\theta\left(\frac{x}{|n-m|+4}\right) \\ \qquad\qquad + \cdots + 2\theta\left(\frac{x}{n+m-2}\right) + \theta\left(\frac{x}{n+m}\right) \qquad \text{for } n\neq m, \\ 2\theta\left(\frac{x}{2}\right) + 2\theta\left(\frac{x}{4}\right) + \cdots + 2\theta\left(\frac{x}{2n-2}\right) + \theta\left(\frac{x}{2n}\right) \qquad \text{for } n=m. \end{cases}$$

I_j's are different integers (half-odd integer) for even (odd) $M_1+M_2+\cdots$. This can be written as

$$I_j \equiv M_1 + M_2 + \cdots \qquad (\text{mod } 1). \tag{4·4a}$$

$J_\alpha{}^n$'s are different integers and satisfy the conditions

$$J_\alpha{}^n \equiv N - M_n + \tfrac{1}{2} \qquad (\text{mod } 1), \tag{4·4b}$$

$$|J_\alpha{}^n| \leqq \tfrac{1}{2}(N - 1 - \textstyle\sum t_{nm}M_m), \tag{4·4c}$$

where

$$t_{nm} = 2\,\text{Min}(n, m) - \delta_{nm}.$$

Giving a set of integers $\{I_j, J_\alpha{}^n\}$ which satisfies Eqs. (4·4), we can determine a set of k_j and $\Lambda_\alpha{}^n$ through Eqs. (4·3). For a set of integers $\{I_j, J_\alpha{}^n\}$ there is a set of omitted integers which satisfy Eqs. (4·4) and are not contained in $\{I_j, J_\alpha{}^n\}$. We define holes of k and holes of Λ^n as solutions of

$$Lh(k) = 2\pi \times (\text{omitted } I),$$

$$Lj_n(\Lambda) = 2\pi \times (\text{omitted } J_\alpha{}^n),$$

where

$$h(k)\equiv k+\frac{1}{L}\sum_{n=1}^{\infty}\sum_{\alpha=1}^{M_n}\theta\left(\frac{k-\Lambda_\alpha{}^n}{nc}\right), \tag{4·5a}$$

$$j_n(\Lambda)\equiv\frac{1}{L}\sum_{j=1}^{N}\theta\left(\frac{\Lambda-k_j}{nc}\right)-\frac{1}{L}\sum_{m=1}^{\infty}\sum_{\alpha=1}^{M_m}\Theta_{nm}\left(\frac{\Lambda-\Lambda_\alpha{}^m}{c}\right). \tag{4·5b}$$

Let us consider the case of a very large system. We put the distribution functions of k's and Λ^n's as $\rho(k)$ and $\sigma_n(k)$, and those of holes as $\rho^h(k)$ and $\sigma_n{}^h(k)$. By the definition of holes it is clear that

$$\frac{d}{dk}h(k)=2\pi(\rho(k)+\rho^h(k)), \tag{4·6a}$$

$$\frac{d}{dk}j_n(k)=2\pi(\sigma_n(k)+\sigma_n{}^h(k)). \tag{4·6b}$$

Equations (4·5a) and (4·5b) are rewritten as

$$h(k)=k+\sum_{n=1}^{\infty}\int\theta\left(\frac{k-k'}{nc}\right)\sigma_n(k')\,dk',$$

$$j_n(k)=\int\theta\left(\frac{k-k'}{nc}\right)\rho(k')\,dk'-\int\Theta_{nm}\left(\frac{k-k'}{c}\right)\sigma_m(k')\,dk'.$$

Hereafter we put that $\int dk$ means $\int_{-\infty}^{\infty}dk$. Substituting these into Eqs. (4·6) we have

$$\frac{1}{2\pi}=\rho(k)+\rho^h(k)-\sum_{n=1}^{\infty}[n]\sigma_n(k), \tag{4·7a}$$

$$[n]\rho(k)=\sigma_n{}^h(k)+\sum_{m=1}^{\infty}A_{nm}\sigma_m(k), \tag{4·7b}$$

where $[n]$ is an operator defined by

$$[n]f(k)\equiv\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n|c|}{(nc)^2+(k-k')^2}f(k')\,dk',$$

$$[0]f(k)\equiv f(k),$$

and

$$A_{nm}\equiv[|n-m|]+2[|n-m|+2]+2[|n-m|+4]+\cdots+2[n+m-2]+[n+m].$$

The energy per unit length is

$$E/L=\int(k^2-\mu_0H)\rho(k)\,dk+\sum_{n=1}^{\infty}2n\mu_0H\int\sigma_n(k)\,dk. \tag{4·8a}$$

The entropy per unit length is

$$S/L=\int\{(\rho+\rho^h)\ln(\rho+\rho^h)-\rho\ln\rho-\rho^h\ln\rho^h\}\,dk$$

$$+\sum_{n=1}^{\infty}\int\{(\sigma_n+\sigma_n{}^h)\ln(\sigma_n+\sigma_n{}^h)-\sigma_n\ln\sigma_n-\sigma_n{}^h\ln\sigma_n{}^h\}\,dk\,. \tag{4·8b}$$

The particle density is

$$N/L=\int\rho dk\,. \tag{4·8c}$$

The magnetization to the z-direction is

$$S_z/L=\tfrac{1}{2}\int\rho dk-\sum_n n\int\sigma_n dk\,. \tag{4·8d}$$

At the equilibrium state the thermodynamic potential $\Omega\equiv E-AN-TS$ should be minimized. So the variation of Ω is zero:

$$\begin{aligned}0=\delta\Omega/L=&\int(k^2-A-\mu_0H)\,\delta\rho(k)\,dk+\sum_{n=1}^{\infty}2n\mu_0H\int\delta\sigma(k)\,dk\\&-T\int\left\{\delta\rho\ln\left(\frac{\rho+\rho^h}{\rho}\right)+\delta\rho^h\ln\left(\frac{\rho+\rho^h}{\rho^h}\right)\right\}dk\\&-T\int\left\{\delta\sigma_n\ln\left(\frac{\sigma_n+\sigma_n{}^h}{\sigma_n}\right)+\delta\sigma_n{}^h\ln\left(\frac{\sigma_n+\sigma_n{}^h}{\sigma_n{}^h}\right)\right\}dk\,.\end{aligned} \tag{4·9}$$

From Eq. (4·7) we have

$$\delta\rho^h=-\delta\rho+\sum_{n=1}^{\infty}[n]\,\delta\sigma_n\,,$$

$$\delta\sigma_n{}^h=[n]\,\delta\rho-\sum_{m=1}^{\infty}A_{nm}\delta\sigma_m\,.$$

Substituting these into Eq. (4·9) we have

$$\begin{aligned}\frac{\delta\Omega}{TL}=&\int\left\{\frac{k^2-A-\mu_0H}{T}-\ln\left(\frac{\rho^h}{\rho}\right)-\sum_{n=1}^{\infty}[n]\ln\left(1+\frac{\sigma_n}{\sigma_n{}^h}\right)\right\}\delta\rho dk\\&+\sum_{n=1}^{\infty}\int\left\{\frac{2n\mu_0H}{T}-\ln\left(1+\frac{\sigma_n{}^h}{\sigma_n}\right)-[n]\ln\left(1+\frac{\rho}{\rho^h}\right)\right.\\&\left.+\sum_{m=1}^{\infty}A_{nm}\ln\left(1+\frac{\sigma_m{}^h}{\sigma_m}\right)\right\}\delta\sigma_n dk\,.\end{aligned}$$

Then we have a set of coupled nonlinear integral equations for $\zeta(k)\equiv\rho^h(k)/\rho(k)$ and $\eta_n(k)\equiv\sigma_n{}^h(k)/\sigma_n(k)$ as follows:

$$\ln\zeta(k)=\frac{k^2-A-\mu_0H}{T}-\sum_{n=1}^{\infty}[n]\ln(1+\eta_n{}^{-1}(k))\,, \tag{4·10a}$$

$$\ln(1+\eta_n(k))=\frac{2n\mu_0H}{T}-[n]\ln(1+\zeta^{-1}(k))+\sum_{m=1}^{\infty}A_{nm}\ln(1+\eta_m{}^{-1}(k))\,. \tag{4·10b}$$

Equations (4·7) are rewritten as

$$(1+\zeta(k))\rho(k)=\frac{1}{2\pi}+\sum_{n=1}^{\infty}[n]\sigma_n(k), \tag{4·11a}$$

$$[n]\rho(k)=\eta_n(k)\sigma_n(k)+\sum_{m=1}^{\infty}A_{nm}\sigma_m(k). \tag{4·11b}$$

From thermodynamics the pressure is given by

$$P=-\Omega/L.$$

Using (4·10) and (4·11) one obtains

$$P=T\int\ln(1+\zeta^{-1}(k))\frac{dk}{2\pi}. \tag{4·12}$$

This expression for the pressure is the same as that for bosons obtained by Yang and Yang.[7]

If we can solve Eqs. (4·10a), (4·10b), (4·11a) and (4·11b), we can determine the energy, entropy, particle density, magnetization and pressure for given temperature, chemical potential and magnetic field using (4·8a), (4·8b), (4·8c), (4·8d) and (4·12).

Equations (4·10a) and (4·10b) are equivalent to

$$[1]\{\ln(1+\eta_2)-\ln(1+\zeta^{-1})\}=([0]+[2])\ln\eta_1, \tag{4·13a}$$

$$[1]\{\ln(1+\eta_{n-1})+\ln(1+\eta_{n+1})\}=([0]+[2])\ln\eta_n,\quad n=2,3,\cdots, \tag{4·13b}$$

$$\ln(1+\eta_1)=\frac{2\mu_0H}{T}-[1]\ln(1+\zeta^{-1})+\sum_{m=1}^{\infty}A_{1m}\ln(1+\eta_m^{-1}), \tag{4·13c}$$

$$\ln\zeta=\frac{k^2-A-\mu_0H}{T}-\sum_{m=1}^{\infty}[m]\ln(1+\eta_m^{-1}). \tag{4·13d}$$

(4·13a) is obtained by $[1]\times$(first formula of (4·10b)) $-([0]+[2])\times$(4·10a). (4·13b) is obtained by $[1]\times\{(n-1$-th formula of (4·10b)) $+(n+1$-th formula of (4·10b))$\}-([0]+[2])\times(n$-th formula of (4·10b)). In the same way we can prove easily that Eqs. (4·11a) and (4·11b) are equivalent to

$$[1](\rho+\eta_2\sigma_2)=([0]+[2])(\eta_1+1)\sigma_1, \tag{4·14a}$$

$$[1](\eta_{n-1}\sigma_{n-1}+\eta_{n+1}\sigma_{n+1})=([0]+[2])(\eta_n+1)\sigma_n,\quad n=2,3,\cdots, \tag{4·14b}$$

$$[1]\rho=\eta_1\sigma_1+\sum_{m=1}^{\infty}A_{1m}\sigma_m, \tag{4·14c}$$

$$(1+\zeta)\rho=\frac{1}{2\pi}+\sum_{m=1}^{\infty}[m]\sigma_m. \tag{4·14d}$$

§ 5. Special cases for $c>0$

1) *The limit $c\to0$*

In this limit we can put

$$[n]f(k)=f(k)$$

for an arbitrary function $f(k)$. Then Eqs. (4·12) are written as

$$(1+\eta_2)/(1+\zeta^{-1})=\eta_1^2, \tag{5·1a}$$

$$(1+\eta_{n-1})(1+\eta_{n+1})=\eta_n^2, \tag{5·1b}$$

$$1+\eta_1=z^{-2}(1+\zeta^{-1})^{-1}\prod_{n=2}^{\infty}(1+\eta_n^{-1})^2, \tag{5·1c}$$

$$\zeta=e^{(k^2-A)/T}z\prod_{n=1}^{\infty}(1+\eta_n^{-1})^{-1}, \tag{5·1d}$$

where

$$z=\exp(-\mu_0 H/T).$$

The general solution of (5·1a) and (5·1b) is

$$\eta_n=f^2(n)-1\,, \qquad \zeta=\frac{f^2(0)}{1-f^2(0)}\,,$$

where

$$f(n)=(ba^n-b^{-1}a^{-n})/(a-a^{-1}).$$

The parameters a and b are functions of k and determined by (5·1c) and (5·1d). The results are

$$a=z \qquad \text{and} \qquad b=\sqrt{\left(1+z\exp\frac{k^2-A}{T}\right)\Big/\left(1+z^{-1}\exp\frac{k^2-A}{T}\right)}. \tag{5·2}$$

Equations (4·13) are transformed as

$$\rho+\eta_2\sigma_2=2(\eta_1+1)\sigma_1\,, \tag{5·3a}$$

$$\eta_{n-1}\sigma_{n-1}+\eta_{n+1}\sigma_{n+1}=2(\eta_n+1)\sigma_n\,, \tag{5·3b}$$

$$\rho=\eta_1\sigma_1+\sum_{m=1}^{\infty}A_{1m}\sigma_m\,, \tag{5·3c}$$

$$(1+\zeta)\rho=\frac{1}{2\pi}+\sum_{n=1}^{\infty}\sigma_n\,. \tag{5·3d}$$

The solution is

$$\rho=\frac{1}{2\pi}(b+b^{-1})\{-f(-1)\}, \tag{5·4a}$$

$$\sigma_n=\frac{1}{2\pi}f(0)\{-f(-1)\}\left(\frac{1}{f(n-1)f(n)}-\frac{1}{f(n)f(n+1)}\right). \tag{5·4b}$$

Substituting (5·2) into (5·4a) we have

$$\rho(k)=\frac{1}{2\pi}\left(\left(1+\exp\frac{k^2-A-\mu_0 H}{T}\right)^{-1}+\left(1+\exp\frac{k^2-A+\mu_0 H}{T}\right)^{-1}\right). \tag{5·5}$$

In the limit $c\to 0$ the quasi-momenta are real momenta. So (5·5) coincides with the well-known result.

2) *The limit $c\to\infty$*

In this limit Eqs. (4·10a), (4·10b), (4·11a) and (4·11b) become

$$\ln\zeta(k)=\frac{k^2-A-\mu_0H}{T}-\sum_{n=1}^{\infty}[n]\ln(1+\eta_n^{-1}(k)), \tag{5·6a}$$

$$\ln(1+\eta_n(k))=\frac{2n\mu_0H}{T}+\sum_{m=1}^{\infty}A_{nm}\ln(1+\eta_m^{-1}(k))+O\left(\frac{1}{c}\right), \tag{5·6b}$$

$$(1+\zeta(k))\rho(k)=\frac{1}{2\pi}+O\left(\frac{1}{c}\right), \tag{5·6c}$$

$$[n]\rho(k)=\eta_n(k)\sigma_n(k)+\sum_{m=1}^{\infty}A_{nm}\sigma_m(k). \tag{5·6d}$$

Equation (5·6b) are easily solved because $\eta_n(k)$ are all constants. The solution is

$$\eta_n=f^2(n)-1\,, \tag{5·7a}$$

where

$$f(n)=(z^{n+1}-z^{-n-1})/(z-z^{-1}),\qquad z=\exp(-\mu_0H/T).$$

From (5·6a) we have

$$\zeta(k)=e^{(k^2-A)/T}/(z+z^{-1}), \tag{5·7b}$$

and from (5·6c) we have

$$\rho(k)=\frac{1}{2\pi}\frac{z+z^{-1}}{z+z^{-1}+e^{(k^2-A)/T}}. \tag{5·7c}$$

Using (5·7a) one obtains

$$\sigma_n(k)=\frac{1}{z+z^{-1}}\left\{\frac{1}{f(n-1)f(n)}[n]-\frac{1}{f(n)f(n+1)}[n+2]\right\}\rho(k). \tag{5·7d}$$

So we have

$$S_z/L=N/2L-\sum_{n=1}^{\infty}n\int\sigma_n dk$$
$$=\frac{N}{L}\left[\frac{1}{2}-\sum_{n=1}^{\infty}\frac{n}{z+z^{-1}}\left\{\frac{1}{f(n-1)f(n)}-\frac{1}{f(n)f(n+1)}\right\}\right]=\frac{1}{2}\left(\frac{N}{L}\right)\tanh\frac{\mu_0H}{T}. \tag{5·7e}$$

This shows that the magnetization of the one-dimensional electron gas behaves as that of $\frac{1}{2}$-spins which are free each other when c is infinity.

3) *The limit $T\to 0$*

We put $\varepsilon_n(k)\equiv T\ln\eta_n(k)$ and $\kappa(k)\equiv T\ln\zeta(k)$. One can derive that

$$\varepsilon_n(k)=2\mu_0 H+[2]T\ln\left(1+\exp-\frac{\varepsilon_n(k)}{T}\right)+[1]T\ln\left(1+\exp\frac{\varepsilon_{n-1}(k)}{T}\right)$$
$$+([0]+[2])\sum_{j=1}^{\infty}[j]T\ln\left(1+\exp-\frac{\varepsilon_{j+n}(k)}{T}\right),\quad n=2,3,\cdots,$$

from Eqs. (4·10b). Therefore ε_2, ε_3, $\cdots$ are always positive. So in the limit $T\to 0$ we have a set of equations

$$\kappa(k)=k^2-A-\mu_0 H+[1]\varepsilon_1^-(k), \tag{5·8a}$$

$$\varepsilon_1(k)=2\mu_0 H+[1]\kappa^-(k)-[2]\varepsilon_1^-(k), \tag{5·8b}$$

where the suffices (+) and (−) mean

$$f^+(k)\equiv\begin{cases}f(k) & \text{at } f(k)>0,\\ 0 & \text{at } f(k)\leqq 0,\end{cases}\qquad f^-(k)\equiv\begin{cases}0 & \text{at } f(k)\geqq 0,\\ f(k) & \text{at } f(k)<0.\end{cases}$$

In Appendix A we prove that ε_1 and κ are increasing functions of k^2. So ε_1 and κ are negative in the regions $[B,-B]$ and $[Q,-Q]$, respectively. Then Eqs. (4·11) give

$$\rho(k)=\frac{1}{2\pi}+\frac{1}{\pi}\int_{-B}^{B}\frac{c}{c^2+(k-k')^2}\sigma_1(k')\,dk', \tag{5·9a}$$

$$\frac{1}{\pi}\int_{-Q}^{Q}\frac{c\rho(k')\,dk'}{c^2+(k-k')^2}=\sigma_1(k)+\frac{1}{\pi}\int_{-B}^{B}\frac{2c\sigma_1(k')\,dk'}{4c^2+(k-k')^2}. \tag{5·9b}$$

The energy, particle number and magnetization per unit length are given by

$$E/L=\int_{-Q}^{Q}k^2\rho(k)\,dk, \tag{5·9c}$$

$$N/L=\int_{-Q}^{Q}\rho(k)\,dk, \tag{5·9d}$$

$$S_z/L=\tfrac{1}{2}\int_{-Q}^{Q}\rho(k)\,dk-\int_{-B}^{B}\sigma_1(k)\,dk. \tag{5·9e}$$

These integral equations coincide with those which were obtained by Gaudin[1] and Yang.[2]

§ 6. Derivation of integral equations for the case of an attractive interaction

If there are pairs of two complex k's each of which has a parameter Λ on real axis. We designate these Λ's as Λ_α' and corresponding k's as k_α^1 and k_α^2. By the conjecture 3 we have

$$k_\alpha^1=\Lambda_\alpha'+i|c|+O(\exp(-\delta L)),$$
$$k_\alpha^2=\Lambda_\alpha'-i|c|+O(\exp(-\delta L)).$$

From Eq. (2·8) we have

$$\exp i(k_\alpha{}^1+k_\alpha{}^2)L=\prod_{\beta\neq\alpha} e\left(\frac{\Lambda_\alpha{}'-\Lambda_\beta{}'}{-2|c|}\right)\left\{\prod_{m,\beta,j} e\left(\frac{\Lambda_\alpha{}'-\Lambda_\beta{}^{n,j}}{-2|c|}\right)\right\} e\left(\frac{k_\alpha{}^1-\Lambda_\alpha{}'}{-|c|}\right) e\left(\frac{k_\alpha{}^2-\Lambda_\alpha{}'}{-|c|}\right). \tag{6·1}$$

From Eq. (2·9) we have

$$e\left(\frac{k_\alpha{}^1-\Lambda_\alpha{}'}{-|c|}\right) e\left(\frac{k_\alpha{}^2-\Lambda_\alpha{}'}{-|c|}\right)=\prod_{j=1}^{N-2M'} e\left(\frac{\Lambda_\alpha{}'-k_j}{-|c|}\right)\left\{\prod_{n,\beta,j} e\left(\frac{\Lambda_\alpha{}'-\Lambda_\beta{}^{n,j}}{2|c|}\right)\right\}.$$

Substituting this into Eq. (6·1) we have

$$e^{2i\Lambda_\alpha{}'L}=-\prod_{j=1}^{N-2M'} e\left(\frac{\Lambda_\alpha{}'-k_j}{-|c|}\right)\prod_{\beta=1}^{M'} e\left(\frac{\Lambda_\alpha{}'-\Lambda_\beta{}'}{-2|c|}\right). \tag{6·2}$$

Here we have represented unpaired k as k_j. From Eq. (2·8) one obtains

$$e^{ik_jL}=\prod_{\alpha=1}^{M'} e\left(\frac{k_j-\Lambda_\alpha{}'}{-|c|}\right)\prod_{n=1}^{\infty}\prod_{\alpha=1}^{M_n} e\left(\frac{k_j-\Lambda_\alpha{}^n}{-n|c|}\right). \tag{6·3}$$

And from Eq. (2·9) we have

$$\prod_{j=1}^{N-2M'} e\left(\frac{\Lambda_\alpha{}^n-k_j}{n|c|}\right)=-\prod_{m=1}^{\infty}\prod_{\beta=1}^{M_m} E_{nm}\left(\frac{\Lambda_\alpha{}^n-\Lambda_\beta{}^m}{|c|}\right). \tag{6·4}$$

The logarithms of Eqs. (6·2), (6·3) and (6·4) are

$$2\Lambda_\alpha{}'L=2\pi J_\alpha{}'+\sum_{j=1}^{N-2M'}\theta\left(\frac{\Lambda_\alpha{}'-k_j}{|c|}\right)+\sum_{\beta=1}^{M'}\theta\left(\frac{\Lambda_\alpha{}'-\Lambda_\beta{}'}{2|c|}\right),\quad \alpha=1,2,\cdots,M', \tag{6·5a}$$

$$k_jL=2\pi I_j+\sum_{\alpha=1}^{M'}\theta\left(\frac{k_j-\Lambda_\alpha{}'}{|c|}\right)+\sum_{n=1}^{\infty}\sum_{\alpha=1}^{M_n}\theta\left(\frac{k_j-\Lambda_\alpha{}^n}{n|c|}\right),\quad j=1,\cdots,N-2M', \tag{6·5b}$$

$$\sum_{j=1}^{N-2M'}\theta\left(\frac{\Lambda_\alpha{}^n-k_j}{n|c|}\right)=2\pi J_\alpha{}^n+\sum_{m=1}^{\infty}\sum_{\beta=1}^{M_m}\Theta_{nm}\left(\frac{\Lambda_\alpha{}^n-\Lambda_\beta{}^m}{|c|}\right),\quad \begin{array}{l}\alpha=1,2,\cdots,M_n,\\ n=1,2,\cdots.\end{array} \tag{6·5c}$$

Here $J_\alpha{}'$ is integer (half-odd integer) for $N-M'$ odd (even), I_j is integer (half-odd integer) for $M'+M_1+M_2+\cdots$ even (odd) and $J_\alpha{}^n$ is integer (half-odd integer) for $N-M_n$ odd (even). $J_\alpha{}^n$ should satisfy the condition

$$|J_\alpha{}^n|\leqq\tfrac{1}{2}(N-2M'-\sum_{m=1}^{\infty}t_{nm}M_m).$$

Following §4 we define $j'(\Lambda')$, $h(k)$ and $j_n(\Lambda^n)$:

$$j'(\Lambda')\equiv 2\Lambda'-\frac{1}{L}\left\{\sum_{j=1}^{N-2M'}\theta\left(\frac{\Lambda'-k_j}{|c|}\right)+\sum_{\beta=1}^{M'}\theta\left(\frac{\Lambda'-\Lambda_\beta{}'}{2|c|}\right)\right\}, \tag{6·6a}$$

$$h(k)\equiv k-\frac{1}{L}\left\{\sum_{\alpha=1}^{M'}\theta\left(\frac{k-\Lambda_\alpha{}'}{|c|}\right)+\sum_{n=1}^{\infty}\sum_{\alpha=1}^{M_n}\theta\left(\frac{k-\Lambda_\alpha{}^n}{n|c|}\right)\right\}, \tag{6·6b}$$

$$j_n(\Lambda^n)\equiv\frac{1}{L}\left\{\sum_{j=1}^{N-2M'}\theta\left(\frac{\Lambda^n-k_j}{n|c|}\right)-\sum_{m=1}^{\infty}\sum_{\beta=1}^{M_m}\Theta_{nm}\left(\frac{\Lambda^n-\Lambda_\beta{}^m}{|c|}\right)\right\}. \tag{6·6c}$$

Holes of Λ', k and Λ^n are defined as solutions of

$$j'(\Lambda') = 2\pi \times (\text{omitted } J'),$$

$$h(k) = 2\pi \times (\text{omitted } I),$$

$$j_n(\Lambda^n) = 2\pi \times (\text{omitted } J^n).$$

In the limit of a very large system we define the distribution functions of Λ', k_j, Λ_α^n as $\sigma'(k)$, $\rho(k)$, $\sigma_n(k)$ and those of holes as $\sigma'^h(k)$, $\rho^h(k)$, $\sigma_n^h(k)$. Using the relations

$$\frac{dj'(k)}{dk} = 2\pi(\sigma'(k) + \sigma'^h(k)),$$

$$\frac{dh(k)}{dk} = 2\pi(\rho(k) + \rho^h(k)),$$

$$\frac{dj_n(k)}{dk} = 2\pi(\sigma_n(k) + \sigma_n^h(k)),$$

we have equations for σ', ρ, σ_n, σ'^h, ρ^h and σ_n':

$$\frac{1}{\pi} = \sigma' + \sigma'^h + [2]\sigma' + [1]\rho\,, \tag{6·7a}$$

$$\frac{1}{2\pi} = \rho + \rho^h + [1]\sigma' + \sum_n [n]\sigma_n\,, \tag{6·7b}$$

$$[n]\rho = \sigma_n^h + \sum_m A_{nm}\sigma_m\,. \tag{6·7c}$$

The definitions of $[n]$ and A_{nm} were given in § 4.

The energy per unit length is

$$E/L = \int (k^2 - \mu_0 H)\rho dk + \int 2(k^2 - c^2)\sigma' dk' + \sum_{n=1}^{\infty} 2n\mu_0 H \int \sigma_n dk\,. \tag{6·8a}$$

The entropy per unit length is

$$S/L = \int \{(\rho + \rho^h)\ln(\rho + \rho^h) - \rho \ln \rho - \rho^h \ln \rho^h\} dk$$
$$+ \int \{(\sigma' + \sigma'^h)\ln(\sigma' + \sigma'^h) - \sigma' \ln \sigma' - \sigma'^h \ln \sigma'^h\} dk$$
$$+ \sum_n \int \{(\sigma_n + \sigma_n^h)\ln(\sigma_n + \sigma_n^h) - \sigma_n \ln \sigma_n - \sigma_n^h \ln \sigma_n^h\} dk\,. \tag{6·8b}$$

The magnetization per unit length is

$$S_z/L = \tfrac{1}{2}\int \rho dk - \sum_{n=1}^{\infty} n \int \sigma_n dk\,. \tag{6·8c}$$

The particle density is

$$N/L = \int \rho dk + 2\int \sigma' dk\,. \tag{6·8d}$$

The thermodynamic potential $\Omega \equiv E - TS - AN$ should be minimized. So we have

$$\begin{aligned}0 = \delta\Omega/L = &\int 2(k^2 - c^2 - A)\delta\sigma' dk + \int (k^2 - A)\delta\rho dk \\ &+ \sum_{n=1}^{\infty} 2n\mu_0 H \int \delta\sigma_n dk - T\int\left\{\delta\rho \ln\left(1 + \frac{\rho^h}{\rho}\right) + \delta\rho^h \ln\left(1 + \frac{\rho}{\rho^h}\right)\right\} dk \\ &- T\int\left\{\delta\sigma' \ln\left(1 + \frac{\sigma'^h}{\sigma'}\right) + \delta\sigma'^h \ln\left(1 + \frac{\sigma'}{\sigma'^h}\right)\right\} dk \\ &- T\sum_{n=1}^{\infty}\int\left\{\delta\sigma_n \ln\left(1 + \frac{\sigma_n{}^h}{\sigma_n}\right) + \delta\sigma_n{}^h \ln\left(1 + \frac{\sigma_n}{\sigma_n{}^h}\right)\right\} dk\,. \end{aligned} \tag{6·9}$$

From Eqs. (6·7a), (6·7b) and (6·7c) we have

$$\delta\sigma'^h = -\delta\sigma' - [2]\delta\sigma' - [1]\delta\rho\,,$$

$$\delta\rho^h = -\delta\rho - [1]\delta\sigma' - \sum_{n=1}^{\infty}[n]\delta\sigma_n\,,$$

$$\delta\sigma_n{}^h = [n]\delta\rho - \sum_{m=1}^{\infty} A_{nm}\delta\sigma_m\,.$$

Substituting these into Eq. (6·9) we have a set of coupled-nonlinear integral equations for $\zeta = \rho^h/\rho$, $\eta' = \sigma'^h/\sigma'$ and $\eta_n = \sigma_n{}^h/\sigma_n$ as follows:

$$\ln\eta' = \frac{2(k^2 - A - c^2)}{T} + [2]\ln(1 + \eta'^{-1}) + [1]\ln(1 + \zeta^{-1}), \tag{6·10a}$$

$$\ln\zeta = \frac{k^2 - A - \mu_0 H}{T} + [1]\ln(1 + \eta'^{-1}) - \sum_{n=1}^{\infty}[n]\ln(1 + \eta_n{}^{-1}), \tag{6·10b}$$

$$\ln(1 + \eta_n) = \frac{2n\mu_0 H}{T} + [n]\ln(1 + \zeta^{-1}) + \sum_{m=1}^{\infty} A_{nm}\ln(1 + \eta_m{}^{-1}). \tag{6·10c}$$

Equations (6·7) are rewritten as

$$\frac{1}{\pi} = (1 + \eta')\sigma' + [2]\sigma' + [1]\rho\,, \tag{6·11a}$$

$$\frac{1}{2\pi} = (1 + \zeta)\rho + [1]\sigma' + \sum_{n=1}^{\infty}[n]\sigma_n\,, \tag{6·11b}$$

$$[n]\rho = \eta_n\sigma_n + \sum_m A_{nm}\sigma_m\,. \tag{6·11c}$$

The pressure P and thermodynamic potential Ω are given by

$$P = -\Omega/L = T\int \ln(1 + \eta'^{-1})\frac{dk}{\pi} + T\int \ln(1 + \zeta^{-1})\frac{dk}{2\pi}\,. \tag{6·12}$$

Here we have used Eqs. (6·10) and (6·11).

The integral equations (6·10) are transformed as

$$[1]\{\ln(1+\eta_1)-\ln(1+\eta')\}=([0]+[2])\ln\zeta^{-1}, \tag{6·13a}$$

$$[1]\{\ln(1+\zeta^{-1})+\ln(1+\eta_2)\}=([0]+[2])\ln\eta_1, \tag{6·13b}$$

$$[1]\{\ln(1+\eta_{n-1})+\ln(1+\eta_{n+1})\}=([0]+[2])\ln\eta_n, \quad n=2,3,\cdots, \tag{6·13c}$$

$$\ln\eta'=\frac{2(k^2-A-c^2)}{T}+[2]\ln(1+\eta'^{-1})+[1]\ln(1+\zeta^{-1}), \tag{6·13d}$$

$$\ln\zeta=\frac{k^2-A-\mu_0 H}{T}+[1]\ln(1+\eta'^{-1})-\sum_{n=1}^{\infty}[n]\ln(1+\eta_n^{-1}). \tag{6·13e}$$

Equations (6·11) are transfomed as follows:

$$[1](\eta'\sigma'+\eta_1\sigma_1)=([0]+[2])(1+\zeta)\rho, \tag{6·14a}$$

$$[1](\rho+\eta_2\sigma_2)=([0]+[2])(1+\eta_1)\sigma_1, \tag{6·14b}$$

$$[1](\eta_{n-1}\sigma_{n-1}+\eta_{n+1}\sigma_{n+1})=([0]+[2])(1+\eta_n)\sigma_n, \quad n=2,3,\cdots, \tag{6·14c}$$

$$\frac{1}{\pi}=(1+\eta'+[2])\sigma'+[1]\rho, \tag{6·14d}$$

$$\frac{1}{2\pi}=(1+\zeta)\rho+[1]\sigma'+\sum_{n=1}^{\infty}[n]\sigma_n. \tag{6·14e}$$

§ 7. Special cases for $c<0$

1) *The limit $c\to 0$*

In this limit we can put $[n]=[0]$. Therefore Eqs. (6·13) become

$$(1+\eta_1)/(1+\eta')=\zeta^{-2},$$

$$(1+\zeta^{-1})(1+\eta_2)=\eta_1^2,$$

$$(1+\eta_{n-1})(1+\eta_{n+1})=\eta_n^2, \quad n=2,3,\cdots,$$

$$\eta_1^2/(1+\eta_1)=e^{2(k^2-A)/T}(1+\zeta^{-1}),$$

$$\xi=ze^{(k^2-A)/T}(1+\eta'^{-1})/\prod_{n=1}^{\infty}(1+\eta_n^{-1}).$$

And the solutions are

$$\eta_n=f^2(n)-1, \tag{7·1a}$$

$$\zeta=(f^2(0)-1)^{-1}, \tag{7·1b}$$

$$\eta'=f^2(-1)-1, \tag{7·1c}$$

where

$$f(n)=\frac{bz^n-b^{-1}z^{-n}}{z-z^{-1}}, \qquad b=z^2\sqrt{\frac{1+z^{-1}e^{(k^2-A)/T}}{1+ze^{(k^2-A)/T}}}.$$

Equations (6·13) become

$$\eta'\sigma'+\eta_1\sigma_1=2(1+\zeta)\rho\,,$$
$$\rho+\eta_2\sigma_2=2(1+\eta_1)\sigma_1\,,$$
$$\eta_{n-1}\sigma_{n-1}+\eta_{n+1}\sigma_{n+1}=2(1+\eta_n)\sigma_n\,,\qquad n=2,3,\cdots,$$
$$\frac{1}{\pi}=(2+\eta')\sigma'+\rho\,,$$
$$\frac{1}{2\pi}=(1+\zeta)\rho+\sigma'+\sum_{n=1}^{\infty}\sigma_n\,.$$

Using (7·1a), (7·1b) and (7·1c) we have a solution for these linear equations:

$$\rho=\frac{1}{2\pi}\frac{(b+b^{-1})f(-1)(-f(-2))}{f(0)}\,, \tag{7·2a}$$

$$\sigma'=\frac{1}{2\pi}\frac{f^2(-1)}{f(0)}(bz^{-1}+b^{-1}z), \tag{7·2b}$$

$$\sigma_n=\frac{1}{2\pi}f(-1)(-f(-2))\left\{\frac{1}{f(n-1)f(n)}-\frac{1}{f(n)f(n+1)}\right\}. \tag{7·2c}$$

One can calculate $\rho+2\sigma'$ which is the distribuion of real momenta in the limit $c\to 0$:

$$\rho+2\sigma'=\frac{1}{2\pi}((e^{(k^2-A-\mu_0H)/T}+1)^{-1}+(e^{(k^2-A-\mu_0H)/T}+1)^{-1}). \tag{7·3}$$

This result coincides with the well-known facts and suggests that our theory is correct.

2) *The limit T→0*

We prove that $\varepsilon_n(k)>0$ from Eq. (6·10c). Therefore in the limit $T\to 0$ we see $\eta_n=\infty$ and $\sigma_n=0$ for $n=1, 2, \cdots$. ε' and κ are determined by

$$\varepsilon'(k)=2(k^2-A-c^2)-[2]\varepsilon'^-(k)-[1]\kappa^-(k), \tag{7·4a}$$

$$\kappa(k)=k^2-A-\mu_0H-[1]\varepsilon'^-(k), \tag{7·4b}$$

and are monotonically increasing functions of k^2 as will be shown in Appendix B. We define the parameters B and Q by $\varepsilon'(B)=0$ and $\kappa(Q)=0$. η' and ζ are zero in the region $[B,-B]$ and $[Q,-Q]$, respectively, and infinity outside these regions. So one obtains a set of coupled linear equations in the limt $T\to 0$.

$$\frac{1}{\pi}=\sigma'(k)+\frac{1}{\pi}\int_{-B}^{B}\frac{2|c|\sigma'(k')dk'}{4|c|^2+(k-k')^2}+\frac{1}{\pi}\int_{-Q}^{Q}\frac{|c|\rho(k')dk'}{|c|^2+(k-k')^2}\,,$$

$$\frac{1}{2\pi}=\rho(k)+\frac{1}{\pi}\int_{-B}^{B}\frac{|c|\sigma'(k')dk'}{|c|^2+(k-k')^2}\,,$$

$$E/L = \int_{-B}^{B} 2(k^2 - c^2)\sigma'(k)\,dk + \int_{-Q}^{Q} k^2\rho(k)\,dk\,,$$

$$N/L = 2\int_{-B}^{B} \sigma'(k)\,dk + \int_{-Q}^{Q} \rho(k)\,dk\,,$$

$$S_z/L = \int_{-Q}^{Q} \rho(k)\,dk\,.$$

These equations are equivalent to those which were obtained by Gaudin.[1]

§ 8. Discussion

Our equations are non-linear and have infinite unknown functions. But the author believes that the numerical calculation of physical quantities can be done if we use a high-speed computer.

It is possible to calculate the excitation spectra from the thermodynamically equilibrium state as Yang and Yang[7] did for one-dimensional bosons.

In Ref. 6) the author discussed the analytic properties of the energy at zero temperature. But if one uses our integral equations it is possible to investigate the analytic properties of thermodynamic quantities at finite temperature.

Our theory is based on the three conjectures of § 3. So it is necessary to prove these conjectures strictly.

We have obtained the integral equations for two-component bosons, namely, the case of a wave function which transforms as an irreducible representation of S_N with two rows. The integral equations for the ground state energy was derived by Yang.[2] We put a chemical potential for first-kind of bosons as $A + \mu_0 H$ and one for second-kind of bosons as $A - \mu_0 H$. In the case of repulsive interactions (4·10a), (4·10b) and (4·11a) are replaced by

$$\ln\zeta + [2]\ln(1+\zeta^{-1}) = \frac{k^2 - A - \mu_0 H}{T} - \sum_{n=1}^{\infty}[n]\ln(1+\eta_n{}^{-1})\,,$$

$$\ln(1+\eta_n) = \frac{2n\mu_0 H}{T} + [n]\ln(1+\zeta^{-1}) + \sum_{m=1}^{\infty} A_{nm}\ln\,(1+\eta_m{}^{-1})\,,$$

$$(1+\zeta)\rho = \frac{1}{2\pi} + [2]\rho - \sum_{n=1}^{\infty}[n]\sigma_n\,,$$

respectively.

Appendix A

Equation (5·8) is transformed as

$$\varepsilon_1(k) = \mu_0 H + \int \frac{1}{4c}\,\mathrm{sech}\,\frac{\pi(k-k')}{2c}\,\kappa^-(k')\,dk' + \int \frac{1}{c} R\left(\frac{k-k'}{c}\right)\varepsilon_1{}^+(k')\,dk'. \tag{A1}$$

So we consider a series of functions $\{\varepsilon_1{}^{(n)}\}$ and $\{\kappa^{(n)}\}$:

$$\varepsilon_1^{(n+1)}(k)=\mu_0 H+\int\frac{1}{4c}\operatorname{sech}\frac{\pi(k-k')}{2c}\kappa^{(n)-}(k')\,dk'+\int\frac{1}{c}R\left(\frac{k-k'}{c}\right)\varepsilon_1^{(n)+}(k')\,dk', \tag{A2}$$

$$\kappa^{(n+1)}(k)=k^2-A-\mu_0 H+[1]\varepsilon_1^{(n)-}(k), \tag{A3}$$

$$\kappa^{(1)}(k)=k^2-A-\mu_0 H, \tag{A4}$$

$$\varepsilon_1^{(1)}(k)=2\mu_0 H. \tag{A5}$$

We prove the following lemma by mathematical induction.
Lemma 1.

a) $\varepsilon_1^{(n)}\geqq -A-\mu_0 H,\ \kappa_1^{(n)}\geqq -2A-2\mu_0 H.$

b) $\varepsilon_1^{(n)}\geqq\varepsilon_1^{(n+1)},\ \kappa^{(n)}\geqq\kappa^{(n+1)}.$

c) $\varepsilon_1^{(n)}$ and $\kappa^{(n)}$ are monotonically increasing functions (MIF) of k^2.

[Proof] It is clear from (A4) and (A5) that a) and c) are valid for $n=1$. From

$$\varepsilon_1^{(2)}=2\mu_0 H+\int\frac{1}{4c}\operatorname{sech}\frac{\pi(k-k')}{2c}\kappa^{(n)-}(k')\,dk'\leqq\varepsilon_1^{(1)}$$

and

$$\kappa^{(1)}=\kappa^{(2)}.$$

We see that b) is valid for $n=1$. It is clear from (A2) and (A3) that if a), b) and c) is valid for $n=k$, a), b) and c) are valid for $n=k+1$. [Q.E.D.]

From a) and b) we see that the limit $\varepsilon_1=\lim_{n\to\infty}\varepsilon_1^{(n)}$ and $\kappa=\lim_{n\to\infty}\kappa^{(n)}$ exist. These two functions ε_1 and κ are solutions of (5·8a) and (5·8b) and MIF's of k^2.

Appendix B

The Equations (7·4a) and (7·4b) are transformed as

$$\kappa(k)=-\mu_0 H+\int\frac{dk'}{4|c|}\operatorname{sech}\frac{\pi(k-k')}{2|c|}\varepsilon'^{+}(k')+\int\frac{dk'}{|c|}R\left(\frac{k-k'}{|c|}\right)\kappa^{-}(k'), \tag{B1}$$

$$\varepsilon'(k)=k^2-A-2c^2+\mu_0 H+\{k^2+[1](\kappa^{+}(k)-k^2)\}. \tag{B2}$$

So we consider the series of functions defined by

$$\varepsilon'^{(1)}(k)=2(k^2-A-c^2), \tag{B3}$$

$$\kappa^{(1)}(k)=k^2-A-\mu_0 H, \tag{B4}$$

$$\varepsilon'^{(n+1)}(k)=k^2-A-2c^2+\mu_0 H+\{k^2+[1](\kappa^{(n)+}(k)-k^2)\}, \tag{B5}$$

$$\kappa^{(n+1)}(k)=-\mu_0 H+\int\frac{1}{|c|}R\left(\frac{k-k'}{|c|}\right)\kappa^{(n)-}(k')\,dk' + \int\frac{1}{4|c|}\operatorname{sech}\frac{\pi(k-k')}{2|c|}\varepsilon'^{(n+1)+}(k')\,dk'. \tag{B6}$$

Lemma 2.

a) $\kappa^{(n)} \leqq \kappa^{(n+1)}$, $\varepsilon'^{(n)} \leqq \varepsilon'^{(n+1)}$.

b) $\kappa^{(n)}(k) \leqq k^2 + c^2$, $\varepsilon'^{(n)}(k) \leqq 2k^2 + 2\mu_0 H$.

c) $\kappa^{(n)}$ and $\varepsilon'^{(n)}$ are MIF's of k^2.

[Proof] For $n=1$ b) and c) are easily proved by (B3) and (B4). From (B5) we have

$$\varepsilon'^{(2)}(k) \geqq k^2 - A - 2c^2 + \mu_0 H + \{k^2 + [1](\kappa^{(1)}(k) - k^2)\} = \varepsilon'^{(1)}(k). \qquad \text{(B7)}$$

Substituting (B5) into (B6) we have

$$\kappa^{(2)}(k) = k^2 - \frac{A + \mu_0 H}{2} + \int \frac{1}{|c|} R\left(\frac{k-k'}{|c|}\right)(\kappa^{(1)}(k') - k'^2)\, dk'$$

$$- \int \frac{1}{4|c|} \operatorname{sech} \frac{\pi(k-k')}{2|c|} \varepsilon'^{(2)-}(k')\, dk' \geqq k^2 - \frac{A + \mu_0 H}{2}$$

$$+ \int \frac{1}{|c|} R\left(\frac{k-k'}{|c|}\right)(\kappa^{(1)}(k') - k'^2)\, dk' = \kappa^{(1)}(k).$$

From Eqs. (B5) and (B6) it can be easily proved that a) and b) are valid for $n = k+1$ if they are for $n = k$. One can prove that

$$\int \frac{1}{|c|} R\left(\frac{k-k'}{|c|}\right) f(k')\, dk',$$

$$\int \frac{1}{4|c|} \operatorname{sech} \frac{\pi(k-k')}{2|c|} f(k')\, dk'$$

and

$$k^2 + [1](f(k) - k^2)$$

are MIF's of k^2 if $f(k)$ is an MIF of k^2. From this fact we have that c) is valid for $n = k+1$ if it is for $n = k$. [Q.E.D.]

From a) and b) we see that there exist the limits

$$\varepsilon'(k) = \lim_{n\to\infty} \varepsilon'^{(n)}(k) \qquad \text{and} \qquad \kappa(k) = \lim_{n\to\infty} \kappa^{(n)}(k).$$

It is clear that these two functions satisfy (7·4a) and (7·4b) are MIF's of k^2.

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Note added in proof: C. K. Lai gave equations which are equivalent to those of ours for fermions with repulsive and attractive interactions. See C. K. Lai, Phys. Rev. Letters **26** (1971), 1472.