GLASNIK MATEMATIČKI
Vol. 33(53) (1998), 199–208

# ONE–DIMENSIONAL FLOW OF A COMPRESSIBLE VISCOUS MICROPOLAR FLUID: A GLOBAL EXISTENCE THEOREM

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*Abstract.* An initial-boundary value problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid is considered. It is assumed that the fluid is thermodinamicaly perfect and politropic. A global-in-time existence theorem is proved. The proof is based on a local existence theorem, obtained in the previous paper [4].

## 1. Statement of the problem and the main result

In this paper we consider an initial-boundary value problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid, being in thermodinamical sense perfect and politropic (see [4] and references therein).

Let $\rho$, $v$, $\omega$ and $\theta$ denotes respectively the mass density, velocity, microrotation velocity and temperature in the Lagrangean description. Then the problem that we consider has the formulation as follows:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\Big(\rho \frac{\partial v}{\partial x}\Big) - K \frac{\partial}{\partial x}(\rho\theta), \tag{1.2}$$

$$\rho \frac{\partial \omega}{\partial t} = A\Big[\rho \frac{\partial}{\partial x}\Big(\rho \frac{\partial \omega}{\partial x}\Big) - \omega\Big], \tag{1.3}$$

$$\rho \frac{\partial \theta}{\partial t} = -K\rho^2\theta \frac{\partial v}{\partial x} + \rho^2\Big(\frac{\partial v}{\partial x}\Big)^2 + \rho^2\Big(\frac{\partial \omega}{\partial x}\Big)^2 + \omega^2 + D\rho \frac{\partial}{\partial x}\Big(\rho \frac{\partial \theta}{\partial x}\Big) \tag{1.4}$$

in $]0, 1[\times \mathbf{R}_+$,

$$v(0,t) = v(1,t) = 0, \tag{1.5}$$

$$\omega(0,t) = \omega(1,t) = 0, \tag{1.6}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0, \tag{1.7}$$

for $t \in \mathbf{R}_+$,

$$\rho(x,0) = \rho_0(x), \tag{1.8}$$

$$v(x,0) = v_0(x), \tag{1.9}$$

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*Mathematics subject classification* (1991): 35K55, 35Q35, 76N10.

*Key words and phrases:* Micropolar fluid, viscousity, compressibility, boundary value problem, global existence.

$$\omega(x,0) = \omega_0(x), \tag{1.10}$$
$$\theta(x,0) = \theta_0(x) \tag{1.11}$$

for $x \in ]0,1[$. Here $K, A$ and $D$ are given positive constants; $\rho_0$, $v_0$, $\omega_0$ and $\theta_0$ are given functions, satisfying the conditions

$$\rho_0, \theta_0 > 0 \quad \text{in} \quad [0,1]. \tag{1.12}$$

Let $T \in \mathbf{R}_+$; a generalised solution of the problem (1.1)–(1.11) in the domain $Q_T = ]0,1[\times]0,T[$ is a function

$$(x,t) \to (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_T, \tag{1.13}$$

where

$$\rho \in L^\infty(0,T;H^1(]0,1[)) \cap H^1(Q_T), \tag{1.14}$$
$$v, \omega, \theta \in L^\infty(0,T;H^1(]0,1[)) \cap H^1(Q_T) \cap L^2(0,T;H^2(]0,1[)), \tag{1.15}$$

that satisfies the equations (1.1)–(1.4) a.e. in $Q_T$, the conditions (1.5)-(1.11) in the sense of traces and the conditions

$$\inf_{Q_T} \rho > 0. \tag{1.16}$$

From embedding and interpolation theorems ([3]) one can conclude that from (1.14) and (1.15) it follows:

$$\rho \in C([0,T], L^2(]0,1[)) \cap L^\infty(0,T;C([0,1])), \tag{1.17}$$
$$v, \omega, \theta \in L^2(0,T;C^{(1)}([0,1])) \cap C([0,T], H^1(]0,1[)), \tag{1.18}$$
$$v, \omega, \theta \in C(\overline{Q}_T). \tag{1.19}$$

Specially, the condition (1.16) has a sense.

Assuming the conditions

$$\rho_0, \theta_0 \in H^1(]0,1[), v_0, \omega_0 \in H_0^1(]0,1[) \tag{1.20}$$

and the inequalities (1.12), in the previous paper [4] we proved a uniqueness of a generalised solution and the following local existence theorem: there exists $T_0 \in \mathbf{R}_+$, such that in the domain $Q_{T_0} = ]0,1[\times]0,T_0[$ there exists a generalised solution, having the property

$$\theta > 0 \text{ in } \overline{Q}_{T_0}. \tag{1.21}$$

With the use of that theorem, in this paper we shall prove the following result.

THEOREM 1.1. *Let the conditions (1.20) and (1.12) be fulfiled. Then for each $T \in \mathbf{R}_+$, in the domain $Q_T$ there exists a generalised solution (1.13) of the problem (1.1)–(1.11), having the property*

$$\theta > 0 \textit{ in } \overline{Q}_T.$$

In our proof we apply the method of the book [1], where the Theorem 1.1 was proved for the classical fluid ($\omega = 0$); for this case see also [2].

## 2. The proof of Theorem 1.1

Because of the local existence result, Theorem 1.1 is an immediate consequence of the following statement.

PROPOSITION 2.1. *Let $T \in \mathbf{R}_+$ and let a function*

$$(x, t) \to (\rho, v, \omega, \theta)(x, t), \quad (x, t) \in Q_T \tag{2.1}$$

*satisfies the condition:*

*for each $T' \in ]0, T[$, (2.1) is a generalised solution of the problem (1.1)–(1.11) in the domain $Q_{T'} = ]0, 1[\times]0, T'[$ and the inequality $\theta > 0$ in $\overline{Q}_{T'}$ holds true.*

*Then (2.1) is a generalised solution of the same problem in the domain $Q_T$ and inequality $\theta > 0$ in $\overline{Q}_T$ holds true.*

The above statement is a consequence of results below. In that what follows we assume that the function (2.1) satisfies the condition of the Proposition 2.1. By $C \in \mathbf{R}_+$ we denote a generic constant, having possibly different values at different places; we also use the notation $\|f\| = \|f\|_{L^2(]0,1[)}$. Because of the fact that equations (1.2) and (1.3) don't contain the function $\omega$, some of our considerations are identical to that of classical fluid. In these cases we omit proofs or details of proofs, making reference to correspondent pages of the book [1].

LEMMA 2.1. *It holds*

$$v, \omega \in L^\infty(0, T; L^2(]0, 1[)), \tag{2.2}$$

$$\theta \in L^\infty(0, T; L^1(]0, 1[)), \tag{2.3}$$

*Proof.* Multiplying the equations (1.2), (1.3) and (1.4) respectively by $v, A^{-1}\rho^{-1}\omega$ and $\rho^{-1}$, integrating over $]0, 1[$ and making use of (1.5)–(1.7), after addition of the obtained equalities we find that

$$\frac{d}{dt}\int_0^1 \left(\frac{1}{2}v^2 + \frac{1}{2A}\omega^2 + \theta\right)dx = 0 \text{ on } ]0, T[. \tag{2.4}$$

Integrating over ]0,t[, $t \in ]0, T[$ and making use of (1.9)–(1.11), we obtain

$$\int_0^1 \left(\frac{1}{2}v^2 + \frac{1}{2A}\omega^2 + \theta\right)dx = \frac{1}{2}\|v_0\|^2 + \frac{1}{2A}\|\omega_0\|^2 + \|\theta_0\|_{L^1(]0,1[)} \text{ on } ]0, T[, \tag{2.5}$$

or

$$\|v\|^2 + \|\omega\|^2 + \|\theta\|_{L^1(]0,1[)} \leqslant C \text{ on } ]0, T[. \tag{2.6}$$

From (2.6) there follow the statements (2.2) and (2.3). □

LEMA 2.2. ([1], pp. 47–48, 50–52). *Let* $t \in ]0,T[$ *and*

$$M_\theta(t) = \max_{[0,1]} \theta(\cdot, t), \tag{2.7}$$

$$m_\rho(t) = \min_{[0,1]} \rho(\cdot, t), \tag{2.8}$$

$$I_1(t) = \int_0^1 \rho(x,t)\Big(\frac{\partial\theta}{\partial x}(x,t)\Big)^2 dx, \tag{2.9}$$

$$I_2(t) = \int_0^t I_1(\tau)d\tau. \tag{2.10}$$

*Then there exist* $C \in \mathbf{R}_+$ *and (for each* $\varepsilon > 0$*)* $C_\varepsilon \in \mathbf{R}^+$, *such that for each* $t \in ]0,T[$ *the inequalities*

$$M_\theta^2(t) \leqslant \varepsilon I_1(t) + C_\varepsilon(1 + I_2(t)), \tag{2.11}$$

$$m_\rho(t) \geqslant C\Big(I + \int_0^t M_\theta(\tau)d\tau\Big)^{-1} \tag{2.12}$$

*hold true.*

LEMA 2.3. *It holds*

$$\inf_{Q_T} \theta > 0, \tag{2.13}$$

$$\rho \in L^\infty(Q_T). \tag{2.14}$$

*Proof.* Let $W = \theta^{-1}$ and $p > 1$. Multiplying the equation (1.4) by $2p\rho^{-1}W^{2p+1}$ and integrating over $]0,1[$ we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 W^{2p}dx = &\int_0^1 \Big[2DpW^{2p-1}\frac{\partial}{\partial x}\Big(\rho\frac{\partial W}{\partial x}\Big) - 2p\Big(2D\rho\theta\Big(\frac{\partial W}{\partial x}\Big)^2 + \rho W^2\Big(\frac{\partial v}{\partial x} - \frac{K\theta}{2}\Big)^2 \\ &+ \frac{\omega^2}{\rho}W^2 + \rho W^2\Big(\frac{\partial \omega}{\partial x}\Big)^2\Big)W^{2p-1} + \frac{K^2p}{2}\rho W^{2p-1}\Big]dx \\ \leqslant &\int_0^1 \Big[2DpW^{2p-1}\frac{\partial}{\partial x}\Big(\rho\frac{\partial W}{\partial x}\Big) + \frac{K^2p}{2}\rho W^{2p-1}\Big]dx \text{ on } ]0,T[.\end{aligned} \tag{2.15}$$

Integrating the first term on right-hand side by parts and making use of (1.7), we find that

$$\frac{d}{dt}\int_0^1 W^{2p}dx \leqslant \int_0^1 \Big[-2Dp(2p-1)W^{2p-2}\Big(\frac{\partial W}{\partial x}\Big)^2 + \frac{K^2p}{2}\rho W^{2p-1}\Big]dx, \tag{2.16}$$

or

$$\frac{d}{dt}\int_0^1 W^{2p}dx \leqslant \frac{pK^2}{2}\int_0^1 \rho W^{2p-1}dx \text{ on } ]0,T[. \tag{2.17}$$

The conclusions (2.13) and (2.14) follow now from (2.17) as in the case of classical fluid ([1], pp. 48–50). □

LEMMA 2.4. *It holds*

$$M_\theta \in L^2(]0,T[), \tag{2.18}$$

$$\inf_{Q_T} \theta > 0, \tag{2.19}$$

$$\theta \in L^\infty(0,T;L^2(]0,1[)) \cap L^2(0,T;H^1(]0,1[)). \tag{2.20}$$

*Proof.* Let

$$\Phi = \frac{1}{2}v^2 + \frac{1}{2A}\omega^2 + \theta. \tag{2.21}$$

Multiplying the equations (1.2), (1.3) and (1.4) respectively by $v\Phi$, $A^{-1}\rho^{-1}\omega\Phi$ and $\rho^{-1}\Phi$, integrating over $]0,1[$ and making use of (1.5)–(1.7), after addition of the obtained equations, we find that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Phi^2 dx + \int_0^1 \rho\left(\frac{\partial\Phi}{\partial x}\right)^2 dx + (D-1)\int_0^1 \rho\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x}dx$$
$$+\left(1-\frac{1}{A}\right)\int_0^1 \rho\omega\frac{\partial\omega}{\partial x}\frac{\partial\Phi}{\partial x}dx - K\int_0^1 \rho\theta v\frac{\partial\Phi}{\partial x}dx = 0 \text{ on } ]0,T[, \tag{2.22}$$

or

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Phi^2 dx + \int_0^1 \rho\left(\frac{\partial\Phi}{\partial x}\right)^2 dx + (D-1)\int_0^1 \rho\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x}dx$$
$$\leqslant L\int_0^1 \rho\left|\omega\frac{\partial\omega}{\partial x}\frac{\partial\Phi}{\partial x}\right|dx + K\int_0^1 \rho\theta\left|v\frac{\partial\Phi}{\partial x}\right|dx \text{ on } ]0,T[, \tag{2.23}$$

where $L = |1-A^{-1}|$. Applying on the right-hand side the Young inequality with a parameter $\delta > 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Phi^2 dx + \int_0^1 \rho\left[(1-2\delta)\left(\frac{\partial\Phi}{\partial x}\right)^2 + (D-1)\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x}\right]dx$$
$$\leqslant C\delta^{-1}\int_0^1 \rho\left[\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2 + \theta^2 v^2\right]dx \text{ on } ]0,T[. \tag{2.24}$$

One can easily see that the following inequality holds true

$$(1-2\delta)\left(\frac{\partial\Phi}{\partial x}\right)^2+(D-1)\frac{\partial\theta}{\partial x}\frac{\partial\Phi}{\partial x} \geqslant (D-6\delta)\left(\frac{\partial\theta}{\partial x}\right)^2-\left(4\delta+\frac{(1-4\delta+D)^2}{8\delta}\right)v^2\left(\frac{\partial v}{\partial x}\right)^2$$
$$-\frac{1}{4\delta}\left((1-2\delta)^2+\frac{1}{2}(1-4\delta+D)^2\right)\frac{\omega^2}{A^2}\left(\frac{\partial\omega}{\partial x}\right)^2. \quad (2.25)$$

Let $\delta = 24^{-1}\min\{1, D\}$. From (2.24) and (2.25) it follows the inequality

$$\frac{d}{dt}\int_0^1 \Phi^2 dx + \frac{3D}{2}\int_0^1 \rho\left(\frac{\partial\theta}{\partial x}\right)^2 dx$$
$$\leqslant C_1\int_0^1 \rho\left[v^2\left(\frac{\partial v}{\partial x}\right)^2+\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2+\theta^2v^2\right]dx \text{ on } ]0,T[, \quad (2.26)$$

where

$$C_1 = 2\max\{4\delta+\frac{(1-4\delta+D)^2}{8\delta}, \frac{C}{\delta}+\frac{2(1-2\delta)^2+(1-4\delta+D)^2}{8\delta}, \frac{C}{\delta}\}.$$

Multiplying (1.2) and (1.3) respectively by $v^3$ and $\rho^{-1}\omega^3$, integrating over $]0,1[$ and making use of (1.5) and (1.6), after applying the Young inequality we obtain the inequalities

$$\frac{d}{dt}\int_0^1 v^4 dx + \int_0^1 \rho v^2\left(\frac{\partial v}{\partial x}\right)^2 dx \leqslant 6K^2\int_0^1 \rho\theta^2v^2 dx \text{ on } ]0,T[, \quad (2.27)$$

$$\frac{d}{dt}\int_0^1 \omega^4 dx + A\int_0^1 \rho\omega^2\left(\frac{\partial\omega}{\partial x}\right)^2 dx \leqslant 0 \text{ on } ]0,T[. \quad (2.28)$$

Multiplying (2.27) by $C_1$ and (2.28) by $C_2 = A^{-1}C_1$, after addition of the obtained inequalities with (2.26), we find that

$$\frac{d}{dt}\int_0^1\left(\Phi^2+C_1v^4+C_2\omega^4\right)dx+D\int_0^1\rho\left(\frac{\partial\theta}{\partial x}\right)^2dx \leqslant C\int_0^1\rho\theta^2v^2\,dx \text{ on } ]0,T[ \quad (2.29)$$

or, taking into account (2.2), (2.14) and (2.11),

$$\frac{d}{dt}\left(\int_0^1(\Phi^2+C_1v^4+C_2\omega^4)dx+DI_2\right) \leqslant C(1+DI_2)$$
$$\leqslant C\left(1+\int_0^1(\Phi^2+C_1v^4+C_2\omega^4)dx+DI_2\right) \text{ on } ]0,T[. \quad (2.30)$$

From (2.30) it follows the inequality

$$\int_0^1(\Phi^2+C_1v^4+C_2\omega^4)dx+DI_2 \leqslant C \text{ on } ]0,T[ \quad (2.31)$$

and therefore it holds

$$I_2 \in L^\infty(]0,T[), \tag{2.32}$$

$$\Phi \in L^\infty\big(0,T;L^2(]0,1[)\big). \tag{2.33}$$

From (2.32) and (2.11) we conclude that (2.18) holds true. The inequality (2.19) follows now from (2.18) and (2.12); the inclusion (2.20) follows from (2.33), (2.19) and (2.32). □

LEMMA 2.5. ([1], pp. 53–54) *It holds*

$$\rho \in L^\infty\big(0,T;H^1(]0,1[)\big) \cap H^1(Q_T). \tag{2.34}$$

LEMMA 2.6. ([1], pp. 53–54) *It holds*

$$v \in L^\infty\big(0,T;H^1(]0,1[)\big) \cap H^1(Q_T) \cap L^2\big(0,T;H^2(]0,1[)\big). \tag{2.35}$$

LEMMA 2.7. *It holds*

$$\omega \in L^\infty\big(0,T;H^1(]0,1[)\big) \cap H^1(Q_T) \cap L^2\big(0,T;H^2(]0,1[)\big). \tag{2.36}$$

*Proof.* Multiplying the equation (1.3) by $\rho^{-1}\omega$, integrating over $]0,1[$ and making use of (1.6), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2 + A\int_0^1 \Big[\rho\Big(\frac{\partial\omega}{\partial x}\Big)^2 + \frac{\omega^2}{\rho}\Big]dx = 0 \quad \text{on} \quad ]0,T[, \tag{2.37}$$

or

$$\frac{1}{2}\|\omega(.,t)\|^2 + A\int_0^t d\tau\int_0^1 \Big[\rho\Big(\frac{\partial\omega}{\partial x}\Big)^2 + \frac{\omega^2}{\rho}\Big](x,\tau)dx$$
$$= \frac{1}{2}\int_0^1 \omega_0^2(x)dx \leqslant C, \quad t\in]0,T[. \tag{2.38}$$

Using (2.19), we conclude that

$$\omega \in L^2\big(0,T;H^1(]0,1[)\big). \tag{2.39}$$

Multiplying (1.3) by $A^{-1}\rho^{-1}\dfrac{\partial^2\omega}{\partial x^2}$ and integrating over $]0,1[$, after integration by parts on the left-hand side and making use of (1.6), we find that

$$\frac{1}{2A}\frac{d}{dt}\Big\|\frac{\partial\omega}{\partial x}\Big\|^2 + \int_0^1 \rho\Big(\frac{\partial^2\omega}{\partial x^2}\Big)^2 dx = \int_0^1 \Big(\frac{\omega}{\rho}\frac{\partial^2\omega}{\partial x^2} - \frac{\partial\rho}{\partial x}\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2}\Big)dx \quad \text{on} \quad ]0,T[. \tag{2.40}$$

In that what follows we use the inequalities

$$|f|^2 \leqslant 2\|f\| \left\|\frac{\partial f}{\partial x}\right\|, \quad \left|\frac{\partial f}{\partial x}\right| \leqslant 2\left\|\frac{\partial f}{\partial x}\right\| \left\|\frac{\partial^f}{\partial x^2}\right\|, \tag{2.41}$$

valid for a function $f$ vanishing at $x = 0$ and $x = 1$ or having derivatives that vanish at the same points.

With the help of (2.19) and (2.41) and using the Young inequality with a parameter $\delta > 0$, for the terms on the right-hand side of (2.40) we find estimates on $]0, T[$ as follows:

$$\left|\int_0^1 \frac{\omega}{\rho}\frac{\partial^2\omega}{\partial x^2}dx\right| \leqslant \delta\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2 + C\|\omega\|^2, \tag{2.42}$$

$$\left|\int_0^1 \frac{\partial\rho}{\partial x}\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2}dx\right| \leqslant 2\left\|\frac{\partial\omega}{\partial x}\right\|^{\frac{1}{2}}\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^{\frac{1}{2}}\int_0^1\left|\frac{\partial^2\omega}{\partial x^2}\frac{\partial\rho}{\partial x}\right|dx$$

$$\leqslant 2\left\|\frac{\partial\omega}{\partial x}\right\|^{\frac{1}{2}}\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^{\frac{3}{2}}\left\|\frac{\partial\rho}{\partial x}\right\| \leqslant \delta\left\|\frac{\partial^2\omega}{\partial x^2}\right\| + C\left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2\left\|\frac{\partial\rho}{\partial x}\right\|^4. \tag{2.43}$$

Using again (2.19), from (2.40), (2.42) and (2.43) we obtain (making use of (1.10))

$$\left\|\frac{\partial\omega}{\partial x}(.,t)\right\|^2 + \int_0^t\left\|\frac{\partial^2\omega}{\partial x^2}(.,\tau)\right\|^2 d\tau \leqslant \|\omega_0'\|^2 + C\int_0^t\left(\|\omega\|^2 + \left\|\frac{\partial\omega}{\partial x}\right\|^2\left\|\frac{\partial\rho}{\partial x}\right\|^4\right)d\tau$$

$$\leqslant C\left(1 + \int_0^t\left(\|\omega\|^2 + \left\|\frac{\partial\omega}{\partial x}\right\|^2\left\|\frac{\partial\rho}{\partial x}\right\|^4\right)d\tau\right), \quad t \in ]0, T[. \tag{2.44}$$

With the help of (2.34) and (2.39), from (2.44) we find that

$$\left\|\frac{\partial\omega}{\partial x}(.,t)\right\|^2 + \int_0^t\left\|\frac{\partial^2\omega}{\partial x^2}(.,\tau)\right\|^2 d\tau \leqslant C, \quad t \in ]0, T[. \tag{2.45}$$

Using (2.14) and (2.19), from (1.3) we obtain

$$\left\|\frac{\partial\omega}{\partial t}\right\|^2 \leqslant C\left(\|\omega\|^2 + \left\|\frac{\partial\omega}{\partial x}\right\|^2 + \left\|\frac{\partial^2\omega}{\partial x^2}\right\|^2\right) \quad \text{on } ]0, T[. \tag{2.46}$$

and, because of (2.39) and (2.45),

$$\int_0^t\left\|\frac{\partial\omega}{\partial t}(.,\tau)\right\|^2 d\tau \leqslant C, \quad t \in ]0, T[. \tag{2.47}$$

The conclusion (2.36) follows from (2.45) and (2.47). □

LEMMA 2.8. *It holds*

$$\theta \in L^{\infty}\big(0, T; H^1(]0,1[)\big) \cap H^1(Q_T) \cap L^2(0, T; H^2(]0,1[)\big). \tag{2.48}$$

*Proof.* Multiplying (1.4) by $\rho^{-1}\dfrac{\partial^2\theta}{\partial x^2}$ and integrating over $]0,1[$, after integration by parts on the left-hand side and making use of (1.7), we obtain

$$\frac{1}{2}\frac{d}{dt}\Big\|\frac{\partial\theta}{\partial x}\Big\|^2 + D\int_0^1 \rho\Big(\frac{\partial^2\theta}{\partial x^2}\Big)^2 dx = K\int_0^1 \rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx - \int_0^1 \rho\Big(\frac{\partial v}{\partial x}\Big)^2\frac{\partial^2\theta}{\partial x^2}dx$$

$$-\int_0^1 \rho\Big(\frac{\partial\omega}{\partial x}\Big)^2\frac{\partial^2\theta}{\partial x^2}dx - \int_0^1 \frac{\omega^2}{\rho}\frac{\partial^2\theta}{\partial x^2}dx - D\int_0^1 \frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx \quad \text{on } ]0,T[. \tag{2.49}$$

With the help of (2.14), (2.35), (2.41) and (2.36) and using the Young inequality with a parametar $\delta > 0$, for the terms on the right-hand side of (2.49) we find estimates on $]0,T[$ as follows:

$$\Big|\int_0^1 \rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant CM_\theta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + CM_\theta^2, \tag{2.50}$$

$$\Big|\int_0^1 \rho\Big(\frac{\partial v}{\partial x}\Big)^2\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant C\Big\|\frac{\partial v}{\partial x}\Big\|^{\frac{3}{2}}\Big\|\frac{\partial^2 v}{\partial x^2}\Big\|^{\frac{1}{2}}\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big\|\frac{\partial^2 v}{\partial x^2}\Big\|$$

$$\leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big(1 + \Big\|\frac{\partial^2 v}{\partial x^2}\Big\|^2\Big), \tag{2.51}$$

$$\Big|\int_0^1 \rho\Big(\frac{\partial\omega}{\partial x}\Big)^2\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant C\Big\|\frac{\partial\omega}{\partial x}\Big\|^{\frac{3}{2}}\Big\|\frac{\partial^2\omega}{\partial x^2}\Big\|^{\frac{1}{2}}\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big\|\frac{\partial^2\omega}{\partial x^2}\Big\|$$

$$\leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big(1 + \Big\|\frac{\partial^2\omega}{\partial x^2}\Big\|^2\Big), \tag{2.52}$$

$$\Big|\int_0^1 \frac{\omega^2}{\rho}\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant C\|\omega\|\Big\|\frac{\partial\omega}{\partial x}\Big\|\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C, \tag{2.53}$$

$$\Big|\int_0^1 \frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx\Big| \leqslant 2\Big\|\frac{\partial\theta}{\partial x}\Big\|^{\frac{1}{2}}\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^{\frac{3}{2}}\Big\|\frac{\partial\rho}{\partial x}\Big\| \leqslant \delta\Big\|\frac{\partial^2\theta}{\partial x^2}\Big\|^2 + C\Big\|\frac{\partial\theta}{\partial x}\Big\|^2. \tag{2.54}$$

Using again (2.19), from (2.49)–(2.54) (making use of (1.11)) we obtain

$$\Big\|\frac{\partial\theta}{\partial x}(.,t)\Big\|^2 + \int_0^t \Big\|\frac{\partial^2\theta}{\partial x^2}(.,\tau)\Big\|^2 d\tau \leqslant \|\theta_0'\|^2 + C\Big(1 + \int_0^t \Big(M_\theta^2(\tau)$$

$$+ \Big\|\frac{\partial^2 v}{\partial x^2}(.,\tau)\Big\|^2 + \Big\|\frac{\partial^2\omega}{\partial x^2}(.,\tau)\Big\|^2 + \Big\|\frac{\partial\theta}{\partial x}(.,\tau)\Big\|^2\Big)d\tau\Big). \tag{2.55}$$

With the help of (2.18), (2.35), (2.36) and (2.20), from (2.55) we find that

$$\left\| \frac{\partial \theta}{\partial x}(.,t) \right\|^2 + \int_0^t \left\| \frac{\partial^2 \theta}{\partial x^2}(.,\tau) \right\|^2 d\tau \leqslant C, \quad t \in ]0,T[. \tag{2.56}$$

Using (2.14), (2.19), (2.34), (2.35), (2.36), (2.41) and (2.53), from (1.4) we obtain

$$\left\| \frac{\partial \theta}{\partial t} \right\|^2 \leqslant C\left(1 + M_\theta^2 + \left\| \frac{\partial^2 v}{\partial x^2} \right\|^2 + \left\| \frac{\partial^2 \omega}{\partial x^2} \right\|^2 + \left\| \frac{\partial^2 \theta}{\partial x^2} \right\|^2\right) \quad \text{on } ]0,T[ \tag{2.57}$$

and, because of (2.18), (2.35), (2.36) and (2.56),

$$\int_0^t \left\| \frac{\partial \theta}{\partial t}(.,\tau) \right\|^2 d\tau \leqslant C, \quad t \in ]0,T[. \tag{2.58}$$

The conclusion (2.48) follows from (2.56) and (2.58). □

The Proposition 2.1 follows immediately from (2.13), (2.19), (2.34), (2.35), (2.36) and (2.48).

*Aknowledgement.* I wish to thank Professor I. Aganović for encouraging me to write this paper and for his valuable remarks.

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(Received January 13, 1998)
(Revised May 25, 1998)

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