

ONE-DIMENSIONAL K -TYPES IN FINITE DIMENSIONAL REPRESENTATIONS OF SEMISIMPLE LIE GROUPS: A GENERALIZATION OF HELGASON'S THEOREM

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1. Introduction.

Let G be a semisimple connected noncompact real Lie group, and let K be a maximal compact subgroup. Let (π, V) be a finite dimensional irreducible representation of G . A renowned theorem due to S. Helgason gives the condition in terms of the highest weight of π under which π is class one. This means that V contains a nonzero vector fixed by $\pi(K)$, or in other words that π contains the trivial K -type in its decomposition into irreducible representations of K . In this paper we generalize this theorem to give a complete description in terms of the highest weight of π of all one-dimensional K -types contained in π .

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra \mathfrak{g} of G , let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} and let \mathfrak{j} be a Cartan subalgebra of \mathfrak{g} containing \mathfrak{a} . Then $\mathfrak{j} = \mathfrak{t}^+ \oplus \mathfrak{a}$ where $\mathfrak{t}^+ = \mathfrak{j} \cap \mathfrak{k}$. Assume that G is contained in the complex simply connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$. Since we are dealing only with finite dimensional representations of G this assumption causes no loss of generality. Choose compatible orderings of \mathfrak{a} and \mathfrak{j} , and let $\lambda \in \mathfrak{j}^*$ be a complex linear form on \mathfrak{j} .

The precise content of Helgason's theorem is as follows (cf. [2, III § 3]): If the restriction of λ to \mathfrak{t}^+ is zero and if for all positive roots α of \mathfrak{a} in \mathfrak{g} the number $(\lambda, \alpha)/(\alpha, \alpha)$ is a nonnegative integer, then λ is the highest weight of a finite dimensional class one representation of G , and all finite dimensional class one representations of G occur in this way.

If K is semisimple then the trivial representation is its only one-dimensional representation, and therefore no more can be said about one-dimensional K -types in π . However, if K is not semisimple, or equivalently if G/K is Hermitian symmetric, then there are other one-dimensional K -types than the trivial. It is for such groups G our generalization of Helgason's theorem applies.

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Let $\mathfrak{f}_1 = [\mathfrak{f}, \mathfrak{f}]$ be the semisimple part of \mathfrak{f} . The condition on the highest weight λ of π for one-dimensional K -types to occur is very similar to that of Helgason's theorem: the restriction of λ to $\mathfrak{t}^+ \cap \mathfrak{f}_1$ has to be zero, and furthermore λ has to satisfy a certain integrality condition (see Theorem 7.2). The method of our investigation is by reduction to the rank-one case in the manner of S. G. Gindikin and F. I. Karpelevič ([1]).

The paper is organized as follows: First the restricted root theorem of C. C. Moore is stated. In Section 3 we study the structure of K and determine the complete set of one-dimensional K -types. In the next section the centralizer of \mathfrak{a} in K and its intersection with the semisimple part of K are considered. In Section 5 the rank-one subgroups of G are studied, and in the succeeding section we look upon $SU(n, 1)$ ($n \geq 1$), which are the only rank-one groups in which K is not semisimple. Finally in Section 7 the main theorem is stated and proved.

The problem of generalizing Helgason's theorem in this direction emerged in [8]. Though we will not go into that here we point out, that Theorem 7.2 in combination with the results of [8] can be applied to the construction of interesting unitary representations, in particular of some exceptional groups.

2. Root structure.

Let G be a connected real simple noncompact Lie group, and let \mathfrak{g} be its Lie algebra. Assume that $G \subset G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is a simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and let K be the corresponding maximal compact subgroup of G . Let $\mathfrak{f}_1 = [\mathfrak{f}, \mathfrak{f}]$ and assume that $\mathfrak{f}_1 \neq \mathfrak{f}$, that is G/K is Hermitian symmetric (cf. [3, Ch. VIII]).

Let \mathfrak{t} be a Cartan subalgebra of \mathfrak{f} , then \mathfrak{t} is also a Cartan subalgebra of \mathfrak{g} . Let $\Delta \subset i\mathfrak{t}^*$ consist of the roots of \mathfrak{t} in \mathfrak{g} , and let $\mathfrak{g}_{\gamma} \subset \mathfrak{g}_{\mathbb{C}}$ for $\gamma \in \Delta$ denote the γ root space. Let Δ_c , respectively Δ_n be the set of compact, respectively noncompact roots, i.e. those roots γ for which $\mathfrak{g}_{\gamma} \subset \mathfrak{k}_{\mathbb{C}}$, respectively $\mathfrak{g}_{\gamma} \subset \mathfrak{p}_{\mathbb{C}}$. Then $\Delta = \Delta_c \cup \Delta_n$.

Let \mathfrak{z} be the center of \mathfrak{f} , then $\dim \mathfrak{z} = 1$. As is well-known, we can choose an element $Z_0 \in \mathfrak{z}$ such that $\gamma(Z_0) = \pm i$ for all $\gamma \in \Delta_n$. Choose an ordering of Δ such that

$$\Delta_n^+ = \{ \gamma \in \Delta \mid \gamma(Z_0) = i \},$$

where $\Delta_n^+ = \Delta^+ \cap \Delta_n$. Let $\Delta_c^+ = \Delta^+ \cap \Delta_c$.

For $\varphi \in \mathfrak{t}_{\mathbb{C}}^*$ let $H_{\varphi} \in \mathfrak{t}_{\mathbb{C}}$ be defined by $\varphi(H) = (H_{\varphi}, H)$ for all $H \in \mathfrak{t}$, where (\cdot, \cdot) denotes the Killing form. Let $\{\gamma_1, \dots, \gamma_r\} \subset \Delta_n$ be a maximal strongly orthogonal subset, such that γ_j is the highest element of Δ_n strongly orthogonal to $\{\gamma_{j+1}, \dots, \gamma_r\}$, for $j = r, \dots, 1$ (cf. [3, p. 386]). Let

$$t^- = \sum_{j=1}^r \mathbf{R}iH_{\gamma_j}$$

and

$$t^+ = \{H \in t \mid \gamma_j(H)=0, j=1, \dots, r\},$$

then $t = t^+ \oplus t^-$. Identify γ_j with its restriction to t^- ($j=1, \dots, r$).

THEOREM 2.1. (C. C. Moore). *The set of nonzero restrictions of the elements of Δ^+ to t^- is one of the following two sets:*

Case I: $\{\gamma_i, \frac{1}{2}(\gamma_j \pm \gamma_k) \mid 1 \leq i \leq r, 1 \leq k < j \leq r\}$

Case II: $\{\frac{1}{2}\gamma_i, \gamma_i, \frac{1}{2}(\gamma_j \pm \gamma_k) \mid 1 \leq i \leq r, 1 \leq k < j \leq r\}$.

Furthermore the nonzero restrictions of compact roots have the form $\frac{1}{2}\gamma_i$ or $\frac{1}{2}(\gamma_j - \gamma_k)$, and the restrictions of noncompact roots have the form $\frac{1}{2}\gamma_i, \gamma_i$ or $\frac{1}{2}(\gamma_j + \gamma_k)$.

The roots $\gamma_1, \dots, \gamma_r$ do all belong to the set of longest roots in Δ . In Case II, only one root length occurs in Δ .

Unless when $t^+ = 0, \gamma_1, \dots, \gamma_r$ are the only restricted roots of multiplicity one.

PROOF. This can be verified case by case from the diagrams [3, pp. 532–34], or it can be proved by combinatorial arguments, cf. [6].

REMARK 2.2. In [4] it is shown that Case I is necessary and sufficient for G/K to be a tube domain. Here is the classification of the possible algebras \mathfrak{g} (cf. [3]):

Case I: $\mathfrak{su}(n, n)$ ($n \geq 2$), $\mathfrak{so}(n, 2)$ ($n \geq 5$), $\mathfrak{so}^*(4n)$ ($n \geq 3$),
 $\mathfrak{sp}(n, \mathbf{R})$ ($n \geq 1$) and $\mathfrak{e}_{7(-25)}$

Case II: $\mathfrak{su}(p, q)$ ($q > p \geq 1$), $\mathfrak{so}^*(4n+2)$ ($n \geq 2$) and $\mathfrak{e}_{6(-14)}$.

Among these, $t^+ = 0$ only happens for $\mathfrak{sp}(n, \mathbf{R})$ ($n \geq 1$).

For each $\gamma \in \Delta_n$ choose $X_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$ subject to $\bar{X}_\gamma = X_{-\gamma}$, and $\gamma([X_\gamma, X_{-\gamma}]) = 2$, where the bar denotes conjugation with respect to the real form \mathfrak{g} of $\mathfrak{g}_\mathbb{C}$. Let

$$\mathfrak{a} = \sum_{j=1}^r \mathbf{R}(X_{\gamma_j} + X_{-\gamma_j}),$$

then \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} . Let c be the automorphism of $\mathfrak{g}_\mathbb{C}$ given by

$$c = \text{Ad exp } \frac{\pi}{4} \sum_{j=1}^r (X_{\gamma_j} - X_{-\gamma_j}).$$

Then c maps \mathfrak{t}^- bijectively to \mathfrak{a} and fixes \mathfrak{t}^+ (cf. [4]).

Let $\mathfrak{j} = \mathfrak{t}^+ + \mathfrak{a}$ and let $c_*: \mathfrak{t}_c^* \rightarrow \mathfrak{j}_c^*$ denote the adjoint of $c^{-1}: \mathfrak{j}_c \rightarrow \mathfrak{t}_c$. Then $c_*\Delta$ consists of the roots of the Cartan subalgebra \mathfrak{j} of \mathfrak{g} . Let Σ denote the set of nonzero restrictions of $c_*\Delta$ to \mathfrak{a} , and let Σ^+ consists of the nonzero restrictions of $c_*\Delta^+$ to \mathfrak{a} . Let $\alpha_j = c_*\gamma_j$, then exchanging the γ 's in Case I and II above with α 's, we get the two possible forms of Σ^+ .

3. The structure of K .

Let K_1 denote the analytic subgroup of K with Lie algebra \mathfrak{k}_1 .

LEMMA 3.1. *Let Φ denote the set of simple roots for Δ^+ , and let $s_\varphi \in \mathbf{R}$ for each $\varphi \in \Phi$. Then*

$$\exp\left(\sum_{\varphi \in \Phi} s_\varphi \frac{2iH_\varphi}{(\varphi, \varphi)}\right) = e$$

if and only if $s_\varphi \in 2\pi\mathbf{Z}$ for all $\varphi \in \Phi$.

PROOF. Let U be the analytic subgroup of G_c with Lie algebra $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Then U is compact and simply connected, and \mathfrak{t} is a Cartan subalgebra of \mathfrak{u} . The lemma then follows from [9, Theorem 4.6.7].

COROLLARY 3.2. K_1 is simply connected.

PROOF. It is easily seen that $\Phi \cap \Delta_c$ consists of the simple roots for Δ_c^+ . From Lemma 3.1

$$\exp\left(\sum_{\varphi \in \Phi \cap \Delta_c} s_\varphi \frac{2iH_\varphi}{(\varphi, \varphi)}\right) = e$$

if and only if $s_\varphi \in 2\pi\mathbf{Z}$ for all $\varphi \in \Phi \cap \Delta_c$, and so K_1 is simply connected, again by [9, Theorem 4.6.7].

Let a denote the length of the longest roots in Δ . The short roots, if there are any, then have length $a/\sqrt{2}$.

LEMMA 3.3 *Let $t_\gamma \in \mathbf{R}$ for each $\gamma \in \Delta_n^+$ and let*

$$x = \exp\left(\sum_{\gamma \in \Delta_n^+} t_\gamma \frac{2iH_\gamma}{(\gamma, \gamma)}\right).$$

Then $x \in K_1$ if and only if

$$\sum_{\gamma \in \Delta_n^+} \frac{t_\gamma}{(\gamma, \gamma)} \in \frac{2\pi}{a^2} \mathbf{Z}.$$

PROOF. Assume that $x \in K_1$. Since x centralizes $\mathfrak{t} \cap \mathfrak{k}_1$, which is a Cartan subalgebra of \mathfrak{k}_1 , $x = \exp Y$ for some $Y \in \mathfrak{t} \cap \mathfrak{k}_1$. Then

$$\exp \left(\sum_{\gamma \in \Delta_n^+} t_\gamma \frac{2iH_\gamma}{(\gamma, \gamma)} - Y \right) = e$$

and it follows from Lemma 3.1 that

$$\sum_{\gamma \in \Delta_n^+} t_\gamma \frac{2iH_\gamma}{(\gamma, \gamma)} - Y = \sum_{\varphi \in \Phi} s_\varphi \frac{2iH_\varphi}{(\varphi, \varphi)}$$

for some $s_\varphi \in 2\pi\mathbf{Z}$ ($\varphi \in \Phi$). Taking inner product with $\frac{1}{2}Z_0$ it follows that

$$\sum_{\gamma \in \Delta_n^+} \frac{t_\gamma}{(\gamma, \gamma)} = \frac{s_\psi}{(\psi, \psi)} \in \frac{2\pi}{a^2} \mathbf{Z}$$

where ψ is the unique simple noncompact root.

Assume conversely that

$$t = \sum_{\gamma \in \Delta_n^+} \frac{t_\gamma}{(\gamma, \gamma)} \in \frac{2\pi}{a^2} \mathbf{Z}.$$

Then

$$\sum_{\gamma \in \Delta_n^+} t_\gamma \frac{2iH_\gamma}{(\gamma, \gamma)} - ta^2 \frac{2iH_{\gamma_1}}{(\gamma_1, \gamma_1)} \in \mathfrak{k}_1$$

since it is orthogonal to Z_0 . However

$$\exp \left(ta^2 \frac{2iH_{\gamma_1}}{(\gamma_1, \gamma_1)} \right) = e$$

by Lemma 3.1, and hence $x \in K_1$.

Let $N = \sum_{\gamma \in \Delta_n^+} a^2/(\gamma, \gamma)$, then N is the number of longest roots in Δ_n^+ plus twice the number of short roots, if there are any. Define $Z \in \mathfrak{t}$ by

$$(3.1) \quad Z = \frac{1}{N} \sum_{\gamma \in \Delta_n^+} \frac{2iH_\gamma}{(\gamma, \gamma)}.$$

PROPOSITION 3.4. Let $t \in \mathbf{R}$

- (i) $Z \in z \setminus \{0\}$,
- (ii) $\exp tZ \in K_1$ if and only if $t \in 2\pi\mathbf{Z}$.

PROOF. (i) From (3.1) it follows that

$$(3.2) \quad (Z, Z_0) = -\frac{2}{a^2}.$$

Therefore $Z \neq 0$. Let $\varphi \in \Delta_c^+$ and $\gamma \in \Delta_n^+$. If $(\varphi, \gamma) = 0$ then γ contributes nothing to $\varphi(Z)$. If $(\varphi, \gamma) \neq 0$ then also the reflected root $\sigma_\varphi\gamma$ is positive and noncompact since $\sigma_\varphi\gamma(Z_0) = i$. From $(\varphi, \gamma + \sigma_\varphi\gamma) = 0$ it then follows that $\varphi(Z) = 0$ for all $\varphi \in \Delta_c^+$, and hence $Z \in z$.

(ii) follows immediately from Lemma 3.3.

Let $l \in \mathbf{Z}$, and define $\chi_l: K \rightarrow \mathbf{C}$ by $\chi_l(k) = 1$ for $k \in K_1$ and $\chi_l(\exp tZ) = e^{itl}$ for $t \in \mathbf{R}$. From Proposition 3.4 we get that χ_l is a well defined one dimensional representation of K , and that all one dimensional representations of K have this form.

REMARK 3.5. In Case I, we can give a simpler formula for Z as follows

$$(3.3) \quad Z = \frac{1}{r} \sum_{j=1}^r \frac{2iH_{\gamma_j}}{(\gamma_j, \gamma_j)}.$$

In fact, let Z' denote the right hand side of (3.3). Then $Z' \in z$ by Theorem 2.1, and $(Z, Z_0) = (Z', Z_0)$ from (3.2), so (3.3) follows. In particular we have $z \subset \mathfrak{t}^-$ in Case I (cf. also [4, Proposition 3.12]).

4. The structure of M .

Let M denote the centralizer of \mathfrak{a} in K , and let \mathfrak{m} be its Lie algebra. Then \mathfrak{t}^+ is a Cartan subalgebra of \mathfrak{m} . For any Lie group F , let F_0 denote its identity component. It is well-known (cf. [2, p. 75]) that

$$(4.1) \quad M = M_0 \cdot (\exp i\mathfrak{a} \cap K)$$

and also that if $H_\alpha \in \mathfrak{a}$ for $\alpha \in \mathfrak{a}^*$ is defined by $(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{a}$, then (cf. [2, p. 77])

$$(4.2) \quad \left\{ H \in \mathfrak{a} \mid \exp iH \in K \right\} = \left\{ \sum_{\alpha \in \Sigma^+} s_\alpha \frac{H_\alpha}{(\alpha, \alpha)} \mid s_\alpha \in 2\pi\mathbf{Z} \text{ for all } \alpha \in \Sigma^+ \right\}.$$

From the description of Σ^+ by Theorem 2.1 it follows that (4.2) can be restated as follows:

$$(4.3) \quad \left\{ H \in \mathfrak{a} \mid \exp iH \in K \right\} = \left\{ \sum_{j=1}^r s_j \frac{H_{\alpha_j}}{(\alpha_j, \alpha_j)} \mid s_j \in 2\pi\mathbf{Z} \text{ for } j=1, \dots, r \right\}.$$

LEMMA 4.1.

$$\exp \frac{2\pi i H_{\alpha_j}}{(\alpha_j, \alpha_j)} = \exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)}$$

for $j = 1, \dots, r$.

PROOF. There is a homomorphism $\mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}_{\mathbb{C}}$ for which

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \frac{2H_{\alpha_j}}{(\alpha_j, \alpha_j)} \quad \text{and} \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \rightarrow \frac{2iH_{\gamma_j}}{(\gamma_j, \gamma_j)}.$$

The lemma then follows, since in $SL(2, \mathbb{C})$

$$\exp \pi i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \exp \pi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

LEMMA 4.2. *In Case II, M is connected.*

PROOF. By (4.3) and Lemma 4.1, it suffices to prove that for $j = 1, \dots, r$:

$$\exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)} \in M_0.$$

Let $\varphi \in \Delta^+$ be a root whose restriction to \mathfrak{t}^- is $\frac{1}{2}\gamma_j$. Then obviously $H_{\gamma_j} - 2H_{\varphi} \in \mathfrak{t}^+$. From Lemma 3.1 we have

$$\exp \left(2\pi \frac{2iH_{\varphi}}{(\varphi, \varphi)} \right) = e$$

and therefore

$$\exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)} = \exp \frac{2\pi i}{a^2} (H_{\gamma_j} - 2H_{\varphi}) \in M_0.$$

Let W be the Weyl group of Δ , and let $\theta = \sigma_{\gamma_1} \dots \sigma_{\gamma_r} \in W$. Then $\theta(S + T) = S - T$ for $S \in \mathfrak{t}^+$, $T \in \mathfrak{t}^-$. Let $\Xi \subset \Delta^+$ denote the set of roots whose restriction to \mathfrak{t}^- is $\frac{1}{2}\gamma_j$ for some j , and let $\Xi_n = \Xi \cap \Delta_n$ and $\Xi_c = \Xi \cap \Delta_c$. Note that $\theta(\Xi_n) = -\Xi_c$, since if $\xi \in \Xi_n$ with $\xi|_{\mathfrak{t}^-} = \frac{1}{2}\gamma_j$, then $\theta\xi = \xi - \gamma_j \in -\Xi_c$.

Let R denote the number of elements of Ξ_n (or Ξ_c). Define $X \in \mathfrak{t}$ by $X = 0$ in Case I and in Case II:

$$(4.4) \quad X = \frac{1}{R} \left[\sum_{\xi \in \Xi_n} \frac{2iH_{\xi}}{(\xi, \xi)} - \sum_{\xi \in \Xi_c} \frac{2iH_{\xi}}{(\xi, \xi)} \right].$$

LEMMA 4.3. *In Case II, the following holds:*

- (i) $Z + \theta Z = \frac{R}{N} X$.
- (ii) $X \in \mathfrak{t}^+$, $X \perp \mathfrak{t}^+ \cap \mathfrak{k}_1$
- (iii) $X \neq 0$.
- (iv) $Z - X \in \mathfrak{k}_1$.
- (v) *For $t \in \mathbf{R}$: $\exp tX \in K_1$ if and only if $t \in 2\pi\mathbf{Z}$.*

PROOF. (i) is clear from (4.4) since $\theta(\Xi_n) = -\Xi_c$.

(ii) follows from (i). From (4.4) it follows that

$$(4.5) \quad (X, Z_0) = -\frac{2}{a^2}$$

and from this (iii) is obvious.

(iv) follows from (4.5) and (3.2).

(v) is obvious from (iv) and Proposition 3.4 (ii).

Note that it follows that $z \notin \mathfrak{t}^-$ in Case II (cf. also [4, Section 4]).

LEMMA 4.4. *In Case II, $M \cap K_1$ is connected.*

PROOF. Since M is connected and $\mathfrak{m} = (\mathfrak{m} \cap \mathfrak{k}_1) \oplus \mathbf{R}X$ by the preceding lemmas, it suffices to prove that $\exp \mathbf{R}X \cap K_1 \subset (M \cap K_1)_0$. By Lemma 4.3 (v) it then suffices to prove that $\exp 2\pi X \in (M \cap K_1)_0$.

Choose $\xi \in \Xi_n$, then $\theta\xi \in \Delta_c$. Therefore $(H_\xi + \theta H_\xi, Z_0) = i$, and from (4.5) it follows that

$$X - \frac{2i}{a^2} (H_\xi + \theta H_\xi) \in \mathfrak{t}^+ \cap \mathfrak{k}_1.$$

However, by Lemma 3.1

$$\exp \left[2\pi \frac{2i}{a^2} (H_\xi + \theta H_\xi) \right] = e$$

and hence $\exp 2\pi X \in \exp \mathfrak{t}^+ \cap \mathfrak{k}_1$.

We can now state an analogue of (4.1) and (4.3).

PROPOSITION 4.5. (i) $M \cap K_1 = (M \cap K_1)_0(\exp ia \cap K_1)$.

(ii) $\{H \in \mathfrak{a} \mid \exp iH \in K_1\}$
 $= \left\{ \sum_{j=1}^r s_j \frac{H_{\alpha_j}}{(\alpha_j, \alpha_j)} \mid s_j \in 2\pi\mathbb{Z} \text{ for } j=1, \dots, r \text{ and } \sum_{j=1}^r s_j \in 4\pi\mathbb{Z} \right\}.$

PROOF. (i) In Case I, $M_0 \subset K_1$ and hence $(M \cap K_1)_0 = M_0$. Therefore (i) follows from (4.1). In Case II (i) is obvious from Lemma 4.4.

(ii) follows from (4.3), Lemma 4.1, and Lemma 3.3.

Later on, we need the following lemma:

- LEMMA 4.6. (i) If $\gamma \in \Delta^+ \setminus \Xi$, then $\gamma(X) = 0$.
 (ii) If $\gamma \in \Xi_m$, then $\gamma(X) = -a^2(X, X)i/4$,
 (iii) If $\gamma \in \Xi_c$, then $\gamma(X) = a^2(X, X)i/4$.

PROOF. Assume Case II. Let $b \in \mathbb{R}$ be given by $Z = bZ_0$.

(i) Let $\gamma \in \Delta^+ \setminus \Xi$. If γ is compact, then $\theta\gamma$ is also compact by Theorem 2.1, and hence $(\gamma + \theta\gamma)(Z) = 0$. If γ is noncompact, then $\theta\gamma$ is also noncompact but negative, and hence $(\gamma + \theta\gamma)(Z) = ib - ib = 0$. Then $\gamma(X) = 0$ by Lemma 4.3 (i).

(ii)–(iii) If $\gamma \in \Xi_m$, then $\theta\gamma \in -\Xi_c$ and hence $\gamma(Z + \theta Z) = ib$. Therefore $\gamma(X) = ibN/R$, and since $\theta X = X$, we then have $-\theta\gamma(X) = -ibN/R$. But then by (4.4)

$$(X, X) = \frac{1}{R} \left[\sum_{\xi \in \Xi_m} \frac{2i\xi(X)}{(\xi, \xi)} - \sum_{\xi \in \Xi_c} \frac{2i\xi(X)}{(\xi, \xi)} \right] = -\frac{4bN}{Ra^2}$$

so $bN/R = -a^2(X, X)/4$.

5. The rank-one reduction.

Let $\alpha \in \Sigma^+ \setminus 2\Sigma^+$ and let \mathfrak{g}^α be the subalgebra of \mathfrak{g} generated by the root spaces \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$. Let G^α be the corresponding analytic subgroup of G . Then G^α is a simple Lie group of real rank one, and $\mathfrak{g}^\alpha = \mathfrak{k}^\alpha \oplus \mathfrak{p}^\alpha$ is a Cartan decomposition, where $\mathfrak{k}^\alpha = \mathfrak{k} \cap \mathfrak{g}^\alpha$ and $\mathfrak{p}^\alpha = \mathfrak{p} \cap \mathfrak{g}^\alpha$. Therefore $K^\alpha = K \cap G^\alpha$ is a maximal compact subgroup of G^α . (For these well-known facts, see [3, pp. 407–409].) Let $m(\alpha)$ denote the multiplicity of α .

LEMMA 5.1. In Case I, G^α/K^α is Hermitian symmetric if and only if $m(\alpha) = 1$, and then $\mathfrak{g}^\alpha \cong \mathfrak{su}(1, 1)$. In Case II, G^α/K^α is Hermitian symmetric if and only if $\alpha = \frac{1}{2}\alpha_j$ for some $j \in \{1, \dots, r\}$, and then $\mathfrak{g}^\alpha \cong \mathfrak{su}(n, 1)$, where $n = 1 + \frac{1}{2}m(\alpha)$.

PROOF. According to classification the only rank one Hermitian symmetric

spaces are $SU(n, 1)/S(U(n) \times U(1))$ ($n \in \mathbf{N}$). If G^α/K^α is Hermitian symmetric, it follows therefore that $\mathfrak{g}^\alpha \cong \mathfrak{su}(n, 1)$ for some $n \in \mathbf{N}$. In Case I, 2α is not a root, and hence $\mathfrak{g}^\alpha \cong \mathfrak{su}(1, 1)$. Obviously this happens if and only if $m(\alpha) = 1$. In Case II, if α is one of the roots $\frac{1}{2}(\alpha_i \pm \alpha_j)$, then $m(\alpha) > 1$ and $2\alpha \notin \Sigma$, so \mathfrak{g}^α cannot be isomorphic to $\mathfrak{su}(n, 1)$ for any $n \in \mathbf{N}$. On the other hand, if $\alpha = \frac{1}{2}\alpha_j$, then $m(2\alpha) = 1$, and therefore $\mathfrak{g}^\alpha \cong \mathfrak{su}(n, 1)$ with $n + \frac{1}{2}m(\alpha)$ by the classification of real rank one algebras.

Let $l \in \mathbf{Z}$. We will determine the restriction of χ_l to K^α , and assume therefore that K^α is not semisimple.

In Case II, $\mathfrak{g}^\alpha \cong \mathfrak{su}(1, 1)$ and in this identification

$$\frac{2ic^{-1}H_\alpha}{(\alpha, \alpha)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

By Remark 3.5

$$\frac{1}{(Z, Z)} \left(\frac{2ic^{-1}H_{\alpha'}}{(\alpha, \alpha)}, Z \right) = \sum_{j=1}^r \frac{(\alpha, \alpha_j)}{(\alpha, \alpha)}$$

and therefore

$$(5.1) \quad \chi_l \left(\exp t \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) = \begin{cases} 1 & \text{if } \alpha = \frac{1}{2}(\alpha_i - \alpha_j) \\ e^{ilt} & \text{if } \alpha = \alpha_i \\ e^{i2lt} & \text{if } \alpha = \frac{1}{2}(\alpha_i + \alpha_j) \quad (i \neq j). \end{cases}$$

In Case II we have $\alpha = \frac{1}{2}\alpha_j$ and $\mathfrak{g}^\alpha \cong \mathfrak{su}(n, 1)$. Let $\Delta_n^+(\alpha)$ denote the set of noncompact positive roots of $\mathfrak{t} \cap \mathfrak{g}^\alpha$ in \mathfrak{g}^α . Note that the cardinality of $\Delta_n^+(\alpha)$ is n , and put

$$(5.2) \quad Z(\alpha) = \frac{1}{n} \sum_{\gamma \in \Delta_n^+(\alpha)} \frac{2iH_\gamma}{(\gamma, \gamma)}.$$

Then, in the identification with $\mathfrak{su}(n, 1)$, $Z(\alpha)$ is the diagonal matrix with i/n in the first n entries and $-i$ in the last entry. From (5.2) and (3.2) it follows that $Z - Z(\alpha) \in \mathfrak{k}_1$. Therefore

$$(5.3) \quad \chi_l(\exp tZ(\alpha)) = e^{ilt}.$$

6. A lemma concerning $SU(n, 1)$.

Let $G = KAN$ be the Iwasawa decomposition of G corresponding to Σ^+ , and define maps $\varkappa: G \rightarrow K$ and $H: G \rightarrow \mathfrak{a}$ by

$$x \in \varkappa(x) \exp H(x)N$$

for $x \in G$. Let $\varrho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m(\alpha)\alpha$ and let \bar{N} be the group opposite to N , i.e.

$$\bar{N} = \exp\left(\sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}\right).$$

Let $d\bar{n}$ denote some Haar measure on \bar{N} . Let $n \in \mathbf{N}$ and $k \in]0, \infty[$. Let $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$.

LEMMA 6.1. Assume $G = \text{SU}(n, 1)$, and let $\beta \in \Sigma^+$ be the root for which $2\beta \notin \Sigma$. Put $\nu = \frac{1}{2}k\beta$. Then the integral

$$(6.1) \quad \int_{\bar{N}} e^{\langle -\nu - \varrho, H(\bar{n}) \rangle} \chi_l(x(\bar{n})) d\bar{n}$$

converges absolutely for all $l \in \mathbf{Z}$, and it is nonzero if and only if $|l| \notin k+n + 2\mathbf{Z}_+$.

PROOF. We have

$$K = \left\{ \begin{pmatrix} U & 0 \\ 0 & \det U^{-1} \end{pmatrix} \mid U \in U(n) \right\}$$

and

$$\chi_l \begin{pmatrix} U & 0 \\ 0 & \det U^{-1} \end{pmatrix} = (\det U)^l.$$

We put $H = E_{1, n+1} + E_{n+1, 1}$, where E_{ij} denotes the $n+1$ square matrix with 1 on the i, j th entry and all other entries 0. With $\alpha = \mathbf{R}H$ we get $\beta(H) = 2$ and $\varrho(H) = n$.

The root spaces are given as follows

$$\mathfrak{g}_\beta = \text{Ri}(E_{1,1} - E_{1, n+1} + E_{n+1, 1} - E_{n+1, n+1})$$

$$\mathfrak{g}_{\frac{1}{2}\beta} = \left\{ \sum_{j=1}^{n-1} (z_j E_{1, j+1} - \bar{z}_j E_{j+1, 1} + \bar{z}_j E_{j+1, n+1} + z_j E_{n+1, j+1}) \mid z_1, \dots, z_{n-1} \in \mathbf{C} \right\}$$

$$\mathfrak{g}_{-\frac{1}{2}\beta} = \left\{ \sum_{j=1}^{n-1} (z_j E_{1, j+1} - \bar{z}_j E_{j+1, 1} - \bar{z}_j E_{j+1, n+1} - z_j E_{n+1, j+1}) \mid z_1, \dots, z_{n-1} \in \mathbf{C} \right\}$$

$$\mathfrak{g}_{-\beta} = \text{Ri}(E_{1,1} + E_{1, n+1} - E_{n+1, 1} - E_{n+1, n+1}).$$

From the first two of these equations it is easily seen that $H(x)$ and $\chi_l(x(x))$ for $x \in \text{SU}(n, 1)$ can be computed as follows: Let $\eta(x)$ denote the sum of the first and the last element in the last row of x , then

$$H(x) = \log |\eta(x)|H$$

$$\chi_l(\chi(x)) = \left(\frac{\eta(x)}{|\eta(x)|} \right)^{-l}.$$

From the expressions for $g_{-\frac{1}{2}\beta}$ and $g_{-\beta}$, it then follows that the integral (6.1) except for a constant (nonzero) factor equals

$$(6.2) \quad \int_{\mathbf{C}^{n-1}} \int_{\mathbf{R}} [(1+|z|^2)^2 + s^2]^{-\frac{k+n}{2}} \left(\frac{1+|z|^2 - is}{|1+|z|^2 - is|} \right)^{-l} ds dz.$$

If $n > 1$ we use polar coordinates in \mathbf{C}^{n-1} and get

$$\int_0^\infty \int_{\mathbf{R}} [(1+r^2)^2 + s^2]^{-\frac{k+n}{2}} \left(\frac{1+r^2 - is}{|1+r^2 - is|} \right)^{-l} ds r^{2n-3} dr.$$

Let $c_n = \int_0^\infty (1+r^2)^{-k-n+1} r^{2n-3} dr$ if $n > 1$ and $c_1 = 1$ (note that this integral converges because $k > 0$). Substitution of $s = (1+r^2)tgt$ if $n > 1$ and $s = tgt$ if $n = 1$ gives the following integral instead of (6.2):

$$c_n \int_{-\pi/2}^{\pi/2} (\cos t)^{k+n-2} e^{ilt} dt.$$

This integral can in fact be computed in terms of the gamma function since $k > n + 1$, and the result is

$$c_n \frac{\pi \Gamma(k+n-1)}{2^{k+n-2} \Gamma(\frac{1}{2}(k+n+l)) \Gamma(\frac{1}{2}(k+n-l))}$$

(cf. [7, p. 158 (5)–(7)]). The lemma now follows since the denominator has poles precisely when $|l| \in k+n+2\mathbf{Z}_+$.

7. The main theorem.

Let $\lambda \in \mathfrak{j}_{\mathbf{C}}^*$, let $m_0 = \lambda(iX)$ and

$$m_j = \frac{2(\lambda, \alpha_j)}{(\alpha_j, \alpha_j)} \quad (j = 1, \dots, r).$$

Note that in Case I, $m_0 = 0$.

PROPOSITION 7.1. *If $\lambda|_{\mathfrak{t}^+ \cap \mathfrak{t}_1} = 0$, then λ is dominant integral (with respect to $c_* \Delta^+$) if and only if the following three conditions hold:*

- (i) m_0, m_1, \dots, m_r are integers satisfying $|m_0| \leq m_1 \leq \dots \leq m_r$.
- (ii) In Case I, if $\mathfrak{t}^+ \neq 0$, then $(-1)^{m_1} = \dots = (-1)^{m_r}$.
- (iii) In Case II, $(-1)^{m_0} = (-1)^{m_1} = \dots = (-1)^{m_r}$.

PROOF. If $t^+ = 0$ the statement is obvious. Assume $t^+ \neq 0$ and Case I, then $t^+ \subset \mathfrak{k}_1$. Let β be a root of \mathfrak{j} in $\mathfrak{g}_{\mathbb{C}}$ which is not supported on \mathfrak{a} and has restriction $\frac{1}{2}(\alpha_i \pm \alpha_j)$. Then necessarily β is a long root, and hence

$$(7.1) \quad \frac{2(\lambda, \beta)}{(\beta, \beta)} = \frac{1}{2}(m_i \pm m_j).$$

The statement then follows from (7.1) and Theorem 2.1.

Assume next Case II. From Lemma 4.6, if β is a root of \mathfrak{j} in $\mathfrak{g}_{\mathbb{C}}$ and if $\beta|_{\mathfrak{a}} = \frac{1}{2}(\alpha_i \pm \alpha_j)$, then (7.1) holds again. On the other hand if $\beta|_{\mathfrak{a}} = \frac{1}{2}\alpha_i$ then

$$\frac{2(\lambda, \beta)}{(\beta, \beta)} = \frac{2(\lambda|_{\mathfrak{a}}, \beta|_{\mathfrak{a}})}{(\beta, \beta)} + \frac{2\lambda(X)\beta(X)}{(X, X)(\beta, \beta)} = \begin{cases} \frac{1}{2}(m_i + m_0) & \text{if } \beta \in c_*\Xi_n \\ \frac{1}{2}(m_i - m_0) & \text{if } \beta \in c_*\Xi_c. \end{cases}$$

With that the proposition follows.

Assume now that π is a finite dimensional irreducible representation of G having λ as its highest weight. It is well known (cf. [9, Lemma 8.5.3]) that the space of N -fixed vectors for π is invariant under M , and that this representation δ of M is irreducible. Note that $\lambda|_{\mathfrak{k}_1}$ is a highest weight of δ .

THEOREM 7.2. *The following three conditions are equivalent.*

- (i) $\lambda|_{\mathfrak{k}_1 \cap \mathfrak{k}_1} = 0$ and $(-1)^{m_1} = \dots = (-1)^{m_r}$.
- (ii) $\delta|_{M \cap K_1}$ is trivial.
- (iii) π has nonzero K_1 -fixed vectors.

If these conditions hold, then π contains precisely the following one dimensional K-types, each contained once:

In Case I: χ_l for $l = -m_1, -m_1 + 2, \dots, m_1 - 2, m_1$.

In Case II: χ_{m_0} .

PROOF. First the equivalence of (i) and (ii) is proved. Obviously δ is trivial on $(M \cap K_1)_0$ if and only if $\lambda|_{\mathfrak{k}_1 \cap \mathfrak{k}_1} = 0$. We have

$$(7.2) \quad \delta\left(\exp \frac{2\pi i H_{\alpha_j}}{(\alpha_j, \alpha_j)}\right) = \exp\left(\frac{2\pi i(\lambda, \alpha_j)}{(\alpha_j, \alpha_j)}\right) = (-1)^{m_j}$$

and from Proposition 4.5 it follows therefore that (i) and (ii) are equivalent.

Let $\nu = \varrho + \lambda|_{\mathfrak{a}} \in \mathfrak{a}^*$ and let V be the representation space of δ . Consider the principal series representation $I_{\delta, \nu}^P$ of G , the definition of which we recall:

Let $C_{\delta, \nu}^P$ denote the space of V valued C^∞ -functions f on G satisfying $f(gman)$

$= a^{-v-e}\delta(m^{-1})f(g)$ for all $g \in G, m \in M, a \in A,$ and $n \in N. I_{\delta,v}^P$ is then the representation by left translation on this space.

For the opposite minimal parabolic subgroup $\bar{P} = MA\bar{N}$ we define similarly $I_{\delta,v}^{\bar{P}}$ as the representation of G on

$$C_{\delta,v}^{\bar{P}} = \{f \in C^\infty(G, V) \mid f(gma\bar{n}) = a^{-v+e}\delta(m^{-1})f(g), \forall g \in G, m \in M, a \in A, \bar{n} \in \bar{N}\} .$$

It is well known that π is equivalent to a subrepresentation of $I_{\delta,v}^P$ (cf. [9, Lemma 8.5.7]), and also that this subrepresentation can be realized as follows:

For each $f \in C_{\delta,v}^P$ and $x \in G$ the integral

$$Af(x) = \int_N f(x\bar{n}) d\bar{n}$$

is absolutely convergent and defines a G -homomorphism $A: C_{\delta,v}^P \rightarrow C_{\delta,v}^P$ (cf. [9, Section 8.10]), whose image is equivalent to π (cf. [5, p. 75]).

Let $l \in \mathbb{Z}$. By Frobenius reciprocity χ_l occurs in the K -decomposition of $I_{\delta,v}^P$ if and only if $\delta = \chi_l|_M$. If χ_l occurs in π it also occurs in $C_{\delta,v}^P$, and therefore (iii) implies (ii), and also χ_l has multiplicity at most one in π .

Assume (ii). We will first determine the set of $l \in \mathbb{Z}$ such that $\delta = \chi_l|_M$. In Case I, $m \in \mathfrak{k}_1$ and hence $\delta|_{M_0} = \chi_l|_{M_0} \equiv 1$ for all $l \in \mathbb{Z}$. By (4.3) and Lemma 4.1, $\delta = \chi_l|_M$ if and only if

$$(7.3) \quad \delta\left(\exp \frac{2\pi i H_{\gamma_l}}{(\gamma_j, \gamma_j)}\right) = \chi_l\left(\exp \frac{2\pi i H_{\gamma_l}}{(\gamma_j, \gamma_j)}\right) .$$

By (5.1) the right hand side of (7.3) is $(-1)^l$. Comparing with (7.2) we see that $\delta = \chi_l|_M$ if and only if l has the same parity as m_1, \dots, m_r . In Case II, M is connected, and since both δ and χ_l are trivial on $M \cap K_1$ it follows from Lemma 4.3 that $\delta = \chi_l|_M$ if and only if

$$\delta(\exp tX) = \chi_l(\exp tX) \quad \text{for all } t \in \mathbb{R} .$$

However $\delta(\exp tX) = e^{imt}$ and $\chi_l(\exp tX) = e^{ilt}$ by Lemma 4.3 (iv). Therefore $\delta = \chi_l|_M$ if and only if $l = m_0$.

Assume now that l is such that $\delta = \chi_l|_M$. As mentioned χ_l then occurs in $I_{\delta,v}^P$. In fact, if we define for $g \in G$

$$f_l(g) = e^{-(v+e)(H(g))} \chi_l(\varkappa(g))^{-1} ,$$

then $f_l \in C_{\delta,v}^P$ and $f_l(k^{-1}g) = \chi_l(k)f_l(g)$ for $k \in K, g \in G$, so f_l generates the K -type χ_l in $I_{\delta,v}^P$. Therefore π contains χ_l if and only if $Af_l \neq 0$.

From Iwasawa decomposition $G = KA\bar{N}$ it follows that $Af_l \neq 0$ if and only if $Af_l(e) \neq 0$, i.e. if and only if

$$(7.4) \quad \int_N e^{-(v+\varrho)(H(\bar{n}))} \chi_l(\chi(\bar{n}))^{-1} d\bar{n} \neq 0.$$

Note that if $l=0$, (7.4) is obvious. This implies that π contains the trivial K -type χ_0 if and only if δ is trivial (this is the main step in the proof of Helgason's theorem, cf. [2, III Corollary 3.8]).

By the method of Gindikin and Karpelevič (see [9, Proof of Theorem 8.10.16]) the problem of proving (7.4) is reduced to the real rank-one case. Thus (7.4) holds if and only if

$$(7.5) \quad \int_{N^\alpha} e^{-(v+\varrho)(H(\bar{n}))} \chi_l(\chi(\bar{n}))^{-1} d\bar{n} \neq 0$$

for all $\alpha \in \Sigma^+ \setminus 2\Sigma^+$, where $\bar{N}^\alpha = G^\alpha \cap \bar{N}$.

When K^α is semisimple (7.5) is clear, so we may assume that $\mathfrak{g}^\alpha = \mathfrak{su}(n, 1)$ (cf. Lemma 5.1).

In Case I, we have $\mathfrak{g}^\alpha \cong \mathfrak{su}(1, 1)$ and $\chi_l|_{K^\alpha}$ is determined by (5.1). If $\alpha = \frac{1}{2}(\alpha_i - \alpha_j)$, then $\chi_l|_{K^\alpha}$ is trivial and (7.5) is obvious. If $\alpha = \alpha_j$, then by Lemma 6.1 we get that (7.5) holds precisely when

$$(7.6) \quad |l| \notin \frac{2(v, \alpha)}{(\alpha, \alpha)} + 1 + 2\mathbb{Z}_+ = m_j + 2\mathbb{N}.$$

(It is easily seen that the conclusion of Lemma 6.1 also holds for any group covered by $SU(n, 1)$, as long as χ_l is well defined on this group.) Finally, if $\alpha = \frac{1}{2}(\alpha_i + \alpha_j)$ we get that (7.5) holds when

$$(7.7) \quad |2l| \notin \frac{2(v, \alpha)}{(\alpha, \alpha)} + 1 + 2\mathbb{Z}_+ = m_i + m_j + 2\mathbb{N}.$$

By Proposition 7.1 (i), we see that (7.6) and (7.7) holds precisely when $|l| \leq m_1$, and thus the theorem follows in Case I.

In Case II we have $\alpha = \frac{1}{2}\alpha_j$ and $\chi_l|_{K^\alpha}$ is determined by (5.3). From Lemma 6.1 we get that (7.5) holds when

$$|l| \leq \frac{2(v, \alpha_j)}{(\alpha_j, \alpha_j)} = m_j.$$

However by our assumption that $\delta = \chi_l|_M$ we have $l = m_0$, and therefore (7.5) holds by Proposition 7.1 (i).

REMARK 7.3. From the proof of Lemma 6.1 it follows that the integral

$$c(v, l) = \int_{\bar{N}} e^{-(v+\varrho)(H(\bar{n}))} \chi_l(\chi(\bar{n}))^{-1} d\bar{n}$$

for $G = \text{SU}(n, 1)$ takes the following value

$$(7.8) \quad \frac{2^{n-k} \Gamma(n) \Gamma(k)}{\Gamma(\frac{1}{2}(k+n+l)) \Gamma(\frac{1}{2}(k+n-l))}.$$

Here $d\bar{n}$ is so normalized that $c(\varrho, 0)$ equals one. From the proof of Theorem 7.2 it then follows that $c(v, l)$ for arbitrary G can be given an explicit formula as product of expressions like (7.8) and the usual factors in the product formula for the c -function (cf. [1]).

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