One-dimensional nonlinear chromatography system and delta-shock waves

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Abstract. The Riemann problem for the nonlinear chromatography system is considered. Existence and admissibility of δ-shock type solution in both variables are established for this system. By the interactions of δ-shock wave with elementary waves, the generalized Riemann problem for this system is presented, the global solutions are constructed, and the large time-asymptotic behavior of the solutions are analyzed. Moreover, by studying the limits of the solutions as perturbed parameter ε tends to zero, one can observe that the Riemann solutions are stable for such perturbations of the initial data.

1. Introduction

In this paper, we are concerned with the following conservation laws:

$$\begin{cases} \partial_t u + \partial_x \left(\dfrac{u}{1-u+v} \right) = 0, \\[2mm] \partial_t v + \partial_x \left(\dfrac{v}{1-u+v} \right) = 0, \end{cases} \tag{1.1}$$

where u, v are the nonnegative functions of the variables $(x,\ t) \in R \times R_+$, which express the concentrations of the two adsorbing species and $u \geq 0$, $v \geq 0$, $1-u+v > 0$. Our main purpose is to investigate the existence and admissibility of nonclassical solutions and the generalized Riemann problem for the nonlinear chromatography system. The motivation comes from the fact that the delta-shock wave was captured numerically and experimentally by Mazzotti et al. [22–24] in the Riemann solutions for the local equilibrium model of two-component nonlinear chromatography, which consists of the following conservation laws

$$\begin{cases} \partial_t(u + \dfrac{a_1 u}{1-u+v}) + \partial_x u = 0, \\[2mm] \partial_t(u + \dfrac{a_2 v}{1-u+v}) + \partial_x v = 0, \end{cases} \tag{1.2}$$

where u and v are the concentrations of the adsorbing species, and $u \geq 0$, $v \geq 0$, $1-u+v > 0$, $a_2 > a_1 > 0$.

There is substantial difference between the nonlinear chromatography system (1.1) and the following chromatography equations

$$\begin{cases} \partial_t u + \partial_x \left(\dfrac{u}{1+u+v} \right) = 0, \\[3mm] \partial_t v + \partial_x \left(\dfrac{v}{1+u+v} \right) = 0, \end{cases} \tag{1.3}$$

where u, v are nonnegative functions of the variables $(x, t) \in R \times R_+$. That is, the fact that $1 - u + v > 0$ in (1.1) and $1 + u + v > 1$ in (1.3), which results in the essential difference of the Riemann solutions, i.e., the former contains delta-shock solutions. For the Riemann solutions to the system (1.1), we can refer to [9] and the references cited therein. The system (1.1) belongs to so-called Temple class [39] and the shock curves coincide with the rarefaction curves in the phase plane. On the basis of the fact, we call the system (1.1) a generalized Temple class and the results about the Temple class are available for the system (1.1). Thanks to its features, well-posedness results for Temple systems are available for a much larger class of initial data compared to general systems of conservation laws. For the related results about Temple systems, we can refer to [1–4,6] and the references cited therein.

Delta-shock wave is a kind of nonclassical nonlinear waves on which at least one of the state variables becomes a singular measure. Korchinski [18] introduced the concept of the Dirac function into the classical weak solution when he studied the Riemann problem for the following system

$$\begin{cases} u_t + \left(\frac{1}{2} u^2 \right)_x = 0, \\ v_t + \left(\frac{1}{2} uv \right)_x = 0, \end{cases} \tag{1.4}$$

in his unpublished Ph.D. thesis in 1977. In fact, the concept of the δ-shock solution and the corresponding Rankine–Hugoniot condition were also presented by Zeldovich and Myshkis [43] in the case of the continuity equation. Tan et al. [38] considered the system

$$\begin{cases} u_t + (u^2)_x = 0, \\ v_t + (uv)_x = 0, \end{cases} \tag{1.5}$$

and discovered that the form of Dirac delta functions supported on shocks was used as parts in their Riemann solutions for certain initial data. There is another well-known example, i.e. the transport equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.6}$$

which are called the one-dimensional system of pressureless Euler equations. The transport equations (1.6) have been analyzed extensively, see [5,8,14,15,19–21,35,41] and so on. Recently, the weak asymptotics method was widely used to study the δ-shock wave type solution by Danilov et al. [12,13,29,32,40] in the case of systems which are linear with respect to one of unknown functions. In the same papers, it is introduced a concept allowing functions of the form

$$u(x,t) = u^0(x,t) + e(t)\delta(x - x(t)), \tag{1.7}$$

to represent a solution to the considered systems. This concept is extended in [16] on systems which are nonlinear with respect to both variables. we also see papers [17,25,27,34,42,43] for the related equations and results.

The paper is organized as follows. In Sect. 2, following [9], we consider the elementary waves to the system (1.1). In Sect. 2, the definition of δ-shock wave type solution is given. Furthermore, under the generalized Rankine–Hugoniot condition and δ-entropy condition, the δ-shock wave solutions are obtained. In Sect. 4, we consider the initial value problem with three constant states. With the help of the interactions of the δ-shock and elementary waves, the global solutions are constructed. Moreover, we prove that the solutions of the perturbed initial value problem converge to the corresponding Riemann solutions as $\varepsilon \to 0$, which shows the stability of the Riemann solutions for the small perturbation, and analyze the large time-asymptotic behavior of the solutions.
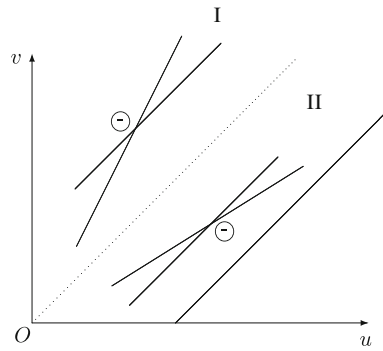
FIG. 1. Curves of elementary waves

2. Elementary waves

In this section, we will consider the Riemann problem of system (1.1) with the following Riemann initial data

$$(u, v)(x, 0) = (u_\pm, v_\pm), \quad \pm x > 0, \tag{2.1}$$

where $u_\pm, v_\pm$ are constants and satisfy

$$u_\pm \geq 0, \quad v_\pm \geq 0, \quad 1 - u_\pm + v_\pm > 0. \tag{2.2}$$

For details about the corresponding Riemann solutions, we can refer the reader to [9] and references cited therein, also see [37].

The eigenvalues of system (1.1) are

$$\lambda_1 = \frac{1}{1 - u + v}, \quad \lambda_2 = \frac{1}{(1 - u + v)^2}, \tag{2.3}$$

and the corresponding right eigenvectors are

$$\overrightarrow{r_1} = (1, 1)^T, \quad \overrightarrow{r_2} = (u, v)^T. \tag{2.4}$$

By simple calculations, we have

$$\nabla \lambda_1 \cdot \overrightarrow{r_1} = 0, \quad \nabla \lambda_2 \cdot \overrightarrow{r_2} \neq 0,$$

where ∇ denotes the gradient with respect to (u, v). Hence, λ_1 is linearly degenerate, λ_2 is genuinely nonlinear, and system (1.1) is nonstrictly hyperbolic for $1 - u + v = 1$. It is easy to obtain that the Riemann invariants of system (1.1) are

$$w = v - u, \quad z = v/u. \tag{2.5}$$

If $1 - u + v = 1$, one can obtain $\lambda_1 = \lambda_2 = 1$, and we divide the phase plane into two parts (see Fig. 1)

$$\text{I} : \{(u, v) | v > u, u \geq 0\} \quad \text{and} \quad \text{II} : \{(u, v) | u - 1 < v < u, v \geq 0\}. \tag{2.6}$$

Riemann problems allow to consider the so-called self-similar solution, that is, the solution depending only on the self-similar variable $\xi = \frac{x}{t}$. By using self-similar transformation, system (1.1) becomes

$$\begin{cases} -\xi u_\xi + \left(\dfrac{u}{1 - u + v} \right)_\xi = 0, \\ -\xi v_\xi + \left(\dfrac{v}{1 - u + v} \right)_\xi = 0, \end{cases} \tag{2.7}$$

and the Riemann initial data (2.1) can be changed into the following infinity boundary value

$$(u,v)(\pm\infty) = (u_\pm, v_\pm). \tag{2.8}$$

For smooth solutions, system (2.7) can be rewritten into

$$A(U)U_\xi = 0, \tag{2.9}$$

where $U = (u,v)^T$ and

$$A(u,v) = \begin{pmatrix} -\xi + \dfrac{1+v}{(1-u+v)^2} & \dfrac{-u}{(1-u+v)^2} \\ \dfrac{v}{(1-u+v)^2} & -\xi + \dfrac{1-u}{(1-u+v)^2} \end{pmatrix}. \tag{2.10}$$

For a fixed left state (u_-, v_-), the possible states which can be connected to (u_-, v_-) on the right by a rarefaction wave lie on a curve, which is given as follows

$$R(u_-, v_-): \begin{cases} \xi = \dfrac{1}{(1-u+v)^2}, \\ \dfrac{v}{u} = \dfrac{v_-}{u_-}, \\ v < v_-, \ u < u_-, \quad \text{if } (u_-, v_-) \in \mathrm{I}, \quad \text{or} \ v > v_-, u > u_-, \quad \text{if } (u_-, v_-) \in \mathrm{II}. \end{cases} \tag{2.11}$$

In is well known that the Rankine–Hugoniot condition for a bounded discontinuity takes the form

$$\begin{cases} \sigma[u] = \left[\dfrac{u}{1-u+v}\right], \\ \sigma[v] = \left[\dfrac{v}{1-u+v}\right], \end{cases} \tag{2.12}$$

where $[\cdot]$ denotes the jump across the discontinuity and $\xi = \sigma$ denotes the speed of the discontinuity.

Let $P = 1 - u + v$, we have

$$\begin{cases} \sigma = \dfrac{1}{P_r P_\ell} + \dfrac{u_r v_\ell - u_\ell v_r}{P_r P_\ell (u_r - u_\ell)}, \\ \sigma = \dfrac{1}{P_r P_\ell} + \dfrac{u_r v_\ell - u_\ell v_r}{P_r P_\ell (v_r - v_\ell)}. \end{cases} \tag{2.13}$$

where $P_\ell = 1 - u_\ell + v_\ell$ and $P_r = 1 - u_r + v_r$. From Eq. (2.13), we have

$$(u_r v_\ell - u_\ell v_r)[(u_r - u_\ell) - (v_r - v_\ell)] = 0. \tag{2.14}$$

Thus, we have $(u_r v_\ell - u_\ell v_r) = 0$ or $u_r - v_r = u_\ell - v_\ell$. The former gives

$$\frac{v_r}{u_r} = \frac{v_\ell}{u_\ell}. \tag{2.15}$$

We also obtain

$$\sigma = \frac{1}{P_\ell P_r}.$$

For a shock wave, the Lax entropy conditions imply $1/P_r < 1/P_\ell$ which means $P_r > P_\ell$. Therefore, for a fixed left state (u_-, v_-), the possible states which can be connected to (u_-, v_-) on the right by a shock wave satisfy

$$S(u_-, v_-): \begin{cases} \sigma = \dfrac{1}{PP_-} = \dfrac{1}{(1-u+v)(1-u_-+v_-)}, \\ \dfrac{v}{u} = \dfrac{v_-}{u_-}, \\ u > u_-, v > v_- \quad \text{if } (u_-, v_-) \in \mathrm{I} \ \text{ or } \ u < u_-, v < v_- \quad \text{if } (u_-, v_-) \in \mathrm{II}. \end{cases} \tag{2.16}$$

The equality $v_r - u_r = v_\ell - u_\ell$ implies $1/(1 - u_r + v_r) = 1/(1 - u_\ell + v_\ell)$, which means a contact discontinuity. Then, the possible states that can be connected to (u_-, v_-) on the right by a contact discontinuity lie on the curve

$$J: \quad \frac{1}{1 - u_- + v_-} = \frac{1}{1 - u + v}. \tag{2.17}$$

Using these classical elementary waves, one can construct the solutions as follows:

(a)   The solution is $R + J$ if $(u_-, v_-) \in \mathrm{I}$ and $-u_+ + v_+ < -u_- + v_-$;
(b)   The solution is $S + J$ if $(u_-, v_-) \in \mathrm{I}$ and $-u_+ + v_+ > -u_- + v_-$;
(c)   The solution is $R + (0,0) + R$ if $(u_-, v_-) \in \mathrm{I}$ and $(u_+, v_+) \in \mathrm{II}$;
(d)   The solution is $J + R$ if $(u_-, v_-) \in \mathrm{II}$ and $-u_+ + v_+ < -u_- + v_-$;
(e)   The solution is $J + S$ if $(u_-, v_-) \in \mathrm{II}$ and $-u_+ + v_+ > -u_- + v_-$;
(f) $(u_-, v_-) \in \mathrm{II}, (u_+, v_+) \in \mathrm{I}$ (or $1 - u_+ + v_+ = 1$) or $1 - u_- + v_- = 1, (u_+, v_+) \in \mathrm{I}$. In this case (f), one can see

$$\lambda_2^+ = \frac{1}{(1 - u_+ + v_+)^2} \le \lambda_1^+ = \frac{1}{1 - u_+ + v_+}$$
$$< \lambda_1^- = \frac{1}{1 - u_- + v_-} \le \lambda_2^- = \frac{1}{(1 - u_- + v_-)^2}, \tag{2.18}$$

which implies that the solution cannot be constructed by applying these classical waves described above. Hence, the Riemann solution containing a weighted δ-measure supported on a line should be constructed in order to established the existence.

3. The nonclassical solutions

In this section, we will consider the nonclassical solutions for system (1.1). Following [12, 13, 16], we have the following definitions.

Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a graph in the closed upper half-plane $\{(x,t) : x \in \mathbb{R}, t \in [0, +\infty)\} \subset \mathbb{R}^2$ containing smooth arc $\gamma_i = \{(x,t) : S_i(x,t) = 0\}$, $i \in I$, and I is a finite set. Let I_0 be subset of I such that an arc γ_i for $i \in I_0$ starts from the points of the x-axis; $\Gamma_0 = \{x_k^0; i \in I_0\}$ is the set of initial points of arc γ_i, $i \in I_0$.

Consider δ-shock wave type initial data $(u^0(x), v^0(x))$, where

$$u^0(x) = u_0(x) + e_u^0 \delta(\Gamma_0), \quad v^0(x) = v_0(x) + \omega_v^0 \delta(\Gamma_0),$$

$u_0, v_0 \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, $e_u^0 \delta(\Gamma_0) = \sum\limits_{i \in I} e_i^0 \delta(x - x_i^0)$ and $\omega_v^0 \delta(\Gamma_0) = \sum\limits_{i \in I} \omega_i^0 \delta(x - x_i^0)$, e_i^0 and ω_i^0 are constants for $i \in I_0$.

Definition 3.1. A pair of distributions $(u(x,t), v(x,t))$ and a graph Γ, where $(u(x,t), v(x,t))$ has the form of the sum

$$u(x,t) = \widehat{u}(x,t) + e(x,t)\delta(\Gamma) \quad \text{and} \quad v(x,t) = \widehat{v}(x,t) + \omega(x,t)\delta(\Gamma),$$

$\widehat{u}, \widehat{v} \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ and $e(x,t)\delta(\Gamma) = \sum\limits_{i \in I} e_i(x,t)\delta(\gamma_i)$, $\omega(x,t)\delta(\Gamma) = \sum\limits_{i \in I} \omega_i(x,t)\delta(\gamma_i)$, $e_i(x,t), \omega_i(x,t) \in C(\Gamma)$ for $i \in I$, is called a generalized δ-shock wave type solution of system (1.1) with the δ-shock wave type initial data $(u^0(x), v^0(x))$ if the integral identities

$$
\begin{cases}
\displaystyle\iint_{\mathbb{R}_+}\int_{\mathbb{R}} \left(\widehat{u}\phi_t + \frac{\widehat{u}}{1 - \widehat{u} + \widehat{v}}\phi_x\right) \mathrm{d}x\mathrm{d}t + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\phi(x,t)}{\partial\ell}\mathrm{d}\ell \\
\quad + \displaystyle\int_{\mathbb{R}} u_0(x)\phi(x,0)\mathrm{d}x + \sum_{i\in I_0} e_i^0\phi(x_i^0,0) = 0, \\
\displaystyle\iint_{\mathbb{R}_+}\int_{\mathbb{R}} \left(\widehat{v}\phi_t + \frac{\widehat{v}}{1 - \widehat{u} + \widehat{v}}\phi_x\right) \mathrm{d}x\mathrm{d}t + \sum_{i\in I}\int_{\gamma_i} \omega_i(x,t)\frac{\partial\phi(x,t)}{\partial\ell}\mathrm{d}\ell \\
\quad + \displaystyle\int_{\mathbb{R}} v_0(x)\phi(x,0)\mathrm{d}x + \sum_{i\in I_0} \omega_i^0\phi(x_i^0,0) = 0,
\end{cases}
\tag{3.1}
$$

hold for all test functions $\phi(x,t) \in \mathcal{D}(R \times R_+)$, where $\dfrac{\partial\phi(x,t)}{\partial\ell}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i}\mathrm{d}\ell$ is a line integral over the arc γ_i.

Theorem 3.2. *For the Cauchy problem* (1.1) *and* (2.1), *in case (f), system* (1.1) *has δ-shock wave type solution*

$$
u(x,t) = u_- + [u]H(x - x(t)) + e(t)\delta(x - x(t)),
$$
$$
v(x,t) = v_- + [v]H(x - x(t)) + \omega(t)\delta(x - x(t)),
$$

which satisfies the integral identities (3.1) *in the sense of Definition* 3.1, *where* $\Gamma = \{(x,t) : x = x(t) = \sigma_\delta t, \ t \geq 0\}$,

$$
\int_\Gamma e(x,t)\frac{\partial\phi(x,t)}{\partial\ell} = \int_0^\infty e(x,t)\frac{\mathrm{d}\phi(x,t)}{\mathrm{d}t}\mathrm{d}t, \quad \int_\Gamma \omega(x,t)\frac{\partial\phi(x,t)}{\partial\ell} = \int_0^\infty \omega(x,t)\frac{\mathrm{d}\phi(x,t)}{\mathrm{d}t}\mathrm{d}t,
$$

and $H(x)$ is the Heaviside function

$$
H(x) = \begin{cases} 0, & \text{for} \quad x < 0, \\ 1, & \text{for} \quad x > 0. \end{cases}
$$

In addition

$$
\begin{cases}
e(t) = \omega(t) = \dfrac{u_-v_+ - v_-u_+}{(1 - u_- + v_-)(1 - u_+ + v_+)}\ t, \\
\sigma_\delta = \dfrac{1}{(1 - u_- + v_-)(1 - u_+ + v_+)}.
\end{cases}
\tag{3.2}
$$

Proof. We need to check that the constructed δ-measure solution satisfies the Definition 3.1 in the sense of distributions, that is,

$$
\iint_{\mathbb{R}_+}\int_{\mathbb{R}} \left(\widehat{u}\phi_t + \frac{\widehat{u}}{1 - \widehat{u} + \widehat{v}}\phi_x\right)\mathrm{d}x\mathrm{d}t + \int_0^\infty e(t)\frac{\mathrm{d}\phi(x(t),t)}{\mathrm{d}t}\mathrm{d}t + \int_{\mathbb{R}} u_0(x)\phi(x,0)\mathrm{d}x = 0, \tag{3.3}
$$

$$
\iint_{\mathbb{R}_+}\int_{\mathbb{R}} \left(\widehat{v}\phi_t + \frac{\widehat{v}}{1 - \widehat{u} + \widehat{v}}\phi_x\right)\mathrm{d}x\mathrm{d}t + \int_0^\infty \omega(t)\frac{\mathrm{d}\phi(x(t),t)}{\mathrm{d}t}\mathrm{d}t + \int_{\mathbb{R}} v_0(x)\phi(x,0)\mathrm{d}x = 0, \tag{3.4}
$$

for all the test functions $\phi(x,t) \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$, where $u_0(x) = u_- + [u]H(x)$, $v_0(x) = v_- + [v]H(x)$, $\widehat{u}(x,t) = u_- + [u]H(x - x(t))$ and $\widehat{v}(x,t) = v_- + [v]H(x - x(t))$.

Denote by $\mathcal{A}$ the left-hand side of (3.3), we have

$$\mathcal{A} = \int_0^\infty \int_{-\infty}^{x(t)} (u_-\phi_t + \frac{u_-}{1 - u_- + v_-}\phi_x)\mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{x(t)}^{+\infty} (u_+\phi_t + \frac{u_+}{1 - u_+ + v_+}\phi_x)\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\infty e(t)\frac{\mathrm{d}\phi(x(t),t)}{\mathrm{d}t}\mathrm{d}t + \int_{-\infty}^0 u_-\phi(x,0)\mathrm{d}x + \int_0^\infty u_+\phi(x,0)\mathrm{d}x. \tag{3.5}$$

Without loss of generality, we assume $\sigma_\delta > 0$, then the first term on the right-hand side of (3.5) equals

$$\int_0^\infty \int_{-\infty}^0 u_-\phi_t\mathrm{d}x\mathrm{d}t + \int_0^\infty \int_0^{x(t)} u_-\phi_t\mathrm{d}x\mathrm{d}t + \int_0^\infty \int_{-\infty}^{x(t)} \frac{u_-}{1 - u_- + v_-}\phi_x\mathrm{d}x\mathrm{d}t$$

$$= -\int_{-\infty}^0 u_-\phi(x,0)\mathrm{d}x + \int_0^\infty \int_0^{x(t)} u_-\phi_t\mathrm{d}x\mathrm{d}t + \int_0^\infty \frac{u_-}{1 - u_- + v_-}\phi(x(t),t)\mathrm{d}t$$

$$= -\int_{-\infty}^0 u_-\phi(x,0)\mathrm{d}x + \int_0^\infty \mathrm{d}x \int_{t(x)}^\infty u_-\phi_t\mathrm{d}t + \int_0^\infty \frac{u_-}{1 - u_- + v_-}\frac{1}{\sigma_\delta}\phi(x,t(x))\mathrm{d}x$$

$$= -\int_{-\infty}^0 u_-\phi(x,0)\mathrm{d}x + \int_0^\infty \Big(\frac{1}{\sigma_\delta(1 - u_- + v_-)} - 1\Big)u_-\phi(x,t(x))\mathrm{d}x. \tag{3.6}$$

The second term on the right-hand side of (3.5) equals

$$\int_0^\infty \mathrm{d}x \int_0^{t(x)} u_+\phi_t\mathrm{d}t - \int_0^\infty \frac{u_+}{1 - u_+ + v_+}\phi(x(t),t)\mathrm{d}t$$

$$= -\int_0^\infty u_+\phi(x,0)\mathrm{d}x + \int_0^\infty \Big(1 - \frac{1}{\sigma_\delta(1 - u_+ + v_+)}\Big)u_+\phi(x,t(x))\mathrm{d}x. \tag{3.7}$$

The third term on the right-hand side of (3.5) equals

$$\big(\sigma_\delta[u] - [\frac{u}{1 - u + v}]\big)t\phi(x(t),t)\Big|_0^{+\infty} - \big(\sigma_\delta[u] - [\frac{u}{1 - u + v}]\big)\int_0^{+\infty} \phi(x(t),t)\mathrm{d}t$$

$$= -\big(\sigma_\delta[u] - [\frac{u}{1 - u + v}]\big)\frac{1}{\sigma_\delta}\int_0^\infty \phi(x,t(x))\mathrm{d}x. \tag{3.8}$$

It is clear from (3.5)–(3.8) that

$$\mathcal{A} = 0.$$

A similar argument gives (3.4). It is easy to obtain (3.2). So, we complete the proof. $\square$

Using Definition 3.1 and repeating the proof of Theorem 3.2 almost word-for-word, one can derive the generalized Rankine–Hugoniot condition for δ-shock wave type solution of system (1.1).

Theorem 3.3. *Suppose that $\Omega \subset R \times R_+$ is some region cut by a smooth curve $\Gamma = \{(x,t) : x = x(t)\}$ into a left- and right-hand parts $\Omega_\pm = \{(x,t) : \pm(x - x(t)) > 0\}$, $(u(x,t)), v(x,t))$, is a generalized δ-shock wave type solution of system* (1.1), *functions $\widehat{u}(x,t), \widehat{v}(x,t)$ are smooth in $\Omega_\pm$, and have one-side limits $\widehat{u}_\pm, \widehat{v}_\pm$ on the curve Γ. Then, the generalized Rankine–Hugoniot condition for δ-shock is*

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \sigma_\delta, \\[2mm] \dfrac{\mathrm{d}e(t)}{\mathrm{d}t} = \left(\dot{x}(t)[u] - [\dfrac{u}{1 - u + v}]\right)\big|_{x=x(t)}, \\[2mm] \dfrac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \left(\dot{x}(t)[v] - [\dfrac{v}{1 - u + v}]\right)\big|_{x=x(t)}, \end{cases} \tag{3.9}$$

where $e(t) \doteq e(x(t), t)$, $\omega(t) \doteq \omega(x(t), t)$ and $\dot{(\cdot)} = \frac{\mathrm{d}}{\mathrm{d}t}(\cdot)$.

In addition to the generalized Rankine–Hugoniot conditions (3.9), to guarantee uniqueness, the discontinuity should satisfy

$$\lambda_1^+ = \frac{1}{(1 - u_+ + v_+)^2} \leq \lambda_2^+ = \frac{1}{1 - u_+ + v_+} \leq \sigma_\delta$$
$$\leq \lambda_1^- = \frac{1}{1 - u_- + v_-} \leq \lambda_2^- = \frac{1}{(1 - u_- + v_-)^2}, \tag{3.10}$$

where $u_\pm$ and $v_\pm$ are the respective left- and right-hand limit values of $u(x,t)$ and $v(x,t)$ on the discontinuity curve. Condition (3.10) is called as **δ-entropy condition**. It is overcompressive and means that all the characteristic lines on both sides of the discontinuity are not out-coming. A discontinuity satisfying (3.9), (3.10), and (3.2) will be called a δ-shock wave to system (1.1).

So, we complete the construction of the Riemann solutions to system (1.1).

4. Interactions of δ-shock wave with elementary waves

To start off, we consider the initial value problem with three pieces constant states

$$(u,v)(x,0) = \begin{cases} (u_-, v_-), & x < 0, \\ (u_m, v_m), & 0 < x < \varepsilon, \\ (u_+, v_+), & x > \varepsilon, \end{cases} \tag{4.1}$$

where $\varepsilon > 0$ is arbitrarily small. The data (4.1) is a perturbation of the Riemann initial data (2.1). Our interest is to investigate whether the Riemann solutions of (1.1) and (2.1) are the limits of the solutions of (1.1) and (4.1) as $\varepsilon \to 0$. In this section, we only consider the interactions of the δ-shock and elementary waves. For the interactions of elementary waves, we refer the readers to the book of Smoller [36] and the monograph of Chang and Hsiao [7]. Also see [33] for the recent work about the interactions of elementary waves. For a comprehensive survey, we can see the books written by Dafermos [10] and Serre [31]. For interaction with δ-shocks, we can see [28] and the references cited therein. The problem can be divided into eleven cases as follows:

$$J + R \oplus \delta, \quad J + S \oplus \delta, \quad R + R \oplus \delta, \quad \delta \oplus \delta, \quad R + J \oplus \delta,$$
$$S + J \oplus \delta, \quad \delta \oplus R + J, \quad \delta \oplus S + J, \quad \delta \oplus J + R, \quad \delta \oplus J + S, \quad \delta \oplus R + R.$$

When the Riemann solution $\mathcal{R}(\ominus, \widehat{\mathrm{m}})$ is of type $S + J$, we have $(u_-, v_-), (u_m, v_m) \in \mathrm{I}$, which implies that the δ-shock cannot appear between left state (u_m, v_m) and right state (u_+, v_+). Thus, $S + J \oplus \delta$ is impossible for this situation. Similarly, $\delta \oplus J + S$ is also impossible. We will discuss the first five cases in detail and the other four cases can be discussed in a similar way.
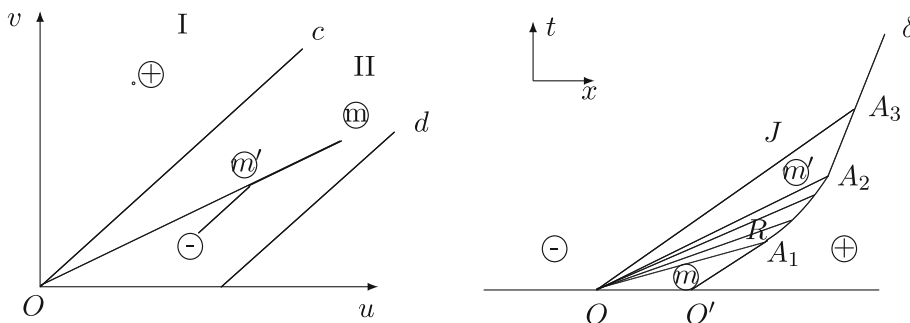
FIG. 2. Case 1.1: $(u_-, v_-) \in \mathrm{II}, (u_+, v_+) \in \mathrm{I}$

For all the first three cases, we have $(u_m, v_m) \in \mathrm{II}$.

Case 1. $J + R \oplus \delta$-shock. According to the location of states (u_-, v_-) and (u_+, v_+), there are three sub-cases: (1.1) $(u_-, v_-) \in \mathrm{II}, (u_+, v_+) \in \mathrm{I}$; (1.2) $1 - u_- + v_- = 1, (u_+, v_+) \in \mathrm{I}$; (1.3) $1 - u_- + v_- = 1$ and $1 - u_+ + v_+ = 1$. In the following, we discuss them in detail.

Case 1.1. $(u_-, v_-) \in \mathrm{II}, (u_+, v_+) \in \mathrm{I}$, see Fig. 2. A contact discontinuity and a rarefaction wave start at point O. The speed of the δ-shock is $\sigma_\delta = 1/(P_m P_+) < 1/P_m^2$, where $P_m = 1 - u_m + v_m$ and the rarefaction wave will overtake the δ-shock at finite time. The intersection point $A_1(x_1, t_1)$ is determined by

$$
\begin{cases}
\dfrac{x}{t} = \dfrac{1}{P_m^2}, \\
\dfrac{x - \varepsilon}{t} = \dfrac{1}{P_m P_+},
\end{cases}
\tag{4.2}
$$

which implies

$$
(x_1, t_1) = \left(\frac{P_+}{P_+ - P_m} \varepsilon, \frac{P_+ P_m^2}{P_+ - P_m} \varepsilon \right).
\tag{4.3}
$$

We have the following fact:

Lemma 4.1. *The δ-shock can penetrate the rarefaction wave R and interacts with the contact discontinuity J, the interaction of the delta wave and the discontinuity yields a new δ-shock.*

Proof. When $\frac{1}{P_{m'}} \le \frac{1}{P} \le \frac{1}{P_m}$, we have $\frac{1}{P_+} < 1 \le \frac{1}{P_{m'}} \le \frac{1}{P} \le \frac{1}{P_m}$, then the δ-shock entropy condition

$$
\frac{1}{P_+^2} < \frac{1}{P_+} \le \sigma_\delta = \frac{1}{P P_+} \le \frac{1}{P} \le \frac{1}{P^2}
$$

is satisfied, where the intermediate state $(u_{m'}, u_{m'})$ is expressed as

$$
\begin{cases}
\dfrac{1}{1 - u_{m'} + v_{m'}} = \dfrac{1}{1 - u_- + v_-}, \\
\dfrac{v_{m'}}{u_{m'}} = \dfrac{v_m}{u_m}.
\end{cases}
\tag{4.4}
$$

The δ-shock begins interacting with the rarefaction wave R at the point A_1 and starts to bend. The bending of δ-shock is determined by

$$
\begin{cases}
\dfrac{x}{t} = \dfrac{1}{P^2}, \\[2mm]
\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{PP_+}, \quad \dfrac{1}{P_{m'}} \le \dfrac{1}{P} \le \dfrac{1}{P_m}, \\[2mm]
x|_{t=t_1} = \dfrac{P_+}{P_+ - P_m}\varepsilon.
\end{cases}
\tag{4.5}
$$

Substituting the first equation of (4.5) into the second equation, we obtain

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{P_+}\sqrt{\frac{x}{t}}.
\tag{4.6}
$$

Differentiating Eq. (4.6) with respect to t, we get

$$
\frac{d^2 x}{\mathrm{d}t^2} = \frac{1}{2P_+\sqrt{xt}}\left(\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{x}{t}\right).
\tag{4.7}
$$

Combining the first equation with the second equation in (4.5), it is easy to get

$$
\frac{d^2 x}{\mathrm{d}t^2} = \frac{1}{2PP_+\sqrt{xt}}\left(\frac{1}{P_+} - \frac{1}{P}\right) < 0
\tag{4.8}
$$

for $\frac{1}{P_+} < \frac{1}{P} < \frac{1}{P_{m'}}$. Equation (4.8) means that the δ-shock decelerates during the process of penetration.

Integrating (4.6), we have

$$
\sqrt{x} = \sqrt{x_1} + \frac{1}{P_+}(\sqrt{t} - \sqrt{t_1}), \quad t > t_1,
\tag{4.9}
$$

which together with $x = \frac{1}{P_{m'}^2}\, t$ determines the ending point $A_2(x_2, t_2)$ of the penetration. A direct calculation leads to

$$
\begin{cases}
x_2 = \dfrac{P_+(P_+ - P_m)}{(P_+ - P_{m'})^2}\varepsilon, \\[3mm]
t_2 = \dfrac{P_+(P_+ - P_m)P_{m'}^2}{(P_+ - P_{m'})^2}\varepsilon.
\end{cases}
\tag{4.10}
$$

After penetrating the R, the propagating speed of the new δ-shock is $\frac{1}{P_{m'}P_+}$, which is less than the speed of the contact discontinuity $\lambda = \frac{1}{P_{m'}} = \frac{1}{P_-}$. So, the contact discontinuity J will overtake the δ-shock at a finite time. Their intersection point $A_3(x_3, t_3)$ is determined by

$$
\begin{cases}
\dfrac{x}{t} = \dfrac{1}{P_-} = \dfrac{1}{P_{m'}}, \\[2mm]
\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{P_{m'}P_+} = \dfrac{1}{P_-P_+}, \\[2mm]
x|_{t=t_2} = x_2,
\end{cases}
\tag{4.11}
$$

which means

$$
\begin{cases}
x_3 = \left(\dfrac{P_-}{t_2}\right)^{\frac{1}{P_+ - 1}} x_2^{\frac{P_+}{P_+ - 1}}, \\[3mm]
t_3 = (P_- x_2)^{\frac{P_+}{P_+ - 1}} t_2^{\frac{1}{1 - P_+}}.
\end{cases}
\tag{4.12}
$$

After the time t_3, the new δ-shock is determined by

$$
\begin{cases}
\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{P_- P_+}, \\[2mm]
x|_{t=t_3} = x_3,
\end{cases}
\tag{4.13}
$$

FIG. 3. Case 1.2: $1 - u_- + v_- = 1, (u_+, v_+) \in \mathrm{I}$

which implies

$$x = \frac{1}{P_- P_+}(t - t_3) + x_3. \tag{4.14}$$

□

From Fig. 2, we can see that as $t \to \infty$, the time-asymptotic solution can be described as

$$(u_-, v_-) + \delta \text{ shock} + (u_+, v_+). \tag{4.15}$$

Moreover, letting $\varepsilon \to 0$, one can easily see that the limit of the solution (1.1) and (4.1) is the corresponding Riemann solution in this case.

We now seek the strength of the δ-shock. Before the δ-shock interacting with the R, the strength of the δ-shock is determined by

$$\begin{cases} u_\delta = \frac{1}{P_m P_+}[(1 - P_m)u_+ + (P_+ - 1)u_m]t, \\ v_\delta = \frac{1}{P_m P_+}[(1 - P_m)v_+ + (P_+ - 1)v_m]t, \end{cases} \tag{4.16}$$

where $t \le t_1$. When $t_1 \le t \le t_2$, the strength of the δ-shock is determined by the following ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}\omega_u}{\mathrm{d}t} = \dfrac{1}{P P_+}[(1 - P)u_+ + (P_+ - 1)u], \quad P_m \le P \le P_{m'}, \\[2mm] \dfrac{\mathrm{d}\omega_v}{\mathrm{d}t} = \dfrac{1}{P P_+}[(1 - P)v_+ + (P_+ - 1)v], \quad P_m \le P \le P_{m'}, \\[2mm] \omega_u|_{t=t_1} = \frac{1}{P_m P_+}[(1 - P_m)u_+ + (P_+ - 1)u_m]t_1, \\[2mm] \omega_v|_{t=t_1} = \frac{1}{P_m P_+}[(1 - P_m)v_+ + (P_+ - 1)v_m]t_1, \end{cases} \tag{4.17}$$

where $P = \frac{\sqrt{t}}{\sqrt{x_1} + (\sqrt{t} - \sqrt{t_1})/P_+}$, derived from Eqs. (4.5) and (4.9).

Case 1.2. $1 - u_- + v_- = 1$, $(u_+, v_+) \in \mathrm{I}$, see Fig. 3.

The δ-shock interacts with the rarefaction wave R and the contact discontinuity J. The picture is shown in Fig. 3. Different from the above subcase 1.1, the δ-shock penetrates the R at point A_2 and meanwhile interacts with the J, then generates another new δ-shock, see subcase 1.1 for detail.

As $t \to \infty$, the time-asymptotic solution can be described as

$$(u_-, v_-) + \delta\text{-shock} + (u_+, v_+). \tag{4.18}$$

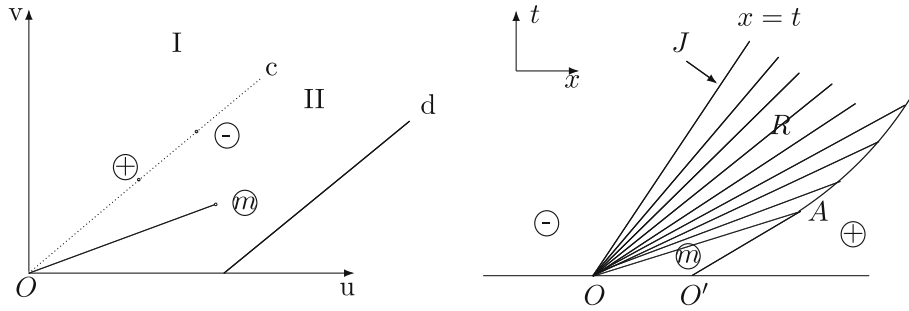We can see that the limit of the solution (1.1) and (4.1) is the corresponding Riemann solution in this case.

FIG. 4. Case 1.3: $1 - u_- + v_- = 1$, and $1 - (u_+ + v_+) = 1$

Case 1.3. $1 - u_- + v_- = 1$ and $1 - u_+ + v_+ = 1$.

We have the following properties.

Lemma 4.2. *The δ-shock interacts with the rarefaction wave R, but cannot penetrate the R. Moreover, it has $x = t$ as its asymptote, see Fig. 4.*

Proof. The δ-shock begins to interact with the rarefaction wave R at point $A(x_1, t_1)$ and changes into another new δ-shock (a curve) which is calculated by

$$\begin{cases} \dfrac{x}{t} = \dfrac{1}{P^2}, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{PP_+}, \quad 1 = \dfrac{1}{P_+} = \dfrac{1}{P_-} \leq \dfrac{1}{P} \leq \dfrac{1}{P_m}, \\ x|_{t=t_1} = x_1. \end{cases} \tag{4.19}$$

From (4.9), we have

$$\sqrt{x} = \sqrt{x_1} + \frac{1}{P_+}(\sqrt{t} - \sqrt{t_1}). \tag{4.20}$$

The first equation in (4.19) implies $\sqrt{x} = \sqrt{t}/P$. Substituting it into (4.20), we get $\sqrt{t} = (\sqrt{x_1} - \sqrt{t_1}/P_+)/(\frac{1}{P} - \frac{1}{P_+})$ which implies $t \to +\infty$, as $P \to P_+ = P_-$. The fact tells us that the new δ-shock cannot penetrate the R and has $x = t/P_-^2 = t$ as its asymptote. $\qquad \square$

As $t \to \infty$, the speed of the δ-shock tends to $\sigma_\delta = 1/(P_- P_+) = 1/P_- = 1/P_+ = 1$ and the δ-shock decelerates during the process of penetration. For large time, the solution can be expressed as

$$(u_-, v_-) + J + (u_+, v_+). \tag{4.21}$$

So, as $\varepsilon \to 0$, we can see that the limit of the solution (1.1) and (4.1) is the corresponding Riemann solution in this case.

Case 2. $J + S \oplus \delta$-shock, see Fig. 5. In this case, we can describe the interactions by the following lemma.

Lemma 4.3. *The shock overtakes the δ-shock at a point $A_1(x_1, t_1)$ and generates a new δ-shock, which is overtook by the contact discontinuity J at finite time as well.*

Proof. The propagation speed of the shock S is $\sigma_1 = \frac{1}{P_{m'} P_m}$, which is greater than that of the δ-shock, $\sigma_2 = \frac{1}{P_m P_+}$. Then, the S intersects with the δ-shock at a point $A_1(x_1, t_1)$ which is calculated by

$$\begin{cases} x = \frac{1}{P_{m'} P_m} t, \\ x = \frac{1}{P_m P_+} t + \varepsilon. \end{cases} \tag{4.22}$$
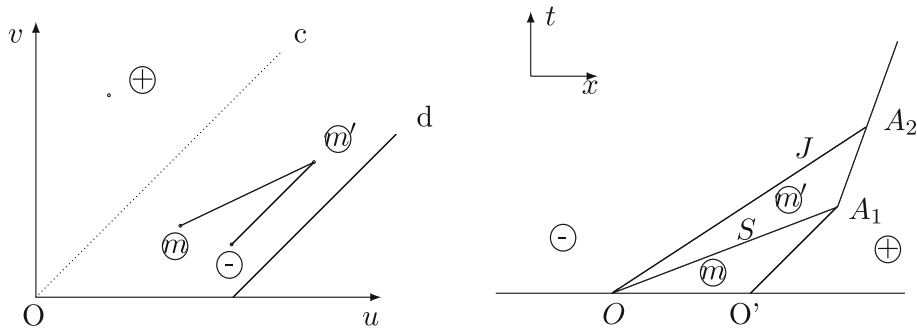
FIG. 5. Case 2: $J + S \oplus \delta$-shock

Using $P_- = P_{m'}$, we have

$$(x_1, t_1) = \left(\frac{P_+}{P_+ - P_-} \varepsilon, \frac{P_- P_m P_+}{P_+ - P_-} \varepsilon \right). \tag{4.23}$$

After the time t_1, the J will overtake the new δ-shock at point $A_2(x_2, t_2)$, which is determined by

$$\begin{cases} \dfrac{x}{t} = \dfrac{1}{P_-} = \dfrac{1}{P_{m'}}, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{P_{m'} P_+}, \\ x|_{t=t_1} = x_1, \end{cases} \tag{4.24}$$

which implies

$$(x_2, t_2) = \left(\left(\frac{P_-}{t_1} \right)^{\frac{1}{P_+ - 1}} x_1^{\frac{P_+}{P_+ - 1}}, (P_- x_1)^{\frac{P_+}{P_+ - 1}} t_1^{\frac{1}{1 - P_+}} \right). \tag{4.25}$$

We know that when the new δ-shock interacts with the J, it keeps the same speed, i.e., $\sigma = \frac{1}{P_- P_+} = \frac{1}{P_{m'} P_+}$. Moreover, after the time t_2, the new δ-shock is calculated by

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{P_- P_+}, \\ x|_{t=t_2} = x_2. \end{cases}$$

$\square$

Thus, as $t \to \infty$, the solution can be described as

$$(u_-, v_-) + \delta\text{-shock} + (u_+, v_+). \tag{4.26}$$

Case 3. $R_1 + R_2 \oplus \delta$-shock. By Riemann solutions $\mathcal{R}(\ominus, \oplus)$ between state (u_-, v_-) and (u_+, v_+), we divide this case into two subcases: (3.1) $\mathcal{R}(\ominus, \oplus)$ is $R + J$ and (3.2) $\mathcal{R}(\ominus, \oplus)$ is $S + J$.

Case 3.1. $\mathcal{R}(\ominus, \oplus)$ is $R + J$. There are two kinds: (i), (u_-, v_-), $(u_+, v_+) \in$ I and (ii) $(u_-, v_-) \in$ I, $1 - u_+ + v_+ = 1$.

(i). (u_-, v_-), $(u_+, v_+) \in$ I, see Fig. 6. By the above discussion, it is clear that the δ-shock cannot penetrate the rarefaction wave R_2 at point $A_2(x_2, t_2)$, since when $\frac{1}{P_-} \leq \frac{1}{P} < 1$, the δ-shock entropy condition is not satisfied and its strength is equal to zero. At the time $t = t_2$, we again have a new Riemann problem with data $(u_\ell, v_\ell) = (0, 0)$ and $(u_r, v_r) = (u_+, v_+)$, which is resolved by a shock S and
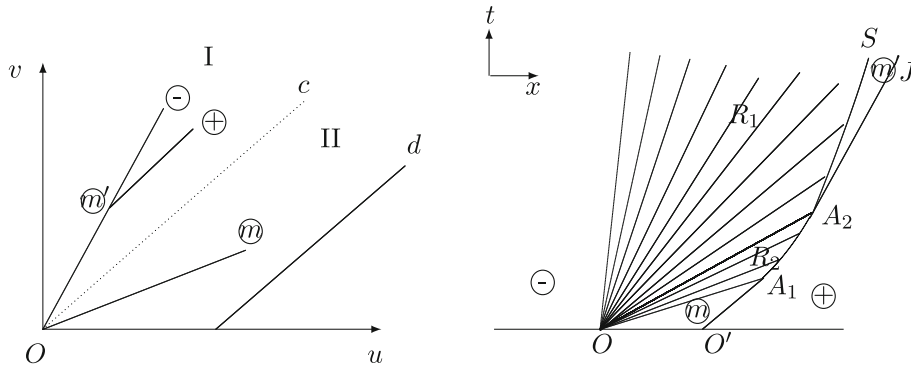
Fig. 6. Case 3.1: $\mathcal{R}(\ominus, \oplus)$ is $R + J, (i)$. $(u_-, v_-), (u_+, v_+) \in \mathrm{I}$
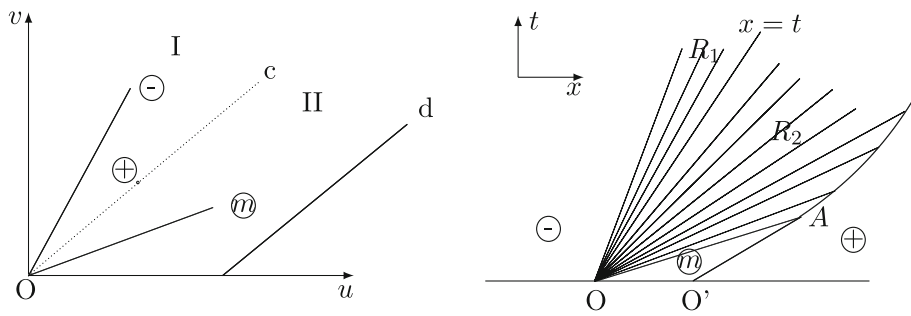

Fig. 7. Case 3.1: $\mathcal{R}(\ominus, \oplus)$ is $R + J, (i)$. $(u_-, v_-) \in \mathrm{I}, 1 - u_+ + v_+ = 1$

a contact discontinuity J. We now consider the interaction between the shock S and the rarefaction wave R_1. We can know that the shock S is determined by

$$\begin{cases} \dfrac{x}{t} = \dfrac{1}{P^2}, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{PP_{m'}}, \\ x|_{t=t_2} = x_2, \end{cases} \tag{4.27}$$

where the point $A_2(x_2, t_2)$ can be calculated by above methods.

So, we have the following lemma

Lemma 4.4. *The δ-shock cannot penetrate the rarefaction wave R_2 and changes into a contact discontinuity J and a shock S at point A_2 after the interaction with R_2. The shock S, determined by (4.27), cannot penetrate the rarefaction wave R_1 and has $x = \frac{1}{P_{m'}^2}t$ as its asymptote.*

According to the Lemma 4.4, as $t \to \infty$, the solution can be described as

$$(u_-, v_-) + R + (u_{m'}, v_{m'}) + J + (u_+, v_+). \tag{4.28}$$

(ii). $(u_-, v_-) \in \mathrm{I}$, $1 - u_+ + v_+ = 1$, see Fig. 7. In this kind, the δ-shock interacts with the rarefaction wave R_2 at point $A(\overline{x}, \overline{t})$, which is calculated by (4.3), then changes into a new δ-shock. Similar to Lemma 4.2, we also have the following lemma.
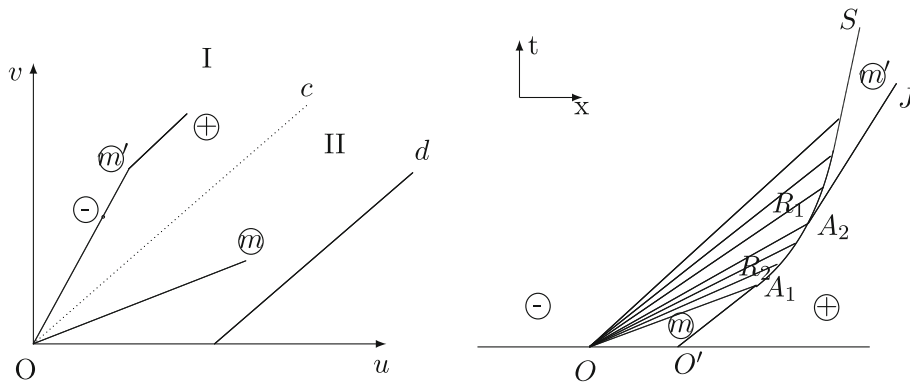
FIG. 8. Case 3.2: $\mathcal{R}(\ominus, \oplus)$ is $S + J$

Lemma 4.5. *The new δ-shock cannot penetrate the rarefaction wave R_2 and has $x = t$ as its asymptote.*

As $\varepsilon \to 0$, the solution can be described as

$$(u_-, v_-) + R_1 + J + (u_+, v_+). \tag{4.29}$$

We can see that the limit of the solution of (1.1) and (4.1) is the corresponding Riemann solution of (1.1) and (2.1).

Case 3.2. $\mathcal{R}(\ominus, \oplus)$ is $S + J$, see Fig. 8. Unlike subcase 3.1 (i), the δ-shock penetrates the rarefaction wave R_2, then interacts with the R_1 and penetrates it. We will prove the following lemma.

Lemma 4.6. *After penetrating the rarefaction wave R_2, the δ-shock changes into a contact discontinuity J and a shock S. Moreover the shock interacts with the rarefaction wave R_1 and penetrates it.*

Proof. As the mentioned above, when the δ-shock penetrates the R_2 at point $A_2(x_2, t_2)$, the δ-shock entropy condition is not satisfied and its strength is zero, and we have a new Riemann problem, which is solved by a contact discontinuity J and a shock S, where the shock S is determined by (4.27), $\frac{1}{P_-} \leq \frac{1}{P} \leq 1$. The propagating speed of the shock is $\sigma = \frac{dx}{dt} = \frac{1}{PP_{m'}}$ and one can obtain $\frac{d^2x}{dt^2} < 0$, which means the σ decelerate in the penetration. We also have $\frac{1}{P_- P_{m'}} < \frac{1}{P_-^2}$, which implies that the shock can penetrate the R_1. $\square$

When $t \to \infty$, it is easy to obtain that the solution can be described as

$$(u_-, v_-) + S + (u_{m'}, v_{m'}) + J + (u_+, v_+). \tag{4.30}$$

which is the same as the corresponding Riemann solution.

Case 4. δ-shock $\oplus$ δ-shock.

In this case, we have $(u_-, v_-) \in \text{II}, 1 - u_m + v_m = 1$ and $(u_+, v_+) \in \text{I}$, see Fig. 9. We can check that the propagating speed of the δ_1-shock $\sigma_1 = \frac{1}{P_- P_m}$ is greater than the speed of the δ_2-shock $\sigma_2 = \frac{1}{P_m P_+}$. The δ_1-shock will overtake the δ_2-shock at point $A(\overline{x}, \overline{t})$, which can be expressed as follows

$$\begin{cases} \dfrac{x}{t} = \dfrac{1}{P_- P_m}, \\ \dfrac{x - \varepsilon}{t} = \dfrac{1}{P_m P_+}. \end{cases} \tag{4.31}$$
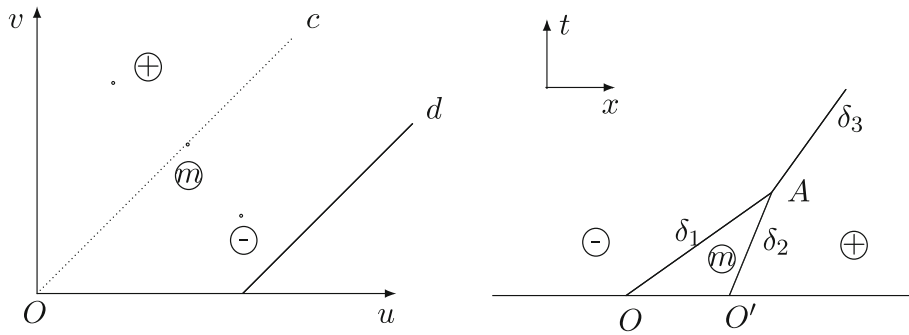
FIG. 9. Case 4: δ-shock $\oplus$ δ-shock

So, we obtain

$$(\overline{x}, \overline{t}) = \left(\frac{P_+}{P_+ - P_-}\varepsilon, \frac{P_- P_m P_+}{P_+ - P_-}\varepsilon \right). \tag{4.32}$$

After the time $t = \overline{t}$, the δ-shock entropy condition is satisfied

$$\lambda_2^+ = \frac{1}{P_+^2} \leq \lambda_1^+ = \frac{1}{P_+} \leq \sigma_3 = \frac{1}{P_- P_+} \leq \lambda_1^- = \frac{1}{P_-} \leq \lambda_2^- = \frac{1}{P_-^2},$$

and a new δ-shock is formed, denoted by δ_3-shock, which is determined by

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{P_- P_+}, \\ x|_{t=\overline{t}} = \overline{x}. \end{cases} \tag{4.33}$$

Thus, as $t \to \infty$, the result of interaction of two δ-shock waves is still a single δ-shock wave. It is easy to see that as $\varepsilon \to 0$, the limit of the solution of (1.1) and (4.1) is the corresponding Riemann solution of (1.1) and (2.1).

Case 5. $R + J \oplus \delta$-shock. In this case, on the basis of the fact that $\mathcal{R}(\ominus, \textcircled{m})$ is $R + J$ and $\mathcal{R}(\textcircled{m}, \oplus)$ is a nonclassical δ-shock, we have $1 - u_m + v_m = 1$ and $(u_-, v_-), (u_+, v_+) \in$ I. So, we divide this case into two subcases.

Case 5.1. $\mathcal{R}(\ominus, \oplus)$ is $R + J$. As discussed above, after the δ-shock interacts with the right-contact discontinuity J at the point A, we can see that $\frac{1}{P_-} \leq \frac{1}{P} \leq 1$ and the δ-shock entropy condition is not satisfied and its strength is zero. Hence, we have a new Riemann problem and the δ-shock changes into $S + J$. The shock begins interacting with the R at the point A. We derive the equation of the shock $S : x = x(t)$ as

$$\begin{cases} \dfrac{x}{t} = \dfrac{1}{P^2}, \quad \dfrac{1}{P_{m'}} \leq \dfrac{1}{P} \leq 1, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{P P_{m'}}, \\ x(\overline{t}) = \overline{x}. \end{cases} \tag{4.34}$$

By virtue of the Lemma 4.4, we obtain that the shock S cannot penetrate the rarefaction wave R and has $x = \frac{1}{P_{m'}^2}t$ as its asymptote (Fig. 10).

We can see that as $\varepsilon \to 0$, the solution can be described as

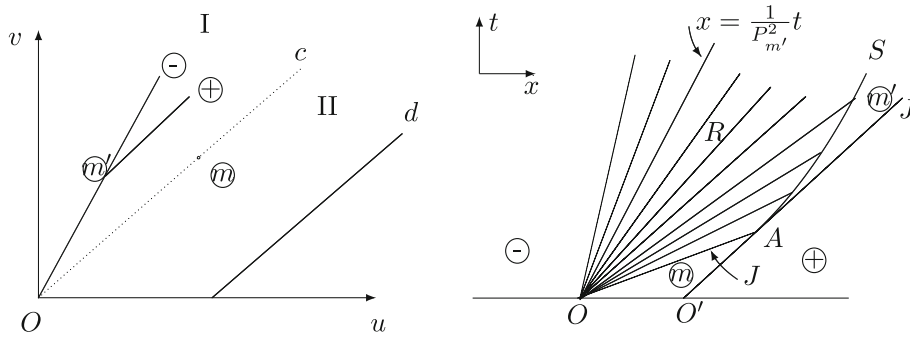$$(u_-, v_-) + R + (u_{m'}, v_{m'}) + J + (u_+, v_+). \tag{4.35}$$

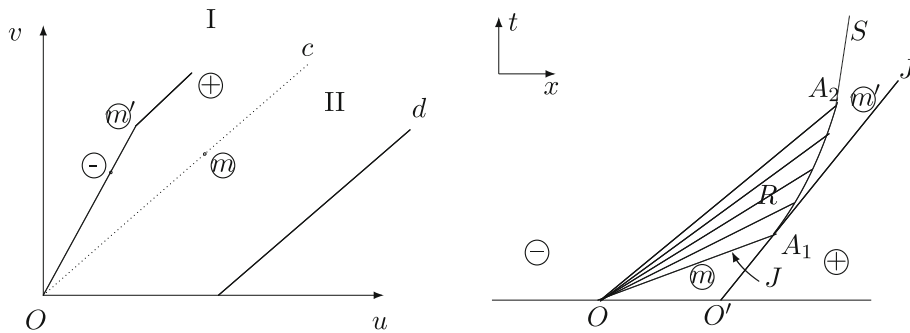FIG. 10. Case 5.1: $\mathcal{R}(\ominus, \oplus)$ is $R + J$

FIG. 11. Case 5.2: $\mathcal{R}(\ominus, \oplus)$ is $S + J$

Case 5.2. $\mathcal{R}(\ominus, \oplus)$ is $S + J$. Similarly, we obtain that the δ-shock interacts with the contact discontinuity at the point A_1, then changes into a shock S and a new contact discontinuity. By Lemma 4.6, we obtain that the shock S can penetrate the rarefaction wave R at some point A_2, the solutions to this subcase are shown in Fig. 11.

We can see that as $t \to \infty$, the solution can be described as

$$(u_-, v_-) + S + (u_{m'}, v_{m'}) + J + (u_+, v_+), \tag{4.36}$$

which is the same as the corresponding Riemann solution.

So far, we have finished the discussion for the interactions of the δ-shock and the elementary waves and the global solutions for the perturbed initial value problem (1.1) and (4.1) have been constructed. We summarize our results in the following.

Theorem 4.7. *The limits of the perturbed Riemann solution of* (1.1) *and* (4.1) *are exactly the corresponding Riemann solution of* (1.1) *and* (2.1). *The Riemann solutions of* (1.1) *and* (2.1) *are stable and admissible with respect to such small perturbations of the initial data.*

Acknowledgments

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