ONE-DIMENSIONAL PERTURBATIONS OF OPERATORS

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The purpose of this note is three-fold. First, to show that certain rank one operators of small norm will split off a unitary piece from the shift; second, to apply this technique to the "Berg construction"; and, third, to exhibit a normal operator whose eigenspace structure is drastically altered by the addition of a rank one operator.

Throughout this paper $\mathscr{L}(\mathscr{H})$ will denote the algebra of bounded linear operators on a separable Hilbert space $\mathscr{H}$.

1. We begin with a proposition which tells us which rank one operators preserve an isometry.

PROPOSITION 1. *Let S be any isometry. Let $F = a(\cdot, g)h$ where $\|g\| = \|h\| = 1$. Then $S + F$ is isometric if and only if*

(1)
$$\begin{cases} S: g \to e^{-i\theta}h \\ \qquad\qquad\qquad \text{for some } \theta \text{ real} \\ S^*: h \to e^{i\theta}g \end{cases}$$

and

(2)
$$2 \operatorname{Re} e^{-i\theta}a = -|a|^2.$$

Proof. To show $S + F$ is isometric we must show

$$I = (S + F)^*(S + F) = I + S^*F + F^*S + F^*F$$

or equivalently $S^*F + F^*S + F^*F = 0$. Note that $F^* = \bar{a}(\cdot, h)g$. If $f \in \mathscr{H}$ then
  (a) $S^*Ff = ae^{i\theta}(f, g)g$,
  (b) $F^*Sf = \bar{a}e^{-i\theta}(f, g)g$, and
  (c) $F^*Ff = |a|^2(f, g)g$.
Thus

$$(S^*F + F^*S + F^*F)f = \left[(ae^{i\theta} + \bar{a}e^{-i\theta}) + |a|^2\right](f, g)g = 0$$

if (2) holds.

To prove the necessity assume $S + F$ is isometric whence $S^*F + F^*S + F^*F = 0$. The operator S^*F is 1-dimensional and thus in some basis

$$S^*F = \begin{vmatrix} \alpha & 0 & \vdots & 0 \\ \beta & 0 & \vdots & \\ \hline & 0 & \vdots & 0 \end{vmatrix} \quad \text{and} \quad F^*S = \begin{vmatrix} \alpha & \beta & \vdots & 0 \\ 0 & 0 & \vdots & \\ \hline & 0 & \vdots & 0 \end{vmatrix}$$

But $S^*F + F^*S = -F^*F \leq 0$ and it is clear that $S^*F + F^*S$ cannot be negative unless $\beta = 0$. Hence

$$S^*F: \{g\} \to \{g\} \quad \text{and} \quad S^*F: \{g\}^\perp \to 0.$$

Set $F = a(\cdot, g)h$ so $F^* = \bar{a}(\cdot, h)g$. Since $F^*S: \{g\}^\perp \to 0$ if $f \perp g$, then $F^*Sf = a(Sf, h)g = 0$, which implies $Sf \perp h$. Thus $S: \{g\}^\perp \to \{h\}^\perp$ and since S is isometric it follows that $Sg = e^{-i\theta}h$ for some θ real. Hence $S^*h = e^{i\theta}S^*Sg = e^{i\theta}g$. To see that (2) is necessary we need only repeat the argument from the first half of the proof.

Let S be the unilateral shift on $H^2(S: f(z) \to zf(z))$ and assume M is an invariant subspace for S. Then it is well known [2] that $M = \phi H^2$ for some inner function ϕ in H^2. Fix ϕ for the moment and let P be the self-adjoint projection of H^2 onto $M^\perp$. Let $Wf = PSf$ for $f \in M^\perp$. It had been observed by Ahern and Clark (see [4]) that there exists a one-dimensional perturbation of W (regarded as an operator on $M^\perp$) which is unitary. This was pointed out to me by Bernie Morrel. It had also been observed earlier by Ron Douglas (see [3] Corollary 2.3 for example) that one can split off unitary direct summands from the shift by compact perturbations.

Our next result can be said to complement both these results by accomplishing their task in an economical fashion. Thus in Clark's case both the restriction *and* the perturbation are achieved simultaneously. We denote unitary equivalence by $\cong$.

THEOREM 2. *As above, let S denote the unilateral shift on H^2. For any inner function ϕ, there exists a one-dimensional operator F such that*

$$S + F \cong U \oplus V \quad on \ (\phi H^2)^\perp \oplus \phi H^2$$

where U is unitary and $V \cong S$. Moreover, given $\varepsilon > 0$, we can choose a ϕ_ε such that $S + F_\varepsilon \cong U \oplus V$ and $\|F_\varepsilon\| < \varepsilon$.

Proof. Note first that $M = \phi H^2$ is spanned by the orthonormal basis $\{z^k\phi\}$ for $k = 0, 1, \ldots$. Clearly $S|M$ is unitarily equivalent to S on H^2. Set $g = (\overline{\phi(0)}\phi - 1)/\|\overline{\phi(0)}\phi - 1\|$. Observe that $g \perp \phi$. (If $\phi(0) = 0$ then $1 \perp \phi$ which greatly facilitates matters, but leads to an uninteresting situation.)

We define our one-dimensional operator F as follows:

$$F = (\cdot, S^*\varphi)(cg - \phi)$$

where

$$|c + \phi(0)/\|\bar{\phi}(0)\phi - 1\| |^2 = 1 + |\phi(0)|^2/\|\bar{\phi}(0)\phi - 1\|^2.$$

Later in the proof we settle on one choice for c. However, we need this more general formulation later on.

Several steps are needed to show F has the desired properties.

1. *Claim*: $SM^{\perp} \subset M^{\perp} + \{\phi\}$. This follows immediately from the observation that S is a contraction and S^* is isometric on $M \ominus \phi$. Or note that $S^*: M \ominus \phi \rightarrow M$.

(2) If we can show $((S + F)f, \phi) = 0$ for $f \in M^{\perp}$ then $S + F$ is clearly $U \oplus V$ on $M^{\perp} \oplus M$, where U is as yet unspecified. But

$$((S + F)f, \phi) = (SF, \phi) + (f, S^*\phi)((cg - \phi), \phi)$$
$$= (Sf, \phi) - (f, S^*\phi) = 0.$$

(3) *Claim*: $S + F|M^{\perp}$ is unitary. To show this we will prove $S + F$ is isometric. The conclusion then follows since

$$-1 = \text{index}(S + F) = \text{index } U + \text{index } V = \text{index } U + (-1)$$

whence index $U = 0$. Hence U must be onto and thus is unitary.

(4) $S + F$ is isometric on $M^{\perp}$. Let P be the self-adjoint projection of H^2 onto $M \ominus g$. Then for $f \in M^{\perp}$,

$$\|(S + F)f\|^2 = |((S + F)f, g)|^2 + \|P(S + F)f\|^2$$
$$= |(Sf, g) + (f, S^*\phi)(cg - \phi, g)|^2 + \|PSf\|^2$$
$$= |(f, S^*\phi)\phi(0)\|\bar{\phi}(0)\phi - 1\|^{-1} + c(f, S^*\phi)|^2 + \|PSf\|^2$$
$$= |(f, S^*\phi)|^2(1 + |\phi(0)|^2\|\bar{\phi}(0)\phi - 1\|^{-2}) + \|PSf\|^2$$
$$= |(Sf, \phi)|^2 + |(Sf, g)|^2 + \|PSf\|^2 = \|Sf\|^2 = \|f\|^2.$$

(5) Finally we show that we can choose ϕ so that the associated F has small norm. One can specify the dimension of $M^{\perp}$ in advance and still achieve this goal as will become obvious in a moment. If ϕ is a singular inner function then

$$\phi(z) = \exp \int \frac{z + e^{i\theta}}{z - e^{i\theta}} d\mu(\theta)$$

and conversely any singular measure μ generates such an inner function. It is well known that for such ϕ's, the associated space $(\phi H^2)^{\perp}$ is infinite

dimensional. Let us consider this case first. Choose μ such that $\mu(\partial \mathbf{D}) = t$. Then $\phi(0) = e^{-t}$. It is at this point, we must exercise some care in the choice of c. The most efficient choice (in terms of size) requires that

$$\arg c = \arg \phi(0)$$

and

$$|c| = \left\{ 1 + |\phi(0)|^2 \|\bar{\phi}(0)\phi - 1\|^{-2} \right\}^{-1/2} - |\phi(0)| \|\bar{\phi}(0)\phi - 1\|^{-1}$$

$$= (1 - |\phi(0)|)[1 - |\phi(0)|^2]^{-1/2}.$$

The latter equality follows from the fact that $\|\bar{\phi}(0)\phi - 1\|^2 = (1 - |\phi(0)|^2)$. Note that with this choice, $|c| \leq 1$. Hence

$$\|F\| \leq \|S^*\phi\|(1 + |c|) \leq 2[1 - |\phi(0)|^2]^{1/2}$$

$$= 2[1 - e^{-2t}]^{1/2}.$$

Thus for suitably chosen t, it follows that $\|F\| \leq \varepsilon$. To handle finite-dimensional $(\phi H^2)^{\perp}$, choose ϕ to be a finite Blaschke product where $|\phi(0)|$ is close to 1.

REMARK. Although one-dimensional perturbations can achieve somewhat surprising results they are limited in certain respects. While it follows from [3] that a compact perturbation of the shift can have the form $S \oplus I$, no one-dimensional perturbation can accomplish this feat as we shall show.

LEMMA. *Let $T \in \mathscr{L}(\mathscr{H})$ and let F_0 be an n-dimensional operator. Then*

$$\dim \ker(T + F_0 - \lambda) \leq n + \dim \ker(T - \lambda)$$

for each $\lambda \in \mathbf{C}$.

Proof. Since F_0 is n-dimensional it can be written as $F_0 = \sum_{j=1}^n (\cdot, g_n)h_n$. Let $G = \text{clm}\{ g_j : j = 1, \ldots, n \}$. Fix $\lambda \in \mathbf{C}$ and assume $\dim \ker(T + F_0 - \lambda) > n + \dim \ker(T - \lambda)$. It follows from an easy argument that there exists a non-zero vector $f \in \ker(T + F_0 - \lambda)$ where $f \perp G$ and $f \perp \ker(T - \lambda)$. Thus

$$0 = (T + F_0 - \lambda)f = (T - \lambda)f + \sum (f, g_j)h_j = (T - \lambda)f$$

which is a contradiction.

COROLLARY. *There exists no finite dimensional operator F such that*

$$S + F \cong U \oplus V$$

where U is a unitary operator on an infinite-dimensional space and $\sigma(U)$ is a finte set. In particular $S + F$ cannot be unitarily equivalent to $I \oplus S$.

Proof. Assume such a U exists with $\sigma(U) = \{\lambda_1, \ldots, \lambda_k\}$. Then $\dim \ker(U - \lambda_j)$ must be infinite for some λ_j, which contradicts the previous lemma.

EXAMPLE. Since we have shown above that no rank one perturbation of S produces a unitary with singleton spectrum it is interesting to see just what sort of a U is obtained by using the simplest possible singular inner function. So let us take $\phi(z) = e^{(z+1)/(z-1)}$. We now compute the spectrum of U by using Theorem 3.2 from [4], assuming we choose c so that $\arg c = \arg \phi(0)$.

Caution: Clark's notation varies from ours. We have used S to denote the unilateral shift while Clark uses S to denote the *restriction* of the shift to a star invariant subspace. Thus his S and ours do not even act on the same space. Also, because of a slight variation in notation, his w in Theorem 3.2 is equal to $-e^{\arg c}$ when we choose c with $\arg c = \arg \phi(0)$.

Continuing, we see that for $\phi(z) = \exp((z + 1)/(z - 1))$ the resulting U has spectrum equal to

$$\{1\} \cup \{-[1 + (2k + 1)\pi i]/[1 - (2k + 1)\pi i] : k = 0, \pm 1, \ldots\}.$$

By the lemma above, each spectral point different from 1 is an eigenvalue of multiplicity one. Clark actually exhibits the associated eigenvectors.

COROLLARY. *Let Q be isometric but not purely unitary. Then there exists a one-dimensional operator F such that*

$$Q + F \cong U \oplus Q'$$

where Q' is unitarily equivalent to Q, U is unitary and F has arbitrarily small norm.

Proof. Use the Wold decomposition.

2. An application: the Berg process. In Theorem 1 of [1], Berg shows how to pull a unitary piece out of the middle of a unilateral shift while leaving the shift "intact". (Trevor West has christened this process the "Seagull Construction".) Our approach also enables us to do this in a simple geometric fashion. In particular, we require only a rank one perturbation of arbitrarily small norm.

Rather than stating a theorem, we shall simply describe the process and its end result. Let S be our shift on H^2. Choose a finite Blaschke

product ϕ with n factors. We will later specify $\phi(0)$. We define our first perturbation to be

$$F_1 = -1(\,\cdot\,, S^*\phi)\phi.$$

Thus $S + F_1$ splits as

$$\overline{\underset{(\phi H^2)^\perp}{\rightarrow}\ \ \underset{\phi H^2}{\rightarrow}}\quad S + F_1.$$

(Also see the matrix at the end of this section.) Of course $S + F_1$ is no longer isometric. We select a second Blaschke product ψ of m factors. We set

$$F_2 = -1(\,\cdot\,, S^*\phi\psi)\phi\psi.$$

Thus $S + F_1 + F_2$ has the representation

$$\overline{\underset{(\phi H^2)^\perp}{\rightarrow}\ \ \underset{\phi H^2 \ominus \phi\psi H^2}{\rightarrow}\ \ \underset{\phi\psi H^2}{\rightarrow}}\quad S + F_1 + F_2.$$

We define $F_3 = (\,\cdot\,, S^*\phi)\phi\psi$. With this choice of F_3, it should be clear by construction that $S + F_1 + F_2 + F_3$ is the unilateral shift on $(\phi H^2)^\perp \oplus \phi\psi H^2$ and leaves $\phi H^2 \ominus \phi\psi H^2$ invariant as indicated below:
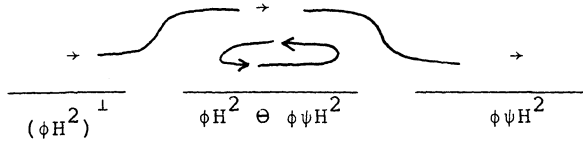

Finally we wish to modify our operator to be unitary on $\phi H^2 \ominus \phi\psi H^2$. To this end define $g = \phi(\overline{\psi}(0)\psi - 1)/\|\phi(\overline{\psi}(0)\psi - 1)\|$. Note that $g \in \phi H^2 \ominus \phi\psi H^2$ and $\|\phi(\overline{\psi}(0)\psi - 1)\|^2 = 1 - |\psi(0)|^2$.

Set $F_4 = (\,\cdot\,, S^*\phi\psi)cg$ where

$$(\alpha)\quad |c + \psi(0)\|\phi(\overline{\psi}(0)\psi - 1)\|^{-1}|^2 = 1 + |\psi(0)|^2\|\phi(\overline{\psi}(0)\psi - 1)\|^{-2}.$$

It is easy to see, by repeating the argument in Theorem 2, that $S + F_1 + F_2 + F_3 + F_4$, which is reduced by $\phi H^2 \ominus \phi\psi H^2$, is unitary on that

subspace if and only if (α) is satisfied. Our operator $S + F_1 + F_2 + F_3 + F_4$ can now be represented as

$$
\underbrace{\qquad\qquad}_{(\phi H^2)^\perp} \quad \underbrace{\qquad\qquad}_{\phi H^2 \ominus \phi\psi H^2} \quad \underbrace{\qquad\qquad}_{\phi\psi H^2}
$$

It is this final result which we view as the Berg construction.

However we are far from finished. Certainly $F_1 + F_2 + F_3 + F_4$ does not seem to be a rank one operator. Let us group the terms as

$$F_1 + F_3 = (\,\cdot\,, S^*\phi)(\phi\psi - \phi),$$

and

$$F_2 + F_4 = (\,\cdot\,, S^*\phi\psi)(cg - \phi\psi).$$

Clearly $F = F_1 + F_2 + F_3 + F_4$ will be a rank one operator if and only if $(\phi\psi - \phi)$ equals $(cg - \phi\psi)$ modulo the scalars. Performing the obvious arithmetic we see that F is rank one if and only if

$$(\beta) \qquad\qquad c = \|\phi(\overline{\psi}(0)\psi - 1)\|(\overline{\psi}(0) - 1)^{-1}.$$

At this point we would like to satisfy (α) and (β) simultaneously. Remarkably enough (β) $\Rightarrow$ (α). Indeed if we observe that

$$\|\phi(\overline{\psi}(0)\psi - 1)\|^2 = (1 - |\psi(0)|^2),$$

then substituting in the left-hand side of (α) we obtain

$$|(1 - |\psi(0)|^2)^{1/2}(\overline{\psi}(0) - 1)^{-1} + \psi(0)(1 - |\psi(0)|^2)^{-1/2}|^2$$
$$= (1 - |\psi(0)|^2)^{-2} = 1 + |\psi(0)|^2\|\phi(\overline{\psi}(0)\psi - 1)\|^{-2}.$$

Thus if we choose c to satisfy (β) then our operator F is rank one, and $S + F$ has the required form. To complete matters we need only show that F has any preassigned norm. Note that $\|F_1\| = \|S^*\phi\| = (1 - |\phi(0)|^2)^{1/2}$. Thus the norm of F_1 is small for $|\phi(0)|$ close to 1, and F_2 and F_3 can be handled in similar fashion. Since

$$\|F_4\| = \|S^*\phi\psi\|\,|c|$$

and $c = (1 - |\psi(0)|^2)^{1/2}(\overline{\psi}(0) - 1)^{-1}$, to handle this case we simply choose $\psi(0)$ to be negative or imaginary and close to 1 in absolute value. Note that $\dim(\phi H^2)^\perp = n$ and $\dim(\phi H^2 - \phi\psi H^2) = m$ so we may specify the size of the pieces if we wish.

$$S = \qquad \text{on } (\phi H^2)^\perp \oplus (\phi H^2 \ominus \phi\psi H^2) \oplus \phi\psi H^2$$
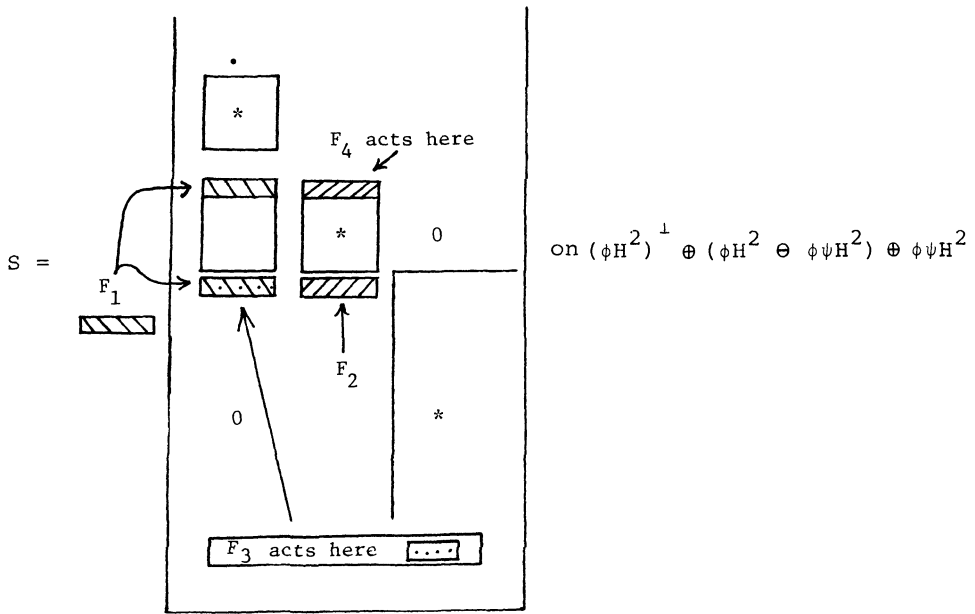
3. In this section we will construct a one-dimensional perturbation of arbitrarily small norm of a diagonal normal operator D which has no eigenvalues. The spectrum of D will be the unit square. This answers a question raised by Hong W. Kim.

Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. The operator D is defined by

$$De_j = \lambda_j e_j \quad \text{for } n = 1, 2, \dots.$$

We designate our one-dimensional operator by F where $Ff = (f, g)h$ for all $f \in \mathcal{H}$. The vectors g, h can be expressed as $\Sigma a_i e_i$ and $\Sigma b_i e_i$ respectively. We will now define the λ_i's and b_i's in stages.

Consider the unit square s with sides on $\operatorname{Re} z = \pm \frac{1}{2}$ and $\operatorname{Im} z = \pm \frac{1}{2}$. At stage 1 we divide it as follows and assign $\lambda_1, \dots, \lambda_4$ as indicated.
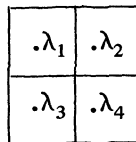
At the second step we subdivide each of the boxes in the last subdivision thus obtaining
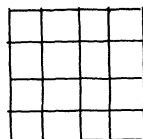
The next 16 λ_i's are taken to be the midpoints of the indicated squares. We continue in this fashion.

Thus at the nth stage we have

(a) 2^n subintervals per side

(b) length of a subinterval is 2^{-n}

(c) there are 2^{2n} boxes in the subdivision

(d) we set the next 2^{2n} b_i's equal to $1/n2^n$ at this stage.

First observe that $h = \Sigma b_i e_i \in \mathcal{H}$ since

$$\|h\| = \sum_{\text{stages}} \sum_{\text{vector in } n\text{th stage}} \frac{1}{2^{2n}n^2} = \sum_{\text{stages}} \frac{1}{n^2} < \infty.$$

(The penultimate step follows from the observation that there are 2^{2n} vectors in the nth stage.)

We do not define the a_i's for the moment but observe that $a_i \neq 0$ for $i = 1, 2, \ldots$. Assume that $(\lambda - D - F)f = 0$ for $\lambda \in S$, the unit square. Then $(\lambda - D)f = (f, g)h$. To rule out the case $(f, g) = 0$, observe that it would then follow that $f = e_{j_0}$ whence $(f, g) = a_{j_0} \neq 0$, a contradiction. Set $f = \Sigma\mu_i e_i$. Then $\Sigma(\lambda - \lambda_i)\mu_i e_i = (f, g)\Sigma b_i e_i$ and hence $\mu_i = (b_i/(\lambda - \lambda_i))(f, g)$ for $i = 1, 2, \ldots$. Without loss of generality we may take $(f, g) = 1$. We claim that $f \notin \mathcal{H}$ for $\lambda \in S$. In our computation of the norm of f, it makes a slight difference where λ is located in S. For clarity take λ as indicated.
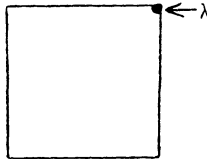
We evaluate $\Sigma|b_j|^2/|\lambda - \lambda_j|^2$ in stages again. Within a given stage, we group the λ_j's into disjoint sets L_k as indicated in the picture.
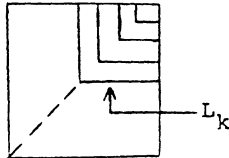
Then we estimate $1/|\lambda - \lambda_j|$ in each L in the same way. Thus

$$\|f\|^2 = \sum_{j \geq 1}^{\infty} \frac{|b_j|^2}{|\lambda - \lambda_j|^2} = \sum_{\text{stages}} \sum_{\text{vectors in } n\text{th stage}} \frac{|b_j|^2}{|\lambda - \lambda_j|^2}$$

$$\geq \sum_{\text{stages}} \sum_{\substack{k=1 \\ \text{vectors in } n\text{th stage}}}^{2^{n-1}} \frac{1}{2^{2n}n^2} 2k \left(\frac{2^n}{k}\right)^2$$

(The term $1/2^{2n}n^2$ is just $|b_j|^2$ and $2^n/k$ minimizes the distance from λ to λ_j's in the kth L.)

$$\geq \sum_{\text{stages}} n^{-2} \sum_{k=1}^{2^{n-1}} \frac{1}{k}$$

$$\geq \sum_{\text{stages}} n^{-2} \log 2^{n-1} \geq \sum \frac{1}{n} = +\infty.$$

Note that this estimate is valid for *any* placement of $\lambda \in S$. Thus $D + F$ has no eigenvalues for $\lambda \in S$.

To handle the case $\lambda \notin S$ we note that if $(\lambda - D)f = Ff$ then f must be a scalar multiple of $(\lambda - D)^{-1}h$. Thus we set $f_\lambda = (\lambda - D)^{-1}h = \Sigma b_j e_j/(\lambda - \lambda_j)$. Then $\lambda \notin S$ is an eigenvalue of $(D + F)$ if and only if $(f_\lambda, g) = 1$. We let $g = \Sigma a_i e_i$ where $a_i > 0$ and choose the a_i's so small that $0 < \Sigma a_i b_i / \text{Re}(1/2 - \lambda_i) < 1$. But

$$\text{Re}(f, g) = \sum \text{Re}\frac{a_i b_i}{(\lambda - \lambda_i)} \leq \sum \frac{a_i b_i}{\text{Re}(1/2 - \lambda_i)} < 1$$

for $\text{Re}\,\lambda \geq \frac{1}{2}$, hence there are no eigenvalues in that half plane.

On the other hand $\text{Re}(f_\lambda, g) < 0$ for $\text{Re}\,\lambda \leq -1/2$ so there are no eigenvalues there either.

For $\text{Im}\,\lambda \geq 1/2$ we see that $\text{Im}(f_\lambda, g) > 0$ and for $\text{Im}\,\lambda < -1/2$ we have $\text{Im}(f_\lambda, g) < 0$ so there are no eigenvalues in those regions. We have exhausted S'. Thus $D + F$ has no eigenvalues. As constructed,

$$\|F\| = \|g\|\,\|h\| = \|g\|\pi/\sqrt{6} = \left[\sum |a_i|^2\right]^{1/2} \pi/\sqrt{6}.$$

Thus we may make $\|F\|$ as small as desired by simply choosing the a_i's to be sufficiently small.

REMARK. By a similar construction we can define a self-adjoint diagonal operator S whose spectrum is $[0, 1]$, and a one-dimensional self-adjoint operator F such that $S + F$ has no eigenvalues. We omit the details.

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