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# One Dimensional State Space Approach to Thermoelastic Interactions with Viscosity

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## ABSTRACT

Present work is concerned with the solution of a one dimensional problem in generalized thermoviscoelastic medium with fractional order strain. The formulation is applied in the context of Green-Naghdi theory of thermoelasticity with energy dissipation. State space approach together with Laplace transform technique is adopted to obtain the general solution. Numerical inversion technique is used to derive the expressions of different field variables in the physical domain. Numerical results are given and illustrated graphically.
**Keywords :** GN theory, Viscosity, Fractional Order Strain.

## I. INTRODUCTION

Thermoelastic wave propagation is of much importance in different fields such as earthquake engineering, nuclear reactors, aeronautics and astronautics etc. The coupling between thermal and strain fields gave rise to the coupled theory of thermoelasticity. The theories of generalized thermoelasticity were developed to amend the classical thermoelasticity theory. By providing sufficient basic modifications to the constitutive equations, Green and Naghdi [1-3] produced a theory which was divided into three different parts, referred to as G-N theory of type I, II and III. Type I is same as classical heat conduction theory (based on Fouriers law of heat conduction). Type II predicts the finite speed of heat propagation involving no energy dissipation. In type III, constitutive equations are derived by including thermal displacement gradient in addition to temperature gradient among constitutive variables.

In the last few years, fractional calculus theory has been employed successfully in theories of thermoelasticity and several models of fractional order generalized thermoelasticity are established by many authors. Sherief et al. [4] introduced the fractional order theory of thermoelasticity by using the methodology of fractional calculus, proved uniqueness theorem and derived variational principle and reciprocity theorem. Ezzat [5] constructed a new mathematical model of fractional heat conduction law in which the generalized Fouriers law of heat conduction is modified by using the new Taylors series expansion of time fractional order developed by Jumarie [6]. Recently, Youssef [7] derived a new theory of thermoelasticity with fractional order strain which is considered as a new modification to Duhamel-Neumanns stress-strain relation. In this paper, the author postulated a new unified system of equations that govern seven different models of thermoelasticity in the context of one-temperature and two-temperature and one dimensional problem for an isotropic and homogeneous elastic half-space.

This investigation studies the one dimensional problem of linear, isotropic solid in thermoviscoelastic medium subjected to mechanical load. The application of the present work can not be ruled out in geophysics and earthquake engineering due to the importance of thermoviscoelastic properties. State space approach is employed for the general solution of the problem. The variations of the considered field variables with the distance are presented graphically.

## II. NOMENCLATURE

| | |
|---|---|
| $\sigma_{ij}$ | Components of stress tensor |
| $e_{ij}$ | Components of strain tensor |
| $u_i(i = x,y,z)$ | Components of displacement vector |
| $\theta = T - T_0$ | Temperature |
| $T$ | Absolute temperature |

| | |
|---|---|
| $T_0$ | The temperature of medium in its natural state assumed to be $\left\lvert \frac{\theta}{T_0} \right\rvert << 1$ |
| $\rho$ | Density of medium |
| $\tau$ | Mechanical relaxation time |
| $\beta$ | Fractional strain parameter |

$$\lambda^* = \lambda_e \left(1 + \alpha_0 \frac{\partial}{\partial t}\right)$$

$$\mu^* = \mu_e \left(1 + \alpha_1 \frac{\partial}{\partial t}\right)$$

$$\beta_1^* = \beta_{1e} \left(1 + \beta_1 \frac{\partial}{\partial t}\right)$$

$$\beta_{1e} = (3\lambda_e + 2\mu_e)\alpha_t$$

$$\beta_1 = \frac{(3\lambda_e\alpha_0 + 2\mu_e\alpha_1)\alpha_t}{\beta_{1e}}$$

| | |
|---|---|
| $\lambda_e, \mu_e$ | Lame's elastic constants |
| $\alpha_0, \alpha_1$ | Viscoelastic relaxation times |
| $\alpha_t$ | Coefficient of linear thermal expansion |
| $c_E$ | Specific heat at constant strain |
| $k$ | Thermal conductivity |
| $K^* = \frac{c_E(\lambda_e + 2\mu_e)}{4}$ | Material characteristic of GN theory |

## III. BASIC EQUATIONS

The constitutive relation and governing equations for a generalized viscothermoelastic problem under the purview of GN III theory with fractional order strain are:

$$\sigma_{ij} = 2\mu^*(1+\tau^\beta D_t^\beta)e_{ij} + \lambda^*(1+\tau^\beta D_t^\beta)e_{kk}\delta_{ij} - \beta_1^*\theta\delta_{ij} \quad (1)$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2)$$

$$\sigma_{ji,j} = \rho\ddot{u}_i \quad (3)$$

$$K^*\theta_{,ii} + k\dot{\theta}_{,ii} = \rho c_E\ddot{\theta} + \beta_1^* T_0(1+\tau^\beta D_t^\beta)\ddot{e} \quad (4)$$

Here, a dot over a variable denotes derivative with respect to time $t$, a comma refers to a spatial derivative and the tensor convention of summing over repeated indices is used.

## IV. PROBLEM FORMULATION

In the consideration of one dimensional problem, the occupied region is $-\infty < x < \infty$, whose state depends only on the space variable $x$ and time $t$. So, the displacement vector $\vec{u}$ and temperature $\theta$ can be expressed in the following form

$$u_x = u(x,t),\ u_y = 0,\ u_z = 0,\ \theta = \theta(x,t). \quad (5)$$

The governing equations (1)-(4) in one-dimensional case assume the shape

$$\sigma = \sigma_{xx} = (\lambda^* + 2\mu^*)(1+\tau^\beta D_t^\beta)e - \beta_1^*\theta, \quad (6)$$

$$e = e_{xx} = \frac{\partial u}{\partial x}, \quad (7)$$

$$\frac{\partial \sigma}{\partial x} = \rho\ddot{u}, \quad (8)$$

$$K^*\frac{\partial^2\theta}{\partial x^2} + k\frac{\partial^3\theta}{\partial t\partial x^2} = \rho c_E\ddot{\theta} + \beta_1^* T_0(1+\tau^\beta D_t^\beta)\ddot{e}. \quad (9)$$

Equation (8) can be expressed as

$$\frac{\partial^2\sigma}{\partial x^2} = \rho\ddot{e}. \quad (10)$$

Proceeding with the analysis, we introduce dimensionless variables defined by the expressions:

$$x' = \frac{\bar{\omega}}{c_1}x,\ u' = \frac{\rho c_E\bar{\omega}}{\beta_{1e}T_0}u,\ \sigma' = \frac{1}{\beta_{1e}T_0}\sigma,$$
$$(t', \tau', \alpha_0', \alpha_1', \beta_1') = \bar{\omega}(t, \tau, \alpha, \alpha, \beta),\ \theta' = \frac{\theta}{T_0}, \quad (11)$$

where $\omega = \frac{c_E(\lambda_e + 2\mu_e)}{k}$ and $c_1^2 = \frac{\lambda_e + 2\mu_e}{\rho}$.

Substituting these non dimensional values in equations (6), (9) and (10), we get following non dimensional equations (suppressing the primes):

$$\sigma=\left(1+\delta_0\frac{\partial}{\partial t}\right)(1+\tau^\beta D_t^\beta)e-\left(1+\beta_1\frac{\partial}{\partial t}\right)\theta, \quad (12)$$

$$\left(1+\varepsilon\frac{\partial}{\partial t}\right)\frac{\partial^2\theta}{\partial x^2}=\varepsilon\frac{\partial^2\theta}{\partial t^2}+\zeta\left(1+\beta_1\frac{\partial}{\partial t}\right)(1+\tau^\beta D_t^\beta)\frac{\partial^2 e}{\partial t^2}, \quad (13)$$

$$\frac{\partial^2\sigma}{\partial x^2}=\frac{\partial^2 e}{\partial t^2}, \quad (14)$$

where $\delta_0=\frac{\lambda_e\alpha_0+2\mu_e\alpha_1}{\rho c_1^2}, \varepsilon=\frac{\bar{\omega}k}{K^*}, \zeta=\frac{\beta_{1e}^2T_0}{\rho K^*}$
are the coupling parameters.
Using the Laplace transformation defined as

$$\bar{f}(x,s)=L\{f(x,t)\}=\int_0^\infty f(t)e^{-st}dt, \quad (15)$$

over the equations (12)-(14) and using the homogeneous initial conditions, we get the following equations

$$\bar{\sigma}=(1+\delta_0 s)(1+\tau^\beta s^\beta)\bar{e}-(1+\beta_1 s)\bar{\theta},$$

$$(1+\varepsilon s^2)D^2\bar{\theta}=\varepsilon s^2\bar{\theta}+\zeta(1+\beta_1 s)(1+\tau^\beta s^\beta)s^2\bar{e},$$

$$D^2\bar{\sigma}=s^2\bar{e}, \quad (16)$$

where $D\equiv\frac{\partial}{\partial x}$.
Eliminating $\bar{e}$ from (16), we arrive at the following system of differential equations

$$D^2\bar{\theta}=L_1\bar{\theta}+L_2\bar{\sigma}, \quad D^2\bar{\sigma}=M_1\bar{\theta}+M_2\bar{\sigma}, \quad (17)$$

where

$$M_1=\frac{s^2(1+\beta_1 s)}{(1+\delta_0 s)(1+\tau^\beta s^\beta)},$$

$$M_2=\frac{s^2}{(1+\delta_0 s)(1+\tau^\beta s^\beta)},$$

$$L_1=\frac{\varepsilon s^2+\zeta(1+\beta_1 s)(1+\tau^\beta s^\beta)M_1}{1+\varepsilon s},$$

$$L_2=\frac{\zeta(1+\beta_1 s)(1+\tau^\beta s^\beta)M_2}{1+\varepsilon s}. \quad (18)$$

## V. STATE SPACE FORMULATION

Having chosen the temperature $\bar{\theta}$ and stress $\bar{\sigma}$ component as state variables, equations (17) may be presented in matrix form as

$$D^2\bar{V}(x,s)=A(s)\bar{V}(x,s), \quad (19)$$

where

$$\bar{V}(x,s)=\begin{bmatrix}\bar{\theta}(x,s)\\ \bar{\sigma}(x,s)\end{bmatrix}, \; A(s)=\begin{bmatrix}L_1 & L_2\\ M_1 & M_2\end{bmatrix}. (20)$$

The formal solution of the differential equation (19) may be written as

$$\bar{V}(x,s)=\exp[-\sqrt{A(s)x}]\bar{V}(0,s), \quad (21)$$

where $\bar{V}(0,s)=\begin{bmatrix}\bar{\theta}(0,s)\\ \bar{\sigma}(0,s)\end{bmatrix}=\begin{bmatrix}\bar{\theta}_0\\ \bar{\sigma}_0\end{bmatrix}$ and $I$ is an identity matrix of second order. The terms containing exponents of growing nature in the space variable $x$ have been discarded due to the regularity condition at infinity.
The characteristic equation of matrix $A(s)$ is obtained as

$$\lambda^2-(L_1+M_2)\lambda+L_1M_2-L_2M_1=0, \quad (22) \quad (22)$$

where the roots $\lambda_1$, $\lambda_2$ of equation (23) must satisfy

$$\lambda_1+\lambda_2=L_1+M_2, \quad (23)$$

$$\lambda_1\lambda_2=L_1M_2-L_2M_{1.} \;. \quad (24)$$

The Taylor series expansion of the matrix exponential has the form

$$\exp[-\sqrt{A(s)}x]=\sum_{n=0}^{\infty}\frac{[-\sqrt{A(s)}x]^n}{n!}. (25)$$

Making use of the well-known Cayley-Hamilton theorem, we can express $A^2$ and higher orders of the matrix $A$ in terms of $I$ and $A$.
Thus the infinite series in (25) can be truncated as

$$\exp[-\sqrt{A(s)x}] = a_0(x,s)I + a_1(x,s)A, \qquad (26)$$

where $a_0$ and $a_1$ are constants depending on $x$ and $s$.
Again by Cayley-Hamilton theorem, the characteristic roots $\lambda_1$ and $\lambda_2$ of the matrix $A$ must satisfy equation (26). Therefore, we have

$$\begin{aligned} \exp[-\sqrt{\lambda_1}x] &= a_0 + a_1\lambda_1, \\ \exp[-\sqrt{\lambda_2}x] &= a_0 + a_1\lambda_2. \end{aligned} \qquad (27)$$

On solving the above linear system of equations, we obtain

$$\begin{aligned} a_0 &= \frac{\lambda_1 e^{-\sqrt{\lambda_2}x} - \lambda_2 e^{-\sqrt{\lambda_1}x}}{\lambda_1 - \lambda_2} \\ a_1 &= \frac{e^{-\sqrt{\lambda_1}x} - e^{-\sqrt{\lambda_2}x}}{\lambda_1 - \lambda_2}. \end{aligned} \qquad (28)$$

Substituting the values of $a_0$ and $a_1$ along with $I$ and $A$ into equation (26) , we have

$$\exp[-\sqrt{A(s)x}] = \Gamma_{ij}(x,s), (i, j = 1, 2). \qquad (29)$$

where the components $\Gamma_{ij}(x,s)$ are given by

$$\begin{aligned} \Gamma_{11} &= \frac{e^{-\sqrt{\lambda_2}x}(\lambda_1 - L_1) - e^{-\sqrt{\lambda_1}x}(\lambda_2 - L_1)}{\lambda_1 - \lambda_2}, \\ \Gamma_{12} &= \frac{L_2(e^{-\sqrt{\lambda_1 x}} - e^{-\sqrt{\lambda_2 x}})}{\lambda_1 - \lambda_2}, \\ \Gamma_{21} &= \frac{M_1(e^{-\sqrt{\lambda_1 x}} - e^{-\sqrt{\lambda_2 x}})}{\lambda_1 - \lambda_2}, \\ \Gamma_{22} &= \frac{e^{-\sqrt{\lambda_2}x}(\lambda_1 - M_2) - e^{-\sqrt{\lambda_1}x}(\lambda_2 - M_2)}{\lambda_1 - \lambda_2} \end{aligned} \qquad (30)$$

.

Hence solution (21) can be written as

$$\bar{V}(x,s) = \Gamma_{ij}\bar{V}(0,s). \qquad (31)$$

Plugging the values of $\bar{V}(x,s)$ and $\left[A(s) + \frac{s^2}{v^2}I\right]^{-1}$ into (31) and after some straightforward calculation, the expressions for conductive temperature and stress are evaluated as

$$\bar{\theta}(x,s) = \frac{1}{\lambda_1 - \lambda_2}[(\lambda_1\bar{\theta}_0 - L_1\bar{\theta}_0 - L_2\bar{\sigma}_0)e^{-\sqrt{\lambda_2}x} -$$

$$(\lambda_2\bar{\theta}_0 - L_1\bar{\theta}_0 - L_2\bar{\sigma}_0)e^{-\sqrt{\lambda_1}x}], \qquad (32)$$

$$\bar{\sigma}(x,s) = \frac{1}{\lambda_1 - \lambda_2}[(\lambda_1\bar{\sigma}_0 - M_2\bar{\sigma}_0 - M_1\bar{\theta}_0)e^{-\sqrt{\lambda_2}x} +$$

$$(M_2\bar{\sigma}_0 - \lambda_2\bar{\sigma}_0 + M_1\bar{\theta}_0)e^{-\sqrt{\lambda_1}x}]. \qquad (33)$$

## VI. APPLICATION

We consider a homogeneous isotropic viscoelastic medium occupying the region $x \geq 0$ with quiescent initial state and boundary conditions in the following forms:

1. Mechanical boundary condition

We will suppose that the medium is subjected to a mechanical shock at $x = 0$ as follows:

$$\sigma(0,t) = \sigma_0 = -\sigma^* H(t), \qquad (34)$$

where $\sigma^*$ is a constant.
By applying Laplace transform defined in (15), we obtain

$$\bar{\sigma}(0,s) = \bar{\sigma}_0 = -\frac{\sigma^*}{s}. \qquad (35)$$

2. Thermal boundary condition

The medium at $x = 0$ is kept at reference temperature $T_0$, i.e.

$$\theta(0,t) = \theta_0 = 0. \qquad (36)$$

Operating Laplace transform on the above equation, one can obtain

$$\theta(0,s) = \bar{\theta}_0 = 0. \qquad (37)$$

Hence, we can utilize the values of $\bar{\sigma}_0$ and $\bar{\theta}_0$ from (35) and (37) in (32) and (33) to finally achieve the solutions in the Laplace transform domain as

$$\bar{\theta} = \frac{1}{\lambda_1 - \lambda_2}\frac{\sigma^*}{s}[L_2(e^{-\sqrt{\lambda_2}x} - e^{-\sqrt{\lambda_1}x})], \qquad (38)$$

$$\bar{\sigma} = \frac{1}{\lambda_1 - \lambda_2}\frac{\sigma^*}{s}[(M_2 - \lambda_1)e^{-\sqrt{\lambda_2}x} + (\lambda_2 - M_2)e^{-\sqrt{\lambda_1}x}]. \qquad (39)$$

Using dimensionless variables and Laplace transform in (8), the displacement component may be evaluated as

$$\bar{u} = \frac{1}{\alpha_1 s^2}\frac{\partial\bar{\sigma}}{\partial x}, \qquad (40)$$

where $\alpha_1 = \dfrac{\beta_{1e} T_0}{\rho c_1^2}$.

Substitution of $\bar{\sigma}$ from (39) into the above equation yields

$$\bar{u} = \frac{-1}{\alpha_1(\lambda_1-\lambda_2)}\frac{\sigma^*}{s^3}[\sqrt{\lambda_2}(M_2-\lambda_1)e^{-\sqrt{\lambda_2}x} + \sqrt{\lambda_1}(\lambda_2-M_2)e^{-\sqrt{\lambda_1}x}]. \quad (41)$$

## VII. NUMERICAL INVERSION OF THE TRANSFORM

The equations (38), (39) and (41) provide the expressions for temperature, stress and displacement in Laplace transform domain. To determine these in physical domain, Laplace inversion is applied with the help of numerical technique based on Fourier expansion of functions performed by Honig and Hirdes [8].

Let $\bar{f}(s)$ be the Laplace transform of function $f(t)$. The inversion formula of Laplace transform states that

$$f(t) = \frac{1}{2\pi\iota}\int_{d-\iota\infty}^{d+\iota\infty} e^{st}\bar{f}(s)ds,$$

where $d$ is an arbitrary real number greater than all the real parts of singularities of $\bar{f}(s)$. Taking $s = d + \iota y$ and using Fourier series in the interval $[0,2T]$, we get the approximate formula

$$f(t) \cong f_N(t) = \frac{1}{2}c_0 + \sum_{k=1}^{N} c_k, \text{for } 0 \le t \le 2T \quad (42)$$

where

$$c_k = \frac{e^{dt}}{T}\mathrm{Re}\left[e^{\frac{\iota k\pi t}{T}}\bar{f}\left(d+\frac{\iota k\pi}{T}\right)\right] \quad (43)$$

and $N$ is a sufficiently large integer representing the number of terms in the truncated Fourier series, chosen such that

$$e^{dt}\,\mathrm{Re}\left[e^{\frac{iN\pi t}{T}}\bar{f}\left(d+\frac{ik\pi}{T}\right)\right] \le \varepsilon_1, \quad (44)$$

where $\varepsilon_1$ is a prescribed small positive value that corresponds to the degree of accuracy to be achieved.

## VIII. NUMERICAL RESULTS AND DISCUSSION

With an aim to illustrate the contribution of fractional strain parameter, mechanical relaxation time and viscosity coefficients and heat source on field quantities, a numerical analysis is carried out. For this purpose, we have taken the following values of relevant parameters:

$$\lambda_e = 7.76\times10^{10}\,Kg\,m^{-1}s^{-2},\ \mu_e = 3.86\times10^{10}\,Kg\,m^{-1}s^{-2},$$
$$\rho = 8954\,kg\,m^{-3}, c_E = 383.1\,Jkg^{-1}K^{-1}, T_0 = 293\,K,$$
$$k = 386\,W\,m^{-1}K^{-1}, \alpha_t = 1.78\times10^{-5}\,K^{-1}, \tau = 0.01s,$$
$$t = 0.1s, \alpha_0 = 0.6s, \alpha_1 = 0.9s, \beta = 0.2, \sigma^* = 1.$$

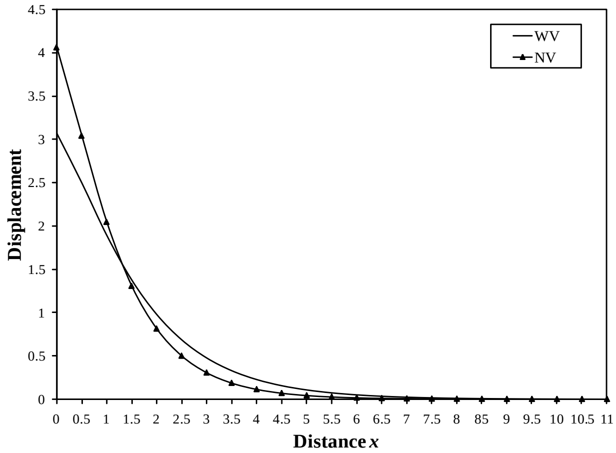


**Figure1:** Effect of viscosity on displacement

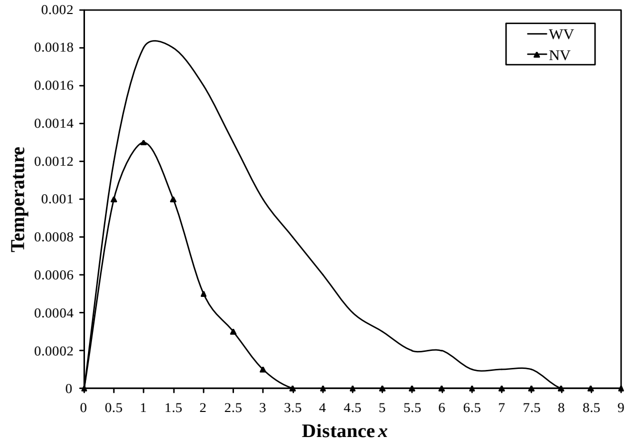


**Figure2:** Effect of viscosity on temperature

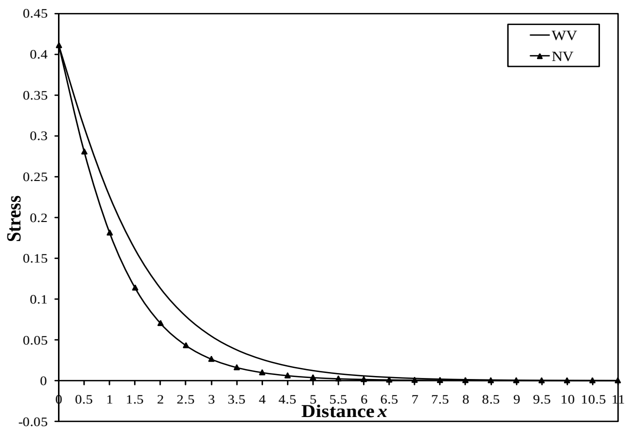


**Figure3:** Effect of viscosity on stress

In figures 1-3, we have studied the effect of viscosity parameter on different fields in thermoelastic medium under the case with viscosity (WV) and no viscosity (NV). Fig. 1 is drawn to observe the effect of viscosity on displacement. Viscosity has decreasing effect on displacement in the region $0 \leq x \leq 1.3$ while increasing effect after $x = 1.3$. Viscosity parameter exhibits an increasing effect on temperature and stress fields which is clear from Fig. 2 and 3 respectively. Initially, temperature is zero, which is in accordance with the boundary conditions. Ultimately, all the curves tend to zero which is physically admissible.

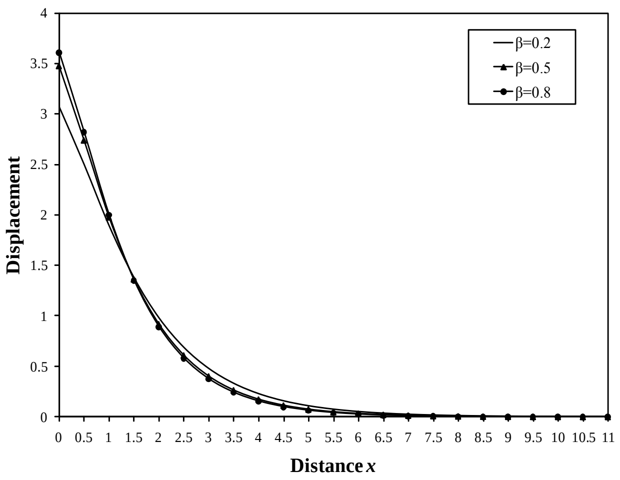


**Figure4:** Effect of fractional parameter on displacement

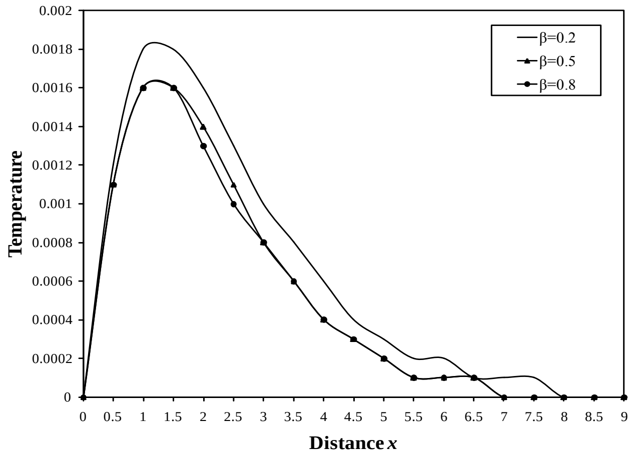


**Figure5:** Effect of fractional parameter on temperature

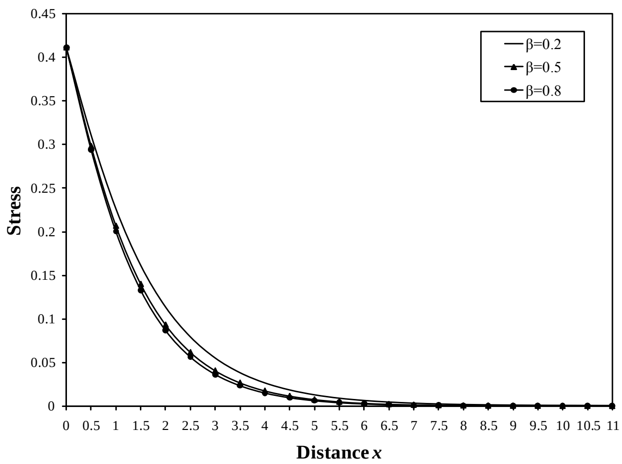


**Figure6:** Effect of fractional parameter on stress

Figures 4-6 are plotted to show the effect of fractional parameter $\beta$ on displacement, temperature and stress fields respectively. Numerically, displacement increases for $0 \leq x \leq 1.5$ while decreases for $x > 1.5$ with the increment of values of $\beta$ untill it becomes zero. It is noticed from Fig. 4 that the profile of temperature is almost same for $\beta = 0.5$ and 0.8 except the region $1.5 \leq x \leq 3$ and here, temperature is larger in the case of $\beta = 0.5$ as compared to the case of $\beta = 0.8$. Numerical values of temperature are larger for $\beta = 0.2$ among all the considered values of $\beta$. Fractional parameter exhibits decreasing effect on stress which can be varified from Fig. 6.

## IX. CONCLUSIONS

According to above analysis, we can conclude the follwing points:

1. All the fields are restricted in a limited region which is in accordance with the notion of generalized thermoelasticity theory.
2. Viscosity parameter has increasing effect on the considered fields except displacement (here viscosity shows mixed effect).
3. The effect of fractional parameter on all the studied fields is very much significant.

The results presented in this paper should prove useful for researchers in material science and designers of new materials.

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