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One-factorizations of the complete graph K_{p+1} arising from parabolas

György Kiss
Eötvös Loránd University, gyorgy.kiss@ttk.elte.hu

Nicola Pace
ITK Engineering GmbH, nicolaonline@libero.it

Angelo Sonnino
Università degli Studi della Basilicata, angelo.sonnino@icloud.com

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Abstract

There are three types of affine regular polygons in $\mathrm{AG}(2,q)$: ellipse, hyperbola and parabola. The first two cases have been investigated in previous papers. In this note, a particular class of geometric one-factorizations of the complete graph K_n arising from parabolas is constructed and described in full detail. With the support of computer aided investigation, it is also conjectured that up to isomorphisms this is the only one-factorization where each one-factor is either represented by a line or a parabola.

1 Introduction

For a positive even integer n, a one-factorization of the complete graph K_n is a partition of the edge set into $n-1$ one-factors—each consisting of $\frac{n}{2}$ edges partitioning the vertex set.

One-factorizations of complete graphs play a crucial role in many practical applications, like for instance scheduling tournaments, where a round robin tournament is to be played in the minimum number of sessions. Besides applications, one-factorizations have strong connections to Design Theory; see for instance [13].

Our approach to the problem of constructing one-factorizations of complete graphs is essentially geometric, as in [3, 6, 9, 10], and is based on techniques that have previously been used to find one-factorizations of multigraphs; see for instance [2, 4, 7, 11].

Basically, there are three types of affine regular polygons in the finite affine plane $\mathrm{AG}(2,q)$. One-factorizations arising from ellipses and hyperbolas have already been addressed in [6, 9]. In this paper the remaining case, the parabola, is investigated.

Our main result is the construction of a parabolic one-factorization—that is, a one-factorization where all one-factors except one are represented by parabolas, and the remaining one is represented by a line—for every complete graph K_{p+1} with p an odd prime. We may also provide a classification of parabolic one-factorizations.

Our notation is standard. For general information about one-factorizations of complete graphs see for instance [8, 12, 13].

2 Preliminaries

Henceforth we assume that $p \geq 3$ is a prime number. We fix a projective frame in $\mathrm{PG}(2,p)$ with homogeneous coordinates $(X_0{:}X_1{:}X_2)$, and consider $\mathrm{PG}(2,p)$ as $\mathrm{AG}(2,p) \cup \ell_\infty$ where ℓ_∞ has equation $X_0 = 0$. As usual, the points of $\mathrm{AG}(2,p)$ are written as (X,Y) with $X = \frac{X_1}{X_0}$ and $Y = \frac{X_2}{X_0}$.

In $\mathrm{AG}(2,p)$, let $\mathcal{P}_a$ be the parabola with affine equation $Y = X^2 + a$, where a varies in $\mathbb{Z}_p$, and $V_\infty = (0{:}0{:}1)$ the point at infinity of the line $X_1 = 0$. Note that, in the projective closure of $\mathrm{AG}(2,p)$, any two parabolas $\mathcal{P}_a$ and $\mathcal{P}_b$, with $a \neq b$, meet at the point V_∞ only.

Let $V_i = (i, i^2)$ denote the points on $\mathcal{P}_0$ for $i = 0, 1, \ldots, p-1$. For $k = 1, 2, \ldots, \frac{p-1}{2}$, let P_i^k denote the pole of the line $\overline{V_i V_{i+k}}$ with respect to $\mathcal{P}_0$. The equation of the tangent line t_i to $\mathcal{P}_0$ at V_i is

$$t_i\colon \quad Y = 2iX - i^2,$$

hence the coordinates of the point $P_i^k = t_i \cap t_{i+k}$ are

$$P_i^k = \left(i + \frac{k}{2}, i^2 + ik\right);$$

see Figure 1. Further, let P_i^∞ denote the point at infinity of the line t_i, that is, $P_i^\infty = (0{:}1{:}2i)$.

Lemma 2.1. *For a fixed k, the points $P_0^k, P_1^k, \ldots, P_{p-1}^k$ are on the parabola $\mathcal{P}_{-\frac{k^2}{4}}$.*

Proof. The claim follows from the equality

$$i^2 + ik = \left(i + \frac{k}{2}\right)^2 - \frac{k^2}{4}.$$

□

The vertices of the complete graph K_{p+1} correspond to the points of $\mathcal{P}_0 \cup \{V_\infty\}$, while the edges of K_{p+1} correspond to the points of type P_i^k, with $k = 1, 2, \ldots, \frac{p-1}{2}, \infty$. Thus the set of edges of K_{p+1} corresponds to the set of points

$$\mathcal{E} = \left(\bigcup_{k=1}^{\frac{p-1}{2}} \mathcal{P}_{-\frac{k^2}{4}}\right) \cup \left(\ell_\infty \setminus \{V_\infty\}\right).$$

These points are called *external points* with respect to $\mathcal{P}_0$.

In this setting, a one-factor of K_{p+1} is a set consisting of $\frac{p+1}{2}$ points of type P_i^k, for $i \in \{0, 1, \ldots, p-1\}$ and $k \in \left\{1, 2, \ldots, \frac{p-1}{2}\right\} \cup \{\infty\}$, satisfying the *tangent property*, that is, no tangent to $\mathcal{P}_0$ meets the set in more than one point; see [6]. Then, a one-factorization of K_{p+1} is just a partition of all the points of type P_i^k into p one-factors.

3 Results

Remark that a parabola of type $\mathcal{P}_a$ cannot contain any point of type P_j^∞, therefore a subset of its points satisfying the tangent property consists of at most $\frac{p-1}{2}$ points. If the line ℓ is not a tangent to $\mathcal{P}_0$, then ℓ is called a *secant* if $|\ell \cap \mathcal{P}_0| = 2$ and ℓ is called an *external line* if $|\ell \cap \mathcal{P}_0| = 0$. It is well known (see e.g. [5, Lemma 6.14]) that a secant contains $\frac{p-1}{2}$ points of $\mathcal{E}$ and an external line contains $\frac{p+1}{2}$ points of $\mathcal{E}$. These motivate the following definitions.

Definition 3.1. *A one-factor represented by a parabola $\mathcal{P}_a$ is a set of $\frac{p-1}{2}$ points of type P_j^k on $\mathcal{P}_a$, together with a suitable point at infinity. A one-factor so defined is referred to as a* parabolic one-factor*.*

Definition 3.2. *A* one-factor represented by a secant line *ℓ of $\mathcal{P}_0$ is a set consisting of $\frac{p-1}{2}$ points of $\mathcal{E}$ on ℓ, plus the pole of ℓ with respect to $\mathcal{P}_0$.*

A one-factor represented by an external line *ℓ of $\mathcal{P}_0$ is a set consisting of $\frac{p+1}{2}$ points of $\mathcal{E}$ on ℓ.*

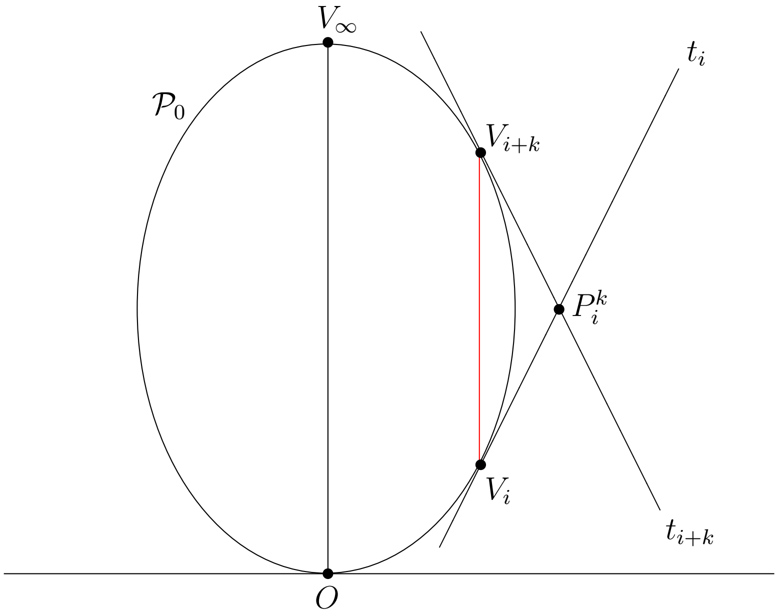

Figure 1: Representation of the edge V_iV_{i+k} of K_{p+1} on the parabola $\mathcal{P}_0$

Definition 3.3. *A one-factorization of* K_{p+1} *is called a* parabolic one-factorization *if* $p-1$ *of its one-factors are represented by parabolas and one of its one-factors is represented by a line.*

Theorem 3.4. *Let* p *be an odd prime. Then the complete graph* K_{p+1} *has a parabolic one-factorization.*

Proof. The proof is constructive. Let

$$F_0 = \left\{ P^k_{-\frac{k}{2}} : \ k = 1, 2, \dots, \frac{p-1}{2} \right\} \cup \{P_0^\infty\}.$$

The set F_0 is a one-factor represented by the secant line of $\mathcal{P}_0$ of equation $X = 0$, and P_0^∞ is its pole with respect to $\mathcal{P}_0$.

For $k = 1, 2, \dots, \frac{p-1}{2}$, define the following sets of points:

$$G_k = \left\{ P^k_{\frac{k}{2}+2jk} : \ j = 0, 1, \dots, \frac{p-3}{2} \right\} \cup \left\{ P^\infty_{-\frac{k}{2}} \right\},$$
$$H_k = \left\{ P^k_{\frac{k}{2}+(2j+1)k} : \ j = 0, 1, \dots, \frac{p-3}{2} \right\} \cup \left\{ P^\infty_{\frac{k}{2}} \right\}.$$

By Lemma 2.1, $G_k \setminus \left\{ P^\infty_{-\frac{k}{2}} \right\}$ and $H_k \setminus \left\{ P^\infty_{\frac{k}{2}} \right\}$ are disjoint subsets of the parabola $\mathcal{P}_{-\frac{k^2}{4}}$. Both G_k and H_k are one-factors represented by the parabola $\mathcal{P}_{-\frac{k^2}{4}}$ because every tangent to $\mathcal{P}_0$ intersects $\mathcal{P}_{-\frac{k^2}{4}}$ in two points, P_i^k and P_{i+k}^k. One of these points falls in G_k, the other one in H_k, and the claim follows. □

Parabolic one-factorisations are completely characterised in the projective closure of $\mathrm{AG}(2,p)$.

Theorem 3.5. *Let $p > 5$ be an odd prime and $\mathcal{F}$ be a parabolic one-factorization of the complete graph K_{p+1}. Then $\mathcal{F}$ is isomorphic to the one-factorization constructed in Theorem 3.4.*

Proof. Let ℓ be the line representing the unique linear one-factor of $\mathcal{F}$ and L denote the pole of ℓ with respect to $\mathcal{P}_0$. First, we show that ℓ contains the point V_∞. By definition, $\ell \cup \{L\}$ must contain one affine point from each parabola of type $\mathcal{P}_a$. Hence ℓ must be a tangent to at least $\frac{p-1}{2} - 1 > 1$ parabolas of type $\mathcal{P}_a$. Suppose that the affine equation of ℓ is $Y = mX + b$. Then ℓ contains exactly one point of $\mathcal{P}_a$ if and only if the discriminant of the quadratic equation $X^2 - mX + a - b = 0$ is zero, that is,

$$a = \frac{m^2 + 4b}{4}. \tag{1}$$

From (1), the line ℓ would be a tangent to at most one parabola of type $\mathcal{P}_a$, hence it must be assumed that the affine equation of ℓ is of type $X = c$.

Now consider the linear transformation $\varphi \in \mathrm{PGL}(3,p)$ associated to the matrix

$$\begin{pmatrix} 1 & -c & c^2 \\ 0 & 1 & -2c \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(1 : c : c^2)^\varphi = (1 : 0 : 0)$ and $(0 : 0 : 1)^\varphi = (0 : 0 : 1)$. Hence, the unique linear one-factor of $\mathcal{F}^\varphi$ corresponds, by projectivity, to the line $X = 0$, that is, the set of points

$$\left\{ P^k_{-\frac{k}{2}} : k = 1, 2, \ldots, \frac{p-1}{2} \right\} \cup \{P^\infty_0\}.$$

Further, the linear transformation φ fixes every parabola $\mathcal{P}_a$ setwise since $(1 : t : t^2 + a)^\varphi = (1 : t - c : (t - c)^2 + a)$.

For a fixed $k \in \{1, 2, \ldots, \frac{p-1}{2}\}$ let G_k and H_k denote the two one-factors of $\mathcal{F}^\varphi$ which are represented by the parabola $\mathcal{P}_{-\frac{k^2}{4}}$. Consider the point $P^k_{\frac{k}{2}}$. We may assume without loss of generality that it belongs to G_k. Then, by the tangent property, $P^k_{\frac{k}{2}+k}$ must belong to H_k. For $j = 1, \ldots, \frac{p-3}{2}$, the points $P^k_{\frac{k}{2}+2jk}$ must belong to G_k, while the points $P^k_{\frac{k}{2}+(2j+1)k}$ must belong to H_k. Furthermore, $P^\infty_{-\frac{k}{2}}$ is in G_k and $P^\infty_{\frac{k}{2}}$ is in H_k. Thus, $\mathcal{F}^\varphi$ is the one-factorization constructed in Theorem 3.4 and hence $\mathcal{F}$ is isomorphic to $\mathcal{F}^\varphi$. □

We conclude with a conjecture that is supported by our computer aided investigations.

Conjecture 3.6. *Let $p > 7$ be an odd prime, $\mathcal{F}$ be a one-factorization of the complete graph K_{p+1} such that each one-factor of $\mathcal{F}$ is either represented by a line or a parabola. Then $\mathcal{F}$ is either a parabolic one-factorization or each one-factor of $\mathcal{F}$ is represented by a line.*

Conjecture 3.6 can easily verified for the values $p = 11, 13, 17$ using the software Magma [1]. One can start with an exhaustive search for all $(p+1)/2$-factors that are represented either by a line or by a parabola. According to the definition in Section 2, each one of these $(p+1)/2$-factors corresponds to a set of points with the tangent property. At this point, one can construct a graph G where the vertices correspond to these $(p+1)/2$-factors and two vertices are incident if and only if the corresponding sets are disjoint. A p-clique of the graph G corresponds to a 1-factorization where all $(p+1)/2$-factors are represented either by a line or a parabola. Finding all p-cliques of G is the computationally longest part of this verification, however, it can be performed in Magma using the function `AllCliques`. This computation takes only few seconds for the cases $p = 11, 13$ while it takes roughly fifteen minutes on a standard laptop with a 2.70GHz Intel Core i7 processor for the case $p = 17$. Finally, the conjecture can be directly verified for all p-cliques, that is, 1-factorizations obtained in such a way.

4 Examples for small p

The examples described in this section serve to illustrate the results from the previous sections.

4.1 $p = 7$

Let us consider the parabola $\mathcal{P}_0$ of projective equation $X_0X_2 = X_1^2$ in PG(2, 7). The construction in Theorem 3.4 provides the following partition of the points of type P_i^k:

$$\begin{aligned}
F_0 &= \{P_3^1(1{:}0{:}5), P_6^2(1{:}0{:}6), P_2^3(1{:}0{:}3), P_0^\infty(0{:}1{:}0)\},\\
F_1 &= \{P_4^1(1{:}1{:}6), P_6^1(1{:}3{:}0), P_1^1(1{:}5{:}2), P_3^\infty(0{:}1{:}6)\},\\
F_1' &= \{P_5^1(1{:}2{:}2), P_0^1(1{:}4{:}0), P_2^1(1{:}6{:}6), P_4^\infty(0{:}1{:}1)\},\\
F_2 &= \{P_1^2(1{:}2{:}3), P_5^2(1{:}6{:}0), P_2^2(1{:}3{:}1), P_6^\infty(0{:}1{:}5)\},\\
F_2' &= \{P_3^2(1{:}4{:}1), P_0^2(1{:}1{:}0), P_4^2(1{:}5{:}3), P_1^\infty(0{:}1{:}2)\},\\
F_3 &= \{P_5^3(1{:}3{:}5), P_4^3(1{:}2{:}0), P_3^3(1{:}1{:}4), P_2^\infty(0{:}1{:}4)\},\\
F_3' &= \{P_1^3(1{:}6{:}4), P_0^3(1{:}5{:}0), P_6^3(1{:}4{:}5), P_5^\infty(0{:}1{:}3)\}.
\end{aligned}$$

This partition is a parabolic one-factorization, where the one-factors are as follows:

- F_0 is represented by the secant line $X_1 = 0$,
- F_1, F_1' are represented by the parabola $\mathcal{P}_5 : X_0X_2 = X_1^2 + 5X_0^2$,
- F_2, F_2' are represented by the parabola $\mathcal{P}_6 : X_0X_2 = X_1^2 + 6X_0^2$,
- F_3, F_3' are represented by the parabola $\mathcal{P}_3 : X_0X_2 = X_1^2 + 3X_0^2$.

4.2 $p = 11$

Let us consider the parabola $\mathcal{P}_0$ of projective equation $X_0X_2 = X_1^2$ in $\mathrm{PG}(2,11)$. The construction in Theorem 3.4 provides the following partition of the points of type P_i^k:

$$
\begin{aligned}
F_0 &= \{P_{10}^2(1{:}0{:}10), P_5^1(1{:}0{:}8), P_3^5(1{:}0{:}2), P_9^4(1{:}0{:}7), P_4^3(1{:}0{:}6), P_0^\infty(0{:}1{:}0)\},\\
F_1 &= \{P_6^1(1{:}1{:}9), P_8^1(1{:}3{:}6), P_{10}^1(1{:}5{:}0), P_1^1(1{:}7{:}2), P_3^1(1{:}9{:}1), P_5^\infty(0{:}1{:}10)\},\\
F_1' &= \{P_7^1(1{:}2{:}1), P_9^1(1{:}4{:}2), P_0^1(1{:}6{:}0), P_2^1(1{:}8{:}6), P_4^1(1{:}10{:}9), P_6^\infty(0{:}1{:}1)\},\\
F_2 &= \{P_1^2(1{:}2{:}3), P_5^2(1{:}6{:}2), P_9^2(1{:}10{:}0), P_2^2(1{:}3{:}8), P_6^2(1{:}7{:}4), P_{10}^\infty(0{:}1{:}9)\},\\
F_2' &= \{P_3^2(1{:}4{:}4), P_7^2(1{:}8{:}8), P_0^2(1{:}1{:}0), P_4^2(1{:}5{:}2), P_8^2(1{:}9{:}3), P_1^\infty(0{:}1{:}2)\},\\
F_3 &= \{P_7^3(1{:}3{:}4), P_2^3(1{:}9{:}10), P_8^3(1{:}4{:}0), P_3^3(1{:}10{:}7), P_9^3(1{:}5{:}9), P_4^\infty(0{:}1{:}8)\},\\
F_3' &= \{P_{10}^3(1{:}6{:}9), P_5^3(1{:}1{:}7), P_0^3(1{:}7{:}0), P_6^3(1{:}2{:}10), P_1^3(1{:}8{:}4), P_7^\infty(0{:}1{:}3)\},\\
F_4 &= \{P_2^4(1{:}4{:}1), P_{10}^4(1{:}1{:}8), P_7^4(1{:}9{:}0), P_4^4(1{:}6{:}10), P_1^4(1{:}3{:}5), P_9^\infty(0{:}1{:}7)\},\\
F_4' &= \{P_6^4(1{:}8{:}5), P_3^4(1{:}5{:}10), P_0^4(1{:}2{:}0), P_8^4(1{:}10{:}8), P_5^4(1{:}7{:}1), P_2^\infty(0{:}1{:}4)\},\\
F_5 &= \{P_8^5(1{:}5{:}5), P_7^5(1{:}4{:}7), P_6^5(1{:}3{:}0), P_5^5(1{:}2{:}6), P_4^5(1{:}1{:}3), P_3^\infty(0{:}1{:}6)\},\\
F_5' &= \{P_2^5(1{:}10{:}3), P_1^5(1{:}9{:}6), P_0^5(1{:}8{:}0), P_{10}^5(1{:}7{:}7), P_9^5(1{:}6{:}5), P_8^\infty(0{:}1{:}5)\}.
\end{aligned}
$$

This partition is a parabolic one-factorization, where the one-factors are as follows:

- F_0 is represented by the secant line $X_1 = 0$,
- F_1, F_1' are represented by the parabola $\mathcal{P}_8 : X_0X_2 = X_1^2 + 8X_0^2$,
- F_2, F_2' are represented by the parabola $\mathcal{P}_{10} : X_0X_2 = X_1^2 + 10X_0^2$,
- F_3, F_3' are represented by the parabola $\mathcal{P}_6 : X_0X_2 = X_1^2 + 6X_0^2$,
- F_4, F_4' are represented by the parabola $\mathcal{P}_7 : X_0X_2 = X_1^2 + 7X_0^2$,
- F_5, F_5' are represented by the parabola $\mathcal{P}_2 : X_0X_2 = X_1^2 + 2X_0^2$.

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