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ONE GENERALIZATION OF THE DYNAMIC PROGRAMMING PROBLEM

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PART I.

1. INTRODUCTION

The main purpose of this article is to provide an exact theory of the dynamic programming on a sufficiently general basis. One attempt to carry out this task was made in [1]. However, in the theory of [1] there are some inaccuracies in introducing the topology on the set of transformations and in some proofs. We hope that the theory described in this article will be a contribution to the various attempts to give a dynamic programming theory which is both sufficiently general and exact.

2. SOME INTRODUCTORY CONCEPTS

Let M be a compact topological Hausdorff's space (further the abbreviation H-space will be used), let T^M be the set of all continuous transformations of the space M into itself and let Φ be the trasformation of the Cartesian product $M \otimes \tilde{T}^M$ into the space M which is defined by

$$\Phi(x, y) = y(x) \text{ for all } x \in M\,,\ y \in \tilde{T}^M\,.$$

Here $y(x)$ means the point from the space M which represents the result of applying the transformation y to the point x.

3. INTRODUCING THE TOPOLOGY ON THE SET $\tilde{T}^M$

Let us denote by $\boldsymbol{T}^M$ the set of topologies, which can be introduced on $\tilde{T}^M$ and which have the following two properties: If $\mathscr{T} \in \boldsymbol{T}^M$ and $\mathscr{T}$ is introduced on $\tilde{T}^M$, then

(1) $\tilde{T}^M$ is a topological H-space;

(2) Φ is a continuous transformation of the topological product $M \otimes \tilde{T}^M$ into the space M (the definition of the topological product see [2]).

Note 1. The dynamic programming theory described below needs introducing on $\tilde{T}^M$ some topology from the set $\mathbf{T}^M$. The problem of introducing on $\tilde{T}^M$ some topology from the set $\mathbf{T}^M$ is solved in [5]. In [5] a set of topologies $\mathbf{V}^M$ is found such that $\mathbf{V}^M \subset \mathbf{T}^M$. The set $\mathbf{V}^M$ is defined in such a way that its definition provides one rather general prodedure of introducing on $\tilde{T}^M$ topologies from $\mathbf{T}^M$. Moreover the set $\mathbf{V}^M$ contains most of usually used topologies (for example the metric space topology, discrete topology).

4. FORMULATION OF THE DYNAMIC PROGRAMMING PROBLEM

Further we shall assume that on $\tilde{T}^M$ some topology from $\mathbf{T}^M$ is introduced.

Let T^M be a compact subspace of the space $\tilde{T}^M$. A pair $\mathscr{P} = \{M; T^M\}$ is called, the automaton $\mathscr{P}$ with the set of states M and the set of transformations T^M. Let us note that the topology on T^M is the topology induced on T^M by the topology on $\tilde{T}^M$ and thus T^M with regard to this topology is an H-space.

Let $\mathfrak{M} = M \otimes T^M \otimes \ldots \otimes T^M \otimes \ldots$ (The set of factors T^M in this product is a countable set.) Let on $\mathfrak{M}$ Tichonoff's topology be introduced (the definition of Tichonoff's topology see [2], [5]). Then $\mathfrak{M}$ with regard to this topology is a compact topological H-space (see [2], [5]).

Two following assertions may be proved (see [5]).

Assertion 1. Let E be a topological H-space. Let $N \subset E$, let N be a compact set let $f(x)$, $g(x)$ be continuous functions defined on N. Then

$$\max_{x \in N} |f(x) - g(x)| \geqq |\max_{x \in N} f(x) - \max_{x \in N} g(x)|$$ [1])

Assertion 2. Let M_1 be a topological H-space, let M_2 be a compact topological H-space, let M be the topological product of the spaces M_1 and M_2, let $\Psi(x, y)$ be a function defined and continuous on M and let

$$f(x) = \max_{y \in M_2} \Psi(x, y) .$$

Then the function $f(x)$ is continuous on M_1.

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[1]) Note that the continuous function on a compact set assumes on this set its supremum (see [3]).

Let $X = (x_0, y_0, y_1, \ldots, y_n, \ldots)$ be an arbitrary point from $\mathfrak{M}$. Then $x_0 \in M$, $y_i \in T^M$, $i = 0, 1, \ldots$ We define the transformations P and N by the following two relations:

$$PX = (y_0(x_0), y_1, \ldots, y_n, \ldots),$$

$$NX = x_0 .$$

Then P is a transformation of the space $\mathfrak{M}$ into itself and N is a transformation of the space $\mathfrak{M}$ into the space M.

Lemma 1. *The transformation P is continuous on $\mathfrak{M}$ with regard to Tichonoff's topology introduced on $\mathfrak{M}$.*

Proof. Let $\bar{X} = (\bar{x}_0, \bar{y}_0, \bar{y}_1, \ldots)$ be an arbitrary fixed point in $\mathfrak{M}$. Then $P\bar{X} = (\bar{y}_0(\bar{x}_0), \bar{y}_1, \bar{y}_2, \ldots)$. We need to prove the continuity of transformation P at the point X, i.e. that for an arbitrary neighbourhood U of the point $P\bar{X}$ there exists such a neighbourhood $\tilde{U}$ of the point $\bar{X}$ that the following implication holds: $X \in \tilde{U} \Rightarrow PX \in U$. Let us introduce the following notation:

$$P\bar{X} = (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4, \ldots) \text{ so that } \bar{z}_1 = \bar{y}_0(\bar{x}_0) \in M ,$$

$\bar{z}_i = \bar{y}_{i-1} \in T^M$ for all $i \geqq q$. Let the neighbourhood U be generated (in the sense of Tichonoff's topology, see [2], [5]) by the neighbourhoods $\bar{U}_i$ $(i = 1, \ldots, k)$ of the points $\bar{z}_{n_i}$ $(i = 1, \ldots, k)$. We can assume that $\bar{U}_i$ are opened. To this neighbourhood U we shall now construct the corresponding neighbourhood $\tilde{U}$ of the point $\bar{X}$. We introduce the notation $\bar{X} = (\bar{\bar{z}}_1, \bar{\bar{z}}_2, \bar{\bar{z}}_3, \ldots, \bar{\bar{z}}_n, \ldots)$. Let us note that $\bar{\bar{z}}_1 = \bar{x}_0 \in M$, $\bar{\bar{z}}_2 = \bar{y}_0 \in T^M$, $\bar{\bar{z}}_i = \bar{z}_{i-1} = \bar{y}_{i-2} \in T^M$ for all $i > 2$. At first we suppose that $n_i \neq 1$ for $i = 1, \ldots, k$. Then the neighbourhood $\tilde{U}$ can be constructed as a neighbourhood which is generated (in Tichonoff's sense) by the neighbourhoods $\bar{U}_i$ $(i = 1, \ldots, k)$ of the points $\bar{\bar{z}}_{n_i+1} = \bar{z}_{n_i}$ $(i = 1, \ldots, k)$. Let now $n_{i_0} = 1$ where $1 \leq i_0 \leq k$. Without any loss of generality we can assume that $i_0 = 1$. So we have the following situation: there is given a neighbourhood $\bar{U}_1$ of the point $\bar{y}_0(\bar{x}_0)$ and we must construct such neighbourhoods $\bar{O}_1$ of the point $\bar{x}_0 \in M$ and $\bar{O}_2$ of the point $\bar{y}_0 \in T^M$ that the following implication holds:

$$x_0 \in \bar{O}_1 , \quad y_0 \in \bar{O}_2 \Rightarrow y_0(x_0) \in \bar{U}_1 .$$

On $\tilde{T}^M$ (and thus also on T^M) a topology from $\boldsymbol{T}^M$ is introduced and therefore the transformation Φ is continuous on $M \otimes T^M$. Thus for the neighbourhood $\bar{U}_1$ of the point $\bar{y}_0(x_0) = \Phi(\bar{x}_0, \bar{y}_0)$ there exists such a neighbourhood $\bar{O}_{12}$ of the point $(\bar{x}_0, \bar{y}_0)$ which is generated (in Tichonoff's sense) by the neighbourhoods $\bar{O}_1$ of $\bar{x}_0$ and $\bar{O}_2$ of $\bar{y}_0$ that the following implication holds: $(x, y) \in \bar{O}_{12} \Rightarrow \Phi(x, y) = y(x) \in \bar{U}_1$. The neighbourhood $\tilde{U}$ will be now in this case generated by the neighbourhoods $\bar{O}_1$, $\bar{O}_2$, and $\bar{U}_i$ $(i = 2, 3, \ldots, k)$ of the points $\bar{\bar{z}}_{n_i+1}$ $(i = 2, 3, \ldots, k)$.

In the both cases described above the neighbourhood $\tilde{U}$ is constructed in such a way that if $X \in \tilde{U}$, then $PX \in U$. This completes, the proof of continuity of the transformation P on $\mathfrak{M}$.

Let x be an arbitrary point of M. We define the set $\mathfrak{M}^{(x)}$ as follows:

$$\mathfrak{M}^{(x)} = \{X; X \in \mathfrak{M}, NX = x\} .$$

Lemma 2. *$\mathfrak{M}^{(x)}$ is a topological subspace of $\mathfrak{M}$ with Tichonoff's topology induced on $\mathfrak{M}^{(x)}$ by the topology of $\mathfrak{M}$. $\mathfrak{M}^{(x)}$ is with regard to this topology a compact topological H-space.*

Proof. Let us denote by $\{x\}$ the set containing only the element $x \in M$. Then

$$\mathfrak{M}^{(x)} = \{x\} \otimes T^M \otimes T^M \otimes \ldots \otimes T^M \ldots$$

(The set of factors T^M in this product is countable.) Thus $\mathfrak{M}^{(x)}$ is a subspace of $\mathfrak{M}$ with Tichonoff's topology induced on $\mathfrak{M}^{(x)}$ by Tichonoff's topology on $\mathfrak{M}$. $\{x\}$ is a compact topological H-space and the same holds for the space T^M. Thus $\mathfrak{M}^{(x)}$ is also a compact topological H-space (see [2], [5]).

Lemma 3. *Let $\Psi(X)$ be a real function defined and continuous on $\mathfrak{M}$. Let us define on M a real function $f(x)$ by:*

$$f(x) = \max_{X \in \mathfrak{M}^{(x)}} \Psi(X) \text{ for all } x \in M .$$

Then $f(x)$ is continuous on M (with regard to the topology introduced on M).

Proof. Let $X = (x_0, y_0, y_1, \ldots, y_n, \ldots)$ be an arbitrary point of $\mathfrak{M}$. The function $\Psi(X)$ can also be regarded as a function of two variables which is defined on the topological product of two compact topological H-spaces, namely of the space M and of the space $\mathfrak{N} = T^M \otimes T^M \otimes \ldots \otimes T^M \otimes \ldots$ We denote this function of two variables by the symbol $\overline{\Psi}(x, Y)$ so that $\Psi(X) = \overline{\Psi}(x_0, Y_0)$ where $x_0 \in M$, $Y_0 \in \mathfrak{N}$ and further

$$f(x) = \max_{X \in \mathfrak{M}} \Psi(X) = \max_{Y \in \mathfrak{N}} \overline{\Psi}(x, Y) \text{ for all } x \in M .$$

Thus according to Assertion 2 the function $f(x)$ is continuous on M.

We shall further assume that a real continuous function $\Psi(X)$ is given on $\mathfrak{M}$. This function will be called the objective function of the automaton $\mathscr{P}$. Now we can formulate the dynamic programming problem as follows:

Find for all $x \in M$ the element (or elements) $\overline{X} = (x, \bar{y}_0, \bar{y}_1, \ldots) \in \mathfrak{M}^{(x)}$ such that $\Psi(\overline{X}) = f(x)$.

5. EXISTENCE AND UNIQUENESS OF THE DYNAMIC PROGRAMMING PROBLEM SOLUTION

Let us assume that the function $\Psi(X)$ is continuous on $\mathfrak{M}$ and such that for all $X = (x_0, y_0, y_1, \ldots) \in \mathfrak{M}$,

$$\Psi(X) - \Psi(PX) = \theta(x_0, y_0) .$$

Let us note that $\theta(x_0, y_0)$ is a continuous function of its variables because it is a difference of two continuous functious of X (namely of the function $\Psi(X)$ and of the composed function $\Psi(PX)$) and such a difference must be continuous with regard to Tichonoff's topology on $\mathfrak{M}$. Then

$$\Psi(X) = \Psi(X) - \Psi(PX) + \Psi(PX)$$

or

$$\Psi(X) = \theta(x_0, y_0) + \Psi(PX) .$$

We pass on the both sides of this equation to the maximum. On the left hand side we have $f(x_0)$. On the right hand side, this procedure will be carried out in two steps. At first we take the maximum over all such $X \in \mathfrak{M}$ that $N(PX) = y_0(x_0)$ for some $y_0 \in T^M$ and then we take the maximum of these expressions over all $y_0 \in T^M$. The maximum on the left hand side exists because it is the maximum of a continuous function on a compact set. Thus the maximum on the right hand side exists too and it holds:

$$\max_{X \in \mathfrak{M}(x_0)} \Psi(X) = \max_{y_0 \in T^M} [\theta(x_0, y_0) + \max_{X \in \mathfrak{M}, N(PX) = y_0(x_0)} \Psi(PX)]$$

$$f(x_0) = \max_{y_0 \in T^M} [\theta(x_0, y_0) + \max_{\tilde{X} \in \mathfrak{M}(y_0(x_0))} \Psi(\tilde{X})]$$

or

$$f(x_0) = \max_{y_0 \in T^M} [\theta(x_0, y_0) + f(y_0(x_0))] . \tag{1}$$

We have just deduced a functional equation satisfied by $f(x)$. From the existence of the maximal values both on the left and on the right hand sides of this equation the existence of the solution of the equation (1) follows. Now we shall solve the question of the uniqueness of this solution.

We define the sets M_i for $i = 0, 1, \ldots$ as follows:

$$M_0 = M$$

$$M_i = \{y(x); y \in T^M, x \in M_{i-1}\} \quad \text{for} \quad i = 1, 2, \ldots$$

It is $M_i \subset M_{i-1}$ for all i. If $M_r = M_{r+1}$ for some index r then $M_r = M_{r+k}$ for an arbitrary integer $k \geqq 2$.

Lemma 4. *M_i is a compact subset of M for an arbitrary i.*

Proof. We prove the lemma by induction $M_0 = M$ is a compact (and thus closed[2])) set. Let us assume that M_{i-1} is compact for some $i \geqq 1$. We want to prove that M_i is compact. We know that

$$M_i = \{y(x), y \in T^M, x \in M_{i-1}\} = \{\Phi(x, y); y \in T^M, x \in M_{i-1}\}$$

or $M_i = \Phi(M_{i-1} \otimes T^M)$.

Thus M_i is the image of the compact space $M_{i-1} \otimes T^M$ obtained by the continuous transformation Φ. Thus M_i is compact (see [3]). Note that the continuity of Φ follows from the assumption that on $\tilde{T}^M$ a topology from T^M is introduced.

According to the definition of M_i, Lemma 4 and the footnote 2) the sets M_i form a sequence of sets which are closed and have the property that $M_i \subset M_{i-1}$ for all i. Since M is a compact topological space, it holds (see [3])

$$\bigcap_{i=0}^{\infty} M_i \neq \emptyset .$$

Definition 1. *The automaton $\mathscr{P} = \{M; T^M\}$ is called contractive if $\bigcap_{i=0}^{\infty} M_i = \{\hat{x}\}$. The point $\hat{x}$ is then called the stationary state of the automaton $\mathscr{P}$.*

Lemma 5. *Let $f(x)$, $\varphi(x)$ be two solutions of the equation (1), let for all $x \in M_{n+1}$ be*

$$|f(x) - \varphi(x)| < \varepsilon .$$

Then $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in M$.

Proof. The functions $f(x)$, $\varphi(x)$ satisfy the equation (1), thus

$$f(x) = \max_{y \in T^M} [\theta(x, y) + f(y(x))] , \tag{2}$$

$$\varphi(x) = \max_{y \in T^M} [\theta(x, y) + \varphi(y(x))] . \tag{3}$$

Let us assume that $x \in M_n$. Then $y(x) \in M_{n+1}$ and according to our assumption

$$|f(y(x)) - \varphi(y(x))| < \varepsilon .$$

Using Lemma 1 we obtain for $x \in M_n$

$$|f(x) - \varphi(x)| = |\max_{y \in T^M} [\theta(x, y) + f(y(x))] - \max_{y \in T^M} [\theta(x, y) + \varphi(y(x))]| \leqq$$

$$\leqq \max_{y \in T^M} |f(y(x)) - \varphi(y(x))|$$

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[2]) Note that every compact set in an H-space is closed. Thus we prove here that M_i is also a closed subset of M for an arbitrary i.

and further it is

$$\max_{y \in T^M} |f(y(x)) - \varphi(y(x))| \leqq \max_{x \in M_{n+1}} |f(x) - \varphi(x)| < \varepsilon .$$

So we have obtained that $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in M_n$. The procedure applied, n-times yields that $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in M$ q.e.d.

Lemma 6. *Let $\mathcal{P} = \{M; T^M\}$ be a contractive automaton and let $\hat{x}$ be its stationary state. Then for an arbitrary neighbourhood U of the point $\hat{x}$ there is an index n_0 such that for $n \geqq n_0$ there is $M_n \subset U$.*

Proof. Let us introduce the notation

$$G_n = M - M_n \quad \text{for} \quad n = 1, 2, \dots$$

Then

$$\bigcup_{n=1}^{\infty} G_n = M - \{\hat{x}\} .$$

Thus the sets G_n for $n = 1, 2, \dots$ and the set U form an opened cover of the compact space M. There exists a finite subcover of this cover which covers M as well. Let this finite subcover be formed by the sets $G_{n_1}, G_{n_2}, \dots, G_{n_e}, U$ where $n_1, n_2, \dots, n_e$ and let $n_0 = n_e$. Then $\bigcup_{k=1}^{e} G_{n_k} \cup U = M$ and further $G_{n_e} = \bigcup_{k=1}^{e} G_{n_k} \supset M - U$ and thus $M_{n_e} \subset U$. For $M_{n_e} \supset M_{n_e+1} \supset M_{n_e+2} \supset \dots$, there holds:

$$M_n \subset U \quad \text{for} \quad n \geqq n_e = n_0 \, .$$

q.e.d.

Theorem 1. *Let $\mathcal{P} = \{M; T^M\}$ be a contractive automaton and let $\hat{x}$ be its stationary state. Then there exists one and only one solution $f(x)$ of the functional equation (1), which is continuous on M and satisfies the condition $f(\hat{x}) = c$ where c is an arbitrary given real number.*

Proof. First we shall prove the existence of such a solution of the equation (1). We know that there exists at least one solution of this equation (see the considerations when deducing equation (1)). We denote this solution by $\tilde{f}(x)$. For this solution need not be $\tilde{f}(\hat{x}) = c$. Let us assume that $\tilde{f}(\hat{x}) = c_1 \neq c$. The function $f(x) = \tilde{f}(x) + c - c_1$ represents also a solution of (1) and moreover it is $f(\hat{x}) = c$. Thus we have proved the existence of the solution $f(x)$ of (1) for which $f(\hat{x}) = c$. Now we prove the uniqueness of such a solution. We use here the proof by contradiction. Let us assume that there are two solutions of (1) $f(x)$, $\varphi(x)$ such that $f(x) \not\equiv \varphi(x)$, $f(\hat{x}) = \varphi(\hat{x}) = c$ and that $f(x)$, $\varphi(x)$ are continuous on M. Thus $f(x)$, $\varphi(x)$ are continuous also at the point $\hat{x}$. Thus for an arbitrary $\varepsilon > 0$ there exists such a neighbourhood U_ε of the point $\hat{x}$ that for all $x \in U_\varepsilon$

$$|f(x) - f(\hat{x})| < \tfrac{1}{2}\varepsilon \quad \text{and} \quad |\varphi(x) - \varphi(\hat{x})| < \tfrac{1}{2}\varepsilon$$

or

$$|f(x) - c| < \tfrac{1}{2}\varepsilon \quad \text{and} \quad |\varphi(x) - c| < \tfrac{1}{2}\varepsilon .$$

Then for $x \in U_\varepsilon$ it is

$$|f(x) - \varphi(x)| = |f(x) - c + c - \varphi(x)| \leqq |f(x) - c| + |\varphi(x) - c| < \varepsilon .$$

According to Lemma 6 there exists such an index n_0 that for $n \geqq n_0$ there is $M_n \subset U_\varepsilon$. Thus $|f(x) - \varphi(x)| < \varepsilon$ for $x \in M_{n_0+1}$. Then following Lemma 5, $|f(x) - \varphi(x)| < \varepsilon$ holds for all $x \in M$. Taking into account that ε is an arbitrary positive real number we come to the conclusion that $f(x) = \varphi(x)$ for all $x \in M$. But this is a contradiction with the assumpton that $f(x) \not\equiv \varphi(x)$. This contradiction proves our Theorem.

Theorem 2. *Let $\mathscr{P} = \{M; T^M\}$ be a contractive automaton with the stationary state $\hat{x}$. Let $f(x)$ be the solution of (1), continuous on M and such that $f(\hat{x}) = c$ where c is an arbitrary given real number. Let be given a functional sequence $\{f^n(x)\}$ which is constructed as follows:*

(1) $f^0(x)$ is an arbitrary function continuous on M and such that $f^0(\hat{x}) = c$;

(2) $f^{n+1}(x) = \max_{y \in T^M} [\theta(x, y) + f^n(y(x))]$ for $n = 0, 1, 2, \ldots$

Then this sequence is uniformly convergent to the solution $f(x)$ of the equation (1)[3].

Proof. According to Lemma 1 it is for all $x \in M$

$$|f(x) - f^{n+1}(x)| = |\max_{y \in T^M} [\theta(x, y) + f(y(x))] - \max_{y \in T^M} [\theta(x, y) + f^n(y(x))]| \leqq$$
$$\leqq \max_{x \in M_1} |f(x) - f^n(x)| .$$

This inequality holds for all $x \in M$, thus it must hold also for those x where the function on the left hand side of this inequality assumes its maximum, i.e.:

$$\max_{x \in M} |f(x) - f^{n+1}(x)| \leqq \max_{x \in M_1} |f(x) - f^n(x)| .$$

Analogously we obtain

$$\max_{x \in M_1} |f(x) - f^n(x)| \leqq \max_{x \in M_2} |f(x) - f^{n-1}(x)|$$

and thus

$$\max_{x \in M} |f(x) - f^{n+1}(x)| \leqq \max_{x \in M_2} |f(x) - f^{n-1}(x)| .$$

Now we shall this successively diminish the index n in this way and estimate the differences $|f(x) - f^k(x)|$ for $k = n - 2, n - 3, \ldots$ until we come to the difference $|f(x) - f^0(x)|$ and to the relation:

$$\max_{x \in M} |f(x) - f^{n-1}(x)| \leqq \max_{x \in M_{n+1}} |f(x) - f^0(x)| . \tag{4}$$

[3]) Such a solution exists according to Theorem 1.

The functions $f(x), f^0(x)$ are continuous on M and thus they must be continuous also at the point $\hat{x}$. Therefore for an arbitrary $\varepsilon > 0$ there exists a neighbourhood U of the point $\hat{x}$ such that for all $x \in U$

$$|f^0(x) - f^0(\hat{x})| = |f^0(x) - c| < \tfrac{1}{2}\varepsilon$$

and

$$|f(x) - f(\hat{x})| = |f(x) - c| < \tfrac{1}{2}\varepsilon .$$

Thus

$$|f(x) - f^0(x)| \leqq |f(x) - c| + |f^0(x) - c| < \varepsilon .$$

Now we choose such an index n_0 that for $n \geqq n_0$ it is $M_n \subset U$ [4]). From the relation (4) it follows

$$\max_{x \in M} |f(x) - f^{n+1}(x)| < \varepsilon \quad \text{for} \quad n \geqq n_0 .$$

Since the choice of the index n_0 depends only on ε and does not depend on x, we have just proved that the sequence $\{f^n(x)\}$ is uniformly convergent to the solution $f(x)$ of the equation (1) q.e.d.

Note 2. Further, this theory can be developed in the following two directions: The first one consists in giving the objective function more concrete forms and keeping all its other properties which are necessary for the validity of the above described theory. The second one consists in further generalization of the above described theory to a larger set of objective functions (for instance we can require that the relation $\Psi(X) - \Psi(P^k X) = \theta(x_0, y_0, y_1, \ldots, y_{2k-2})$ holds instead of $\Psi(X) - \Psi(PX) = \theta(x_0, y_0)$), or to the case of an automaton whose set of transformations depends on the state of the automaton. These further generalizations can be found in [5] and some of them will be the subject of some other publication.

PART II.

6. SPECIAL FORM OF THE FUNCTION $\Psi(X)$

Lemma 7. *Let* $\Psi(X) = \sum_{i=0}^{\infty} \theta_i(x_i, y_i)$ *where* $x_i = y_{i-1}(x_{i-1})$ *for* $i = 1, 2, \ldots,$ *let the series* $\sum_{i=0}^{\infty} |\theta_i(x_i, y_i)|$ *be uniformly convergent and let the functions* $\theta_i(x_i, y_i)$ *be continuous on* $M \otimes T^M$ *for all* i. *Then the function* $\Psi(X)$ *is continuous on* $\mathfrak{M}$.

Proof. Let $\bar{X} = (\bar{x}_0, \bar{y}_0, \bar{y}_1, \ldots)$ be an arbitrary but fixed point of the space $\mathfrak{M}$. We must prove that for an arbitrary $\varepsilon > 0$ there exists such a neighbourhood $\bar{O}$ of

[4]) This can be done according to Lemma 6.

the point $\bar{X}$ that if $X \in \bar{O}$ then $|\Psi(X) - \Psi(\bar{X})| < \varepsilon$ or

$$\left|\sum_{i=0}^{\infty} \theta_i(x_i, y_i) - \sum_{i=0}^{\infty} \theta_i(\bar{x}_i, \bar{y}_i)\right| < \varepsilon.$$

Let be given an $\varepsilon > 0$. The series $\sum_{i=0}^{\infty} |\theta_i(x_i, y_i)|$ is uniformly convergent, thus for this $\varepsilon > 0$ there exists such an index n_0 that for $n > n_0$ there is $\sum_{i=n}^{\infty} |\theta_i(x_i, y_i)| < \frac{1}{4}\varepsilon$ and at the same time $\sum_{i=n}^{\infty} |\theta_i(\bar{x}_i, \bar{y}_i)| < \frac{1}{4}\varepsilon$. Further

$$|\Psi(X) - \Psi(\bar{X})| \leqq \sum_{i=0}^{\infty} |\theta_i(x_i, y_i) - \theta_i(\bar{x}_i, \bar{y}_i)| \leqq \sum_{i=0}^{n_0} |\theta_i(x_i, y_i) - \theta_i(\bar{x}_i, \bar{y}_i)| +$$

$$+ \sum_{i=n_0+1}^{\infty} |\theta_i(x_i, y_i)| + \sum_{i=n_0+1}^{\infty} |\theta_i(\bar{x}_i, \bar{y}_i)| < \sum_{i=0}^{n_0} |\theta_i(x_i, y_i) - \theta_i(\bar{x}_i, \bar{y}_i)| + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon .$$

The functions $\theta_i(x_i, y_i)$ are for all i continuous at the point $(\bar{x}_i, \bar{y}_i) \in M \otimes T^M$. Therefore for an arbitrarily chosen $\varepsilon > 0$ there exists such a neighbourhood $\bar{O}_i$ of the point $(\bar{x}_i, \bar{y}_i)$ that the following implication holds:

$$(x_i, y_i) \in \bar{O}_i \quad \text{implies} \quad |\theta_i(x_i, y_i) - \theta_i(\bar{x}_i, \bar{y}_i)| < \frac{\varepsilon}{2(n_0 + 1)} .$$

Let $\bar{O}_i$ be generated (in the sense of Tichonoff's topology on $M \otimes T^M$) by the neighbourhoods $O^i_{\bar{x}_0}$ of the point $\bar{x}_0$ and $\bar{O}_{\bar{y}_i}$ of the point $\bar{y}_i$ [5]. Let $\bar{O}_{\bar{x}_0} = \bigcap_{i=0}^{n_0} O^i_{\bar{x}_0}$ and let $\bar{O}_{n_0}$ be the neighbourhood of the point $\bar{X} \in \mathfrak{M}$. which is generated by the neighbourhoods $O_{\bar{x}_0}$, $O_{\bar{y}_i}$ for $i = 0, 1, \ldots, n_0$. Then for $X \in O_{n_0}$ there is

$$|\Psi(X) - \Psi(\bar{X})| < (n_0 + 1) \frac{\varepsilon}{2(n_0 + 1)} + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon = \varepsilon \quad \text{q.e.d.}$$

Further we shall assume in this paragraph that the function $\Psi(X)$ satisfies all the assumptions of Lemma 5 and furthermore that this function is the objective function of the automaton $\mathscr{P} = \{M; T^M\}$. We define functions $\Psi_r(X), f_r(x)$ for $r = 0, 1, 2, \ldots$ as follows:

$$\Psi_r(X) = \sum_{i=0}^{\infty} \theta_{i+r}(x_i, y_i), \quad f_r(x_0) = \max_{X \in \mathfrak{M}(x_0)} \Psi_r(X) ;$$

here $X = (x_0, y_0, y_1, \ldots) \in \mathfrak{M}$, $x_0 \in M_r$. Let us note that $\Psi_0(X) = \Psi(X)$, $f_0(x_0) = f(x_0)$.

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[5]) This may be supposed because for an arbitrary point $X = (x_0, y_1, y_2, \ldots) \in \mathfrak{M}$ it is $\Theta_i(x_i, y_i) = \Theta_i(y_{i-1}(y_{i-2} \cdots (y_0(x_0) \ldots), y_i)$ and this is a continuous function on $M \otimes T^M$.

Analogously as in the preceding paragraph we can deduce that the functions $f_r(x)$ satisfy for $x \in M_r$ and $r = 0, 1, 2, \ldots$ the following system of equations:

(5) $$f_r(x) = \max_{y \in T^M} [\theta_r(x, y) + f_{r+1}(y(x))] \quad r = 0, 1, 2, \ldots$$

Lemma 8. *Let $\mathscr{P} = \{M; T^M\}$ be a contractive automaton with the stationary state $\hat{x}$. Then $y(\hat{x}) = \hat{x}$ for all $y \in T^M$.*

Proof. Let $\bar{y}(\hat{x}) = x_1 \neq \hat{x}$ for some $\bar{y} \in T^M$. M is Hausdorff's space, therefore there exists such a neighbourhood $\hat{O}$ of the point $\hat{x}$ and such a neighbourhood O_1 of the point x_1 that $O \cap O_1 = \emptyset$. The automaton $\mathscr{P}$ is contractive, thus in accordance with Lemma 6 there exists such an index n_0 that $M_{n_0} \subset \hat{O}$. From the definition of the set M_{n_0} it follows that $\hat{x} \in M_{n_0}$ and $\bar{y}(\hat{x}) \in M_{n_0+1} \subset \hat{O}$. On the other hand it is $x_1 = \bar{y}(\hat{x}) \in O_1$ and thus $\bar{y}(\hat{x}) \notin \hat{O}$, which is a contradiction. This completes the proof.

Lemma 9. *Let $\mathscr{P} = \{M; T^M\}$ be a contractive automaton with the stationary state $\hat{x}$. Let the objective function $\Psi(X)$ of the automaton $\mathscr{P}$ satisfy all the assumptions of Lemma 7. Let* $\max \theta_r(\hat{x}, y) = 0$ *for* $r = 0, 1, 2, \ldots$ *Then* $f_r(\hat{x}) = f_{r+1}(\hat{x})$ for $r = 0, 1, 2, \ldots$

Proof. For an arbitrary r it is:

$$f_r(\hat{x}) = \max_{y \in T^M} [\theta_r(\hat{x}, y) + f_{r+1} y(\hat{x})]$$

$$f_r(\hat{x}) = \max_{y \in T^M} [\theta_r(\hat{x}, y) + f_{r+1}(\hat{x})]$$

$$f_r(\hat{x}) = \max_{y \in T^M} [\theta_r(\hat{x}, y)] + f_{r+1}(\hat{x})$$

$$f_r(\hat{x}) = f_{r+1}(\hat{x}) \quad \text{q.e.d.}$$

Lemma 10. *Let $\{f_r\}$, $\{\varphi_r\}$ be two continuous solutions of the system of equations (5). Let for some r_0 be $|f_{r_0}(x) - \varphi_{r_0}(x)| < \varepsilon$ for $x \in M_{r_0}$. Then it is $|f_r(x) - \varphi_r(x)| < \varepsilon$ for $x \in M_r$ and for all $r \leqq r_0$.*

Proof. Let be $x \in M_{r_0-1}$. Then

$$f_{r_0-1}(x) = \max_{y \in T^M} [\theta_{r_0-1}(x, y) + f_{r_0}(y(x))],$$

$$\varphi_{r_0-1}(x) = \max_{y \in T^M} [\theta_{r_0-1}(x, y) + \varphi_{r_0}(y(x))].$$

It follows from our assumptions that

$$|f_{r_0}(y(x)) - \varphi_{r_0}(y(x))| < \varepsilon$$

for all $y \in T^M$ and therefore also $|f_{r_0-1}(x) - \varphi_{r_0-1}(x)| < \varepsilon$ (according to Assertion

2). Repeating this procedure r_0-times we come to the conclusion that for $x \in M_r$ and for $r = 0, 1, 2, \ldots, r_0$ it is

$$|f_r(x) - \varphi_r(x)| < \varepsilon$$

q.e.d.

Theorem 3. *Let $\mathscr{P} = \{M; T^M\}$ be a contractive automaton with the stationary state $\hat{x}$. Let the objective function Ψ of the automaton $\mathscr{P}$ fulfil all the assumptions from Lemma 7 and let $\max_{y \in T^M} \theta_r(\hat{x}, y) = 0$ for all r. Then there exists one and only one solution of the system (5) which is continuous and has the property $f_r(\hat{x}) = c$ for all r (c is a given real number).*

Proof. First we prove the existence of such a solution of system (5). Since a continuous function always assumes its maximum on a compact space, the existence of at least one solution $\{\bar{f}_r(x)\}$ of (5) is clear. But this solution need not in general satisfy the condition that $\bar{f}_r(\hat{x}) = c$ for all r. Let us assume that $\bar{f}_r(\hat{x}) = c_1 \neq c$ for all r.[6]) Then the functions $f_r(x) = \bar{f}_r(x) + c - c_1$ for $r = 0, 1, 2, \ldots$ represent also a solution of (5) and it is $f_r(\hat{x}) = c$ for all r. This completes the proof of existence of at least one solution $\{f_r\}$ of (5) satisfying the condition that $f_r(\hat{x}) = c$ for all r. Now we must prove the uniqueness of this solution of (5). Let $\{f_r(x)\}$, $\{\varphi_r(x)\}$ be two continuous (on M) solutions of (5) and let them satisfy the condition, that $f_r(\hat{x}) = c$, $\varphi_r(\hat{x}) = c$ for all r. It follows from the continuity at the point $\hat{x}$ that for an arbitrary $\varepsilon > 0$ there exists such a neighbourhood O_r of the point $\hat{x}$ that for each r and for $x \in O_r \cap M_r$ there is

$$|f_r(x) - f_r(\hat{x})| = |f_r(x) - c| < \tfrac{1}{2}\varepsilon\,,$$

$$|\varphi_r(x) - \varphi_r(\hat{x})| = |\varphi_r(x) - c| < \tfrac{1}{2}\varepsilon$$

and therefore

$$|f_r(x) - \varphi_r(x)| < \varepsilon\,.$$

Since the automaton $\mathscr{P}$ is contractive, there exists according to Lemma 6 such an index $\bar{n}_0$ that $M_{\bar{n}_0} \subset O_r$. Thus for $x \in M_n$ and for $n \geqq n_0 = \max(\bar{n}_0; r)$ it is $|f_n(x) - \varphi_n(x)| < \varepsilon$ and thus in accordance with Lemma 10 for $x \in M_r$ it is $|f_r(x) - \varphi_r(x)| < \varepsilon$ for all r. Since ε is an arbitrarily chosen real positive number, it follows from these relations that $f_r(x) \equiv \varphi_r(x)$ for $x \in M_r$ and for all r, q.e.d.

Theorem 4. *Let all the assumptions of Theorem 3 be fulfilled. Let $\{f_r(x)\}$ be a continuous solution of the system (5) with the property $f_r(\hat{x}) = c$ for all r (c is a given real number).[7]) For each r let be given a sequence of functions $\{f_r^n(x)\}$ constructed as follows:*

[6]) This may be assumed since according to Lemma 9 $f_r(\hat{x}) = f_{r+1}(\hat{x})$ for all r.

[7]) Such a solution exists and is unique according to Theorem 3.

(1) $f_r^0(x)$ *are for all* r *arbitrary continuous functions defined on* M_r *and such that* $f_r^0(\hat{x}) = c$ *for all* r;

(2) $f_r^{n+1}(x) = \max_{y \in T^M} [\theta_r(x, y) + f_{r+1}^n(y(x))]$ *for* $x \in M_r$ *and* $n = 0, 1, 2, \ldots$

Then the sequence $\{f_r^n(x)\}$ *is uniformly convergent to the function* $f_r(x)$ *for all* r.

Proof. Let x be an arbitrary point from M_r for some fixed r. According to Assertion 2 it is

$$|f_r(x) - f_r^{n+1}(x)| = |\max_{y \in T^M} [\theta_r(x, y) + f_{r+1}(y(x))] - \max_{y \in T^M} [\theta_r(x, y) + f_{r+1}^n(y(x))]) \leqq$$
$$\leqq \max_{x \in M_{r+1}} |f_{r+1}(x) - f_{r+1}^n(x)|$$

and thus also

$$\max_{x \in M_{r+1}} |f_{r+1}(x) - f_{r+1}^n(x)| \leqq \max_{x \in M_{r+2}} |f_{r+2}(x) - f_{r+2}^{n-1}(x)|$$

so that we have

$$\max_{x \in M_r} |f_r(x) - f_r^{n+1}(x)| \leqq \max_{x \in M_{r+2}} |f_{r+2}(x) - f_{r+2}^{n-1}(x)| .$$

We diminish the index n in this way and estimate the differences

$$|f_{r+i}(x) - f_{r+i}^{n-i+1}(x)| \quad \text{for} \quad i = 2, 3, 4, \ldots$$

until we come to the inequality

$$\max_{x \in M_r} |f_r(x) - f_r^{n+1}(x)| \leqq \max_{x \in M_{r+n+1}} |f_{r+n+1}(x) - f_{r+n+1}(x)| .$$

The functions $f_{r+n+1}(x)$, $f_{r+n+1}^0(x)$ are continuous for all r on M_{r+n+1}, thus they are continuous also at the point $\hat{x}$. Therefore for an arbitrary $\varepsilon > 0$ there exists such a neighbourhood O_{r+n+1} of the point $\hat{x}$ that if $x \in O_{r+n+1}$ then it is

$$|f_{r+n+1}(x) - f_{r+n+1}(\hat{x})| = |f_{r+n+1}(x) - c| < \tfrac{1}{2}\varepsilon$$

and at the same time

$$|f_{r+n+1}^0(x) - f_{r+n+1}^0(\hat{x})| = |f_{r+n+1}^0(x) - c| < \tfrac{1}{2}\varepsilon$$

and thus

$$|f_{r+n+1}(x) - f_{r+n+1}^0(x)| < \varepsilon .$$

Since the automaton $\mathscr{P}$ is contractive there exists such an index n_0 that for $k \geqq n_0$ there is $M_k \subset O_{r+n+1}$. We choose for our given fixed r such an index n that $r + n + 1 \geqq n_0$. Then

$$\max_{x \in M_r} |f_r(x) - f_r^{n+1}(x)| \leqq \max_{x \in M_{r+n+1}} |f_{r+n+1}(x) - f_{r+n+1}^0(x)| < \varepsilon .$$

Since the choice of the index n_0 is independent of x, we have also proved the uniform convergence of the sequence $\{f_r^n(x)\}$ to the function $f_r(x)$ for all r. This completes the proof.

Note 3. We have mentioned in Note 2 the possibility of extending the above described theory to the case of an automaton with a variable transformation set. The realization of this extension is a rather complicated task which we will not treat here. In this paper we shall consider only a special case of it, namely

$$\mathfrak{M} = \mathfrak{M}_n = M \otimes T^M \otimes T^M \otimes \ldots \otimes T^M \otimes \{\hat{y}\} \otimes \{\hat{y}\} \otimes \ldots$$

where $\{\hat{y}\}$ is a set consisting of the unique element $\hat{y}$ which represents the identical transformation. Furthermore we shall assume in this special case that $\Psi(X) = \sum_{k=0}^{n-1} \theta_k(x_k, y_k)$, where $\theta_k(x_k, y_k)$ are for all k continuous functions on $M \otimes T^M$.

Under the assumptions mentioned above in Note 3 $\mathfrak{M}_n$ is a compact topological Hausdorff's space, $\Psi(X)$ is continuous on $\mathfrak{M}_n$ and $M_n = \{\hat{x}\}$. Let $X = (x_0, y_0, y_1, \ldots \ldots, y_{n-1}, \hat{y}, \hat{y}, \ldots)$ be an arbitrary point of $\mathfrak{M}_n$. We define X^m for $m = 0, 1, \ldots, n-1$ as follows:

$$X^m = P^m X = P(P(\ldots(PX)\ldots)) = y_{m-1}(y_{m-2}(\ldots(y_0(x_0))\ldots)), y_{m+1}, \ldots, y_{n-1}, \hat{y}, \hat{y}, \ldots) = \\ = (x_m, y_{m+1}, \ldots, y_{n-1}, \hat{y}, \hat{y}, \ldots)$$

where $x_m \in M_m$. Further we define for $i = 0, 1, \ldots, n-1$

$$\mathfrak{M}_{n,i} = \underbrace{M_i \otimes T^M \otimes \ldots \otimes T^M}_{(n-i)\text{ factors}} \otimes \{\hat{y}\} \otimes \{\hat{y}\} \otimes \ldots \otimes \{\hat{y}\} \otimes \ldots$$

Consequently $\mathfrak{M}_{n,0} = \mathfrak{M}_n$. We put

$$\mathfrak{M}_{n,i}^{(x)} = \{X; X \in \mathfrak{M}_{n,i}, NX = x\}.$$

Let us note that if it is $X \in \mathfrak{M}_{n,i}$, then it must be $NX \in M_i$ for $i = 0, 1, \ldots, n-1$. We define the functions $\Psi_m(X^m)$ and $f_m(x)$ for $m = 0, 1, \ldots, n-1$ as follows

$$\Psi_m(X^m) = \sum_{k=m}^{n-1} \theta_k(x_k, y_k), \quad f_m(x) = \max_{X^m \in \mathfrak{M}^{(x)}_{n,m}} \Psi_m(X^m).$$

Hence we have

$$\Psi_0(X^0) = \Psi(X), \quad f_0(x) = f(x).$$

Then $\Psi_m(X^m)$ for all m are continuous on $\mathfrak{M}_{n,m}$, therefore $f_m(x)$ for all m are continuous on M_m and satisfy the following system of n functional equations

$$f_m(x) = \max_{y \in T^M} [\theta_m(x, y) + f_{m+1}(y(x))], \quad m = 0, 1, \ldots, n-1$$

The method of solution of this system was suggested in [1]. It is the so called factorization method. We now describe its main idea.

It is $M_n = \{\hat{x}\}$, $y(\hat{x}) = \hat{x}$ for all $y \in T^M$ and $f_n(x) = \text{const}$ on M_n. Further it is

$$f_{n-1}(x) = \max_{y \in T^M} [\theta_{n-1}(x, y) + f_n(y(x))] \quad \text{for} \quad x \in M_{n-1}$$

or in general

$$f_{n-k}(x) = \max_{y \in T^M} [\theta_{n-k}(x, y) + f_{n-k+1}(y(x))]$$

for $x \in M_{n-k}$ and for $k = 1, 2, \ldots, n$.

By means of this procedure we obtain the functions $f_{n-k}(x)$ defined on M_{n-k} for $k = 1, 2, \ldots, n$ and the transformations $y_{n-1}, y_{n-2}, \ldots, y_0$ at which the maximum is reached in each equation. Now making use of these transformations we can obtain the corresponding states from the relations $x_i = y_{i-1}(x_{i-1})$ for $i = 1, 2, \ldots, n$.

Now we illustrate the theory just described by a short example.

Example 1. Let M be the closed interval $\langle 0; 1 \rangle$, then $\tilde{T}^M$ is the set of all continuous transformations of the interval $\langle 0; 1 \rangle$ into itself and let T^M be the set of transformations of the form

$$y_t(x) = atx + (1 - t)\, bx$$

where a, b are fixed given constants such that $0 \leqq a < 1$, $0 \leqq b < 1$, $a \geqq b$ and and t is some real number of $\langle 0; 1 \rangle$. Then it is $T^M \subset \tilde{T}^M$. We introduce the topology on $\tilde{T}^M$ by means of the following metric:

$$\varrho(y_1, y_2) = \max_{x \in M} |y_1(x) - y_2(x)| \quad \text{for} \quad y_1, y_2 \in T^M .$$

Now we must prove that such a topology introduced on $\tilde{T}^M$ belongs to $\boldsymbol{T}^M$. It is known from the literature that each metric space is Hausdorff's space and that ϱ is a metric. Thus $\tilde{T}^M$ is Hausdorff's space. Now it is necessary to prove that the transformation $\Phi(x, y) = y(x)$ of $M \otimes \tilde{T}^M$ into M is continuous on $M \otimes \tilde{T}^M$ (with regard to Tichonoff's topology on $M \otimes \tilde{T}^M$). Let (x_0, y_0) be an arbitrarily chosen fixed point from $M \otimes \tilde{T}^M$. We want to prove that to an arbitrary chosen $\varepsilon > 0$ there exists a neighbourhood U_ε of $(x_0, y_0) \in M \otimes \tilde{T}^M$ such that if $(x, y) \in U_\varepsilon$ then there is $|y_0(x_0) - y(x)| < \varepsilon$. Let $\varepsilon > 0$ be given. Then

$$|y_0(x_0) - y(x)| = |y_0(x_0) - y_0(x) + y_0(x) - y(x)| \leqq$$
$$\leqq |y_0(x_0) - y_0(x)| + |y_0(x) - y(x)| .$$

It follows from the continuity of y_0 at the point x_0 that for the given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then it is $|y_0(x_0) - y_0(x)| < \frac{1}{2}\varepsilon$. Let us suppose that

$$y \in O_{y_0} = \{y, \varrho(y, y_0) < \tfrac{1}{2}\varepsilon, y \in \tilde{T}^M\} , \quad x \in O_{x_0} = \{x; |x - x_0| < \delta, x \in M\}$$

and let U_ε be the neighbourhood of $(x_0, y_0) \in M \otimes T^M$ generated by the neighbourhoods O_{x_0}, O_{y_0}. Then the following implication holds:

$$(x, y) \in U_\varepsilon \quad \text{implies} \quad |y_0(x_0) - y(x)| < \varepsilon .$$

Thus we have just proved that the topology introduced on $\tilde{T}^M$ belongs to $\mathbf{T}^M$.

Now we shall prove that T^M is a compact topological subspace of $\tilde{T}^M$. It suffices to prove that the set T^M is uniformly bounded and equally continuous (in accordance with Arzela's theorem). For an arbitrary $y_t(x) \in T^M$ it is:

$$|y_t(x)| = |atx + b(1 - t)\,x| \leqq |(a - b)\,t + b| \leqq a < 1 .$$

Thus the set T^M is uniformly bounded. Further for arbitrary two functions $y_{t_1}, y_{t_2} \in T^M$ we have:

$$|y_{t_1}(x_1) - y_{t_2}(x_2)| = |at_1x_1 + (1 - t_1)\,bx_1 - at_2x_2 - (1 - t_2)\,bx_2| \leqq$$
$$\leqq |ax_1 - bx_2| = K|x_1 - x_2| \quad \text{where} \quad K = \max\,(a; b) = a .$$

Now if it is $|x_1 - x_2| < \delta = \varepsilon/K$ for an arbitrarily chosen $\varepsilon > 0$, then it is $|y_{t_1}(x_1) - y_{t_2}(x_2)| < \varepsilon$ and thus the set T^M is uniformly continuous. Therefore in accordance with Arzela's theorem T^M is a compact topological subspace of $\tilde{T}^M$. We put

$$\mathfrak{M} = M \otimes T^M \otimes \ldots \otimes T^M \otimes \ldots$$

We introduce now the following notation:

$$y_i(x) = at_ix + b(1 - t_i)\,x \quad \text{for} \quad i = 0, 1, 2, \ldots ;$$

$$x_i = y_{i-1}(x_{i-1}) = at_{i-1}x_{i-1} + b(1 - t_{i-1})\,x_{i-1} \quad \text{for} \quad i = 1, 2, \ldots ;$$

$$X = (x_0, y_0, y_1, \ldots) , \quad \text{then} \quad X \in \mathfrak{M} .$$

Then it is

$$x_i = x_0 \prod_{j=0}^{i-1} [at_j + b(1 - t_j)] .$$

$\mathscr{P} = \{M; T^M\}$ is a contractive automaton with the stationary state $\hat{x} = 0$. Let $h(x)$, $g(x)$ be two functions defined and continuous on $\langle 0; 1\rangle$ and such that $h(0) = g(0) = 0$ and that the series

$$\Psi(X) = \sum_{i=1}^{\infty} ag(x_{i-1})\,t_{i-1} + bh(x_{i-1})\,(1 - t_{i-1})$$

is uniformly convergent.

If $\Psi(X)$ is an objective function of $\mathscr{P}$ then the corresponding dynamic programming problem is the same as that about distribution of sources described in [4]. Further we shall assume that $h(x) = x$ and $g(x) = x$ for all $x \in \langle 0; 1 \rangle$. Then it is

$$\Psi(X) = \sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} x_0 \{ \prod_{j=0}^{i=1} [a t_j + b(1 - t_j)] \}$$

and further

$$\sum_{i=1}^{\infty} |x_i| \leqq \sum_{i=1}^{\infty} x_0 \prod_{j=0}^{i=1} (|(a - b)\, t_j + b|) \leqq \sum_{i=1}^{\infty} x_0 \prod_{j=0}^{i=1} a = \sum_{i=1}^{\infty} x_0 a^i = x_0 \frac{a}{1 - a} .$$

Thus the series $\sum_{i=1}^{\infty} |x_i|$ is uniformly convergent and according to Lemma 7 the function $\Psi(X)$ is continuous on $\mathfrak{M}$. Besides, the function $\Psi(X)$ satisfies the condition that $\Psi(X) - \Psi(PX) = \theta(x_0, y_0)$, because it is

$$\Psi(X) - \Psi(PX) = (a - b)\, t_0 x_0 + b x_0 ,$$

so that

$$\theta(x_0, y_0) = (a - b)\, t_0 x_0 + b x_0 .$$

So we have shown that $\mathscr{P} = \{M; T^M\}$ is a contractive automaton with the stationary state $\hat{x} = 0$. Let $\Psi(X) = \sum_{i=1}^{\infty} x_i$ be the objective function of this automaton. It follows from the relation $a \geqq b$ that it is

$$f(x_0) = \sum_{i=1}^{\infty} x_0 a^i = x_0 \frac{a}{1 - a} .$$

Now we prove that $f(x_0)$ satisfies the functional equation

$$f(x_0) = \max_{y_0 \in T^M} [\theta(x_0, y_0) + f(y_0(x_0))]$$

or

$$f(x_0) = \max_{0 \leqq t_0 \leqq 1} [(a - b)\, t_0 x_0 + b x_0 + f(y_0(x_0))]$$

and the condition $f(\hat{x}) = 0$.

Indeed,

$$f(y_0(x_0)) = a \,.\, y_0(x_0) \,.\, \frac{1}{1 - a} = \frac{a}{1 - a} [(a - b)\, t_0 x_0 + b x_0] ;$$

therefore

$$\max_{0 \leqq t_0 \leqq 1} [\theta(x_0, y_0) + f(y_0(x_0))] = a \,.\, a \,.\, x_0 \frac{1}{1 - a} + a \,.\, x_0 = \frac{a x_0}{1 - a} = f(x_0)$$

q.e.d. Moreover it is $f(\hat{x}) = f(0) = 0 \,.\, a/(1 - a) = 0$ q.e.d.

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Souhrn

ZOBECNĚNÍ ÚLOHY DYNAMICKÉHO PROGRAMOVÁNÍ

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Článek obsahuje pokus o vytvoření exaktní teorie dynamického programování na dostatečně obecném základě.

Nechť M je kompaktní topologický Hausdorffův prostor (dále zkráceně H-prostor), nechť $\tilde{T}^M$ je množina všech spojitých zobrazení tohoto prostoru do sebe. Nechť na množině $\tilde{T}^M$ je zavedena taková topologie, že $\tilde{T}_M$ je vzhledem k této topologii H-prostorem, a že zobrazení $\Phi(x, y) = y(x)$ prostoru $M \otimes \tilde{T}^M$ do M je spojité. Nechť T^M je kompaktní podprostor prostoru $\tilde{T}^M$ a nechť $\mathfrak{M} = M \otimes T^M \otimes \ldots \otimes T^M \otimes \ldots$, přičemž na $\mathfrak{M}$ je zavedena Tichonovova topologie. Nechť $X = (x_0, y_0, y_1, \ldots) \in \in \mathfrak{M}$. Definujeme zobrazení P a N na $\mathfrak{M}$ takto: $PX = (y_0(x_0), y_1, y_2, \ldots)$, $NX = x_0$; nechť $\mathfrak{M}^{(x_0)} = \{X; X \in \mathfrak{M}, NX = x_0\}$, nechť Ψ je spojitá funkce na $\mathfrak{M}$ a nechť $f(x_0) = \max_{X \in \mathfrak{M}^{(x_0)}} \Psi(X)$. Úlohu dynamického programování lze nyní zformulovat takto: Pro všechna $x \in M$ najít prvek (resp. prvky) $\bar{X} \in \mathfrak{M}^{(x)}$, pro nějž (resp. pro něž) platí: $\Psi(\bar{X}) = f(x)$.

V článku se dokazuje existence a jednoznačnost řešení této úlohy a navrhuje se její řešení metodou postupných aproximací pro případ, že $\Psi(X) - \Psi(PX) = \theta(x_0, y_0)$. Dále se řeší úloha dynamického programování v případě, že $\Psi(X) = \sum_{i=1}^{\infty} \theta_i(x_i, y_i)$, kde $\theta_i(x_i, y_i)$ jsou spojité funkce na $M \otimes T^M$, $x_i = y_{i-1}(x_{i-1})$, $i = 1, 2, \ldots$, a řada $\sum_{i=0}^{\infty} |\theta_i(x_i, y_i)|$ stejnoměrně konverguje. Uvádí se malý ilustrativní příklad.

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