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One-Loop Effective Potential in Anti-de Sitter Space

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A scheme is given to compute the one-loop effective potential in a general class of curved spacetimes which are homogeneous and globally static. The renormalized effective potential for scalar field theories in anti-de Sitter background is evaluated by this method. The computation based on the S^p background is disproved by demonstrating that this prescription does not respect the correct boundary condition.

Quantum effects in anti-de Sitter (AdS) space have recently been an important issue in connection with possible breakdown of supersymmetry.[1)~3)] AdS space, which emerges as a natural background geometry in supergravity and Kaluza-Klein theories, is homogeneous and globally static.[*)] These nice properties give this space a special role as a background spacetime.

Several different approaches have been proposed to obtain the one-loop effective potential in AdS space.[3),4)] However, none of them have been demonstrated on a general ground to yield the correct result. In this letter we develop a general scheme to compute the one-loop effective potential for scalar field theories in a homogeneous and globally static spacetime. This scheme is applied to the case of AdS space to evaluate renormalized effective potential. It is demonstrated that the previously known prescription based on the S^p background[3),4)] does not respect the boundary condition in AdS space and yields a result different from that obtained here.

Extension of the present method to supersymmetric theories will be made in a forthcoming letter,[5)] in which we discuss spontaneous breakdown of supersymmetry in the $O(N)$ symmetric Wess-Zumino model in two-dimensional AdS space.

This communication is a brief report of the present work. The details, including some of the proofs of the statements made here, will be reported elsewhere.

We consider scalar field theories in a homogeneous and globally static manifold M (of dimensions $p\geq 2$) with a Lorentizian signature. M is isomorphic to a coset space G/H, where G is the isometry group of M and H is the isotropy group.

To construct a field theory on M, the function space $\mathscr{H}$ on M must be furnished with an inner product. A natural one for scalar fields is

$$(\phi,\psi)\overset{d}{=}\int_M \phi^*(x)\psi(x)\sqrt{|g(x)|}\,dx\,,$$
$$\phi,\psi\in\mathscr{H}\,, \tag{1}$$

where $g(x)$ is the determinant of the metric tensor $g_{\mu\nu}$ on M. The Killing vector fields ξ on M are derivative operators on $\mathscr{H}$. If M is compact, they are anti-selfadjoint with respect to the inner product (1), and hence G is realized as a group of unitary operators on $\mathscr{H}$. In the case of a non-compact manifold M with which we are concerned, ξ becomes anti-selfadjoint, if

$$\phi(x)|g(x)|^{1/4}\sim 0 \text{ (weakly)} \tag{2}$$

at infinities for every $\phi\in\mathscr{H}$.[5)] This condition, in fact, ensures that field quantization can be consistently carried out when M is globally static even though it is not globally hyperbolic.[6),7)]

For a general curved spacetime, the correspondence between the path integral and the canonical quantization is obscure. In the case of a globally static spacetime, perturbative equivalence between these two approaches can be established.[8),9)] Hence the usual trace calculation gives the correct one-loop effective potential. Since the D'Alembertian $\Box$ on a homogeneous spacetime M is a Casimir operator of G, evaluation of the one-loop effective potential involves calculation of a Casimir operator $\hat{O}$ on $\mathscr{H}$. The volume of M (which may be infinite) can be shown to be factored out of $\mathrm{tr}(\hat{O})$. As a consequence, the

*) A manifold is globally static, if it has a global timelike Killing vector and (spacelike) hypersurfaces perpendicular to it.

one-loop effective potential per unit volume is given by

$$V(\phi_0)=-\frac{i}{2}\sum_k\phi^*{}_k(x_0)$$
$$\times[\log(\Box+M^2(\phi_0)-i\varepsilon)\phi_k(x_0)], \qquad (3)$$

where $\{\phi_k\}$ is an orthonormal basis of $\mathcal{H}$. The mass term M^2 depends on the constant background field ϕ_0. The $i\varepsilon$ term is added to choose the correct branch of the logarithm corresponding to the boundary condition of the Feynman propagator.[5)]

The spacetime manifold M in consideration is globally static and hence can be embedded in a larger manifold by complexifying its timelike Killing coordinate. This means that the Lorentzian manifold M can be continued analytically to a Riemannian manifold M_E by the "Wick-rotation" $t\to-i\tau$. The Green function G_E on M_E is unique, and it can be shown[5),8),9)] that G_E gives the Feynman propagator when Wick-rotated back. Therefore perturbation calculations in M_E yield the same results as those on M. The effective potential in M_E is

$$V(\phi_0)=\frac{1}{2}\sum_k\phi_{Ek}{}^*(x_{E0})$$
$$\times[\log(-\Delta_E+M^2{}_{(\phi_0)})\phi_{Ek}(x_{E0})]$$
$$=\frac{1}{2}\int_{M^2(\phi_0)}^{\infty}G_E(x_{E0},x_{E0};m^2)dm^2, \qquad (4)$$

where Δ_E is the Laplacian on M_E, and $\{\phi_{Ek}\}$ is an orthonormal basis of $\mathcal{H}_E$, the function space on M_E. The Killing vectors in M_E are automatically anti-selfadjoint on $\mathcal{H}_E$. Δ_E is a quadratic form in Killing vectors and hence is selfadjoint. In this case, the heat kernel expansion can be employed[8)] to evaluate the divergent terms of $V_E(\phi_0)$. According to the general theory of heat kernels, it can be concluded that the divergent terms of $V_E(\phi_0)$ are polynomials in ϕ_0, and no pathological term appears. This settles the question raised in Ref. 3).

Whereas the general expression of the divergent terms of $V_E(\phi_0)$ can be directly read off from the heat kernel expansion, the computation of the finite remainder will, in general, involves laborious mode sums. In the case of p-dimensional AdS space (AdS^p), however, the Green function in the RHS of Eq. (4) can be obtained in a closed form.

AdS^p space is a single-leaf hyperboloid embedded in R^{p+1},

$$(x^0)^2-(x^1)^2-\cdots-(x^{p-1})^2+(x^p)^2=R^2 \qquad (5)$$

with the metric

$$ds^2=(dx^0)^2-(dx^1)^2-\cdots-(dx^{p-1})^2+(dx^p)^2. \qquad (6)$$

This space is isomorphic to $SO(p-1,2)/SO(p-1,1)$ and has the following parametrization:

$$x^0=R\cosh(r/R)\cos(t/R),$$
$$x^p=R\cosh(r/R)\sin(t/R),$$
$$x^j=R\sinh(r/R)\times(\text{coordinates on } S^{p-2})$$
$$(1\le j\le p-1), \qquad (7)$$
$$ds^2=\cosh^2(r/R)dt^2-dr^2-R^2\sinh^2(r/R)$$
$$\times(\text{line element of } S^{p-1}). \qquad (8)$$

This shows that AdS^p space is globally static and t is a timelike Killing coordinate. In order to unwrap the closed timelike curve, it is customary to consider the universal covering space of AdS^p (CAdS^p). In this case t takes values from $-\infty$ to $+\infty$, instead of 0 to $2\pi R$.

Since an elliptic operator is easier to deal with than a hyperbolic operator, we employ the Wick-rotation and set $t=-i\tau$. This procedure is legitimate because CAdS^p space is globally static. (This should not be confused with the S^p prescription, which we shall discuss later.) If we set $t=-i\tau$ in Eq. (7), CAdS^p space is transformed into a single connected leaf of a two-leaf hyperboloid (EAdS^p).

$$(y^0)^2-(y^1)^2-\cdots-(y^p)^2=R^2. \qquad (9)$$

As can be easily read off from this, EAdS^p space is isomorphic to $SO(p,1)/\mathrm{SO}(p)$. The Green function G_E in EAdS^p satisfies the equation:

$$(\Delta_E-m^2)G_E(x_E,y_E;m^2)=\delta(x_E,y_E). \qquad (10)$$

This equation is a second order ordinary differential equation in terms of σ, the geodesic distance between x_E and y_E. Its solution which ensures the conservation and the positivity of energy[7),10)] when analytically continued back to that in CAdS^p space is

$$G_E=C(p)\cosh^{-2\alpha}\left(\frac{\sigma}{R}\right)F\left(\alpha,\alpha+\frac{1}{2};\right.$$
$$\left.2\alpha+\frac{3}{2}-\frac{1}{2}p;\frac{1}{\cosh^2(\sigma/R)}\right), \qquad (11)$$

where F is the hypergeometric function and $C(p)$ is given by

$$C(p)\equiv-\frac{1}{2^{2\alpha+1}\pi^{(p-1)/2}}\frac{1}{R^{p-2}}\frac{\Gamma(2\alpha)}{(2\alpha+3/2-p/2)}. \qquad (12)$$

In Eq. (11) the dependence on the mass m is contained in α, which is determined such that the appropriate boundary condition at infinite distances $\sigma=\infty$ is met.

$$\alpha=\begin{cases}\frac{1}{4}[p-1+\sqrt{(p-1)^2+4m^2R^2}, \\ \quad \left(m^2R^2>-\frac{1}{4}(p-1)^2\right) \\ \frac{1}{4}[p-1-\sqrt{(p-1)^2+4m^2R^2}]. \\ \quad \left(1-\frac{1}{4}(p-1)^2>m^2R^2>-\frac{1}{4}(p-1)^2\right)\end{cases} \tag{13}$$

The Feynman propagator G_F in CAdS^p space is obtained by Wick-rotating back Eq. (11).

$$G_F(t, r; t'=r'=0; m^2) = C(p)(\cos(t/R)\cosh(r/R))^{-2\alpha} \times F\left(\alpha, \alpha+\frac{1}{2}; 2\alpha+\frac{3}{2}-\frac{1}{2}p; \frac{1}{\cos^2(t/R)\cosh^2(r/R)}-i\varepsilon\right). \tag{14}$$

The $i\varepsilon$ term specifies the branch of the hypergeometric function and ensures that G_F obeys the Feynman boundary condition. This fact confirms the preceding statement that the Wick-rotation from CAdS^p to EAdS^p is legitimate.

We are now ready to compute the effective potential by taking the coincidence limit $x_E \to y_E$ of $G_E(x_E, y_E)$ in Eq. (4). Needless to say, G_E diverges in this limit, thus we temporarily separate x_E from y_E by an infinitesimal geodesic distance. The expansion of Eq. (11) around $\sigma=0$ in the case of even dimensions $p=2n(n=1, 2, 3\cdots)$ is

$$\begin{aligned}G_E(\sigma; m^2)=&-\frac{1}{4\pi^n R^{2(n-1)}}\cosh^{-2\alpha}(\sigma/R) \\ &\times\left\{(n-2)!\sum_{k=0}^{n-2}\frac{(\alpha+1-n)_k\left(\alpha+\frac{3}{2}-n\right)_k}{k!(2-n)_k}\left(\frac{1}{\tanh(\sigma/R)}\right)^{2(n-k-1)}\right. \\ &+\frac{(-)^n}{2^{2n-3}(n-1)!}\frac{\Gamma(2\alpha)}{\Gamma(2(\alpha-n+1))}\left[\log(\tanh(\sigma/R))\right. \\ &\left.\left.+\psi(2\alpha)+\gamma-\frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k}-\log 2\right]\right\}+O(\sigma/R),\end{aligned} \tag{15}$$

where $\psi(x)$ and γ are the poly-gamma function and the Euler constant respectively. A similar expression holds in odd dimensions. For $p=2$, Eq. (15) is reduced to*)

$$G_E(\sigma; m^2)=-\frac{1}{2\pi}\log\left(\frac{R}{\sigma}\right) +\frac{1}{2\pi}\{\psi(2\alpha)+\gamma-\log 2\}+O(\sigma/R). \tag{16}$$

The logarithmically divergent term in Eq. (16) can be handled by the usual renormalization prescription. The finite remainder, when integrated with respect to m^2, gives the renormalized effective potential.

We now look at the behaviour of G_E in the limit $R\to\infty$. For example for $p=4$ Eq. (15) becomes

$$G_E(\sigma; m^2) \underset{R\to+\infty}{=} -\frac{1}{4\pi^2}\frac{1}{\sigma^2}+\frac{m^2}{8\pi^2} \times\left[\log\left(\frac{1}{m\sigma}\right)-\gamma+\frac{1}{2}+\log 2\right]+O(m\sigma). \tag{17}$$

Therefore the effective potential in CAdS^p space has a smooth flat-spacetime limit, contrary to the observation in Ref. 3).

Both AdS^p and EAdS^p can be embedded in a larger complex manifold,

$$(z^0)^2+(z^1)^2+\cdots+(z^p)^2=R^2, \quad z^j\in C. \tag{18}$$

This manifold also contains a p-sphere S^p as its submanifold, and AdS^p (or CAdS^p) space can be transformed into S^p by complexifying its spacelike coordinates by setting $r=\pm i\rho$. Although no justification has been claimed, several authors[3),4)] make use of S^p as a background geometry in the computation of quantum effects in AdS^p space. In order to clarify the relation between this S^p approach and the EAdS^p approach described above, we compare the Green functions in the respective spaces and examine carefully the boundary

*) G_E for $p=2$ has also been derived by Bardeen and Freedman.[2)]

condition satisfied in each case.

If the two approaches were equivalent, the Green function G_F evaluated in CAdS^p would coincide with that in S^p after the analytic continuation. Inserting $r=\pm i\rho$, Eq.(14) becomes

$$G_F(t,\ r=\pm i\rho;\ t'=r'=0;\ m^2)$$
$$=C(p)(\cos(t/R)\cos(\rho/R))^{-2\alpha}$$
$$\times F\left(\alpha,\ \alpha+\frac{1}{2};\ 2\alpha+\frac{3}{2}-\frac{1}{2}p;\ \frac{1}{\cos^2(t/R)\cos^2(\rho/R)}-i\varepsilon\right). \quad (19)$$

Besides the singular point $t=\rho=0$, this function has a singularity at the antipode, which should not be present if Eq. (19) gave the Green function in S^p.

Let us illustrate the disagreement between G_F and G_{S^2} in the case $p=2$. In the coincidence limit G_{S^2} is given by

$$-\frac{1}{2\pi}\ln\left(\frac{R}{\sigma'}\right)-\frac{1}{2\pi}\{\psi(2\alpha)+\gamma-\log 2\}$$
$$+\frac{1}{2}\cot(2\pi\alpha)+O(\sigma/R), \quad (20)$$

where $\sigma'(\rightarrow 0)$ is the geodesic distance in S^2. Of the three terms in Eq. (20), the first two find their counterparts in Eq. (16), whereas the last term does not. This term, $\cot(2\pi\alpha)$, has a nontrivial m^2 dependence and cannot be discarded. In the case of higher dimensions, this kind of disagreement appears in non-leading divergent terms as well.

From these observations, it is concluded that the evaluation of the quantum effects using the S^p background[4] does not incorporate the correct boundary condition in CAdS^p spacetime, and hence is not guaranteed to give the correct result.

The present scheme of computing the effective potential in AdS background spacetime can be extended to supersymmetric theories and enables us to study supersymmetry breaking in AdS space.[5]

In the course of writing this letter, we have received a preprint by Burgess and Lütken,[11] who have reached virtually the same results as ours. However, the approach we employed is different from theirs and is applicable to field theories in other types of homogeneous spaces.

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