Progress of Theoretical Physics, Vol. 79, No. 1, January 1988

One-Loop Mass Shifts in $O(32)$ Open Superstring Theory

Hisashi YAMAMOTO*)

Research Institute for Theoretical Physics
Hiroshima University, Takehara, Hiroshima 725

(Received September 10, 1987)

One-loop amplitudes of $O(N)$ open superstring with emission of massive bosons are studied. Divergences appearing at $\lambda=0$ (λ: the overall Teichmüller parameter) are shown to be canceled if $N=32$ just as in the massless case. We explicitly evaluate the two-point on-shell amplitudes for all the levels of bosons lying on the leading ($m^2=2l, J=l+1$, m: mass J: spin l: level number of an excited state) and the next-to-leading ($m^2=2l, J=l$) Regge trajectories and observe that they are nonvanishing even at $N=32$. This implies that $O(32)$ open superstring one-loop amplitudes with massive bosons generally suffer from external-line divergences. Further the obtained expressions of on-shell self-energies (mass shifts $\delta m^2(l)$) seem to have nontrivial dependences on l (being not proportional to l), although mass degeneracies remain. This strongly suggests that the Regge trajectories form a set of parallel polygonal lines at one-loop level so that the mass shifts cannot be absorbed by the shift of the slope parameter. The divergences would have to be cured by the vertex operator renormalizations at every excited level.

§ 1. Introduction

The finiteness issue is a matter of great importance for superstring theory. In a recent perturbation approach, attentions are largely shifted to the analysis of multiloop amplitudes.[1] It may be too premature to say, however, that we have already acquired the complete understanding of one-loop physics in string theory. A study of one-loop amplitudes is still an important task for the deep understanding of the quantum effect of string theory.

In previous one-loop analyses interest has been mostly focused on amplitudes with only massless external particles.[2] For example, one-loop amplitudes $A(M)$ with any number M of external massless bosons for type-I superstring are found to be finite if the theory posseses $O(N=2^{D/2}=32)$ internal gauge symmetry (D is the space-time dimension$=10$)[3),4)] under an appropriate regularization.[5] In particular, for $M\leq 3$, amplitudes vanish irrespective of the gauge group that is, mass or coupling constant undergoes no radiative correction (nonrenormalization theorem).[6] The divergence in a massless amplitude occurs at the boundary of integration region of the overall Teichmüller parameter (λ for the open and τ for the closed string) and it is interpreted as the divergence which appears when the hole of an annulus or a Möbius strip (or a torus for closed string) surface shrinks to the vacuum.

However, as discussed by Weinberg,[7] there can be another type of divergences which may occur at the boundaries of integration regions of $M-1$ relative Teichmüller parameters ν_i, namely, divergences which appear when the positions of $M-1$ vertex operators are close together. These divergences are proportional to the two-point amplitude evaluated on mass shell, therefore to the mass shift of the external particle. Weinberg argued that these external-line divergences first appear

*) Fellow of the Japan Society for the Promotion of Science.

for the tachyon amplitudes in a bosonic string and that they can be absorbed into the wavefunction renormalization of vertex operators consistent with the one-loop unitarity. In superstring theory, since the massless two-point amplitude vanishes after the sum over spin structure,[8] such a problem does not occur for massless amplitudes.[6] However, once we consider the amplitudes involving massive particles, the problem of external-line divergences seems to be unavoidable even in superstring theory, unless the higher-loop two-point amplitudes miraculously vanish due to the supersymmetry. But we know of no complete calculation demonstrating the (non-)existence of the mass shifts of massive particles in superstring theory[9] or their dependence on the particle species, if any. What we can say at best is that they *can* exist since there is no plausible reason to expect otherwise such as chirality or gauge symmetry for the massless particle cases.

It is one of the purposes of this paper to exemplify the existence of such mass shifts of massive particles and whether massive one-loop amplitudes are finite or not. The mass shifts require a renormalization procedure for removing the divergences in any loop diagrams.[7] Moreover, the detailed renormalization procedure would depend on the properties of the mass shifts, such as their dependence on mass, spin, etc.

After all, in order to establish the one-loop finiteness for the massive amplitudes, two conditions must be satisfied. First, the divergence appearing at the boundary of the overall Teichmüller parameter region must be absent also in massive amplitudes if the gauge group or regularization are so chosen as massless amplitudes could be finite.[2)~5)] Second, two-point amplitudes must vanish for avoiding external-line divergences.[7] In addition to these, meromorphicity of the amplitude, which means the absence of unitarity violating logarithmic terms (non-leading divergences), must be verified.[10]

For simplicity, we consider here the parity-even part of type-I superstring amplitudes including only open strings, in which only manifolds of annulus and Möbius topologies appear as one-loop diagrams. The calculation is based on covariant operator formalism with the help of the old technique due to Clavelli and Shapiro.[3),11)]

In § 2, following a standard procedure of superconformal field theory,[12] vertex operators describing the emissions of all bosons lying on the leading and the next-to-leading Regge trajectories are constructed as the superconformal primary fields with unit conformal weight. In § 3, using the trace reduction technique of Clavelli-Shapiro,[11] we formulate the parity-conserving M-point one-loop amplitudes involving massive external bosons in the Lorentz covariant operator formalism. Section 4 is devoted to the proof of meromorphicity and finiteness at $\lambda=0$ of massive one-loop amplitudes, which is a massive extension of Ref. 3). In § 5, two-point amplitudes of massive bosons lying on all the levels of the two trajectories are explicitly evaluated. Summary and discussion are given in § 6.

§ 2. Vertex operators

A vertex operator $V(z)$ for superstring should be constructed as a superconformal primary field with unit conformal weight, that is, the state $\lim_{z\to 0}[V(z)|0\rangle]$ should be a physical state satisfying the super Virasoro conditions.[12] This condition for $V(z)$ is characterized by operator product expansions (O.P.E) with the super stress energy tensor:

$$
\begin{aligned}
W(z, \theta) &= -\frac{1}{2} D\hat{X}^{\mu} D^2 \hat{X}_{\mu} \\
&= -\frac{1}{2}[\Psi \cdot \partial X + \theta(\partial X \cdot \partial X - \Psi \cdot \partial \Psi)] \\
&= \frac{1}{2} J(z) + \theta T(z), \qquad (2\cdot1)
\end{aligned}
$$

where the superfield $\hat{X}^{\mu}(z, \theta) = X^{\mu}(z) + \theta \Psi^{\mu}(z)$ and the supercovariant derivative $D = \partial_{\theta} + \theta \partial_z$ are introduced. The vertex operator is also a superfield:

$$V(z, \theta) = V^b(z) + \theta V^f(z). \qquad (2\cdot2)$$

In components the required conditions are

$$T(z) V^b(w) \sim \frac{\frac{1}{2}}{(z-w)^2} V^b(w) + \frac{1}{z-w} \partial_w V^b(w), \qquad (2\cdot3)$$

$$J(z) V^b(w) \sim \frac{1}{z-w} V^f(w). \qquad (2\cdot4)$$

The remaining O.P.E's of $V^f(w)$ with $T(z)$ and $J(z)$ are automatically satisfied when Eqs. (2·3) and (2·4) are satisfied. It is not difficult to see this in modes considering that the mode operators of $T(z)$ and $J(z)$ satisfy the superconformal algebra. $V^b(z)$ and $V^f(z)$ should have the conformal weights 1/2 and 1 respectively.

The two series of solutions for (2·3) and (2·4) are given in superfields by

$$
\begin{aligned}
&\zeta_{\mu}: D\hat{X}^{\mu} e^{ik\cdot\hat{X}}(z, \theta):, \qquad \zeta_{\mu\nu}: D\hat{X}^{\mu} \partial \hat{X}^{\nu} e^{ik\cdot\hat{X}}(z, \theta):, \\
&\zeta_{\mu\nu_1\nu_2}: D\hat{X}^{\mu} \partial \hat{X}^{\nu_1} \partial \hat{X}^{\nu_2} e^{ik\cdot\hat{X}}(z, \theta):, \cdots, \text{etc}. \qquad (2\cdot5)
\end{aligned}
$$

and

$$
\begin{aligned}
&\zeta_{\mu_1\mu_2\mu_3}: D\hat{X}^{\mu_1} D\hat{X}^{\mu_2} D\hat{X}^{\mu_3} e^{ik\cdot\hat{X}}(z, \theta):, \\
&\zeta_{\mu_1\mu_2\mu_3\nu}: D\hat{X}^{\mu_1} D\hat{X}^{\mu_2} D\hat{X}^{\mu_3} \partial \hat{X}^{\nu} e^{ik\cdot\hat{X}}(z, \theta):, \\
&\zeta_{\mu_1\mu_2\mu_3\nu_1\nu_2}: D\hat{X}^{\mu_1} D\hat{X}^{\mu_2} D\hat{X}^{\mu_3} \partial \hat{X}^{\nu_1} \partial \hat{X}^{\nu_2} e^{ik\cdot\hat{X}} \\
&\qquad (z, \theta):, \cdots, \text{etc.}, \qquad (2\cdot6)
\end{aligned}
$$

where : : denotes the normal ordering. These vertex operators describe the emissions of bosons lying on the leading ($J=l+1$) and the next-to-leading ($J=l$) trajectories respectively (Fig. 1), where J is

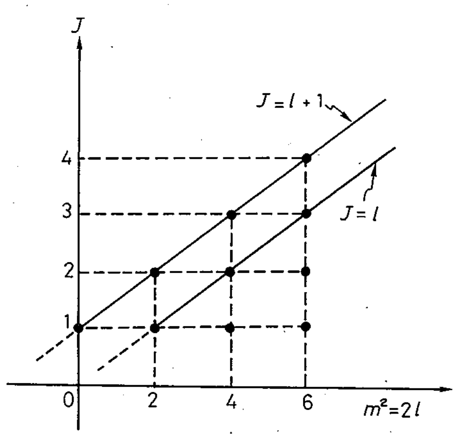

Fig. 1. The leading and the next-to-leading Regge trajectories in open superstring.

the spin of the particle and l is the level number of an excited state ($(\text{mass})^2=m^2=-k^2=2l$). In components ($V^b$: the lower component), solutions are

1. *the leading trajectory* ($m^2=2l, J=l+1, l\geq 0$) $\begin{array}{|c|c|c|c|}\hline \mu & \nu_1 & \cdots & \nu_l \\ \hline \end{array}$

$$V^b_{(2l,l+1)}(z)=\frac{i^{l+1}}{\sqrt{l!}}\zeta_{\mu\nu_1\cdots\nu_l}\Psi^\mu\partial X^{\nu_1}\cdots\partial X^{\nu_l}e^{ik\cdot X}(z) \tag{2·7}$$

with

$$k^\rho\zeta_{\cdot\cdot\rho\cdot\cdot}=\zeta_{\cdot\cdot\rho\cdot\cdot}{}^{\rho}{}_{\cdot\cdot}=0\,, \qquad \zeta_{\mu\nu_1\cdots\nu_l}\zeta^{\mu\nu_1\cdots\nu_l}=1\,, \tag{2·8}$$

and

2. *the next-to-leading trajectory* ($m^2=2l, J=l, l\geq 1$) $\begin{array}{|c|c|c|c|}\hline \mu_1 & \nu_1 & \cdots & \nu_{l-1} \\ \hline \mu_2 \\ \cline{1-1} \mu_3 \\ \cline{1-1} \end{array}$

$$V^b_{(2l,l)}(z)=\frac{i^{l+2}}{\sqrt{3!(l-1)!}}\zeta_{\mu_1\mu_2\mu_3\nu_1\cdots\nu_{l-1}}\Psi^{\mu_1}\Psi^{\mu_2}\Psi^{\mu_3}\partial X^{\nu_1}\cdots\partial X^{\nu_{l-1}}e^{ik\cdot X}(z) \tag{2·9}$$

with

$$k^\rho\zeta_{\cdot\cdot\rho\cdot\cdot}=\zeta_{\cdot\cdot}{}^{\rho}{}_{\cdot\cdot\rho\cdot\cdot}=0\,, \qquad \zeta_{\mu_1\mu_2\mu_3\nu_1\cdots\nu_{l-1}}\zeta^{\mu_1\mu_2\mu_3\nu_1\cdots\nu_{l-1}}=1\,, \tag{2·10}$$

where $e^{ik\cdot X}$ is to be taken as $:e^{ik\cdot X}:$. The polarization tensors are transverse and traceless with respect to any indices. This makes it possible to change the operator ordering freely in V^b or V^f. It is this point that is required when formulating one-loop amplitudes clearly in the method of Clavelli-Shapiro (§ 3). The polarization tensors have (anti)symmetries correspondig to the Young diagrams. For example, in $V^b_{(2l,l)}(z)$, ζ is first symmetrized in $\mu_1, \nu_1, \cdots, \nu_{l-1}$ and then anti-symmetrized in μ_1, μ_2 and μ_3. The normalizations of the vertex operators are determined by imposing

$$\langle b|b\rangle=\langle f|f\rangle=\langle 0|0\rangle=1\,, \tag{2·11}$$

where $|b\rangle=\lim_{z\to 0}[V^b(z)|0\rangle]$ and $|f\rangle=\lim_{z\to 0}[V^f(z)|0\rangle]$. Since we see from Eq. (2·4) that $z^{-3/2}V^f=\{J_{-1/2}, V^b\}$ where J_r(r: half integer) is the mode operator of $J(z)$ in the Neveu-Schwarz sector, i.e., $J(z)=\sum_r z^{-r}J_r$, the first equality is automatically satisfied for the vertex operators (2·7) and (2·9). Further, the hermiticity condition is imposed on the part of the vertex operator except for $e^{ik\cdot X}$. But even in these conditions, the coefficients of vertex operators are still ambiguous up to a factor ± 1. This factor, however, does not affect the later results for the finiteness problem or the values of mass shifts. Although, in principle, the normalization must be determined by the requirement of unitarity at each loop order which we would not concern here, our way of determining the normalization is sufficient for demonstrating the existence of mass shifts and for investigating their qualitative properties, such as their dependence on spin and mass of the particle. We also suppress the explicit dependences on the coupling parameter and on the slope parameter α'.

In practical calculations, we will go in (F2) formalism for the Neveu-Schwarz sector[13] and (R1) formalism for the Ramond sector,[14] i.e., we use V^f(upper component) instead of V^b. They are related to each other through $z^{-3/2}V^f(z)=\{J_{-1/2}, V^b(z)\}$ (Neveu-Schwarz) and $z^{-1/2}V^f(z)=\{F_0, V^b(z)\}$ (Ramond) where F_0 is the usual Dirac-Ramond operator. Using the conventional dimensionless fields $P^\mu(z)=iz\partial_z X^\mu$, $H^\mu(z)=-iz^{1/2}\Psi^\mu(z)$ and $\dot{H}^\mu(z)=z\partial_z H^\mu$, corresponding V^f's for (2·7) and (2·9) are expressed as follows (the Neveu-Schwarz sector):

$$\begin{aligned}V^f_{(2l,l+1)}(z)&=z^{3/2}\{J_{-1/2}, V^b_{(2l,l+1)}(z)\}\\&=\frac{z^{-l-1}}{\sqrt{l!}}\zeta_{\mu\nu_1\cdots\nu_l}[P^\mu P^{\nu_1}\cdots P^{\nu_l}+\sum_{j=1}^{l}\dot{H}^{\nu_j}H^\mu P^{\nu_1}\cdot\cdot\hat{P}^{\nu_j}\cdot\cdot P^{\nu_l}\\&\quad+k\cdot HH^\mu P^{\nu_1}\cdots P^{\nu_l}]e^{ik\cdot X}(z)\,,\end{aligned}\tag{2·12}$$

$$\begin{aligned}V^f_{(2l,l)}(z)&=z^{3/2}\{J_{-1/2}, V^b_{(2l,l)}(z)\}\\&=\frac{z^{-l-1}}{\sqrt{3!(l-1)!}}\zeta_{\mu_1\mu_2\mu_3\nu_1\cdots\nu_{l-1}}[H^{[\mu_1}H^{\mu_2}P^{\mu_3]}P^{\nu_1}\cdots P^{\nu_{l-1}}\\&\quad+\sum_{j=1}^{l-1}\dot{H}^{\nu_j}H^{\mu_1}H^{\mu_2}H^{\mu_3}P^{\nu_1}\cdot\cdot\hat{P}^{\nu_j}\cdot\cdot P^{\nu_{l-1}}\\&\quad+k\cdot HH^{\mu_1}H^{\mu_2}H^{\mu_3}P^{\nu_1}\cdots P^{\nu_{l-1}}]e^{ik\cdot X}\,,\end{aligned}\tag{2·13}$$

where $\hat{}$ means here that the field under this symbol is omitted and the indices in the bracket [ ] are to be antisymmetrized. These are the vertex operators for the Neveu-Schwarz sector with unit conformal weight which are used in (F2) formalism. For the Ramond sector, we get the same expressions except for the replacement $H^\mu(z)\to\Gamma^\mu(z)/i\sqrt{2}$, which are used in (R1) formalism.

§ 3. Formulation of one-loop amplitudes

We formulate here one-loop amplitudes in a compact form which gives the starting point to investigate the meromorphicity or the divergences at $\lambda=0$ (§ 4) and to perform the explicit computation of the mass shifts (§ 5). Parity-conserving M-point annulus (planar) one-loop amplitudes with arbitraly external bosons lying on the leading and the next-to-leading Regge trajectories are defined in an operator formalism as follows:

$$\begin{aligned}A^P(M)&=\frac{1}{2}\mathrm{Tr}[(1+G)\prod_{i=1}^{M}V_{\mathrm{NS}}(k_i,1)\varDelta_{\mathrm{NS}}-\prod_{i=1}^{M}V_{\mathrm{R}}(k_i,1)\varDelta_{\mathrm{R}}]\\&=\frac{1}{2}\int d\omega\,\mathrm{Tr}[w^{L_0^{\mathrm{NS}}+L_0^{gh}-1/2}(1+G)\prod_{i=1}^{M}V_{\mathrm{NS}}(k_i,\rho_i)-w^{L_0^{\mathrm{R}}+L_0^{gh}}\prod_{i=1}^{M}V_{\mathrm{R}}(k_i,\rho_i)]\,,\end{aligned}\tag{3·1}$$

where the notations are usual ones:

$$\varDelta_{\mathrm{NS}}=\frac{1}{L_0^{\mathrm{NS}}+L_0^{gh}-\frac{1}{2}}\,,\qquad \varDelta_{\mathrm{R}}=\frac{1}{L_0^{\mathrm{R}}+L_0^{gh}}\,,$$

$$G=(-)^{\sum_{r=1/2}^{\infty} b_{-r}\cdot b_r}, \qquad V(k_i, \rho_i)=\rho_i^{l+1} V^f(k_i, \rho_i), \tag{3·2}$$

$$\rho_0=1, \qquad \rho_i=x_1x_2\cdots x_i, \qquad w=x_1x_2\cdots x_M,$$

$$\int d\omega=\prod_{i=1}^{M-1}\int_0^{\rho_{i-1}}\frac{d\rho_i}{\rho_i}\int_0^{\rho_{M-1}}\frac{dw}{w}.$$

Here $L_0^{\mathrm{NS}}+L_0^{gh}$ and $L_0^{\mathrm{R}}+L_0^{gh}$ are the zero modes of energy-momentum tensor including the ghost part for the Neveu-Schwarz and Ramond sectors, respectively. V^f is the vertex operator with $k^2=-m^2=-2l$ constructed in § 2, namely, $V^f_{(2l,l+1)}(z)$ (2·12) or $V^f_{(2l,l)}(z)$ (2·13). Using the trace-reduction technique due to Clavelli and Shapiro,[11] Eq. (3·1) and the corresponding one for Möbius strip (nonorientable) diagram can be transformed into the following forms:

$$A^P(M)=\frac{N}{2}\int d\omega[F_{\mathrm{NS}}(w)T_{\mathrm{NS}}^M(\rho, w)+(w\to we^{2\pi i})-F_{\mathrm{R}}(w)T_{\mathrm{R}}^M(\rho, w)], \tag{3·3}$$

$$A^N(M)=-\frac{1}{2}\int d\omega[w\to -w \text{ in non-zero modes}], \tag{3·4}$$

where

$$F_{\mathrm{NS}}(w)=w^{-1/2}\prod_{n=1}^{\infty}\left(\frac{1+w^{n-(1/2)}}{1-w^n}\right)^{D-2}, \qquad F_{\mathrm{R}}(w)=2^{(D/2)-1}\prod_{n=1}^{\infty}\left(\frac{1+w^n}{1-w^n}\right)^{D-2}, \tag{3·5}$$

$$T_{\mathrm{NS}}(\rho, w)=(-\varepsilon\ln|w|)^{-D}\langle 0|\prod_{i=1}^{M} V_{\mathrm{NS}}(\rho_i, d(w), \bar{d}(w))|0\rangle, \tag{3·6}$$

$$T_{\mathrm{R}}(\rho, w)=(-\varepsilon\ln|w|)^{-D}\langle 0|\prod_{i=1}^{M} V_{\mathrm{R}}(\rho_i, d(w), \bar{d}(w)|0\rangle \tag{3·7}$$

with

$$d^n(w)=\frac{a^n}{1\pm w^{n-I}}+a'^{-n}, \qquad \bar{d}^n(w)=a^{-n}+\frac{w^{n-I}a'^n}{1\pm w^{n-I}}, \tag{3·8}$$

the plus and minus signs referring, respectively, to the fermionic and bosonic oscillators which have, $I=-1/2$ (0) and $-\varepsilon$ in the Neveu-Schwarz (Ramond) sector. Here we use the zero-mode prescriptions:[11]

$$q^\mu=\frac{1}{\sqrt{2\varepsilon}}(a^{0\mu\dagger}+a^{0\mu}), \qquad P^\mu=i\sqrt{\varepsilon/2}(a^{0\mu\dagger}+a^{0\mu}). \tag{3·9}$$

We take the limit $\varepsilon\to 0$ after the trace evaluation.

Now, in contrast to the massless case, the general massive case needs some more careful treatments of T_{NS}^M and T_{R}^M. Originally, a vertex operator is defined in the form:

$$V_{\mathrm{NS}}(\rho, a^n, b^r)=:\tilde{V}_{\mathrm{NS}}(\rho)e^{ik\cdot X}(\rho):, \tag{3·10}$$

where : : denotes the normal ordering with respect to $a^n (n\neq 0)$ or b^r. However, in T_{NS}^M or T_{R}^M, it is modified to

$$V_{\rm NS}(\rho, d(w), \bar{d}(w)) = \mathring{:} \tilde{V}_{\rm NS}(\rho, d(w), \bar{d}(w)) e^{ik\cdot X}(\rho, d(w), \bar{d}(w)) \mathring{:} , \tag{3·11}$$

where $\mathring{:}\ \mathring{:}$ means the "normal" ordering with respect to $d^n(w)$ and $\bar{d}^n(w)$ $(n \neq 0)$ which is not the normal ordering with respect to the duplicated oscillators a'^n or b'^r as seen in Eq. (3·8). Hence some extra terms may appear in general when putting it back to the usual normal ordering. For the vertex operators constructed in § 2, due to their superconformal properties, Eq. (3·11) can be rewritten as

$$V_{\rm NS}(\rho, d, \bar{d}) = \mathring{:} e^{ik\cdot X}(\rho, d, \bar{d}) \mathring{:} \tilde{V}_{\rm NS}(\rho, d, \bar{d}) , \tag{3·12}$$

where $\mathring{:}\ \mathring{:}$ cannot be replaced by : : because of the massiveness $(k^2 \neq 0)$. Carefully treating, we get

$$\mathring{:} e^{ik\cdot X}(\rho, d, \bar{d}) \mathring{:} = \exp\left[k^2\left(\frac{1}{2\varepsilon^2 \ln w} + \frac{\ln w}{24} + \sum_{n=1}^{\infty} \ln(1-w^n) + O(\varepsilon)\right)\right]$$
$$\times : e^{ik\cdot X}(\rho, d, \bar{d}) : . \tag{3·13}$$

In $T_{\rm NS}^M$ all of the factors $: e^{ik_i\cdot X}(\rho_i, d, \bar{d}):$ can be pulled to the left since

$$[P^\mu(\rho_i, d, \bar{d}), X^\nu(\rho_j, d, \bar{d})] = -\eta^{\mu\nu}\{G(\rho_{ij}, w) + G(\rho_{ji}, w)\} = 0 , \tag{3·14}$$

where $G(\rho_{ji})$ is the correlation function of P^μ and X^ν defined in the Appendix $(\rho_{ji} \equiv \rho_j/\rho_i)$. Then we have

$$T_{\rm NS}^M = (-\varepsilon \ln|w|)^{-D} \exp\left[\left(\sum_{i=1}^{M} k_i^2\right)\left(\frac{1}{2\varepsilon^2 \ln w} + \frac{\ln w}{24} + \sum_{n=1}^{\infty} \ln(1-w^n)\right) + O(\varepsilon)\right]$$
$$\times \left\langle \prod_{i=1}^{M} : e^{ik_i\cdot X}(\rho_i, d, \bar{d}) : \prod_{i=1}^{M} \tilde{V}_{\rm NS}(\rho_i, d, \bar{d}) \right\rangle . \tag{3·15}$$

Using the relations

$$\prod_{i=1}^{M} : e^{ik_i\cdot X}(\rho_i, d, \bar{d}) : = \prod_{i<j}^{M} \exp[-k_\mu^i k_\nu^j \langle X^\mu(\rho_i, d, \bar{d}) X^\nu(\rho_j, d, \bar{d})\rangle]$$
$$\times : \prod_{i=1}^{M} e^{ik_i\cdot X}(\rho_i, d, \bar{d}) : \tag{3·16}$$

with

$$\langle X^\mu(\rho_i, d, \bar{d}) X^\nu(\rho_j, d, \bar{d})\rangle$$
$$= \eta^{\mu\nu}\left[-\frac{1}{\varepsilon^2 \ln w} - \ln\psi(\rho_{ji}, w) - \frac{\ln w}{12} - 2\sum_{n=1}^{\infty} \ln(1-w^n) + O(\varepsilon)\right], \tag{3·17}$$

$$\psi(x, w) = -2\pi i e^{i\pi v^2/\tau} \frac{\theta_1(v|\tau)}{\theta_1'(0|\tau)} , \tag{3·18}$$

$$v \equiv \frac{\ln x}{2\pi i} , \qquad \tau \equiv \frac{\ln w}{2\pi i} , \qquad \theta_1'(v|\tau) \equiv \frac{\partial}{\partial v} \theta_1(v|\tau) , \tag{3·19}$$

and pulling the annihilation parts of the generalized plane-wave factors to the right, we obtain

$$T_{\mathrm{NS}}^{M}=\left(\frac{\tau}{i}\right)^{-D/2}\delta^{(D)}(\sum_i k_i)\prod_{i<j}\psi(\rho_{ji},\,w)^{k_i\cdot k_j}\langle\prod_{i=1}^{M}\tilde{\tilde{V}}_{\mathrm{NS}}(\rho_i,\,d,\,\bar{d})\rangle\,, \tag{3·20}$$

where

$$\tilde{\tilde{V}}_{\mathrm{NS}}(\rho_i,\,d,\,\bar{d})\equiv\tilde{V}_{\mathrm{NS}}(\rho_i,\,d,\,\bar{d})(P^\mu\to\tilde{P}^\mu\equiv P^\mu+B^\mu) \tag{3·21}$$

with

$$\begin{aligned}B^\mu(\rho_i,\,w)&=\sum_{j=1}^{M}ik_j{}^\nu\langle P^\mu(\rho_i,\,d,\,\bar{d})X_\nu(\rho_j,\,d,\,\bar{d})\rangle\\&=\sum_{j=1}^{M}ik_j{}^\mu G(\rho_{ji},\,w)\,.\end{aligned} \tag{3·22}$$

Here we have also utilized the relation:

$$\begin{aligned}&(-\varepsilon\ln|w|)^{-D}\exp\Big[(\sum_{i=1}k_i{}^2+2\sum_{i<j}k_i\cdot k_j)\\&\times\left(\frac{1}{2\varepsilon^2\ln w}+\frac{\ln w}{24}+\sum_{n=1}^{\infty}\ln(1-w^n)+O(\varepsilon)\right)+\sum_{i<j}k_i\cdot k_j\ln\psi(\rho_{ji},\,w)\Big]\\&\xrightarrow{\varepsilon\to0}\left(\frac{\tau}{i}\right)^{-D/2}\delta^{(D)}(\sum_i k_i)\prod_{i<j}\psi(\rho_{ji},\,w)^{k_i\cdot k_j}\,.\end{aligned} \tag{3·23}$$

A similar expression is obtained for $T_{\mathrm{R}}{}^M$, where $\tilde{\tilde{V}}_{\mathrm{R}}(\rho_i,\,d,\,\bar{d})\equiv\tilde{\tilde{V}}_{\mathrm{NS}}(H^\mu(\rho_i,\,d,\,\bar{d})\to\Gamma^\mu(\rho_i,\,d,\,\bar{d})/i\sqrt{2})$. Inserting them into Eqs. (3·3) and (3·4) and taking also account of the relation:

$$F_{\mathrm{R}}(w)=F_{\mathrm{NS}}(w)+F_{\mathrm{NS}}(we^{2\pi i})\,, \tag{3·24}$$

we finally get for the whole amplitudes:

$$\begin{aligned}A^P(M)=&\frac{N}{2}\delta^{(D)}(\sum k_i)\int d\omega\left(\frac{\tau}{i}\right)^{-D/2}\prod_{i<j}\psi(\rho_{ji},\,w)^{k_i\cdot k_j}\\&\times\{F_{\mathrm{NS}}(w)[U_{\mathrm{NS}}^M(\rho,\,w)-U_{\mathrm{R}}{}^M(\rho,\,w)]+(w\to w^{2\pi i})\}\,,\end{aligned} \tag{3·25}$$

$$\begin{aligned}A^N(M)=&-\frac{1}{2}\delta^{(D)}(\sum k_i)\int d\omega\left(\frac{\tau}{i}\right)^{-D/2}\prod_{i<j}\psi_N(\rho_{ji},\,w)^{k_i\cdot k_j}\\&\times[w\to-w\text{ in non-zero modes}]\,,\end{aligned} \tag{3·26}$$

where

$$U_{\mathrm{NS}}^M(\rho,\,w)\equiv\langle\prod_{i=1}^{M}\tilde{\tilde{V}}_{\mathrm{NS}}(\rho_i,\,d,\,\bar{d})\rangle\,, \tag{3·27}$$

$$U_{\mathrm{R}}{}^M(\rho,\,w)\equiv\langle\prod_{i=1}^{M}\tilde{\tilde{V}}_{\mathrm{R}}(\rho_i,\,d,\,\bar{d})\rangle\,, \tag{3·28}$$

$$\psi_N(x,\,w)\equiv-2\pi ie^{i\pi v^2/\tau}\frac{\theta_1\left(v\middle|\tau+\frac{1}{2}\right)}{\theta_1'\left(0\middle|\tau+\frac{1}{2}\right)}\,. \tag{3·29}$$

§ 4. Meromorphicity and finiteness at $\lambda=0(w=1)$

In this section we study the behavior of the w-integrand in the amplitudes (3·25) and (3·26). Owing to the high inverse power of $\ln w(=2\pi i\tau)$, w-integral converges near $w=0$, that is, amplitudes are infrared finite. So what we are concerned with is their behavior near $w=1$ (ultraviolet limit). As in the massless case,[3] we carry out the Jacobi-transformation:

$$w'=q^2=e^{2\pi i\tau'}=e^{4\pi^2/\ln w}\,, \qquad \rho'=e^{2\pi i\nu_i}=e^{2\pi i\ln\rho_i/\ln w}\,, \qquad \text{(annulus)}$$

$$w''=w'^{1/4}\,, \qquad \rho_i''=\rho_i'^{1/2}=e^{2\pi i\nu_i^N}\,, \qquad \text{(Möbius)} \tag{4·1}$$

by which the singular behavior near $w=1$ is transformed to the one near $w'=0$. The transformation properties in each part of the amplitudes are listed as follows:

measure

$$d\omega=\tau^{M+1}d\omega'\,, \qquad \text{(annulus)}$$

$$=(2\tau)^{M+1}d\omega''\,, \qquad \text{(Möbius)} \tag{4·2}$$

where

$$d\omega'=(2\pi)^{M-1}(-i)^{M+1}\int_0^1\frac{dw'}{w'}\int_0^1\prod_{i=1}^{M-1}\theta(\nu_{i+1}-\nu_i)d\nu_i\,, \tag{4·3}$$

$$d\omega''=(2\pi)^{M-1}(-i)^{M+1}\int_0^1\frac{dw''}{w''}\int_0^1\prod_{i=1}^{M-1}\theta(\nu_{i+1}^N-\nu_i^N)d\nu_i^N\,. \tag{4·4}$$

plane-wave factor

$$\psi(\rho_{ji},w)=\tau\hat{\psi}(\rho'_{ji},w')\,, \qquad \text{(annulus)} \tag{4·5}$$

$$\psi_N(\rho_{ji},w)=2\tau\hat{\psi}(\rho''_{ji},-w'')\,, \qquad \text{(Möbius)} \tag{4·6}$$

where

$$\hat{\psi}(\rho',w')=-2\pi i\theta_1(\nu|\tau')/\theta_1'(0|\tau')$$

$$=-2i\sin\pi\nu\prod_{n=1}^{\infty}\frac{1-2q^{2n}\cos2\pi\nu+q^{4n}}{(1-q^{2n})^2}\,, \tag{4·7}$$

and $\rho'_{ji}\equiv\rho_j'/\rho_i'$. Therefore

$$\prod_{i<j}\psi(\rho_{ji},w)^{k_i\cdot k_j}=\tau^{-(1/2)\Sigma_{i=1}^M k_i^2}\prod_{i<j}\hat{\psi}(\rho'_{ji},w')^{k_i\cdot k_j}\,, \qquad \text{(annulus)} \tag{4·8}$$

$$\prod_{i<j}\psi_N(\rho_{ji},w)^{k_i\cdot k_j}=(2\tau)^{-(1/2)\Sigma_{i=1}^M k_i^2}\prod_{i<j}\hat{\psi}(\rho''_{ji},-w'')^{k_i\cdot k_j}\,, \qquad \text{(Möbius)} \tag{4·9}$$

where we note that the weight factor $(1/2)\Sigma_{i=1}^M k_i^2$ is just equal to the sum of the conformal weights of $:e^{ik_i\cdot X}(z):$, which vanishes in a massless case.[3]

partition function

The transformation property of partition functions for various spin structures is just the same as the one for the pure massless case which is listed in Ref. 3) by Clavelli. We do not mention here again.

U^M_{NS} (or $U_R{}^M$)

U^M, the vaccum expectation value of the product of primary fields, can be rewritten through Wick expansion as the product of two point correlation functions of elementary conformal fields. Hence the transformation properties of the U^M's are determined by those of the two-point correlation functions which are listed in the Appendix. For $U^M_{\rm NS}(\rho, w)$, for instance, the correlation function of each field is simply related by the replacements

$$P^\mu(\rho, d(w), \bar{d}(w)) \to \tau^{-1}\hat{P}^\mu(\rho', d(w'), \bar{d}(w')),$$

$$H^\mu(\rho, d(w), \bar{d}(w)) \to \tau^{-1/2}H^\mu(\rho', d(w'), \bar{d}(w')),$$

$$\dot{H}^\mu(\rho, d(w), \bar{d}(w)) \to \tau^{-3/2}\dot{H}^\mu(\rho', d(w'), \bar{d}(w')), \tag{4·10}$$

where $\hat{P}$ is defined in the Appendix. The weight of the Jacobi transformation of any two-point correlation function is just equal to the sum of the conformal weight of each field. Therefore, using Wick expansion, we obtain

$$U^M_{\rm NS}(\rho, w) = \tau^{(1/2)\Sigma_{i=1}^M k_i^2 - M}\hat{U}^M_{\rm NS}(\rho', w') \tag{4·11}$$

with

$$\hat{U}^M_{\rm NS}(\rho', w') \equiv \langle \prod_{i=1}^M \hat{\hat{V}}_{\rm NS}(\rho_i', d(w'), \bar{d}(w'))\rangle, \tag{4·12}$$

where each P^μ in $\tilde{\tilde{V}}_{\rm NS}$ has been replaced by $\hat{P}^\mu$, to form $\hat{\hat{V}}_{\rm NS}$. Similary we have

$$U^M_{\rm NS}(\rho, we^{2\pi i}) = \tau^{(1/2)\Sigma_{i=1}^M k_i^2 - M}\hat{U}_{\rm R}{}^M(\rho', w'), \tag{4·13}$$

$$U_{\rm R}{}^M(\rho, w) = \tau^{(1/2)\Sigma_{i=1}^M k_i^2 - M}\hat{U}^M_{\rm NS}(\rho', w'e^{2\pi i}), \tag{4·14}$$

$$U^M_{\rm NS}(\rho, -w) = (2\tau)^{(1/2)\Sigma_{i=1}^M k_i^2 - M}\hat{U}^M_{\rm NS}(\rho'', -w''e^{2\pi i}), \tag{4·15}$$

$$U^M_{\rm NS}(\rho, -we^{2\pi i}) = (2\tau)^{(1/2)\Sigma_{i=1}^M k_i^2 - M}\hat{U}^M_{\rm NS}(\rho'', -w''), \tag{4·16}$$

$$U_{\rm R}{}^M(\rho, -w) = (2\tau)^{(1/2)\Sigma_{i=1}^M k_i^2 - M}\hat{U}_{\rm R}{}^M(\rho'', -w''). \tag{4·17}$$

The six spin structures are related to each other. The weight of the transfomation $M-(1/2)\Sigma_{i=1}^M k_i^2$ equals the sum of conformal weights of the constituent conformal fields making up U^M. Collecting Eqs. (4·2), (4·8), (4·9), (4·11) and (4·13)~(4·17), we get the net effect of this transformation on the whole amplitudes (3·25) and (3·26):

$$A^P(M) = \frac{Ni}{2}\delta^{(D)}(\Sigma k_i)\int d\omega' \tau^{M+1-D/2-(1/2)\Sigma_{i=1}^M k_i^2 + D/2 - 1 + (1/2)\Sigma_{i=1}^M k_i^2 - M}$$

$$\times \prod_{i<j}\hat{\psi}(\rho'_{ji}, w')^{k_i \cdot k_j}\{F_{\rm NS}(w')[\hat{U}^M_{\rm NS}(\rho', w') - \hat{U}_{\rm R}{}^M(\rho', w')] + (w' \to w'e^{2\pi i})$$

$$= \frac{Ni^{M+2}}{2}(-2\pi)^{M-1}\delta^{(D)}(\Sigma k_i)\int_0^1 \frac{dw'}{w'}F(w'), \tag{4·18}$$

$$A^N(M)=\frac{2^{D/2}i^M}{2}(-2\pi)^{M-1}\delta^{(D)}(\sum k_i)\int_0^1\frac{dw''}{w''}F(-w''),\tag{4·19}$$

where the functional form of F is given by

$$F(\lambda)=\int_0^1\prod_{i=1}^{M-1}\theta(\nu_{i+1}-\nu_i)d\nu_i\prod_{i<j}^{M}\hat{\psi}(\nu_{ji},\lambda)^{k_i\cdot k_j}$$
$$\times\{F_{\rm NS}(\lambda)[\hat{U}_{\rm NS}^M(\nu,\lambda)-\hat{U}_{\rm R}{}^M(\nu,\lambda)]+(\lambda\to\lambda e^{2\pi i})\},\tag{4·20}$$

in which λ stands for either w' or $-w''$. As seen above the τ dependence in each amplitude is completely canceled for an arbitrary set of external bosons, which is the reflection of the fact that the vertex operator is so constructed as to have a conformal weight 1.

Thus the integrand of each amplitude is meromorphic in $w'(w'')$. Further, as $F(\lambda)$ is not singular at $\lambda=0$, the integrand has only a simple-pole singularity at the origin. The divergence coming from this singularity is of the same type as the one observed in a massless amplitude.[2)~5)] This divergence can be canceled for $N=32$ by combining the two amplitudes (4·18) and (4·19) into a principal-value integral. Namely, defining $w'=\lambda$ and $w''=-\lambda$ we obtain for $N=2^{D/2}$

$$A(M)=A^P(M)+A^N(M)$$
$$=\frac{i^{M+2}}{2}(-2\pi)^{M-1}2^{D/2}\mathrm{PP}\int_{-1}^1\frac{d\lambda}{\lambda}F(\lambda),\tag{4·21}$$

where PP denotes the principal-part prescription.[2)]

Needless to say, this proof of finiteness at $w'(w'')=0$ crucially depends on the identification of variables (4·1) in the two different parts of the amplitudes. This is the same situation as already seen in a massless case.[5)] So the ambiguity on finiteness at $\lambda=0$ remains also in the massive case. But, to be stressed here is the existence of a regularization prescription applicable for amplitudes involving arbitrary external bosons. It is easy to check that the situation can also be realized by a Pauli-Villars regularization method [5)] by assuming the cancelled propagator argument at one-loop level, where a mass ratio between regulators in the two parts of the amplitudes become crucial for finiteness. Finally we note that the discussion given above holds for more general external bosons as long as the corresponding vertex operators allow a rewriting (3·11)→(3·12).

§ 5. Evaluation of two-point amplitudes

The divergence we have considered in the previous section is the one occurring at the boundary of the integration region of the overall Teichmüller parameter ($=\lambda$). It has been found that such a divergence can be canceled between the annulus and Möbius strip amplitudes based on a certain regularization procedure. As referred to in the Introduction, however, there can exist another type of divergence related to the insertion of an on-shell propagator occurring at the boundary of the relative Teichmüller parameters($=\nu_i$). This divergence is therefore proportional to the on-shell two-point amplitudes, i.e., to the mass shift of external particles.

In this section we try to explicitly evaluate the on-shell two-point amplitudes by using vertex operators constructed in § 2. We start again from Eqs. (3·25) and (3·26). What we have to calculate first is

$$U_{\rm NS}^{2\,(2l,J)}(\rho, w)=\langle \tilde{\tilde{V}}_{\rm NS}^{(2l,J)}(\rho_1, d, \bar{d})\,\tilde{\tilde{V}}_{\rm NS}^{(2l,J)}(\rho_2, d, \bar{d})\rangle\,, \quad (J=l+1,\, l,\, l\geq 1) \tag{5·1}$$

where

$$\begin{aligned}\tilde{\tilde{V}}_{\rm NS}^{(2l,l+1)}&=\tilde{V}_{\rm NS}^{(2l,l+1)}[P^\mu\to\tilde{P}^\mu]\\&=\frac{1}{\sqrt{l\,!}}\zeta_{\mu\nu_1\cdot\cdot\nu_l}\{\tilde{P}^\mu\tilde{P}^{\nu_1}\cdot\cdot\tilde{P}^{\nu_l}-H^\mu\sum_{j=1}^{l}\dot{H}^{\nu_j}\tilde{P}^{\nu_1}\cdot\cdot\hat{\tilde{P}}^{\nu_j}\cdot\cdot\tilde{P}^{\nu_l}\\&\quad -H^\mu k\cdot H\tilde{P}^{\nu_1}\cdot\cdot\tilde{P}^{\nu_l}\}\,,\end{aligned} \tag{5·2}$$

$$\begin{aligned}\tilde{\tilde{V}}_{\rm NS}^{(2l,l)}&=\tilde{V}_{\rm NS}^{(2l,l)}(P^\mu\to\tilde{P}^\mu)\\&=\frac{1}{\sqrt{3!(l-1)!}}\zeta_{\mu_1\mu_2\mu_3\nu_1\cdot\cdot\nu_{l-1}}\{H^{[\mu_1}H^{\mu_2}\tilde{P}^{\mu_3]}\tilde{P}^{\nu_1}\cdot\cdot\tilde{P}^{\nu_{l-1}}\\&\quad+\sum_{j=1}^{l-1}\dot{H}^{\nu_j}H^{\mu_1}H^{\mu_2}H^{\mu_3}\tilde{P}^{\nu_1}\cdot\cdot\hat{\tilde{P}}^{\nu_j}\cdot\cdot\tilde{P}^{\nu_{l-1}}+k\cdot HH^{\mu_1}H^{\mu_2}H^{\mu_3}\tilde{P}^{\nu_1}\cdot\cdot\tilde{P}^{\nu_{l-1}}\}\,.\end{aligned} \tag{5·3}$$

Now let us evaluate $U_{\rm NS}^{2(2l,l+1)}$ first. It is sufficient to consider only the parts which include fermion correlations since only a difference $U_{\rm NS}^M-U_{\rm R}^M$ contributes to the amplitude (space-time supersymmetry). Hence we calculate

$$\begin{aligned}U_{\rm NS}^{2(2l,l+1)}(\rho, w)&=\frac{1}{l\,!}\zeta^1_{\mu\nu_1\cdot\cdot\nu_l}\zeta^2_{\rho\sigma_1\cdot\cdot\sigma_l}\\&\quad\times\langle\{H^\mu\sum_{j=1}^{l}\dot{H}^{\nu_j}\tilde{P}^{\nu_1}\cdot\cdot\hat{\tilde{P}}^{\nu_j}\cdot\cdot\tilde{P}^{\nu_l}(1)+H^\mu k_1\cdot H\tilde{P}^{\nu_1}\cdot\cdot\tilde{P}^{\nu_l}(1)\}\\&\quad\times\{H^\rho\sum_{i=1}^{l}\dot{H}^{\sigma_i}\tilde{P}^{\sigma_1}\cdot\cdot\hat{\tilde{P}}^{\sigma_j}\cdot\cdot\hat{\tilde{P}}^{\sigma_i}\cdot\cdot\tilde{P}^{\sigma_l}(2)+H^\rho k_2\cdot H\tilde{P}^{\sigma_1}\cdot\cdot\tilde{P}^{\sigma_l}(2)\}\rangle\\&=\frac{1}{l\,!}\zeta^1_{\mu\nu_1\cdot\cdot\nu_l}\zeta^2_{\rho\sigma_1\cdot\cdot\sigma_l}\\&\quad\times\{\sum_{i,j=1}^{l}\langle H^\mu\dot{H}^{\nu_j}(1)H^\rho\dot{H}^{\sigma_i}(2)\rangle\langle\tilde{P}^{\nu_1}\cdot\cdot\hat{\tilde{P}}^{\nu_j}\cdot\cdot\tilde{P}^{\nu_l}(1)\tilde{P}^{\sigma_1}\cdot\cdot\hat{\tilde{P}}^{\sigma_i}\cdot\cdot\tilde{P}^{\sigma_l}(2)\rangle\\&\quad+\sum_{j=1}^{l}\langle H^\mu\dot{H}^{\nu_j}(1)H^\rho k_2\cdot H(2)\rangle\langle\tilde{P}^{\nu_1}\cdot\cdot\hat{\tilde{P}}^{\nu_j}\cdot\cdot\tilde{P}^{\nu_l}(1)\tilde{P}^{\sigma_1}\cdot\cdot\tilde{P}^{\sigma_l}(2)\rangle\\&\quad+\sum_{i=1}^{l}\langle H^\mu k_1\cdot H(1)H^\rho\dot{H}^{\sigma_i}(2)\rangle\langle\tilde{P}^{\nu_1}\cdot\cdot\tilde{P}^{\nu_l}(1)\tilde{P}^{\sigma_1}\cdot\cdot\hat{\tilde{P}}^{\sigma_i}\cdot\cdot\tilde{P}^{\sigma_l}(2)\rangle\\&\quad+\langle H^\mu k_1\cdot H(1)H^\rho k_2\cdot H(2)\rangle\langle\tilde{P}^{\nu_1}\cdot\cdot\tilde{P}^{\nu_l}(1)\tilde{P}^{\sigma_1}\cdot\cdot\tilde{P}^{\sigma_l}(2)\rangle\}\,,\end{aligned} \tag{5·4}$$

where $H^\mu(j)\equiv H^\mu(\rho_j)$. We have dropped the terms consisting only of boson correlations. Noting that $k_1=-k_2(\equiv k)$, we find after Wick expansion of fermionic vacuum expectation values that the second and third terms vanish because of the transversality of the polarization tensor: $k^\mu\zeta_{\cdot\cdot\mu\cdot\cdot}=0$. The fermionic part of the fourth term is proportional to $k_1\cdot k_2\chi^2(21)$ and does not vanish in itself where $\chi(ji)(\equiv\chi(\rho_{ji}))$

($\chi_0(\rho_{ji})$) is a Neveu-Schwarz (Ramond) two-point correlation function (Appendix). But, in the amplitude, from the sum over spin structure, its contribution vanishes as in a massless two-point amplitude due to the vanishing identity for theta functions. That is, $F_{\mathrm{NS}}(w)[U_{\mathrm{NS}}(w)-U_{\mathrm{R}}(w)]+(w\to we^{2\pi i})$ is proportional to

$$F_{\mathrm{NS}}(w)[\chi^2(21)-\chi_0^2(21)]+(w\to we^{2\pi i})\propto\sum_{\alpha=2}^{4}(-)^{\alpha+1}\theta_\alpha{}^2(0)\theta_\alpha{}^2(21)=0\,, \tag{5·5}$$

where $\theta_\alpha(ji)\equiv\theta_\alpha(v_{ji})\equiv\theta_\alpha(v_{ji}\equiv\ln\rho_{ji}/2\pi i|\tau)$. Hence we only have to consider the first term of Eq. (5·4). Using Wick expansion its fermionic part is evaluated as

$$\langle H^\mu\dot{H}^{\nu_j}(1)H^\rho\dot{H}^{\sigma_i}(2)\rangle=-\eta^{\mu\sigma_i}\eta^{\nu_j\rho}\dot{\chi}^2(21)+\eta^{\mu\rho}\eta^{\nu_j\sigma_i}\ddot{\chi}(21)\chi(21)\,, \tag{5·6}$$

where $\dot{\chi}(\rho,w)\equiv\rho(\partial/\partial\rho)\chi(\rho,w)$ and $\ddot{\chi}(\rho,w)\equiv\rho(\partial/\partial\rho)\dot{\chi}(\rho,w)$. Next let us evaluate the bosonic part. Note that $B^\nu(\rho)$ term included in $\tilde{P}^\nu(\rho)$ does not contribute to U_{NS}^2 or $U_{\mathrm{R}}{}^2$ on account of the transversality and momentum conservation $k_1=-k_2$. Therefore, neglecting these terms and performing Wick expansion, we have

$$\begin{aligned}
&\langle\tilde{P}^{\nu_1}\cdots\hat{\tilde{P}}^{\nu_j}\cdots\tilde{P}^{\nu_l}(1)\tilde{P}^{\sigma_1}\cdots\hat{\tilde{P}}^{\sigma_i}\cdots\tilde{P}^{\sigma_l}(2)\rangle\\
&=\langle P^{\nu_1}\cdots\hat{P}^{\nu_j}\cdots P^{\nu_l}(1)P^{\sigma_1}\cdots\hat{P}^{\sigma_i}\cdots P^{\sigma_l}(2)\rangle\\
&=\sum_{\mathrm{Perm}}\langle P^{\nu_1}(1)P^{\sigma_1}(2)\rangle\cdots\langle \widehat{P^{\nu_j}(1)P^{\sigma_i}}(2)\rangle\cdots\langle P^{\nu_l}(1)P^{\sigma_l}(2)\rangle\\
&=(\chi^P)^{l-1}\{\eta^{\nu_1\sigma_1}\cdots\hat{\eta}^{\nu_j\sigma_i}\cdots\eta^{\nu_l\sigma_l}\\
&\qquad+[(l-1)!-1]\text{ more terms obtained by permutation}\}\,,
\end{aligned} \tag{5·7}$$

where $\eta^{\mu\nu}\chi^P(\rho_{ji},w)\equiv\langle P^\mu(\rho_i,d,\bar{d})P^\nu(\rho_j,d,\bar{d})\rangle$ (Appendix). Recalling that polarization tensor is totally symmetric with respect to all the indices we find that a part of $U_{\mathrm{NS}}^{2(2l,l+1)}$ that contributes to the amplitude is given by

$$\begin{aligned}
&U_{\mathrm{NS}}^{2(2l,l+1)}(\rho,w)\\
&=\frac{l^2(l-1)!}{l!}\zeta^1_{\mu\nu_1\cdots\nu_l}\zeta^{2\mu\nu_1\cdots\nu_l}[\chi(21)\ddot{\chi}(21)-\dot{\chi}^2(21)][\chi^P(21)]^{l-1}\,.
\end{aligned} \tag{5·8}$$

Similar expressions are obtained for $U_{\mathrm{R}}{}^{2(2l,l+1)}$ where χ is replaced by χ_0. Then summing over spin structure after multiplication by partition function and using the expressions of partition and correlation functions in terms of theta functions[11] (Appendix) we get through a straightforward but tedious calculation,

$$\begin{aligned}
&F_{\mathrm{NS}}(w)[U_{\mathrm{NS}}^{2(2l,l+1)}(\rho,w)-U_{\mathrm{R}}{}^{2(2l,l+1)}(\rho,w)]+(w\to we^{2\pi i})\\
&=\frac{l}{8\pi^4}\zeta^1_{\mu\nu_1\cdots\nu_l}\zeta^{2\mu\nu_1\cdots\nu_l}w^{-1/2}f(w)^{-12}\theta_1'^4(0)\left[\frac{\partial^2}{\partial(\ln\rho_{21})^2}(\ln\psi(\rho_{21},w)\right]^{l-1}
\end{aligned} \tag{5·9}$$

with $f(w)\equiv\prod_{n=1}^{\infty}(1-w^n)$ and $\theta_i'\equiv(\partial/\partial v)\theta_i(v|\tau)$ [and $\theta_i''\equiv(\partial^2/\partial v^2)\theta_i(v|\tau)$] ($i=1,2,3,4$). Here we have used again the following identities: $\sum_{\alpha=2}^{4}\theta_\alpha{}^2(0)\theta_\alpha{}^2(v)=0$ and its first and second derivatives. Use is also made of

$$\sum_{\alpha=2}^{4}(-)^{\alpha+1}\theta_\alpha{}^2(0)\theta_\alpha'^2(v)=-\theta_1'^2(0)\theta_1{}^2(v)\,. \tag{5·10}$$

Let us next evaluate $U_{\mathrm{NS}}^{2(2l,l)}$. As in the previous case, because of the transversality and space-time supersymmetry, the terms of $U_{\rm NS}^{2(2l,l)}$ that is likely to remain in the amplitudes are

$$U_{\rm NS}^{2(2l,l)}(\rho,w)=\frac{1}{3!(l-1)!}\zeta^1_{\mu_1\mu_2\mu_3\nu_1\cdots\nu_{l-1}}\zeta^2_{\rho_1\rho_2\rho_3\sigma_1\cdots\sigma_{l-1}}\Big\{\sum_{i,j=1}^{l-1}$$

$$\langle\dot H^{\nu_j}H^{\mu_1}H^{\mu_2}H^{\mu_3}(1)\dot H^{\sigma_i}H^{\rho_1}H^{\rho_2}H^{\rho_3}(2)\rangle\langle P^{\nu_1}\cdots\hat P^{\nu_j}\cdots P^{\nu_{l-1}}(1)P^{\sigma_1}\cdots\hat P^{\sigma_i}\cdots P^{\sigma_{l-1}}(2)\rangle$$

$$+\langle k_1\cdot HH^{\mu_1}H^{\mu_2}H^{\mu_3}(1)k_2\cdot HH^{\rho_1}H^{\rho_2}H^{\rho_3}(2)\rangle\langle P^{\nu_1}\cdots P^{\nu_{l-1}}(1)P^{\sigma_1}\cdots P^{\sigma_{l-1}}(2)\rangle\Big\}\,. \tag{5·11}$$

The fermion part of the first term, however, is evaluated as

$$\zeta^1_{\mu_1\mu_2\mu_3\nu_1\cdots\nu_{l-1}}\zeta^2_{\rho_1\rho_2\rho_3\sigma_1\cdots\sigma_{l-1}}\langle H^{\mu_1}H^{\mu_2}H^{\mu_3}\dot H^{\nu_j}(1)H^{\rho_1}H^{\rho_2}H^{\rho_3}\dot H^{\sigma_i}(2)\rangle$$

$$=-\zeta^1_{\mu_1\mu_2\mu_3\cdots\nu_j\cdots}\zeta^{2[\mu_1\mu_2\mu_3]\cdots\nu_j\cdots}\chi^2(21)[\dot\chi^2(21)-\chi(21)\ddot\chi(21)]\,, \tag{5·12}$$

and the sum over spin structure after multiplication by partition function yields

$$F_{\rm NS}(w)[\chi^2(21)\{\dot\chi^2(21)-\chi(21)\ddot\chi(21)\}-\chi_0^2(21)\{\dot\chi_0^2(21)-\chi_0(21)\ddot\chi_0(21)\}]$$

$$+(w\to we^{2\pi i})$$

$$=\Big[\sum_{\alpha=2}^{4}(-)^{\alpha+1}\{\theta_\alpha^2(21)\theta_\alpha'^2(21)-\theta_\alpha^3(21)\theta_\alpha''(21)\}-\theta_1^3(21)\theta_1''(21)+\theta_1^2(21)\theta_1'^2(21)\}\Big]$$

$$=0\,. \tag{5·13}$$

Here we use the following identities among theta functions

$$\sum_{\alpha=2}^{4}(-)^{\alpha+1}\theta_\alpha^4(v)=-\theta_1^4(v)\,, \tag{5·14}$$

$$\sum_{\alpha=2}^{4}(-)^{\alpha+1}\theta_\alpha^2(v)\theta_\alpha'^2(v)=-\theta_1^2(v)\theta_1'^2(v)\,, \tag{5·15}$$

$$\sum_{\alpha=2}^{4}(-)^{\alpha+1}\theta_\alpha^3(v)\theta_\alpha''(v)=-\theta_1^3(v)\theta_1''(v)\,. \tag{5·16}$$

These and Eqs. (5·10) are all derived from Jacobi's fundamental formula for theta functions.[15] Thus only the second term of (5·11) contributes to the amplitudes. It is evaluated as

$$U_{\rm NS}^{2(2l,l)}(\rho,w)=\frac{-k^2}{3!(l-1)!}\zeta^1_{\mu_1\cdots\nu_{l-1}}\zeta^{2\rho_1\cdots\sigma_{l-1}}[\chi(21)]^4[\chi^P(21)]^{l-1}$$

$$\times(-\delta^{\mu_1}{}_{[\rho_1}\delta^{\mu_2}{}_{\rho_2}\delta^{\mu_3}{}_{\rho_3]})(\delta^{\nu_1}{}_{\sigma_1}\cdots\delta^{\nu_{l-1}}{}_{\sigma_{l-1}}+\text{perm.})$$

$$=-2l\zeta^1_{\mu_1\cdots\nu_{l-1}}\zeta^{2\mu_1\cdots\nu_{l-1}}\chi^4(21)[\chi^P(21)]^{l-1}\,. \tag{5·17}$$

Taking the sum over spin structure we get

$$F_{\rm NS}(w)\{U_{\rm NS}^{2(2l,l)}(\rho,w)-U_{\rm R}^{2(2l,l)}(\rho,w)\}+(w\to we^{2\pi i})$$

$$=\frac{l}{8\pi^4}\zeta^1_{\mu_1\cdots\nu_{l-1}}\zeta^{2\mu_1\cdots\nu_{l-1}}w^{-1/2}f(w)^{-12}\theta_1'^4(0)\Big[\frac{\partial^2}{\partial(\ln\rho_{21})^2}(\ln\psi(\rho_{21},w))\Big]^{l-1}\,, \tag{5·18}$$

where we use again the identity (5·14). We note that Eqs. (5·9) and (5·18) have the same form except for kinematical factors. Inserting them into Eqs. (3·25) and (3·26) and performing Jacobi transformation, we obtain the whole amplitudes ($N=2^{D/2}$)

$$A^{(2l,l+1)}(2)=A^{P(2l,l+1)}(2)+A^{N(2l,l+1)}(2)$$

$$=(2\pi)^3 Nl\zeta^1_{\mu\cdots\nu_l}\zeta^{2^{\mu\cdots\nu_l}}\mathrm{PP}\int_{-1}^{1}\frac{d\lambda}{\lambda}\int_0^1 d\nu\left(\frac{\theta_1}{\theta_1'(0)}\right)^2\left(\frac{\theta_1\theta_1''-\theta_1'^2}{\theta_1'^2(0)}\right)^{l-1}, \quad (5\cdot19)$$

$$A^{(2l,l)}(2)=(2\pi)^3 Nl\zeta^1_{\mu_1\cdots\nu_{l-1}}\zeta^{2^{\mu_1\cdots\nu_{l-1}}}\mathrm{PP}\int_{-1}^{1}\frac{d\lambda}{\lambda}\int_0^1 d\nu\left(\frac{\theta_1}{\theta_1'(0)}\right)^2\left(\frac{\theta_1\theta_1''-\theta_1'^2}{\theta_1'^2(0)}\right)^{l-1}, \quad (5\cdot20)$$

where θ_1, θ_1', θ_1'' are $\theta_1(\nu|\tau'=\ln\lambda/2\pi i)$ and its first and second derivatives with respect to ν and $\theta_1'(0)\equiv\theta_1'(\nu=0)$. Then one loop mass shifts (one-loop shifts in the position of the single pole of $\{k^2+m^2+\Pi(k^2)\}^{-1}$ where Π is the one-loop self-energy) is given by

$$\delta m^2(l)=\Pi(k^2=-m^2)$$

$$\propto\frac{\delta^2 A^{(2l,l+1)}(2)}{\delta\zeta^1_{\mu\cdots\nu_l}\delta\zeta^{2^{\mu\cdots\nu_l}}}=\frac{\delta^2 A^{(2l,l)}(2)}{\delta\zeta^1_{\mu_1\cdots\nu_{l-1}}\delta\zeta^{2^{\mu_1\cdots\nu_{l-1}}}}$$

$$=(2\pi)^3 Nl\mathrm{PP}\int_{-1}^{1}\frac{d\lambda}{\lambda}\int_0^1 d\nu\left(\frac{\theta_1}{\theta_1'(0)}\right)^2\left(\frac{\theta_1\theta_1''-\theta_1'^2}{\theta_1'^2(0)}\right)^{l-1}. \quad (5\cdot21)$$

Mass shifts of massive particles do not vanish at least at the integrand level in sharp contrast with the massless case, although they are finite. Putting $l=1$ into Eq. (5·21), we can rewrite $\delta m^2(1)$ as

$$\delta m^2(1)\propto\int_0^1\frac{d\lambda}{\lambda}[g(\lambda)-g(-\lambda)], \quad (5\cdot22)$$

where

$$g(\lambda)\equiv\int_0^1 d\nu\left[\pi\frac{\theta_1(\nu|\tau)}{\theta_1'(0|\tau)}\right]^2$$

$$=\int_0^1 d\nu\left[\sin\pi\nu\prod_{n=1}^{\infty}\frac{1-2\lambda^n\cos2\pi\nu+\lambda^{2n}}{(1-\lambda^n)^2}\right]^2. \quad (5\cdot23)$$

It is not so difficult to see that $g(\lambda)>g(-\lambda)$ for $0<\lambda<1$. Hence we can explicitly see that δm^2 is really nonvanishing at least for the first excited state.

Furthermore, we find from Eq. (5·21) that $\delta m^2(l)$ has a nontrivial dependence on the level number l in addition to a simple l factor in front of the integral (we can prove that $(\theta_1\theta_1''-\theta_1'^2)/\theta_1'^2(0)\neq1$). If $\delta m^2(l)$ were simply proportional to l, it would mean that Regge trajectory becomes again a straight-line after receiving a one-loop quantum correction. Then one-loop mass shifts could be interpreted as just a one-loop shift of the slope parameter. They might be succesfully absorbed into a slope parameter redefinition: $\alpha'_{1\text{-loop}}=\alpha'_{\text{tree}}+\delta\alpha'$ where α'_{tree} and $\delta\alpha'$ are of order $O(\hbar^0)$ and $O(\hbar)$ respectively. However our result Eq. (5·21) shows that such a program cannot work. Because of the existence of an extra nontrivial l dependence which is purely a one-loop effect, it is impossible to remove the effect of mass shifts only by the slope parameter redefinition, although mass degeneracy between particles with different spins $J=l+1$ and l remains in one-loop approximation at all the mass levels.

§ 6. Summary and discussion

General vertex operators describing the emissions of bosons lying on the leading and the next-to-leading Regge trajectories are constructed and some properties of massive one-loop amplitudes for $O(32)$ open superstring theory are investigated with attention paid especially to the one-loop mass shifts.

First, integrands of massive one-loop amplitudes are generally proved to be meromorphic in λ, namely, a unitarity violating logarithmic cut is absent. Divergences at $\lambda=0$ are canceled if $N=32$ and an appropriate regularization is chosen; the principal part prescription[2] for example. These results are the same as those obtained in a massless case[2)~5)] and are the reflection of the universal fact that the vertex operators have been constructed as superconformal primary fields with unit weight and do not depend on a detailed structure of vertex operators.

Next two-point amplitudes for all the bosons lying on the two trajectories are calculated and the mass shifts formula for them up to a constant factor are obtained. Existence of mass shifts itself is not so surprising. In the new formalism by Green and Schwarz the absence of mass shifts for massless particles is ensured by a trace over the product of a few number of the S_0's, the zero mode of space-time fermion in new formalism. In a massive case, however, there would appear a larger number of the S_0's in a vertex operator, which would not lead to a vanishing trace. Being correspondent to it, in our covariant approach, the integrand of the two-point amplitude is really not reduced to a vanishing identity for theta function in contrast to a massless case.[3] Anyhow, using the factorization,[7] it is established that any massive one-loop amplitudes for $O(32)$ open superstring with more than three external legs are generally divergent although the higher point ($M\geq 3$) amplitudes are not checked directly. The divergence would come from the singularity at the boundary of relative Teichmüller parameters' regions.

With use of relations among theta functions we can see that the mass shifts $\delta m^2(l)$ turned out to be identical for the first two leading trajectories, in other words, the mass degeneracy between them remains even at one-loop level for all the excited levels. For other trajectories, since more fermion fields Ψ^μ appear in the corresponding vertex operators, we are not readily able to reduce the two-point amplitudes to compact forms with the help only of the well-known identities for theta functions and to show whether the mass degeneracy generally holds or not. Hence, though the existence of the mass degeneracy at loop level might have a general reason, for the present it seems rather accidental for the first two leading trajectories.

To be noted is the existence of a nontrivial l dependence of $\delta m^2(l)$. In addition to a simple l factor, l appears in the integrand as an exponent of a certain combination of theta functions. Although we cannot see the l dependence explicitly since carrying out the integration is difficult, this fact strongly suggests that $\delta m^2(l)$ is not proportional to l.

Here we comment on the influence of the normalization ambiguity in vertex operators on $\delta m^2(l)$. The value of mass shift $\delta m^2(l)$ is really dependent on the normalization of each vertex operator. We fixed it in a simple way based on a normalization of a physical state constructed by acting the vertex operator on vaccum. Although, properly speaking, the normalization should be determined by the unitarity, for one-loop calculation only the tree-level unitarity is sufficient. Hence the modification of the normalization, even if it exists, would be a kinematical one[16) and so would never change the qualitative one-loop results, namely, the existence of mass shifts, the mass degeneracy between the first two leading trajectories and the existence of the extra nontrivial l dependence of $\delta m^2(l)$.

Thus, in a strong possibility, we are led to a conclusion that at one-loop level Regge trajectories form parallel polygonal lines instead of straight-lines at least for the first two leading trajectories. The mass shifts cannot be interpreted as the shift of the value of the single parameter α'. Then, as Weinberg discussed in the case of tachyon one-loop amplitudes in the closed bosonic string,[7) we would have to devise a renormalization procedure for vertex operators at every mass level in the superstring theory. Although it is still obscure whether there exists a freedom in a field theory of strings, the properties of mass shifts studied here, such as their dependence on mass and spin, may then be important for such a renormalization.

Anyhow further investigations are required to confirm the perturbative consistency of superstring theory.

After completion of this work we received the paper, Ref. 17), where the three point massive (two massless and one massive scalar) amplitude of type-II closed superstring in a light-cone gauge is constructed by factorizing the four massless graviton amplitudes. There the singularity in the limit $|z_1-z_2|=\varepsilon\to 0$ (z_1 and z_2 are the positions of masless vertex operators) is explicitly observed.

Acknowledgements

The author would like to thank K. Fujikawa, H. Hata, T. Kugo, T. Kuramoto, N. Nakazawa, S. Nojiri, T. Onogi, K. Sakai, N. Sakai, M. Sakamoto, S. Yahikozawa, I. Yamanaka and T. Yoneya for valuable discussions and comments, and R. Sasaki for careful reading of the manuscript and discussion. He is also indebted to the Japan Society for the Promotion of Science for financial aid. This work is partly supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (62790122).

Appendix

We present here the various two-point correlation functions of elementary conformal fields and their properties under Jacobi transformation (4·1).

Bosonic two-point correlation functions

The correlation function of $X^\mu(\rho, d, \bar{d})$ is given by Eq. (3·17). Differentiating it with respect to $\ln\rho_{ji}$, we get

$$\langle P^\mu(\rho_i, d, \bar{d})X^\nu(\rho_j, d, \bar{d})\rangle = -\eta^{\mu\nu}G(\rho_{ji}, w)$$

$$= -i\eta^{\mu\nu}\left[\frac{1}{2}-\frac{\ln\rho_{ji}}{\ln w}+\sum_{n=1}^{\infty}\frac{(\rho_{ji})^n-(w\rho_{ij})^n}{1-w^n}\right] \qquad (\mathrm{A}\cdot 1)$$

for $|\rho_{ji}|<1$, where

$$G(x,w)=-i\frac{\partial}{\partial\ln x}\ln\psi(x,w), \tag{A·2}$$

$$G(x^{-1},w)=-G(x,w)=-G\left(\frac{w}{x},w\right) \tag{A·3}$$

and

$$\langle P^{\mu}(\rho_i,d,\bar{d})P^{\nu}(\rho_j,d,\bar{d})\rangle\equiv\eta^{\mu\nu}\chi^{P}(\rho_{ji},w)$$

$$=i\eta^{\mu\nu}\frac{\partial}{\partial\ln\rho_{ji}}G(\rho_{ji},w)=\eta^{\mu\nu}\left[-\frac{1}{\ln w}+\sum_{n=1}^{\infty}n\frac{(\rho_{ji})^n+(w\rho_{ij})^n}{1-w^n}\right] \tag{A·4}$$

for $|\rho_{ji}|<1$.

Fermionic two-point correlation functions

Two point correlation functions of fermions in the Neveu-Schwarz and the Ramond sectors are given by

$$\langle H^{\mu}(\rho_i,d,\bar{d})H^{\nu}(\rho_j,d,\bar{d})\rangle\equiv\eta^{\mu\nu}\chi(\rho_{ji},w)=\eta^{\mu\nu}\sum_{r=1/2}^{\infty}\frac{(\rho_{ji})^r+(w\rho_{ij})^r}{1+w^r}\quad|\rho_{ji}|<1$$

$$=\frac{i}{2}\eta^{\mu\nu}\frac{\theta_2(0|\tau)\theta_4(0|\tau)\theta_3(v_{ji}|\tau)}{\theta_1(v_{ji}|\tau)} \tag{A·5}$$

and

$$-\frac{1}{2}\langle\Gamma^{\mu}(\rho_i,d,\bar{d})\Gamma^{\nu}(\rho_j,d,\bar{d})\rangle\equiv\eta^{\mu\nu}\chi_0(\rho_{ji},w)=\eta^{\mu\nu}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(\rho_{ji})^n+(w\rho_{ij})^n}{1+w^n}\right]_{|\rho_{ji}|<1}$$

$$=\frac{i}{2}\eta^{\mu\nu}\frac{\theta_3(0|\tau)\theta_4(0|\tau)\theta_2(v_{ji}|\tau)}{\theta_1(v_{ji}|\tau)}, \tag{A·6}$$

where $v_{ji}=\ln\rho_{ji}/2\pi i$ and the vacuum expectation value in (A·6) is defined to include a normalized trace over γ matrices, i.e., $\langle f(\gamma^{\mu})\rangle\equiv2^{-D/2}\mathrm{Tr}f(\gamma^{\mu})$. We also use the relations obtained by differentiating them with respect to $\ln\rho_{ji}$. For example,

$$\langle\dot{H}^{\mu}(\rho_i,d,\bar{d})H^{\nu}(\rho_j,d,\bar{d})\rangle=\eta^{\mu\nu}\frac{\partial}{\partial\ln\rho_i}\chi(\rho_{ji},w)\equiv\eta^{\mu\nu}\dot{\chi}(\rho_{ji},w)$$

$$=-\frac{\eta^{\mu\nu}}{4\pi}\frac{\theta_2(0)\theta_4(0)[\theta_3'(v_{ji})\theta_1(v_{ji})-\theta_3(v_{ji})\theta_1'(v_{ji})]}{\theta_1^{2}(v_{ji})} \tag{A·7}$$

and

$$\langle\dot{H}^{\mu}(\rho_i,d,\bar{d})\dot{H}^{\nu}(\rho_j,d,\bar{d})=-\eta^{\mu\nu}\frac{\partial^2}{\partial(\ln\rho_{ji})^2}\chi(\rho_{ji},w)$$

$$=\frac{i\eta^{\mu\nu}}{8\pi^2}\theta_2(0)\theta_4(0)\frac{\theta_1(\theta_3''\theta_1-\theta_1''\theta_3)-2\theta_1'(\theta_3'\theta_1-\theta_3\theta_1')}{\theta_1^{3}}, \tag{A·8}$$

where $\theta_1\equiv\theta_1(v_{ji})$ and θ_1' and θ_1'' are its first and second derivatives, respectively.

Jacobi transformation

The transformation property of the plane wave factors are given by Eqs. (4·5) and (4·6). $G(\rho_{ji}, w)$ and $\chi^P(\rho_{ji}, w)$ are transformed as

$$\eta^{\mu\nu}G(\rho_{ij}, w)=\eta^{\mu\nu}\tau^{-1}\hat{G}(\rho'_{ij}, w')$$
$$=\langle \hat{P}^\mu(\rho_i', d(w'), \bar{d}(w')\hat{X}^\nu(\rho_j', d(w'), \bar{d}(w')\rangle, \tag{A·9}$$

$$\eta^{\mu\nu}\chi^P(\rho_{ji}, w)=\eta^{\mu\nu}\tau^{-2}\hat{\chi}^P(\rho'_{ji}, w')$$
$$=\langle \hat{P}^\mu(\rho_j', d(w'), \bar{d}(w')\hat{P}^\nu(\rho_i', d(w'), \bar{d}(w')\rangle, \tag{A·10}$$

where

$$\hat{G}(x, w)\equiv -i\frac{\partial}{\partial \ln x}\ln\hat{\psi}(x, w), \tag{A·11}$$

$$\hat{\chi}^P(x, w)\equiv i\frac{\partial}{\partial \ln x}\hat{G}(x, w), \tag{A·12}$$

$$\hat{X}_\mu(\rho, d, \bar{d})\equiv\frac{1}{\sqrt{2\varepsilon}}(a_\mu{}^{0\dagger}\rho^\varepsilon+a_\mu{}^0\rho^\varepsilon)+\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(\bar{d}_\mu{}^n\rho^n+d_\mu{}^n\rho^{-n}), \tag{A·13}$$

$$\hat{P}_\mu(\rho, d, \bar{d})=i\frac{\partial}{\partial \ln\rho}\hat{X}_\mu(\rho, d, \bar{d}). \tag{A·14}$$

Similar relations hold for Möbius part, i.e.,

$$G(\rho_{ji}, -w)=(2\tau)^{-1}\hat{G}(\rho''_{ji}, -w''), \tag{A·15}$$
$$\chi^P(\rho_{ji}, -w)=(2\tau)^{-2}\hat{\chi}^P(\rho''_{ji}, -w''). \tag{A·16}$$

The transformation law of fermion correlation functions is given in Ref. 3) as

$$\begin{aligned}
&\chi(\rho, w)=\tau^{-1}\chi(\rho', w'),\\
&\chi(\rho, we^{2\pi i})=\tau^{-1}\chi_0(\rho', w'),\\
&\chi_0(\rho, w)=\tau^{-1}\chi(\rho', w'e^{2\pi i}),\\
&\chi(\rho, -w)=(2\tau)^{-1}\chi(\rho'', -w''e^{2\pi i}),\\
&\chi(\rho, -we^{2\pi i})=(2\tau)^{-1}\chi(\rho'', -w''),\\
&\chi_0(\rho, -w)=(2\tau)^{-1}\chi_0(\rho'', -w'').
\end{aligned} \tag{A·17}$$

These homogenuity properties in six spin structures are maintained also for the first and second derivatives of such fermion correlation function. Only the "order"[18] of homogenuity changes as the weight of the conformal field changes. For example we have

$$\begin{aligned}
&\dot{\chi}(\rho_{ji}, w)=\tau^{-2}\dot{\chi}(\rho'_{ji}, w'),\\
&\ddot{\chi}(\rho_{ji}, w)=\tau^{-3}\ddot{\chi}(\rho'_{ji}, w').
\end{aligned} \tag{A·18}$$

References

1) See for example, E. D'Hoker and D. H. Phong, Nucl. Phys. **B269** (1986), 205.
2) M. B. Green and J. H. Schwarz, Phys. Lett. **151B** (1985), 21.
P. C. Frampton, P. Moxhay and Y. J. Ng, Phys. Rev. Lett. **55** (1985), 21; Nucl. Phys. **B276** (1986), 599.
P. H. Frampton, Y. Kikuchi and Y. J. Ng, Phys. Lett. **174B** (1986), 262.
S. Yahikozawa, Phys. Lett. **166B** (1986), 2575; UT-493.
D. J. Gross, J. A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. **B267** (1986), 75.
C. S. Lam and Da-Xi Li, Phys. Rev. Lett. **56** (1986), 2575.
L. Clavelli, P. H. Cox and B. Harms, Phys. Rev. **D35** (1987),1908; Nucl. Phys. **B289** (1987), 445.
N. Ohta, Phys. Rev. Lett. **59** (1987), 176.
C. G. Callan, C. Lovalace, C. R. Nappi and S. A. Yost, Nucl. Phys. **B293** (1987), 83, and other references.
3) L. Clavelli, Phys. Rev. **D33** (1986), 1098; Prog. Theor. Phys. Suppl. No. 86 (1986), 135.
4) C. P. Burgess, Nucl. Phys. **B284** (1987), 605.
5) H. Yamamoto, Y. Nagahama and N. Nakazawa, Prog. Theor. Phys. **76** (1986), 1150.
R. Potting and J. Shapiro, Phys. Rev. **D34** (1986), 3260.
L. Clavelli, Phys. Rev. **D34** (1986), 3262.
P. H. Frampton, A. K. Kshirsagar and Y. J. Ng., Phys. Rev. **D34** (1986), 2532.
N. Nakazawa and H. Yamamoto, Prog. Theor. Phys. **77** (1987), 132.
6) J. H. Schwarz, Phys. Rep. **89** (1982), 223.
E. Martinec, Phys. Lett. **171B** (1986), 189.
M. A. Namazie, K. S. Narain and M. H. Sarmadi, Phys. Lett. **B177** (1986), 329.
Y. Tanii, Phys. Lett. **B179** (1986), 251.
M. Dine and N. Seiberg, Phys. Rev. Lett. **57** (1986), 2625.
J. J. Atick and A. Sen, Phys. Lett. **186B** (1987), 339.
S. Carlip, Phys. Lett. **186B** (1987), 141.
7) S. Weinberg, in *Proc. Oregon Meeting* (Eugene, Oregon, August 1985), ed. R. C. Hwa (World Scientific, Singapore, 1986), p. 850.
8) N. Seiberg and E. Witten **B276** (1986), 272.
9) N. Seiberg, Phys. Lett. **187B** (1987), 56.
10) J. A. Shapiro, Phys. Rev. **D11** (1975), 2937.
11) L. Clavelli and J. A. Shapiro, Nucl. Phys. **B57** (1973), 490.
12) For example, D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. **B271** (1986), 93.
R. Sasaki and I. Yamanaka, Phys. Lett. **165B** (1985), 283.
13) A. Neveu and J. H. Schwarz, Nucl. Phys. **B31** (1971), 86.
A. Neveu, J. H. Schwarz and C. B. Thorn, Phys. Lett. **B35** (1971), 529.
14) P. Ramond, Phys. Rev. **D3** (1971), 2415.
15) E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis,* 4th edition reprinted (Cambridge University Press, 1952).
16) N. Sakai and Y. Tanii, Nucl. Phys. **B287** (1987), 457.
17) O. Foda, Phys. Lett. **191B** (1987), 75.
18) L. Clavelli, Alabama Preprint, UA HEP871.