One-loop partition functions of 3D gravity

Simone Giombi

Harvard University

UNC-CH, Dec. 4th 2008

Outline

- Overview and motivation
  - The partition function of pure $3D$ gravity
- Heat Kernel method
- “Warm-up” examples: Scalar and Vector fields
- Gravity one-loop partition function
- Extension to higher genus
- Conclusion

(Based on arXiv:0804.1773, with A. Maloney and X. Yin)

Overview and motivation

- The general problem we want to address is the computation of the Euclidean partition function

$$Z = \int \mathcal{D}\phi e^{-\frac{1}{g^2}S(\phi)} ,$$

where ϕ is a quantum field living on some manifold $\mathcal{M}$ which is *locally* Euclidean $AdS_3 = \mathbb{H}_3$ (3D hyperbolic space) i.e.

$$\mathcal{M} = \mathbb{H}_3/\Gamma , \quad \Gamma \subset SL(2,\mathbb{C}) \text{ (\textit{discrete} subgroup of isometries)}$$

- Our results are quite general, may be extended to supergravities or even to higher dimensions.
One of the most interesting applications, which was the motivation for this work:

Problem of pure 3D Quantum Gravity with $\Lambda < 0$

Pure 3D Gravity

- Pure 3D gravity with negative cosmological constant is described by the action

$$S = -\frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right)$$

Recently there has been a renewed interest in this theory, following the work of *Witten '07*.

- Basic question is: does this minimal theory exist, i.e. make sense as a quantum theory? If yes, can we solve it explicitly? By AdS/CFT, one may alternatively ask what is the CFT dual to pure 3D gravity?

Pure 3D Gravity

- At first sight, the theory may look quite trivial. In $3D$

$$R_{\mu\nu\rho\sigma} \leftrightarrow R_{\mu\nu}$$

Then by Einstein's equations

$$R_{\mu\nu} = -\frac{2}{\ell^2} g_{\mu\nu}$$

every classical solution $\mathcal{M}$ is a hyperbolic manifold, $\mathcal{M} = \mathbb{H}_3/\Gamma$.

It also follows that there are no propagating degrees of freedom!

However things are more subtle:

- No gravitons *in the bulk*, but there are *"boundary excitations"* associated with the asymptotic Virasoro symmetry (sign of dual CFT!) discovered by Brown and Henneaux.

- Moreover, the theory has black hole solutions (BTZ).

The partition function

- So the theory is not so trivial, but maybe simple enough that we can solve it!
  Is it possible to explicitly compute the gravity path integral?
  Note that unlike in higher dimensions, pure 3D gravity is renormalizable.

- To define the gravity path integral, one fixes the asymptotic boundary conditions and then sum over all possible interiors.
  The partition function may be written as

$$Z_{\text{gravity}}\left(\partial\mathcal{M}=\Sigma\right) = \sum_{\text{Class. solutions}\ \mathcal{M}_i} \int_{\partial\mathcal{M}_i=\Sigma} \mathcal{D}g\, e^{-kS(g)} \qquad \left(k\equiv\tfrac{\ell}{16G}\right)$$

$$= \sum_{\mathcal{M}_i} \exp\left[-kS^{(0)}(\mathcal{M}_i) + S^{(1)}(\mathcal{M}_i) + \frac{1}{k}S^{(2)}(\mathcal{M}_i) + \ldots\right] = Z_{\text{CFT}}(\Sigma)$$

The partition function

$$Z_{\text{gravity}} = \sum_{\mathcal{M}_i} \exp\left[-kS^{(0)}(\mathcal{M}_i) + S^{(1)}(\mathcal{M}_i) + \frac{1}{k}S^{(2)}(\mathcal{M}_i) + \ldots\right]$$

- In this talk, I will not be concerned with the sum over geometries and related problems (*Maloney-Witten '07, Yin '07*)
- Instead, we consider fixing a particular geometry and focus on the calculation of the one-loop piece $S^{(1)}(\mathcal{M}_i)$.
- We find that this can be evaluated precisely for a large class of geometries and the results are consistent with the structure expected in a CFT.

Thermal *AdS*

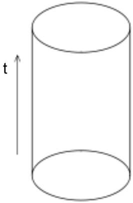
- The easiest case to start with is the one in which $\partial\mathcal{M} = T^2$. The simplest manifold with such boundary is Thermal AdS_3.

Take Euclidean AdS_3

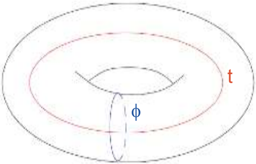
$ds^2 = \cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2$

with the identification $(t, \phi) \sim (t + \beta, \phi + \theta)$

β : Inverse temperature, θ : Angular potential



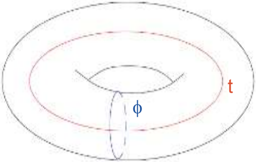
- The resulting space is a solid torus with a metric of constant negative curvature



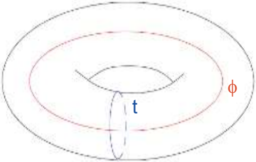
- The boundary $\partial\mathcal{M}$ is a T^2 with modulus $\tau = \frac{1}{2\pi}(\theta + i\beta)$.
- As a quotient, this is just $\mathbb{H}_3/\mathbb{Z}$.

Thermal AdS vs BTZ black hole

- Note that the BTZ black hole can also be described as a solid torus. It is related to thermal AdS simply by $\tau \to -\frac{1}{\tau}$.
- One can do a coordinate transformation so that $\mathrm{Im}\tau$ is the periodicity of a new time variable. The effect is basically to interchange contractible and non-contractible cycles



Thermal AdS



BTZ

- More generally there is a whole $SL(2,\mathbb{Z})$ family of black holes obtained from thermal AdS by

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,.$$

Partition function on thermal *AdS*

- As it is familiar in quantum field theory, the partition function on this geometry (empty AdS_3 with $(t, \phi) \sim (t + \beta, \phi + \theta)$) has a simple hamiltonian interpretation

$$Z(\tau, \bar{\tau}) = \mathrm{Tr}\, e^{-\beta H - i\theta J} \qquad \begin{cases} H: & \text{Energy} \\ J: & \text{Angular momentum} \end{cases}$$

where the trace is taken over small fluctuations around *empty* AdS_3.

- At first sight, there are no excitations (no gravitons!). But from Brown and Henneaux analysis we know that there is an *asymptotic Virasoro symmetry* with generators L_n, $\bar{L}_n$ and central charge $c = \frac{3\ell}{2G} \equiv 24k$.

Partition function on thermal AdS

- This means that the Hilbert space consists of "boundary excitations" which are the Virasoro descendants of the $SL(2,\mathbb{C})$ invariant vacuum (i.e. empty AdS)

$$\prod_{n,m\geq 2} L_{-n}^{a_n} \bar{L}_{-m}^{\bar{a}_m} |0\rangle \,, \qquad a_m \,, \bar{a}_m \geq 0$$

- The partition function then takes the form of a Virasoro character

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \,, \qquad \begin{cases} L_0 + \bar{L}_0 - c/12 = H, \; L_0 - \bar{L}_0 = J \\ q = e^{2\pi i \tau} \end{cases}$$

- The answer is

$$Z(\tau,\bar{\tau}) = |q|^{-2k} \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2}$$

Partition function on thermal AdS

- This was an *indirect* way to obtain the partition function based on Brown and Henneaux results.
- Comparing to the perturbative expansion

$$Z = e^{-kS^{(0)} + S^{(1)} + \frac{1}{k} S^{(2)} + \dots} = |q|^{-2k} \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2}$$

$e^{-kS^{(0)}}$ $e^{S^{(1)}}$ $(S^{(2)}, S^{(3)} \dots = 0)$

- The calculation of the classical action $S^{(0)}$ is well understood. On the other hand $S^{(1)}$ has never been computed directly! (Some attempts were made, e.g. *Carlip-Teitelboim '95, Bytsenko et al '98*, but results appear to be wrong!)
- We will directly obtain $S^{(1)}$ from the relevant functional determinants.
- This gives an independent check of Brown and Henneaux analysis, and moreover can be extended to the case of higher genus conformal boundaries.

The heat kernel method

- The problem of evaluating a one-loop partition function basically reduces to the computation of functional determinants

$$Z_{\text{1-loop}} = \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int_{\mathcal{M}} d^3x \sqrt{g}\phi\Delta\phi} = (\det \Delta)^{-\frac{1}{2}}$$

but in practice this may still be a rather difficult task...

- The most straightforward approach would be to solve the spectral problem

$$\Delta\psi_n = \lambda_n \psi_n \quad \rightarrow \quad \det \Delta = \prod_n \lambda_n$$

but in general this may be very laborious and not very convenient.

The heat kernel method

- Instead we use the *heat kernel approach*. One defines the heat kernel as

$$K(t,x,x') = \langle x|e^{-t\Delta}|x'\rangle\,, \qquad x,x' \in \mathcal{M}$$

- Then it is standard to extract the determinant

$$-\log\det\Delta = \int_0^\infty \frac{dt}{t}\mathrm{Tr}e^{-t\Delta} = \int_0^\infty \frac{dt}{t}\int_{\mathcal{M}} d^3x\sqrt{g}\,K(t,x,x)\,.$$

- The advantage is that one can simply define $K(t,x,x')$ as the solution to the "heat equation"

$$(\partial_t + \Delta_x)\,K(t,x,x') = 0$$
$$K(0,x,x') = \delta^3(x,x')$$

The heat kernel method

- In general it is still a daunting task to solve the heat equation. But it turns out that it is not too difficult to find the solution for $\mathcal{M} = \mathbb{H}_3/\Gamma$.
- One starts with finding the heat kernel on $\mathbb{H}_3$, which can be done because $\mathbb{H}_3$ is maximally symmetric.
- The solution on $\mathcal{M} = \mathbb{H}_3/\Gamma$ can then be obtained by the method of images

$$\boxed{K^{\mathbb{H}_3/\Gamma}(t, x, x') = \sum_{\gamma \in \Gamma} K^{\mathbb{H}_3}(t, x, \gamma x')}$$

- In fact, this is the same idea lying behind the *Selberg trace formula*. However as far as we know this mathematical theory has not been extended to the case of the graviton.
- So our results may also be viewed as a brute force derivation of a "new Selberg trace formula" for gravity.

Scalar Field

- Consider a scalar field on $\mathbb{H}_3$ with action

$$S = \frac{1}{2}\int_{\mathbb{H}_3} d^3x\sqrt{g}\left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right)$$
$$= \frac{1}{2}\int_{\mathbb{H}_3} d^3x\sqrt{g}\phi\left(-\nabla^2 + m^2\right)\phi$$

- So we want to solve

$$\left(\nabla^2 - m^2\right)K(t,x,x') = \partial_t K(t,x,x')\,, \qquad K(0,x,x') = \delta^3(x,x')$$

- It will be convenient to work with the Poincare metric on $\mathbb{H}_3$

$$ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2}$$

Scalar Field heat kernel

- Because $\mathbb{H}_3$ is maximally symmetric the heat kernel depends only on the geodesic distance $r(x, x')$

$$K(t, x, x') = K(t, r(x, x'))$$

where

$$r(x, x') = \text{arccosh}\left(1 + u(x, x')\right) \ , \ \ u(x, x') = \frac{(y - y')^2 + |z - z'|^2}{2yy'}$$

- Then the heat equation becomes a simple differential equation in r, t whose solution is

$$K^{\mathbb{H}_3}(t, r) = \frac{e^{-(m^2+1)t - \frac{r^2}{4t}}}{(4\pi t)^{3/2}} \frac{r}{\sinh r}$$

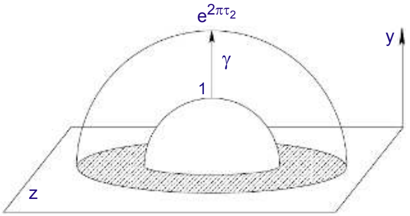
One-loop determinant on $\mathbb{H}_3/\mathbb{Z}$

- In the Poincare coordinates, thermal *AdS* with $\tau = \frac{1}{2\pi}(\theta + i\beta)$ can be described as the quotient of $\mathbb{H}_3$ by the group $\mathbb{Z} \subset SL(2,\mathbb{C})$ generated by

$$\gamma(y,z) \to (|q|^{-1}y\,,\, q^{-1}z)\,, \qquad q = e^{2\pi i\tau}$$

boundary torus: $z = e^{-2\pi i w} \to w \sim w+1 \sim w+\tau$

- Fundamental domain



$$y = \rho\sin\theta,\; z = \rho\cos\theta e^{i\phi}$$
$$1 < \rho < e^{2\pi\tau_2}\,, 0 < \theta < \pi/2\,, 0 < \phi < 2\pi.$$

Scalar one-loop determinant on $\mathbb{H}_3/\mathbb{Z}$

- From the heat kernel on $\mathbb{H}_3$ we can get

$$K^{\mathbb{H}_3/\mathbb{Z}}(t,x,x') = \sum_{n\in\mathbb{Z}} K^{\mathbb{H}_3}(t, r(x,\gamma^n x'))$$

- Then we can compute

$$\begin{aligned} -\log\det\Delta &= \int_0^\infty \frac{dt}{t}\int_{\mathbb{H}_3/\mathbb{Z}} d^3x\sqrt{g}K^{\mathbb{H}_3/\mathbb{Z}}(t,x,x) \\ &= \mathrm{vol}(\mathbb{H}_3/\mathbb{Z})\int_0^\infty \frac{dt}{t}\frac{e^{-(m^2+1)t}}{(4\pi t)^{3/2}} \qquad (n{=}0) \\ &+ \sum_{n\neq 0}\int_0^\infty \frac{dt}{t}\int_{\mathbb{H}_3/\mathbb{Z}} d^3x\sqrt{g}K^{\mathbb{H}_3}(t, r(x,\gamma^n x)) \,. \end{aligned}$$

- The term proportional to the volume is divergent but just corresponds to a renormalization of the cosmological constant. It will be disregarded from now on.

Scalar one-loop determinant on $\mathbb{H}_3/\mathbb{Z}$

- The sum over $n \neq 0$ is more interesting. After a change of variable $r(x, \gamma^n x) \leftrightarrow \theta$ the integral over the manifold can be done

$$
\begin{aligned}
-\log\det\Delta &= 2\sum_{n=1}^{\infty}\frac{(2\pi\tau_2)(2\pi)}{4|\sin\pi n\tau|^2}\int_0^{\infty}\frac{dt}{t}\,\frac{e^{-(m^2+1)t-\frac{(2\pi n\tau_2)^2}{4t}}}{4\pi^{\frac{3}{2}}\sqrt{t}} \\
&= 2\sum_{n=1}^{\infty}\frac{|q|^{2nh}}{n|1-q^n|^2}\,,
\end{aligned}
$$

where $h = \frac{1}{2}(1+\sqrt{1+m^2})$. Note that this is just the standard AdS/CFT formula for the dimension of the primary operator dual to the scalar field!

Scalar partition function

- The scalar one-loop partition function is then

$$Z^{\text{1-loop}}_{\text{scalar}}(\tau,\bar{\tau}) = (\det \Delta)^{-1/2} = \exp\left(\sum_{n=1}^{\infty} \frac{|q|^{2nh}}{n|1-q^n|^2}\right)$$

$$= \boxed{\prod_{\ell,\ell'=0}^{\infty} \frac{1}{1-q^{\ell+h}\bar{q}^{\ell'+h}}}$$

- This is precisely the expected result and has a natural interpretation in the language of the CFT. Indeed it takes the form

$$\mathrm{Tr}\, q^{L_0}\bar{q}^{\bar{L}_0}$$

and it is not difficult to see what are the states that should contribute to the trace.

Scalar partition function

- In AdS/CFT a scalar field with mass m is dual to a primary operator $|\phi\rangle$ with weight (h, h)

$$L_0|\phi\rangle = \bar{L}_0|\phi\rangle = h\,.$$

- One can then construct the $SL(2,\mathbb{C})$-descendants

$$L_{-1}^{\ell}\bar{L}_{-1}^{\ell'}|\phi\rangle\,, \qquad \ell,\ell' \geq 0\,, \qquad \text{weight: } (h+\ell, h+\ell')\,.$$

- These are single-particle states. Summing over all multi-particle insertions one gets the partition function

$$\prod_{\ell,\ell'=0}^{\infty}\sum_{n=0}^{\infty} q^{n(\ell+h)}\bar{q}^{n(\ell'+h)} = \prod_{\ell,\ell'=0}^{\infty}\frac{1}{1-q^{\ell+h}\bar{q}^{\ell'+h}}$$

which indeed is the same as the heat kernel calculation.

Vector Field

- Consider a $U(1)$ gauge field on $\mathbb{H}_3$ (in Feynman gauge)

$$S = \int_{\mathbb{H}_3} d^3x\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\nabla_\mu A^\mu)^2\right)$$
$$= \frac{1}{2}\int_{\mathbb{H}_3} d^3x\sqrt{g}A_\mu\left(-g^{\mu\nu}\nabla^2 + R^{\mu\nu}\right)A_\nu$$

  and there are of course also the usual massless scalar ghosts b, c with standard kinetic operator.

- The heat kernel for the vector kinetic operator is now a *bi-tensor*

$$K_{\mu\mu'}(t,x,x') = \langle \mu\, x|e^{-t\Delta^{(1)}}|\mu'\, x'\rangle\,,$$

  which by the isometries of $\mathbb{H}_3$ is completely specified by 2 scalar functions

$$K_{\mu\mu'}(t,u(x,x')) = \partial_\mu\partial_{\mu'}u\, F(t,u) + \partial_\mu\partial_{\mu'}\, S(t,u)$$

Vector Field heat kernel

- The heat equation for the vector kinetic operator is

$$\left(\nabla^2 + 2\right) K_{\mu\mu'} = \partial_t K_{\mu\mu'} \,, \qquad K_{\mu\mu'}(0, u(x, x')) = g_{\mu\mu'} \delta^3(x, x')$$

- This reduces to 2 coupled differential equations for $F(t, u)$ and $S(t, u)$

$$\begin{aligned} \left(\nabla^2 + 1\right) F(t, u) &= \partial_t F(t, u) \\ \nabla^2 S(t, u) - 2 \int_u^\infty du' \, F(t, u') &= \partial_t S(t, u) \,. \end{aligned}$$

  and it turns out that it is not too difficult to solve these equations to obtain the exact vector heat kernel $K_{\mu\mu'}(t, u)$.

- Incidentally, this exact solution for the vector heat kernel seems to be a new result.

Vector one-loop determinant on $\mathbb{H}_3/\mathbb{Z}$

- Since we have the heat kernel on $\mathbb{H}_3$ we can as before use the method of images to get the heat kernel on $\mathbb{H}_3/\mathbb{Z}$ and compute the vector one-loop determinant on the solid torus.
- The final result is again quite simple

$$-\frac{1}{2}\log\det\Delta^{(1)} = \sum_{n=1}^{\infty} \frac{2\cos(2\pi n\tau_1) + e^{-2\pi n\tau_2}}{4n|\sin\pi n\tau|^2} = \sum_{n=1}^{\infty} \frac{q^n + \bar{q}^n + |q|^{2n}}{n|1-q^n|^2}$$

The last term is the contribution of the longitudinal mode. The rest corresponds to the transverse part of the vector, and precisely agrees with the result of the Selberg trace formula applied to this case.

Gravity

- We start with the Einstein-Hilbert action with $\Lambda < 0$

$$S_{GR} = -\frac{1}{16\pi G}\int d^3x\sqrt{g}(R+2)\,.$$

- We want to path-integrate over metric fluctuations $h_{\mu\nu} \equiv g_{\mu\nu} - g^{(0)}_{\mu\nu}$ around the background solution. It is convenient to separate traceless and trace part of the fluctuations

$$\phi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{3}g_{\mu\nu}h^{\rho}{}_{\rho}, \qquad \phi \equiv h^{\rho}_{\rho}$$

- As usual we have to gauge-fix. We choose

$$S_{GF} = \frac{1}{32\pi G}\int d^3x\sqrt{g}\,\nabla^{\mu}\left(h_{\mu\sigma} - \frac{1}{2}g_{\mu\sigma}h\right)\nabla^{\nu}\left(h_{\nu}{}^{\sigma} - \frac{1}{2}\delta_{\nu}{}^{\sigma}h\right).$$

Gravity

- The gauge-fixed action to quadratic order is

$$-\frac{1}{32\pi G}\int d^3x\sqrt{g}\left\{\frac{1}{2}\phi^{\mu\nu}\left(\nabla^2+2\right)\phi_{\mu\nu}-\frac{1}{12}\phi\left(\nabla^2-4\right)\phi\right\}$$

  here we encounter the usual "wrong sign" problem for the trace mode. We follow the standard prescription and do an analytic continuation $\phi \to i\phi$.

- Along with the gauge fixing procedure, we also have complex valued vector ghosts η_μ with action

$$S_{gh}=-\frac{1}{32\pi G}\int d^3x\sqrt{g}\,\bar{\eta}^\mu\left(\nabla^2-2\right)\eta_\mu$$

  which is the same as the vector field in Feynman gauge with a mass $m^2=4$.

Gravity one-loop partition function

- The one-loop partition function we are after is then the following ratio of determinants

$$Z^{\text{1-loop}}_{\text{gravity}} = \frac{\overbrace{\det \Delta^{(1)}}^{\eta_\mu}}{\sqrt{\underbrace{\det \Delta^{(2)}}_{\phi_{\mu\nu}} \underbrace{\det \Delta^{(0)}}_{\phi}}}$$

- All we need to compute is the heat kernel on $\mathbb{H}_3$ for the kinetic operator $\Delta^{(2)}$ of the symmetric traceless fluctuations $\phi_{\mu\nu}$. This is a rank-two bitensor

$$K_{\mu\nu\,,\mu'\nu'}(t,x,x') = \langle \mu\nu\, x | e^{-t\Delta^{(2)}} | \mu'\nu'\, x' \rangle$$

traceless in each pair of indices $g^{\mu\nu} K_{\mu\nu\,,\mu'\nu'} = g^{\mu'\nu'} K_{\mu\nu\,,\mu'\nu'} = 0$.

Graviton heat kernel

- Because of the isometries of $\mathbb{H}_3$, the heat kernel for the graviton fluctuations can be expressed in terms of 5 scalar functions of t and the chordal distance $u(x, x')$.

$$K^{\mathbb{H}_3}_{\mu\nu,\mu'\nu'}(t,u(x,x')) = (\partial_\mu\partial_{\mu'}u\partial_\nu\partial_{\nu'}u + \partial_\mu\partial_{\nu'}u\partial_\nu\partial_{\mu'}u)G(t,u) + g_{\mu\nu}g_{\mu'\nu'}H(t,u)$$
$$+\nabla_{(\mu}\left[\partial_{\nu)}\partial_{(\mu'}u\partial_{\nu')}u\,X(t,u)\right] + \nabla_{(\mu}\left[\partial_{\nu)}u\partial_{\mu'}u\partial_{\nu'}u\,Y(t,u)\right]$$
$$+\nabla_\mu[\partial_\nu u\,Z(t,u)]g_{\mu'\nu'} + \{\mu\leftrightarrow\mu',\nu\leftrightarrow\nu'\}$$

- The heat equation

$$\left(\nabla^2 + 2\right)K_{\mu\nu,\mu'\nu'} = \partial_t K_{\mu\nu,\mu'\nu'}$$

translates into a system of 5 coupled differential equations for G, H, X, Y, Z (together with the appropriate b.c. at $t = 0$ and a first order constraint coming from the traceless condition).

Graviton heat kernel

- The heat equation gives

$$\nabla^2 G = \partial_t G$$

$$\nabla^2 H - 4H - 4G - 8(u+1)\int_u^\infty du'\, G(t,u') = \partial_t H$$

$$\nabla^2 X + 2(u+1)\partial_u X + 4X + 4(u+1)Y + 4G = \partial_t X$$

$$\nabla^2 Y + 6(u+1)\partial_u Y + 2\partial_u X + 7Y = \partial_t Y$$

$$\nabla^2 Z + 2(u+1)\partial_u Z - Z + 2Y + 4\int_u^\infty du'\, G(t,u') = \partial_t Z.$$

- The solution of these equations is a bit involved, but it can be made completely explicit thus allowing to reconstruct the exact graviton heat kernel.
- As for the vector field, this solution for $K^{\mathbb{H}_3}_{\mu\nu,\mu'\nu'}$ is a new result.

Gravity partition function on $\mathbb{H}_3/\mathbb{Z}$

- We can now apply the method of images to obtain the graviton heat kernel on $\mathbb{H}_3/\mathbb{Z}$ and directly compute the desired partition function.
- After a bit of work, the final result for the one-loop free energy of pure gravity is remarkably simple

$$\begin{aligned}
\log Z^{\text{1-loop}}_{\text{gravity}} &= -\frac{1}{2}\log\det\Delta^{(2)} + \log\det\Delta^{(1)} - \frac{1}{2}\log\det\Delta^{(0)} \\
&= \sum_{n=1}^{\infty} \frac{q^{2n}+\bar{q}^{2n}-|q|^{2n}(q^n+\bar{q}^n)}{n|1-q^n|^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{q^{2n}}{1-q^n}+\frac{\bar{q}^{2n}}{1-\bar{q}^n}\right) \\
&= \boxed{-\sum_{m=2}^{\infty}\log|1-q^m|^2}
\end{aligned}$$

Holomorphically factorized!

Gravity partition function on $\mathbb{H}_3/\mathbb{Z}$

- This is precisely the expected result!

$$Z^{\text{1-loop}}_{\text{gravity}}(\tau,\bar{\tau}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \qquad \checkmark$$

  The functional determinants automatically produce the trace over Virasoro descendants of the vacuum!

- An independent check, purely from a "bulk" calculation, of the Brown and Henneaux analysis. The gravity path integral exhibits the expected structure of a CFT.
- As mentioned before, the calculation applies as well to all solid tori $\mathbb{H}_3/\mathbb{Z}$ related to thermal *AdS* by $SL(2,\mathbb{Z})$ transformations on the boundary modulus τ (e.g. BTZ).

$$Z_{\text{BTZ}}(\tau) = Z_{\text{thermal}}(-1/\tau)$$

Extension to higher genus

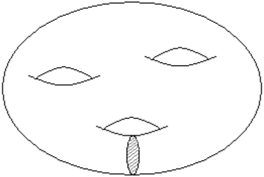
- It is clearly of interest to compute the gravity partition function on more general hyperbolic manifolds $\mathbb{H}_3/\Gamma$ with conformal boundary Σ_g with $g \geq 2$. The knowledge of the partition function for arbitrary g would encode all correlation functions of all operators in the boundary CFT.
- Unlike the genus 1 case, there is no simple hamiltonian route to obtain the partition function, and there is also no argument saying that the partition function is one-loop exact.

$$Z_{\text{gravity}} = \sum_{\mathcal{M}_i \text{ s.t. } \partial\mathcal{M}_i = \Sigma_g} \exp\left[-kS^{(0)}(\mathcal{M}_i) + S^{(1)}(\mathcal{M}_i) + \frac{1}{k}S^{(2)}(\mathcal{M}_i) + \ldots\right]$$

- Our methods are however powerful enough that we can compute the one-loop piece $S^{(1)}$ precisely for a large class of quotients $\mathbb{H}_3/\Gamma$.

Extension to higher genus

- At $g = 1$, the solid torus $\mathbb{H}_3/\mathbb{Z}$ is the only smooth hyperbolic manifold without cusps. If one allows cusps, then there is also $\mathbb{H}_3/\mathbb{Z} \times \mathbb{Z}$ (defined by $z \sim z + 1 \sim z + \tau$, this is the "zero mass BTZ black hole").
- The case $g \geq 2$ is more complicated. The simplest class of such manifolds are the *handlebodies*



- In this case $\Gamma \subset SL(2, \mathbb{C})$ is freely generated by g loxodromic elements $\gamma_1, \gamma_2, \ldots, \gamma_g$.

- We will consider in general hyperbolic manifolds $\mathbb{H}_3/\Gamma$ which are *smooth* and *without cusps*, so we assume
  1. Γ contains only loxodromic elements
  2. Γ does not contain a $\mathbb{Z} \times \mathbb{Z}$ subgroup.

One-loop determinants on $\mathbb{H}_3/\Gamma$

- By the method of images we may again write down

$$-\log\det\Delta = \int_0^\infty \frac{dt}{t} \sum_{\gamma\in\Gamma} \int_{\mathcal{F}} d^3x\sqrt{g}\,K^{\mathbb{H}_3}(t, r(x,\gamma x))$$

  where $\mathcal{F}$ is a fundamental domain for Γ in $\mathbb{H}_3$.

- Under the technical assumption that Γ does not contain $\mathbb{Z}\times\mathbb{Z}$, it can be shown that the sum can be recast as a sum over "primitive conjugacy classes"

$$-\log\det\Delta = \sum_{\gamma\in\mathcal{P}} \int_0^\infty \frac{dt}{t} \sum_{n=1}^{\infty} \int_{\mathcal{F}_\gamma} d^3x\sqrt{g}\,K^{\mathbb{H}_3}(t, r(x,\gamma^n x))$$

  - $\mathcal{P}$ is the set of primitive conjugacy classes (γ is primitive if $\gamma \neq \beta^n$ for $\beta\in\Gamma$ and $n>1$)
  - $\mathcal{F}_\gamma$ is a fundamental domain for the group $\mathbb{Z}$ generated by γ.

The one-loop partition function on $\mathbb{H}_3/\Gamma$

- Now except for the sum over primitive conjugacy classes, we just have the one-loop determinant on the solid torus $\mathbb{H}_3/\langle\gamma\rangle$, which we have already computed! So we get

$$Z^{\mathbb{H}_3/\Gamma} = \prod_{\gamma\in\mathcal{P}} \left(Z^{\mathbb{H}_3/\langle\gamma\rangle} \right)^{\frac{1}{2}}$$

  The derivation applies to scalar, vector as well as graviton fluctuations.
- For γ loxodromic, one can take $\gamma \sim \begin{pmatrix} q_\gamma^{1/2} & 0 \\ 0 & q_\gamma^{-1/2} \end{pmatrix}, \ |q_\gamma|<1.$

  The gravity one-loop partition function is then

$$\boxed{Z^{\mathbb{H}_3/\Gamma}_{\text{gravity}} = \prod_{\gamma\in\mathcal{P}} \prod_{n=2}^{\infty} \frac{1}{|1-q_\gamma^n|}}$$

Conclusion

- We have given the first direct path integral derivation of the $3D$ gravity one-loop partition function around a classical solution. At genus 1, the result agrees with the expected trace over Virasoro descendants of empty AdS_3.
- At higher genus, we have derived a new formula for the one-loop partition function on $\mathbb{H}_3/\Gamma$. It would be interesting to try computing the higher loop corrections from gravity perturbation theory.
- Other possible future directions
  - Extension to supergravity theories
  - Partition function on manifolds with cusps?
  - Application to the "chiral" topological massive gravity of *Li, Song and Strominger '08*