# One more derivation of the Lorentz transformation

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After a criticism of the emphasis put on the invariance of the speed of light in standard derivations of the Lorentz transformation, another approach to special relativity is proposed. It consists of an elementary version of general group-theoretical arguments on the structure of space-time, and makes only use of simple mathematical techniques. The *principle* of relativity is first stated in general terms, leading to the idea of equivalent frames of reference connected through "inertial" transformations obeying a group law. The *theory* of relativity then is constructed by constraining the transformations through four successive hypotheses : homogeneity of space-time, isotropy of space-time, group structure, causality condition. Only the Lorentz transformations and their degenerate Galilean limit obey these constraints. The role and significance of each one of the hypotheses is stressed by exhibiting and discussing counterexamples, that is, transformations obeying all but one of these hypotheses.

## INTRODUCTION

A great many derivations of the Lorentz transformation have already been given, and the subject, because of its pedagogical importance, still receives continuous attention, particularly in this Journal[1]. Most of these analyses, following the original one by Einstein[2], rely on the invariance of the speed of light $c$ as a central hypothesis. That such an hypothesis, firmly based on experimental grounds, has had a crucial historical role cannot be denied. The chronological building order of a physical theory, however, rarely coincides with its logical structure. This is the point of view from which I intend to criticize the overemphasized role of the speed of light in the foundations of the special relativity, and to propose an approach to these foundations that dispenses with the hypothesis of the invariance of $c$. By establishing special relativity on a property of the speed of light, one seems to link this theory to a restricted class of natural phenomena, namely, electromagnetic radiations. However, the lesson to be drawn from more than half a century is that the special relativity up to now seems to rule *all* classes of natural phenomena, whether they depend on electromagnetic, weak, strong, or even gravitational[3] interactions. This theory does not derive from the use of electromagnetic signals for synchronizing clocks, for example, as an ultrapositivistic reading of Einstein's paper[2] might lead one to believe ; quite the contrary, it is the validity of the theory which constrains electromagnetic signals to have their specific propagation properties. We believe that special relativity at the present time stands as a universal theory discribing the structure of a common space-time arena in which all fundamental processes take place. All the laws of physics are constrained by special relativity acting as a sort of "super law[4]," and electromagnetic interactions here have no privilege other than a historical and anthropocentric one. Relativity theory, in fact, is but the statement that all laws of physics are invariant under the Poincaré group (inhomogeneous Lorentz group). The requirement of invariance, when applied to a classification of the possible fundamental particles[5], permits but does not require the existence of zero-mass objects[6].

The evidence of the nonzero mass of the photon would not, as such, shake in any way the validity of the special relativity. It would, however, nullify all its derivations which are based on the invariance of the photon velocity[7]. Fortunately, it may be shown, starting from a few general hypotheses on the properties of space-time, that very little freedom is allowed for the choice of a relativity group, and that the Poincaré group (or the Galilean group, its singular limit) is an almost unique solution to the problem[8]. These analyses make use of more or less elaborate group-theoretical tools, too abstract and general for didactic purposes[9]. This is why I believe it useful to offer here a simple version of the argument, relying only on rather elementary calculus. The discussion will be restricted to the one-dimentional (in space) case.

## PRINCIPLE OF RELATIVITY : INERTIAL FRAMES AND TRANSFORMATIONS

I will take as a starting point the statement of the *principle of relativity* in a very general form : there exists an infinite continuous class of reference frames in space-time which are physically equivalent. In other words, the laws of physics take on the same form when referred to any one of these frames, and no physical effects can distinguish between them. This does not mean, of course, that physical quantities have the same value in every such reference frame : only the relations between them are invariant. The abstract principle of relativity thus a priori is open to many realizations as concrete *theories of relativity*. A theory of relativity tells us how to relate two expressions of the same physical quantity as referred to two of these equivalent frames ; such a theory may then be expressed exactly by the "transformation formulas" connecting different equivalent frames. A theory of relativity then restricts the possible forms of the physical laws which relate various physical quantities in one chosen frame ; it requires the same relationship to hold in a different frame through the use of the transformation formulas. Because of well-known physical considerations, I find it convenient to call "inertial frames" and "inertial transformations" the equivalent reference frames and the

transformations connecting them. Indeed, the very existence of such equivalent reference frames corresponds to the validity of the principle of inertia, namely, that a physical object has no absolute state of motion or rest; for instance, a free body (with no "forces" acting on it) is caracterized by an "inertial motion" which is not entirely determined, since it depends on "initial conditions", that is also to say, on the reference frame considered.

It is natural enough, when trying to establish the nature of inertial transformations, to consider the transformation formulas for specific physical quantities, namely, the spatiotemporal coordinates $(x, t)$ of an arbitrary event in an inertial frame. Now, since we have assumed the existence of an infinite continuous class of inertial frames, the relationship between two of them depends upon a certain number of parameters $\{a_1, a_2, ..., a_N\}$, the values of which caracterize any special inertial transformation. Denoting by $(x', t')$ the coordinates of the same event in another inertial frame, we write the inertial transformation connecting these two sets of coordinates in the general form

$$
\begin{aligned}
x' &= f(x, t; a_1, ..., a_N), \\
t' &= g(x, t; a_1, ..., a_N).
\end{aligned}
\tag{1}
$$

We now appeal to the existence of two special classes of inertial transformations, namely space and time translations, to single out two of the parameters $\{a\}$ and the associated inertial transformations. Indeed, the transformations

$$
\begin{aligned}
x' &= x + \xi, \\
t' &= t + \tau,
\end{aligned}
\tag{2}
$$

amounting to a simple displacement in space and time, are supposed to leave the laws of physics invariant; the space and time origins may be chosen arbitrarily. Using this freedom, we restrict our attention from now on to the class of inertial frames with common space-time origins. In so doing, we dispose of two of our parameters $\{a_1, a_2, ..., a_N\}$—precisely the ones which we called $\xi$ and $\tau$ in (2). We are left with $n = N - 2$ parameters and transformation formulas :

$$
\begin{aligned}
x' &= F(x, t; a_1, ..., a_n), \\
t' &= G(x, t; a_1, ..., a_n),
\end{aligned}
\tag{3}
$$

such that $x' = 0$, $t' = 0$ if $x = 0$, $t = 0$; that is to say,

$$
\begin{aligned}
0 &= F(0, 0; a_1, ..., a_n), \\
0 &= G(0, 0; a_1, ..., a_n).
\end{aligned}
\tag{4}
$$

We may now look upon (3) as giving the transformation formulas for the spatiotemporal *interval* between the events with coordinates $(x, t)$ and the event located at the (common) origin in space and time of the inertial frame. Let us inquire whether such an interval, with coordinates $(x, t)$, may have coordinates $(x', t')$ in an another inertial frame; this is tantamount to asking for the existence of a set of parameters $\{a_1, a_2, ..., a_N\}$ such as (3) holds good. In other words, we consider (3) for given $(x, t)$ and $(x', t')$ as a set of two equations in the $n$ unknowns $\{a_1, a_2, ..., a_N\}$. It is clear that, if $n \geqslant 2$, these equations will, in general, have solutions; an interval between two physical events might then have arbitrary coordinates in a suitably chosen inertial frame, which runs contrary to

simple physical knowledge. The arbitrariness of the time coordinate in particular would seem to preclude any sensible notion of causality. On the other hand, if $n = 0$, there would be no other inertial transformations than space and time translations, and thus no proper theory of relativity. From this argument we may conclude that $n = 1$; that is, the inertial transformation between inertial frames with a common origin depends on only one parameter, and may be written

$$
\begin{aligned}
x' &= F(x, t; a), \\
t' &= G(x, t; a),
\end{aligned}
\tag{5}
$$

under the condition corresponding to (4) :

$$
\begin{aligned}
0 &= F(0, 0; a), \\
0 &= G(0, 0; a).
\end{aligned}
\tag{6}
$$

Another argument leading to the same result is the following. Consider a moving object with equation of motion $x = \varphi(t)$ in the first inertial frame and going through the space-time origin, suitably chosen so that $\varphi(0) = 0$. Its motion as considered in the second frame, obtained through the inertial transformation (3), will be of the form $x' = \varphi'(t'; a_1, ..., a_n)$, with $0 = \varphi'(0; a_1, ..., a_n)$ because of (4). Its speed, acceleration, and higher derivatives of its position with respect to time are obtained by differentiating $\varphi'$. They will depend upon the speed, acceleration, etc., of the object at the origin in the first inertial frame *and* upon the $n$ parameters $\{a_1, ..., a_n\}$. Conversely, for a given motion in a given frame, parameters $\{a_1, ..., a_n\}$ could be chosen in such a way as to find another frame where the object would have arbitrarily preassigned values of its speed, acceleration, and higher time derivatives of its position, up to the $n_{th}$ order. We know from simple physical experience that speed, indeed, is only relative and can be varied from one inertial frame to the other; this is the empirical basis of the principle of relativity. We know, though, that the same is not true for acceleration, which is associated with physical effects differentiating various frames. It follows that $n = 1$.

I am now going to derive a precise functional form for the inertial transformations (5) by relying on a succession of simple and general physical assumptions. Let me state already that the crux of the matter will be the requirement of a group structure for the set of all inertial transformations.

## HYPOTHESIS 1 : HOMOGENEITY OF SPACE-TIME AND LINEARITY OF INERTIAL TRANSFORMATIONS

We assume first that space-time is *homogeneous*, in that it has "everywhere and every time" the same properties. More precisely, the transformation properties of a spatiotemporal interval $(\Delta x, \Delta t)$ depend only on that interval and not on the location of its end points (in the considered reference frame). In other words, the transformed interval $(\Delta x', \Delta t')$ obtained through an inertial transformation (5) is independent of these end points. Looking at an infinitesimal interval $(dx, dt)$, for which

$$
\begin{aligned}
dx' &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt, \\
dt' &= \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt,
\end{aligned}
\tag{7}
$$

this requirement is seen to mean that the coefficients of $dx$ and $dt$ in (7) must be independent of $x$ and $t$, so that $F$ and $G$ are linear functions of $x$ and $t$. We thus write

$$x' = H(a)x - K(a)t, \qquad (8a)$$

$$t' = L(a)t - M(a)x, \qquad (8b)$$

where the minus sign has been introduced for future convenience. Use has been made of the condition on the coincidence of origins to write a *homogeneous* transformation, following (6).

The linearity of inertial transformations has an important physical consequence. Let us call *inertial motions* those motions which are obtained from rest by an inertial transformation; an object has an inertial motion in some reference frame if it is at rest in another equivalent frame. Inertial motions then are caracterized by the same parameter as inertial transformations. Their general equation of motion, according to (8a), is

$$H(a)x - K(a)t = C^{st}. \qquad (9)$$

Inertial motions thus are uniform motions with a velocity $v = K(a)/H(a)$, except in the pathological situation where $K$ identically vanishes[10]. This suggest using the parameter $v$, with the meaning of velocity, instead of the previously undefined parameter $a$. With an adequate change of notation, our general transformation formulas may be rewritten

$$x' = \gamma(v)(x - vt),$$
$$t' = \gamma(v)[\lambda(v)t - \mu(v)x], \qquad (10)$$

depending on three unknown functions $\gamma$, $\lambda$, $\mu$.

### Counterexample 1

It is worthwhile emphasizing the stringency of our Hypothesis 1. The homogeneity of space-time, especially as it concerns time, does *not* hold in every conceivable physical theory. Evolutionary models of the universe do not have a homogeneous time, their inertial transformations, if they exist, are not linear, and their inertial motions are not uniform. Such is the case for the de Sitter space-time or its "nonrelativistic" approximation[8]; the transformation formulas of the last one read

$$x' = x - vT\sinh(t/T),$$
$$t' = t, \qquad (11)$$

Where $T$ is some cosmological time scale (in an oscillating model, a "sin" would replace the "sinh" of this ever expanding universe). These transformations, it is to be stressed, satify all the other hypotheses to be made in the following, as can be checked easily.

In abstract group-theoretical terms, the preceding remarks are related to the connection between the group of inertial transformations and the group of space-time translations whithin the full relativity group of the theory. Only when the translation group is an invariant subgroup of the full relativity group (which is the case for the Galilean and Poincaré group but *not* for the de Sitter one) does linearity of the inertial tansformations hold true.

## HYPOTHESIS 2 : ISOTROPY OF SPACE

Let us go back to the general form (10) of our inertial transformations. We assume that space is nondirectional, so that both orientations of the space axis are physically equivalent. Suppose that $(x, t)$ and $(x', t')$ are two sets of coordinates of a given event related by an inertial transformation (10) with parameter $v$. If the direction of the space axis is arbitrary, $(-x, t)$ and $(-x', t')$ qualify as well for equivalent coordinates of the same event, and must also be related by an inertial transformation of the general form (10) but depend on some parameter $u$, unknown for the time being. In other words,

$$-x' = \gamma(u)(-x - ut),$$
$$t' = \gamma(u)[\lambda(u)t + \mu(u)x]. \qquad (12)$$

By comparing (12) with (10), we obtain

$$\gamma(u) = \gamma(v), \qquad (13a)$$

$$u\gamma(u) = -v\gamma(v), \qquad (13b)$$

$$\lambda(u)\gamma(u) = \lambda(v)\gamma(v), \qquad (13c)$$

$$-\mu(u)\gamma(u) = \mu(v)\gamma(v). \qquad (13d)$$

From (13a) and (13b), we see first that

$$u = -v. \qquad (14)$$

Such a natural result, expressing the relative velocity of the "reversed" reference frames as the opposite of the relative velocity of the initial frames, might have been taken for granted. It is satisfying, however, to derive it from first principles. The remaining relations in (13) now express parity properties of our unknown functions $\gamma$, $\lambda$, $\mu$ :

$$\gamma(-v) = \gamma(v), \qquad (15a)$$

$$\lambda(-v) = \lambda(v), \qquad (15b)$$

$$\mu(-v) = -\mu(v). \qquad (15c)$$

It is simple matter to check that exactly the same results could have been obtained by imposing upon our transformation formulas a similar requirement of symmetry under a reversal of the time axis. Alternately, this symmetry may be taken now as a consequence of the spatial one[11].

### Counterexamples 2

The role of Hypothesis 2 is best illustrated by dropping it. A simple example of inertial transformations respecting the homogeneity of space-time (so that they are linear) and forming a transformation group (which will be our next requirement) is the following one :

$$x' = \exp(\sigma v)(x - vt),$$
$$t' = \exp(\sigma v)t, \qquad (16)$$

for some constant $\sigma$. These transformations approximate the usual Galilean transformation for $v \ll \sigma^{-1}$. Another example of linear group of transformations not satisfaying Hypothesis 2 is given by

$$x' = \frac{x - vt}{1 + \rho v},$$
$$t' = \frac{t - \rho^2 vx}{1 + \rho v}, \qquad (17)$$

where $\rho$ is some characteristic constant. It is an amusing exercice to check the group property to be required in the following and to find the composition law for the velocities[12].

The importance of space and/or time inversion properties should not come as a surprise. Indeed, space inversion in a one-dimensional space plays the important role of rotations in two or more dimensions. It expresses the *isotropy* of space, that is, the physical equivalence of all possible orientations, here reduced to a number of 2. It is perhaps appropriate to remember that such a hypothesis was already made in the first derivation of the Lorentz transformations formulas by Einstein[2].

## HYPOTHESIS 3 : THE GROUP LAW

The physical equivalence of the inertial frames implies a group structure for the set of all inertial transformations (10). This requires the following conditions to be met :

(a) *Identity transformation.* There must exist some parameter $v$ such that $x' = x$ and $t' = t$. This is clearly $v = 0$, and requires

$$\gamma(0) = 1, \tag{18a}$$

$$\lambda(0) = 1, \tag{18b}$$

$$\mu(0) = 0. \tag{18c}$$

Observe that (18c) is already implemented by (15c).

(b) *Inverse transformation.* If $(x', t')$ derives from $(x, t)$ through the transformation (10) parametrized by $v$, the inverse transformation, from $(x', t')$ to $(x, t)$, must be of the same functional form (10), with a different parameter $w$ ; that is to say,

$$x = \gamma(w)(x' - wt'),$$
$$t = \gamma(w)[\lambda(w)t' - \mu(w)x']. \tag{19}$$

Inverting (10), we also obtain

$$x = \frac{1}{\gamma(v)}\left(1 - \frac{v\mu(v)}{\lambda(v)}\right)^{-1}\left(x' + \frac{v}{\lambda(v)}t'\right),$$
$$t = \frac{1}{\gamma(v)}\left(1 - \frac{v\mu(v)}{\lambda(v)}\right)^{-1}\left(\frac{1}{\lambda(v)}t' + \frac{\mu(v)}{\lambda(v)}x'\right). \tag{20}$$

By identifying (20) with (19) we derive the following equations linking the unknown $w$ to $v$ while at the same time constraining the functions $\gamma, \lambda, \mu$ :

$$w = -v/\lambda(v), \tag{21a}$$

$$\lambda(w) = 1/\lambda(v), \tag{21b}$$

$$\mu(w) = -\mu(v)/\lambda(v) \tag{21c}$$

$$\gamma(w) = [1/\gamma(v)][1 - v\mu(v)/\lambda(v)]^{-1}. \tag{21d}$$

Let us restrict our attention to the first two equations (21a) and (21b). They result in a functional equation for $\lambda$,

$$\lambda[-v/\lambda(v)] = 1/\lambda(v). \tag{22}$$

In order to make more acceptable the apparent oddity of the functional equation (22), let us define the associated function :

$$\zeta(v) \overset{df}{=} v/\lambda(v). \tag{23}$$

The condition (22) now reads

$$\zeta[-\zeta(v)] = -v \tag{24}$$

or

$$\zeta^{-1}(-v) = -\zeta(v). \tag{25}$$

But the curves of $\zeta(v)$ and $\zeta^{-1}(v)$, when plotted on the same Cartesian graph, are symmetric with respect to the line $\zeta = v$. Condition (25) then requires the graph of $\zeta(v)$ to be symmetrical with respect to the line $\zeta = -v$. On the other hand, since $\lambda(0) = 1$ [see (18b)], according to (23), one has

$$\left.\frac{d\zeta}{dv}\right|_{v=0} = 1. \tag{26}$$

Any $\zeta$ such that its graph is symmetrical with respect to the line $\zeta = -v$ and is tangent at the origin to the line $\zeta = v$ will give rise to $\lambda(v) = v/\zeta(v)$, which obeys (22). As the simplest example, the reader may check that $\lambda(v) = 1 - kv$ is a solution for any real number $k$.

However, the consequence (15b) of Hypothesis 2, which requires $\lambda$ to be an even function, drastically reduces this arbitrariness. With it taken into account, the functional equation in $\lambda$ reads

$$\lambda[v/\lambda(v)] = 1/\lambda(v). \tag{27}$$

Or using the associated function $\zeta$ (23),

$$\zeta[\zeta(v)] = v; \tag{28}$$

that is,

$$\zeta^{-1}(v) = \zeta(v). \tag{29}$$

The graph of $\zeta(v)$ now has to be symmetrical with respect to the line $\zeta = v$. Because of (18b), on the other hand, it must also obey (26) ; that is, it must be tangent at the origin to this same line. Hence, in view of the continuity of $\lambda$ and $\zeta$[13], the graph of $\zeta(v)$ must in fact be identical to the line $\zeta = v$, so that $\zeta(v) = v$. We may conclude that

$$\lambda(v) = 1. \tag{30}$$

Going back to (21a), it results that the parameter of the transformation (19) inverse to (10) is, naturally enough, given by

$$w = -v, \tag{31}$$

but this has been proven and not assumed. Finally, (21d) with (15a) yields a relationship between the two remaining functions $\gamma$ and $\mu$ :

$$[\gamma(v)]^2[1 - v\mu(v)] = 1. \tag{32}$$

(c) *Composition law.* Let us now perform in succession two transformations of the form (10), taking into account our previous result (30) :

$$x_1 = \gamma(v_1)(x - v_1 t),$$
$$t_1 = \gamma(v_1)[t - \mu(v_1)x], \tag{33}$$

$$x_2 = \gamma(v_2)(x_1 - v_2 t_1),$$
$$t_2 = \gamma(v_2)[t_1 - \mu(v_2)x_1]. \tag{34}$$

The resulting transformation,

$$x_2 = \gamma(v_1)\gamma(v_2)[1 + \mu(v_1)v_2]\left(x - \frac{v_1 + v_2}{1 + \mu(v_1)v_2}t\right),$$
(35a)

$$t_2 = \gamma(v_1)\gamma(v_2)[1 + v_1\mu(v_2)]\left(t - \frac{v_1 + v_2}{1 + v_1\mu(v_2)}x\right),$$
(35b)

is to be identified with a general transformation of the form (10), depending on a new parameter $V$ (a function of $v_1$ and $v_2$) :

$$x_2 = \gamma(V)[x - Vt]$$
$$t_2 = \gamma(V)[t - \mu(V)x].$$
(36)

By identification of the factor $\gamma(V)$ in (35a) and (35b), we obtain the condition

$$\mu(v_1)v_2 = v_1\mu(v_2).$$
(37)

Hence,

$$\mu(v) = \alpha v,$$
(38)

for some constant $\alpha$. According to (32), the last unknown function finally is

$$\gamma(v) = (1 - \alpha v^2)^{-1/2},$$
(39)

where the sign of the square root has been chosen in accordance with (18a). Also, "the law of addition of velocities" directly derives from (35) and (36) :

$$V = (v_1 + v_2)/(1 + \alpha v_1 v_2).$$
(40)

It is clear that the following three different cases now arise, depending on the sign of the constant $\alpha$ or on its vanishing.

(i) $\alpha < 0$. We may write $\alpha = -\kappa^{-2}$, where $\kappa$ has the dimensions of a velocity. The transformation law reads

$$x' = \frac{x - vt}{(1 + v^2/\kappa^2)^{-1/2}},$$
$$t' = \frac{t + vx/\kappa^2}{(1 + v^2/\kappa^2)^{-1/2}}.$$
(41)

Observe that values of the velocity $v$ are allowed in the whole real range. The law of addition of velocities is

$$V = \frac{v_1 + v_2}{1 - v_1 v_2/\kappa^2}.$$
(42)

(ii) $\alpha = 0$. The corresponding formulas are

$$x' = x - vt,$$
(43)
$$t' = t, \qquad \text{(Galilean)}$$

and

$$V = v_1 + v_2.$$
(44)

(iii) $\alpha > 0$. We thus write $\alpha = c^{-2}$, where $c$ is a constant with the dimensions of a velocity, and we have

$$x' = \frac{x - vt}{(1 - v^2/c^2)^{1/2}},$$
(45)
$$t' = \frac{t - vx/c^2}{(1 - v^2/c^2)^{1/2}}, \qquad \text{(Lorentz)}$$

and

$$V = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}.$$
(46)

Velocities here (as parameters of the Lorentz transformation[14]) are restricted to lie within the range $-c \leqslant v \leqslant c$. Clearly, as in case (i), the numerical value of $\alpha$ depends on the initial choice of units for space and time coordinates, so that physically..., there is but one situation here. Before examining these three cases in the light of our last physical hypothesis, it may be worthwhile to mention the following.

### Counterexamples 3

It is easy to exhibit transformation formulas with a "reasonable" look satisfaying, for instance, all previous hypotheses but not having the group property ; any two of them, under composition, do not yield to a transformation belonging to the family. Such would be the transformations

$$x' = x - vt,$$
(47)
$$t' = t - vx/c^2.$$

Also, the set of transformations

$$x' = (1 + \tfrac{1}{2}v^2/c^2)x - vt,$$
(48)
$$t' = (1 + \tfrac{1}{2}v^2/c^2)t - vx/c^2,$$

is a mathematically consistent expansion of the Lorentz formulas (45) to first order in $c^2$ and has been proposed as a "new" relativity group[15] ; unfortunately, it is *not* a group.

### HYPOTHESIS 4 : CAUSALITY

Except in the singular case (ii) above, the time interval between two events depends on the reference frame, since it changes under the transformation (41) or (45). Now, in order to maintain *some* order in the universe, we would like to require the existence of at least a class of spatiotemporal events such that the *sign* of the time interval, that is, the nature of a possible causal relationship, is not changed under inertial transformations.

This requirement is obviously met by the Galilean transformations (43) for *all* time intervals. It is met also in the Lorentz case (iii) for those intervals such that $|\Delta x/\Delta t| \leqslant c$ ("time-like") because of the limited range of values available for $v$. However, case (i) does not agree with our hypothesis, so that it becomes the following.

### Counterexample 4[16]

Indeed, since

$$\Delta t' = \frac{\Delta t + v\Delta x/\kappa^2}{(1 + v^2/\kappa^2)^{1/2}}$$
(49)

for any given $(\Delta t, \Delta x)$, one may always find a $v$, within the unlimited available range, such that $\Delta t'$ has a sign opposite to that of $\Delta t$, which thereby forbids the very existence of a causal relationship. Another paradoxical feature of case (i) appears when we compose two positive velocities, for instance $v_1 = 2\kappa$ and $v_2 = \kappa$, which result in a velocity $V = -3\kappa,...,$ in the opposite direction[17]!

## CONCLUSION

Our four general hypotheses thus suffice to single out the Lorentz transformations and their degenerate Galilean limit as the only possible inertial transformations. The Lorentz case is caracterized by a parameter with the dimensions of a velocity which is a universal constant associated with the very structure of space-time. A further analysis of the possible objects moving in such a space-time shows that this constant *turns out* to be the (invariant) velocity of zero-mass objects.

*Note added in proof :* This work was already submitted for publication when a closely related investigation by A. R. Lee and T. M. Kalotas appeared in this journal [Am. J. Phys. **43**, 434 (1975)]. These authors also point out the very early roots of such considerations, going back to several papers more than sixty years old, and long forgotten or neglected. To the historical references they give, the following one may be added : L. A. Pars, Philos. Mag. **43**, 249 (1921). Despite the similarity of the present paper with the article by Lee and Kalotas, the somewhat greater generality of its assumptions, as well as its offering counterexamples, may still justify its publication.

1. See, for example, the following recent and nonexhaustive samples : A. L. Harvey, Am. J. Phys. **36**, 901 (1968) ; A. B. Kaiser, *ibid.* **37**, 216 (1969) ; J. Rekveld, *ibid.* **37**, 717 (1969) ; L. Parker and G. M. Schmieg, *ibid.* **38**, 218 (1970) ; L. Parker *ibid.* **39**, 223 (1971) ; J. R. Shepanski and R. Simons, *ibid.* **40**, 486 (1972).

2. A. Einstein, Ann. Phys. (Leipz.) **17**, 891 (1905) ; reprinted in English translation in *The Principle of Relativity* (Dover, New York).

3. General relativity *may* be interpreted as a special relativistic theory of a self-coupled tensor field, which turns out to play the role of an effective metric, masking the assumed Minkovskian one : W. Thirring, Ann. Phys. N.Y. **16**, 96 (1961) ; see also R. V. Sexl, Fortschr. Phys. **15**, 269 (1967).

4. E. P. Wigner, Phys. Today **17** (3), 34 (1964).

5. E. P. Wigner, Ann. Math. **40**, 149 (1939).

6. This is the place to stress how mystifying the terminology itself can be. Why call $c$ the speed of light when it is the speed of *all* zero-mass objects, for instance, the speed of neutrinos ? Alternatively, if photons, neutrinos, and gravitons had a very small but nonzero mass (Ref. 7), $c$ would not be the speed of any real object. I do not suggest, of course, a change in the vocabulary of physics as fixed by years of practice ; abuse of language is not only unavoidable, it is probably necessary. At least, let it be recognized for such.

7. A. S. Goldhaber and M. M. Nieto [Rev. Mod. Phys. **43**, 277 (1971)] review the present limits on the photon mass. For the case of a photon with nonzero mass, these authors present a more sophisticated but rather artificial derivation of special relativity from considerations still based on the (now variable) velocity of light (see p. 294 of their paper).

8. H. Bacry and J. M. Levy-Leblond, J. Math. Phys. **9**, 1605 (1968), and additional references therein.

9. I do *not* include in these works the very nice theorem by E. C. Zeeman [J. Math. Phys. **5**, 490 (1964)], deriving Lorentz transformations from a causality principle. Indeed, his starting point is the existence of a causal cone, implying *ab initio* the existence of a limit velocity.

10. This vanishing of $K$, implying absolute rest as the only inertial motion, characterizes the singular kinematics described by the "Static group" or the "Carroll group" studied in Ref. 8.

11. It is perhaps necessary to stress here that, in the present state of our knowledge, the symmetry of space-time under space or time reflection has nothing to do with parity or time-reversal invariance in physical interactions. Parity violation in weak interactions and the failure of time-reversal invariance in super weak ones take place within a relativistic space-time with perfect symmetry under space or time reflection. It might be tempting to find the origin of these breakings of invariance laws for specific dynamics in a distortion of space-time itself. Such attempts have not yet succeded, however.

12. That the same law of addition of velocities appears as in the Lorentz case may be understood by making use of a "rapidity" parameter $\varphi$, such that $\rho v = \tanh \varphi$, to rewrite (17) under the form

$$x' = \exp(-\varphi)(\cosh \varphi x - \sinh \varphi t),$$
$$t' = \exp(-\varphi)(\cosh \varphi t - \sinh \varphi x).$$

This transformation thus appears as a Lorentz transformation combined with a velocity-dependent (and not reflection-invariant) global dilatation.

13. This is *not* a completely rigorous mathematical proof of this point, which requires a degree of sophistication far beyond the pedagogical leved adopted here [J. Morgenstern and M. Zerner (private communication)].

14. This is an opportunity to stress the difference, not always clear enough in some discussions about tachyons, between the relative velocity of two reference frames, parameter of the Lorentz transformation, which is necessarily smaller than $c$, and the velocity of "something" within a given reference frame, which would be larger than $c$ for tachyons and *may be* larger than $c$ for shadows, as the spot of an electron beam on an oscilloscope screen, for instance.

15. V. N. Streltov, Dubna JINR Reports P2-4461, P2-4462, P2-5523, P2-5823, P2-6208 (unpublished).

16. See Eqs. (41) and (42).

17. In more mathematical terms, causality as expressed here requires the noncompactness of the group of inertial transformations (see Ref. 8). The group considered here (counterexample 4) clearly is isomorphic to the compact rotation group in two dimensions, while the two-dimensional Lorentz group is noncompact.