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# ONE-PARAMETER INVERSE SEMIGROUPS

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**Abstract.** This is the second in a projected series of three papers, the aim of which is the complete description of the closure of any one-parameter inverse semigroup in a locally compact topological inverse semigroup. In it we characterize all one-parameter inverse semigroups. In order to accomplish this, we construct the free one-parameter inverse semigroups and then describe their congruences.

0. Let $G$ be a subgroup of the multiplicative group of positive real numbers and let $P$ denote the subsemigroup of $G$ consisting of all $x \in G$ with $x \geqq 1$. Denote by $\mathscr{C}_P$ the class of all inverse semigroups $H$ for which there is a homomorphism $f\colon P \to H$ such that $f(P)$ generates $H$ (no proper inverse subsemigroup of $H$ contains $f(P)$). We shall call such semigroups $H$ *one-parameter inverse semigroups* and denote by $\mathscr{C} = \bigcup_P \mathscr{C}_P$ the class of all one-parameter inverse semigroups.

The class $\mathscr{C}$ contains well-known semigroups. For example, each homomorphic image of a subgroup of $R$, the positive real numbers, is a member of $\mathscr{C}$. Also the bicyclic semigroup $B$ is a member of $\mathscr{C}$, as is seen by noting that $B$ is generated by a copy of the nonnegative integers. Indeed, if $H$ is any elementary inverse semigroup, then $H^1$ is generated by a homomorphic image of the nonnegative integers, and so is a one-parameter inverse semigroup.

The main purpose of this paper is to describe all one-parameter inverse semigroups. In the process of doing this, we shall construct what we term the *free one-parameter inverse semigroups* $F_P$, one for each subgroup $G$ of $R$ and its associated semigroup $P$. The semigroup $F_P$ is the only inverse semigroup (up to isomorphism) generated by a subsemigroup isomorphic with $P$ which has the property that each homomorphism $f\colon P \to S$, an inverse semigroup, extends uniquely to a homomorphism $\bar{f}\colon F_P \to S$. In particular, every $H \in \mathscr{C}_P$ is a homomorphic image of $F_P$. We thus adopt the point of view that by describing $F_P$ and the lattice of congruences of $F_P$ for arbitrary $P$, we will have described all one-parameter inverse semigroups.

We shall assume a certain familiarity with the algebraic theory of semigroups, particularly inverse semigroups. (See Clifford and Preston [1].)

The existence and uniqueness of $F_P$ is a consequence of a theorem due to McAlister [3, Theorem 33]. We were greatly aided in the actual description of $F_P$

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by two results of Gluskin on elementary inverse semigroups [2, p. 24]. For the description of the congruences on $F_P$, the results of Reilly and Scheiblich in [4] proved useful.

Although this paper is primarily algebraic in nature, there is a natural topology on $F_P$ with respect to which $F_P$ is a topological inverse semigroup. This fact, together with several other comments of a topological nature, are included in remarks throughout the paper.

1. **The free inverse semigroup on a set** $X$. In this section we shall review some theory which has already been obtained by McAlister in [3].

If $S$ is an inverse semigroup generated by a subset $X$, then we say that $S$ is *freely generated* by $X$ provided each function from $X$ into an inverse semigroup extends to a homomorphism on $S$. One shows easily, using the fact that homomorphisms on inverse semigroups take inverses to inverses, that if $S$ is freely generated by $X$, then each function from $X$ into an inverse semigroup $T$ extends to a unique homomorphism from $S$ into $T$.

1.1. THEOREM. *For any nonvoid $X$ there is one and only one inverse semigroup (up to isomorphism) $I_X$ freely generated by $X$.*

Although it is not our intention to investigate them here, we remark that many interesting questions arise concerning the structure of $I_X$ and its lattice of congruences. For example, it is not difficult to show that the smallest group congruence on $I_X$ has the free group on $X$ as its quotient semigroup.

Now let $P$ be a fixed semigroup. Consider the class of pairs $(f, S)$ where $S$ is an inverse semigroup and $f$ is a homomorphism from $P$ into $S$ so that $f(P)$ generates $S$. Define two pairs $(f, S)$ and $(g, T)$ to be equivalent provided there is an isomorphism $\phi: S \xrightarrow{\text{onto}} T$ so that $\phi f = g$. This is easily seen to be an equivalence relation on pairs. We call a pair $(f, S)$ a *free pair* provided given any pair $(g, T)$ there is a homomorphism $\phi: S \to T$ such that $\phi f = g$. It follows from the fact that two homomorphisms on an inverse semigroup which agree on a generating set are identical, that the homomorphism $\phi$ above is unique.

The next theorem establishes the existence and uniqueness of a free pair $(f, S)$.

1.2. THEOREM. *There is an inverse semigroup $S$ and a homomorphism $f: P \to S$ such that $(f, S)$ is a free pair. Furthermore any two free pairs are equivalent. The homomorphism $f$ is* 1-1 *if and only if $P$ is embeddable in an inverse semigroup.*

In case $f$ is 1-1 we identify $P$ with $f(P)$ and call $S$ the inverse semigroup freely generated by the subsemigroup $P$ and denote $S$ by $F_P$. Note that $F_P$ is characterized by the property that any homomorphism from $P$ into an inverse semigroup extends to a unique homomorphism on $F_P$. In particular, any inverse semigroup generated by a homomorphic image of $P$ is isomorphic with a quotient semigroup of $F_P$.

2. **The free one-parameter inverse semigroups $F_P$.** Let $G$ be a fixed subgroup of $R$ and let $P=\{x\in G \mid x\geqq 1\}$, $P_0=P\backslash\{1\}$. In this section we shall describe fully the structure of the semigroups $F_P$ and $F_{P_0}$ freely generated by the subsemigroups $P$ and $P_0$ respectively.

First we construct a homomorphic image $B_P$ of $F_P$ which is a generalization of the bicyclic semigroup $B$. This construction is similar to the one found on p. 107 of Vol. 2 of [1]. Let $B_P=P\times P$ with the following operation:

$$(x, y)(z, w) = (xz/y \wedge z, yw/y \wedge z)$$

where $y\wedge z=\min\{y, z\}$. It is easily checked that the product of two elements of $B_P$ is an element of $B_P$. In fact we have the following consequence of Theorems 8.43 and 8.44 of Vol. 2 of [1]:

2.1. THEOREM. *$B_P$ is a bisimple inverse semigroup which is generated by $P_0\times 1$.*

2.2. THEOREM. *The real number 1 is the identity for $F_P$. Furthermore $F_{P_0}$ does not have an identity and in fact is isomorphic with $F_P\backslash\{1\}$. Thus $F_P$ is obtained from $F_{P_0}$ by adjoining an identity.*

**Proof.** Since 1 is the identity of $P$ and $P$ generates $F_P$, 1 is the identity of $F_P$. Let $S$ denote the inverse subsemigroup of $F_P$ generated by $P_0$, and let $f$ be a homomorphism from $P_0$ into an inverse semigroup $T$. We assume $T$ has an identity $e$, for otherwise we could adjoin it. Then $f$ extends to a homomorphism $g: P\to T$ by defining $g(1)=e$. Now $g$ extends to a homomorphism $\bar{g}: F_P\to T$, and $\bar{g}|S$ is clearly the sought extension of $f$ to $S$. Thus $S$ is freely generated by $P_0$; that is, $S=F_{P_0}$. Now suppose $S$ has an identity $i$. Then there exist $x_1, x_2, \ldots, x_n$ in $P_0$ such that $i=x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ where $j_k\in\{1, -1\}$ for $k=1, 2, \ldots, n$. Thus $x_1^{j_1}x_1^{-j_1}=x_1^{j_1}x_1^{-j_1}\cdot i$ $=i$ and hence, for some $x\in P_0$, $i=xx^{-1}$ or $i=x^{-1}x$. Suppose that $i=xx^{-1}$. Let $f: P_0\to P_0\times 1\subseteq B_P$ be given by $f(t)=(t, 1)$. Then $f$ extends to a homomorphism $\bar{f}: S\to B_P$. Further $f(S)=B_P$ since $P_0\times 1$ generates $B_P$. Hence $\bar{f}(i)$ is an identity for $B_P$ and so $\bar{f}(i)=(1, 1)$. But $\bar{f}(i)=\bar{f}(xx^{-1})=\bar{f}(x)\bar{f}(x)^{-1}=(x, 1)(1, x)=(x, x)$ and $x\neq 1$. From this contradiction we conclude that $S=F_{P_0}$ does not have an identity. In particular $1\notin S$. Suppose $x\in F_P\backslash\{1\}$. Then there exist elements $x_1, x_2, \ldots, x_n$ of $P$ so that $x=x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$ where $j_k\in\{1, -1\}$ for $k=1, 2, \ldots, n$. In fact we may assume that $x_k\in P_0$ for $k=1, 2, \ldots, n$ (this is true for at least one value of $k$ since $x\neq 1$). Thus $x\in S$ and we have shown that $S=F_P\backslash\{1\}$. This completes the proof of this theorem.

An elementary inverse semigroup is defined to be an inverse semigroup generated by a single element. An elementary inverse semigroup may or may not be a one-parameter inverse semigroup depending on whether it has an identity; however we do have the following corollary.

2.3. COROLLARY. *Suppose the given subgroup $G$ of $R$ is cyclic. Then $F_{P_0}$ is an elementary inverse semigroup with the property that every elementary inverse semigroup is a homomorphic image of $F_{P_0}$.*

**Proof.** This follows from 2.2 together with the fact that a homomorphism on the positive integers is determined by its value at 1.

2.4. LEMMA. *If* $x \leqq y$ *then*
(i) $xy^{-1}=(y/x)^{-1}yy^{-1}$,
(ii) $y^{-1}x=y^{-1}y(y/x)^{-1}$,
(iii) $yx^{-1}=(y/x)xx^{-1}$,
(iv) $x^{-1}y=x^{-1}x(y/x)$.

**Proof.** To see (i), note that

$$\begin{aligned} xy^{-1} &= x((y/x)x)^{-1} = xx^{-1}(y/x)^{-1} = xx^{-1}(y/x)^{-1}(y/x)(y/x)^{-1} \\ &= (y/x)^{-1}(y/x)xx^{-1}(y/x)^{-1} = (y/x)^{-1}yy^{-1}. \end{aligned}$$

Part (ii) is proved similarly and (iii) and (iv) are trivial.

The next result is, in a sense, an analogue of a theorem of Gluskin [2, Lemma 1.2] and follows immediately from the above lemma.

2.5. LEMMA. *Let* $x, y, z \in P$. *Then the elements* $xy^{-1}z$ *and* $x^{-1}yz^{-1}$ *of* $F_P$ *can also be written as follows*:

(i)
$$\begin{aligned} xy^{-1}z &= xz/y && \text{if } y \leqq x, z, \\ &= (y/x)^{-1}z && \text{if } x \leqq y \leqq z, \\ &= x(y/z)^{-1} && \text{if } z \leqq y \leqq x, \\ &= (y/x)^{-1}y(y/z)^{-1} && \text{if } x, z \leqq y. \end{aligned}$$

(ii)
$$\begin{aligned} x^{-1}yz^{-1} &= (xz/y)^{-1} && \text{if } y \leqq x, z, \\ &= x^{-1}(y/z) && \text{if } z \leqq y \leqq x, \\ &= (y/x)z^{-1} && \text{if } x \leqq y \leqq z, \\ &= (y/x)y^{-1}(y/z) && \text{if } x, z \leqq y. \end{aligned}$$

(iii) *There exist* $a, b, c$ *in* $P$ *such that* $b \geqq a, c$ *and* $x^{-1}yz^{-1}=ab^{-1}c$.

**Proof.** Parts (i) and (ii) follow immediately from Lemma 2.4. Using (ii) we can write $x^{-1}yz^{-1}$ as $ab^{-1}c$ if we choose $a, b$ and $c$ as follows: if $y \leqq x, z$ let $a=1$, $b=xz/y$, $c=1$; if $z \leqq y \leqq x$, let $a=1$, $b=x$, $c=y/z$; if $x \leqq y \leqq z$, let $a=y/x$, $b=z$, $c=1$; and if $x, z \leqq y$, let $a=y/x$, $b=y$, $c=y/z$. In each case $b \geqq a$, $c$, and $a$, $b$ and $c$ are in $P$.

2.6. THEOREM. $F_P=PP^{-1}P=P^{-1}PP^{-1}$ *and* $F_{P_0}=PP_0^{-1}P=P^{-1}P_0P^{-1}$.

**Proof.** It is an immediate consequence of 2.5(i) that $PP^{-1}P \subset P^{-1}PP^{-1}$. Hence $P^{-1}PP^{-1}=(PP^{-1}P)^{-1} \subset (P^{-1}PP^{-1})^{-1}=PP^{-1}P$ and so $PP^{-1}P=P^{-1}PP^{-1}$. Note also that $(PP^{-1}P)^2=(PP^{-1}P)(PP^{-1}P) \subset P(P^{-1}PP^{-1})P=P(PP^{-1}P)P \subset PP^{-1}P$. Hence $PP^{-1}P$ is an inverse subsemigroup of $F_P$. Since $P \subset PP^{-1}P$, we obtain $F_P=PP^{-1}P$. Now suppose $u \in F_{P_0}=F_P \backslash \{1\}$. Then there exist $x, y, z \in P$ such that $u=x^{-1}yz^{-1}$. Now it follows from 2.5(iii) that there exist $a, b, c \in P$ with $b \geqq a, c$ so that $u=x^{-1}yz^{-1}=ab^{-1}c$. However, at least one of $a, b, c$ is not 1, and so $b\neq 1$. This says that $u\in PP_0^{-1}P$. On the other hand, choose $xy^{-1}z$ in $PP_0^{-1}P$. Suppose $1=xy^{-1}z$. Note that $y\neq 1$. If $x=z=1$, then $y^{-1}=1$. So $y=1$ which is a contradiction. Thus, either $x\neq 1$ or $z\neq 1$. Without loss of generality, suppose $x\neq 1$. Now if $z=1$, then $1=xy^{-1}\in F_{P_0}$, which is a contradiction. So $z\neq 1$. Thus none of $x$, $y$, or $z$ is 1. Therefore $1=xy^{-1}z\in F_{P_0}$, another contradiction. Thus $xy^{-1}z\neq 1$; i.e., $xy^{-1}z\in F_{P_0}$. Hence $PP_0^{-1}P=F_{P_0}$.

2.7. THEOREM. *Each element of $F_P$ can be written in one and only one way in the form $xy^{-1}z$ where $x, y, z\in P$ with $x, z\leqq y$. Refer to this as the canonical representation of elements of $F_P$. Then if $u, v\in F_P$ with canonical representations $u=xy^{-1}z$ and $v=rs^{-1}t$, then $uv$ has as its canonical representation*

$$uv = (xzr/y \wedge zr)(yzrs/(y \wedge zr)(zr \wedge s))^{-1}(zrt/zr \wedge s).$$

**Proof.** Let $u\in F_P$. Then by 2.6 there are elements, $a, b, c\in P$ such that $u=a^{-1}bc^{-1}$. Now using 2.5(iii) we can write $u=xy^{-1}z$ where $x, z\leqq y$. To show that the representation is unique, we make use of the semigroup $B_P$ defined earlier. Let $f, g: P\to B_P$ be the homomorphisms given by $f(x)=(x, 1)$ and $g(x)=(1, x)$. Let $\bar{f}$ and $\bar{g}$ be the extensions of $f$ and $g$ respectively to $F_P$. Now suppose that $u\in F_P$ has two representations $xy^{-1}z$ and $rs^{-1}t$ where $x, z\leqq y$ and $r, t\leqq s$. Then $\bar{f}(xy^{-1}z)=f(x)f(y)^{-1}f(z)=(x, 1)(1, y)(z, 1)=(x, y/z)$ and similarly $\bar{f}(rs^{-1}t)=(r, s/t)$, $\bar{g}(xy^{-1}z)=(y/x, z)=\bar{g}(rs^{-1}t)=(s/r, t)$. Hence $r=x$, $s=y$ and $z=t$ and thus the representation is unique.

To establish the rule for multiplication, let $u, v\in F_P$ with representations (not necessarily canonical) $u=xy^{-1}z$ and $v=rs^{-1}t$. It then follows from 3.4(ii) that

$$\begin{aligned} uv &= x(ys/zr)^{-1}t && \text{if } zr \leqq s, y,\\ &= xy^{-1}(zrt/s) && \text{if } s \leqq zr \leqq y,\\ &= (xzr/y)s^{-1}t && \text{if } y \leqq zr \leqq s,\\ &= (xzr/y)(zr)^{-1}(zrt/s) && \text{if } s, y \leqq zr. \end{aligned}$$

Now since $y\wedge zr\leqq xzr$ and $zr\wedge s\leqq zrt$ it follows that $xzr/(y\wedge zr)$, $zrt/(zr\wedge s)$, and $yzrt/((y\wedge zr)(zr\wedge s))$ are all in $P$. It is a simple matter to check using the four cases above that in fact,

$$uv = (xzr/y \wedge zr)[yzrs/(y \wedge zr)(zr \wedge s)]^{-1}(zrt/zr \wedge s).$$

Further, if $xy^{-1}z$ and $rs^{-1}t$ are canonical; i.e. if $x, z\leqq y$ and $r, t\leqq s$ then it is easily checked that $xzr/y\wedge zr$, $zrt/zr\wedge s\leqq yzrs/(y\wedge zr)(zr\wedge s)$ and so the representation for the product above is canonical. This completes the proof.

2.8. COROLLARY. *The elements of $F_{P_0}=F_P\backslash\{1\}$ consist precisely of those elements of $F_P$ whose canonical representation $xy^{-1}z$ is such that $y\neq 1$.*

**Proof.** Let $u\in F_{P_0}$ and let $xy^{-1}z$ be its canonical representation. If $y=1$ then $x=z=1$ and so $u=1$. Hence $y\neq 1$. Conversely, if $xy^{-1}z\in F_P$ with $x, z\leqq y\neq 1$, then $xy^{-1}z\in PP_0^{-1}P=F_{P_0}$, by 2.6. Q.E.D.

Using 2.7 and 2.8 we immediately obtain the following parametrization theorem for $F_P$ and $F_{P_0}$.

2.9. COROLLARY. *Let* $T_P=\{(x,y,z) \mid x,y,z\in P \text{ with } x,z\leqq y\}$. *Define an operation on* $T_P$ *by*

$$(x,y,z)(r,s,t) = (xzr/y\wedge zr, yzrs/(y\wedge zr)(zr\wedge s), zrt/zr\wedge s).$$

*Then the map* $\phi\colon F_P\to T_P$ *defined by* $\phi(u)=(x,y,z)$ *for* $u\in F_P$ *with canonical representation* $u=xy^{-1}z$ *is an isomorphism from* $F_P$ *onto* $T_P$. *Further if* $T_{P_0}=T_P\backslash\{(1,1,1)\}$, *then* $\phi|F_{P_0}$ *is an isomorphism from* $F_{P_0}$ *onto* $T_{P_0}$.

2.10. REMARK. If $T_P$ is given the subspace topology from the product space $P\times P\times P$, where $P$ is given the subspace topology from $R$ with the usual topology, then it is easily seen that the multiplication and inversion on $T_P$ are continuous; that is, $T_P$ is a topological inverse semigroup. This follows from the fact that multiplication and inversion on $R$ and the $\wedge$ operation on $P$ are all continuous operations. Hence there is a natural topology on $F_P$ making $F_P$ into a topological inverse semigroup. Indeed, $F_P$ is freely generated by $P$ even in the topological sense; that is, any continuous homomorphism from $P$ into a topological inverse semigroup $S$ extends to a unique continuous homomorphism from $F_P$ into $S$.

The idempotent structure of $F_P$ is determined next.

2.11. LEMMA. *Let* $u\in F_P$ *with canonical representation* $u=xy^{-1}z$. *Then the canonical representation of* $u^{-1}$ *is* $(y/z)y^{-1}(y/x)$.

**Proof.** Note $y/z$, $y/x\in P$. *Also note* $u^{-1}=z^{-1}yx^{-1}$. Hence by 2.5(ii) $u^{-1}=(y/z)y^{-1}(y/x)$.

For $x\in P$, let $e_x=xx^{-1}$ and $f_x=x^{-1}x$, and let $E=\{e_x \mid x\in P\}$, $F=\{f_x \mid x\in P\}$. Note $E,F\subseteq E_P$, the set of idempotents of $F_P$.

2.12. THEOREM. *Let* $u\in F_P$ *with canonical representation* $xy^{-1}z$. *Then* $u\in E_P$ *if and only if* $y=xz$. *Furthermore, each element of E can be written in one and only one way in the form* $e_xf_x$ *for some* $x,z\in P$. *Thus* $E_P$ *is the direct sum of the two subsemilattices E and F. Also* $e_xf_y\leqq e_uf_v$ *if and only if* $u\leqq x$ *and* $v\leqq y$.

**Proof.** Suppose $u\in E_P$ and $xy^{-1}z$ is the canonical representation of $u$. Then by 2.9, $u=u^{-1}=(y/z)y^{-1}(y/x)$. Hence $(y/z)=x$, that is, $y=xz$. On the other hand, if $y=zx$ then $xy^{-1}z=(xx^{-1})(z^{-1}z)=e_xf_z\in E_P$. Hence to establish the last statement we need only show the uniqueness of the representation. So suppose $x,z,r,t\in P$ with $xx^{-1}z^{-1}z=e_xf_z=e_rf_t=rr^{-1}t^{-1}t$. Then, using the homomorphisms $\bar f$ and $\bar g$ of 2.7 we see that $f(xx^{-1}z^{-1}z)=f(x)f(x)^{-1}f(z)^{-1}f(z)=(x,1)(1,x)(1,z)(z,1)=(x,x)=\bar f(rr^{-1}tt^{-1})=(r,r)$ and similarly $\bar g(xx^{-1}z^{-1}z)=(z,z)=\bar g(rr^{-1}t^{-1}t)=(t,t)$. Hence $x=r$ and $z=t$. The last assertion follows easily upon noting that $e_xe_u=e_{x\vee u}$. 2.13 follows immediately from 2.12 and the fact that $F_{P_0}=F_P\backslash\{1\}$.

2.13. Corollary. *The idempotents of* $F_{P_0}$ *are precisely those elements of* $F_P$ *which can be written* (*uniquely*) *in the form* $e_x f_z$ *where* $\{x, z\} \cap P_0 \neq \varnothing$.

Next we determine Green's relations (confer with [1]) on $F_P$.

2.14. Theorem. *Let* $u, v \in F_P$ *with canonical representations* $u = xy^{-1}z$ *and* $v = rs^{-1}t$. *Then*

(i) $u \mathscr{R} v$ *if and only if* $x = r$ *and* $y = s$,

(ii) $u \mathscr{L} v$ *if and only if* $y = s$ *and* $z = t$,

(iii) $u \mathscr{H} v$ *if and only if* $x = r$, $y = s$ *and* $z = t$,

(iv) $u \mathscr{D} v$ *if and only if* $y = s$.

**Proof.** (i) We know $u \mathscr{R} v$ if and only if $uu^{-1} = vv^{-1}$. But

$$uu^{-1} = (xy^{-1}z)((y/z)y^{-1}(y/x)) = xy^{-1}(y/x)$$

and similarly $vv^{-1} = rs^{-1}(s/t)$. Hence by 2.7 $uu^{-1} = vv^{-1}$ if and only if $x = r$ and $y = s$.

(ii) Analogous to (i).

(iii) Follows immediately from (i) and (ii).

(iv) Suppose $u \mathscr{D} v$. Then there is an element $w$ of $F$ with $u \mathscr{R} w$ and $w \mathscr{L} v$. Let $ab^{-1}c$ be the canonical representation of $w$. Then by (i) $y = b$ and by (ii) $b = s$. Hence $y = s$. On the other hand, if $y = s$ let $w = xy^{-1}t$. Then $u \mathscr{R} w$ by (i) and $w \mathscr{L} v$ by (ii). Hence $u \mathscr{D} v$. This completes the proof of 2.12.

From 2.14 we get that there is a $\mathscr{D}$-class $D_y$ for each element $y$ of $D$: $D_y = \{xy^{-1}z \mid x, z \in P \text{ with } x, z \leqq y\}$. Note also that $E_P \cap D_y = \{e_x f_z \mid xz = y\}$. Hence the $\mathscr{D}$-class $D_y$ can be pictured as in Figure 1.

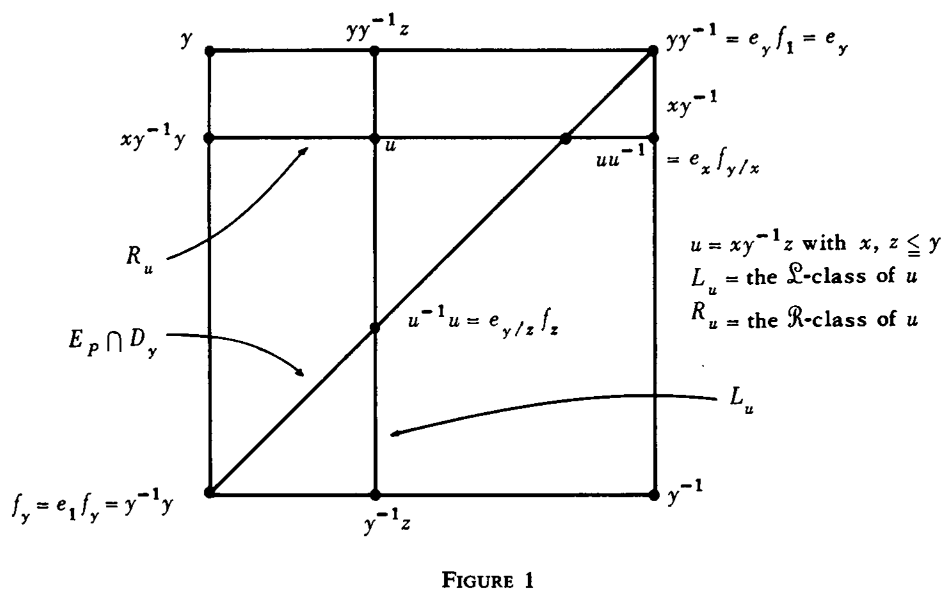



Figure 1

It may be helpful to visualize $F_P$ as in Figure 2.

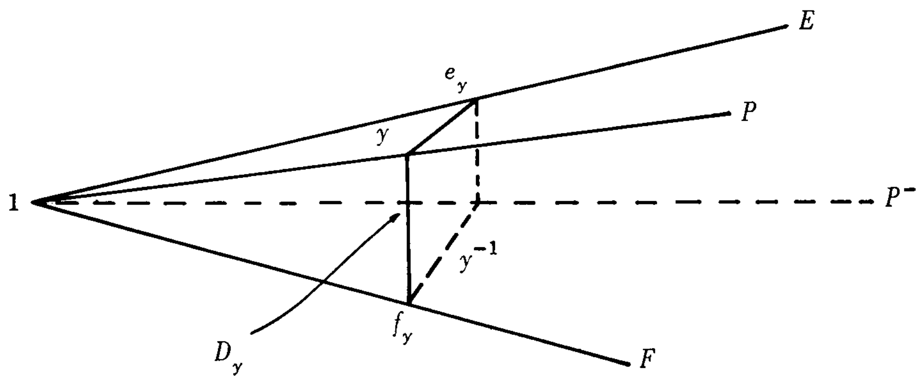



FIGURE 2

Note that the idempotents of $F_P$ lie in a plane which cuts $F_P$ into two pieces.

Next we determine the ideal structure of $F_P$. For $y \in R$, let

$$I_y = \bigcup \{D_t \mid t \geqq y \text{ and } t \in P\}$$

and let

$$I_y^\circ = \bigcup \{D_t \mid t > y \text{ and } t \in P\}.$$

2.15. THEOREM. *For each $y \in P$, $I_y$ and $I_y^\circ$ are ideals of $F_P$. Conversely, if $I$ is an ideal of $F_P$, then there is an element $y \geqq 1$ of $R$ such that $I = I_y$ or $I = I_y^\circ$. Consequently the ideals of $F_P$ are totally ordered with respect to set inclusion.*

**Proof.** The fact that $I_y$ and $I_y^\circ$ are ideals of $F_P$ follows readily from the rule for multiplication expressed in 2.7. On the other hand, if $I$ is an ideal of $F_P$, then let $y$ denote the greatest lower bound of the set of all $t \in P$ such that $D_t \cap I \neq \varnothing$. It is not difficult to show that if $D_t \cap I \neq \varnothing$, then $D_{tt_1} \subset I$ for all $t_1 \in P$, and hence $I = I_y$ if $D_y \cap I \neq \varnothing$ or $I = I_y^\circ$ if $D_y \cap I = \varnothing$. Q.E.D.

2.16. REMARK. If we give $F_P$ the natural topology described in 2.10 then the closed ideals are the ones which can be written in the form $I_y$.

3. **The lattice of congruences on $F_P$.** In this section as in the last, $G$ is an arbitrary subgroup of $R$, the multiplicative group of positive reals, and $P = \{x \in G \mid x \geqq 1\}$. We shall describe here the structure of the lattice of congruences on the free one-parameter inverse semigroup $F_P$, and hence obtain a description of every one-parameter inverse semigroup.

The set $\Lambda(S)$ of congruences on a semigroup $S$ is well known to be a complete lattice with respect to the operations

$$\sigma \wedge \rho = \sigma \cap \rho \quad \text{and} \quad \sigma \vee \rho = \bigcap \{\delta \in \Lambda(S) \mid \cup \rho\, \sigma \subset \delta\}.$$

The largest (resp. smallest) congruence on $S$, which is $S^2 = S \times S$ (resp. $\Delta S^2 = \{(x, x) \mid x \in S\}$), is denoted by 1 (resp. 0). The $\theta$ relation on $\Lambda(S)$, first defined and studied on regular semigroups $S$ by Reilly and Scheiblich [4] provides a useful aid in visualizing $\Lambda(S)$. The relation is defined by $\sigma\, \theta\, \rho$ if and only if $\sigma \cap E^2 =$

$\rho \cap E^2$, where $E$ is the set of idempotents on $S$. It is shown in [4] that if $S$ is an inverse semigroup, then $\theta$ is a lattice congruence on $\Lambda(S)$. The $\theta$-class of 1 is the set of group congruences on $S$; the $\theta$-class of 0 is the set of idempotent-separating congruences; in general, each $\theta$-class is a complete lattice of commuting congruences on $S$.

A congruence $\omega$ on $E$, the idempotents of an inverse semigroup $S$, is *normal* provided whenever $e\,\omega f$, then $xex^{-1}\,\omega\, xfx^{-1}$ for all $x \in S$. The normal congruences on $E$ are precisely those congruences $\omega$ on $E$ such that $\omega = \sigma \cap E^2$ for some $\sigma \in \Lambda(S)$. In fact one sees that $\Lambda(S)/\theta$ is isomorphic with the lattice of normal congruences on $E$, under the map induced by the map from $\Lambda(S)$ to the normal congruences on $E$ given by $\sigma \to \sigma \cap E^2$.

As a first step in describing $\Lambda(F_P)$, we shall determine the normal congruences on $E_P$, the set of idempotents of $F_P$. Recall 2.12, which says that $E_P$ is the direct sum of $E = \{xx^{-1} \mid x \in P\}$ and $F = \{x^{-1}x \mid x \in P\}$.

3.1. LEMMA. *Let $x, y, t \in P$. Then*

(i) $$te_xf_yt^{-1} = \begin{cases} e_{tx}f_{y/t} & \text{if } t \leqq y \\ e_{tx} & \text{if } y \leqq t \end{cases} = e_{xt}f_{y/y\wedge t},$$

(ii) $$t^{-1}e_xf_yt = \begin{cases} e_{x/t}f_{ty} & \text{if } t \leqq x \\ f_{ty} & \text{if } x \leqq t \end{cases} = e_{x/x\wedge t}f_{ty}.$$

**Proof.** This follows from the rule for multiplication expressed in 2.7.

Let $A$ and $B$ denote the relations on $E_P$ defined by $e_xf_y\, A\, e_rf_s$ if and only if $x = r$ and $e_xf_y\, B\, e_rf_s$ if and only if $y = s$. These are clearly congruence relations on $E_P$. Furthermore, it is also clear that $A \vee B = E_P^2$ and $A \wedge B = \Delta E_P^2$. Let $I$ be an ideal of $F_P$, and let $IA = (A \cap I^2) \cup \Delta E_P^2$, $IB = (B \cap I^2) \cup \Delta E_P^2$, and $IE_P^2 = (E_P^2 \cap I^2) \cup \Delta E_P^2$. We see immediately that $IA$, $IB$, and $IE_P^2$ are all congruences on $E_P$ also.

3.2. THEOREM. *Each of the above congruences on $E_P$ is normal. As a set of normal congruences, they form a lattice with the structure as indicated in the diagram below: ($I, J$ ideals of $F_P$ with $J \subseteq I$)*

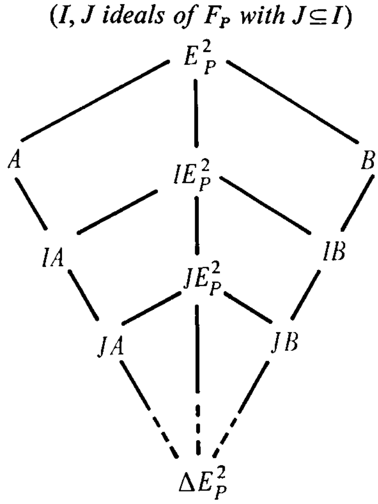



FIGURE 3

**Proof.** If $I$ is an ideal of $F_P$ and $\omega$ is a normal congruence on $E_P$, then $I\omega=(\omega\cap I^2)\cup\Delta E_P^2$ is clearly a normal congruence on $E_P$, since it is the intersection of the two normal congruences $\omega$ and $(I^2\cap E_P^2)\cup\Delta E_P^2$. Hence the only assertion requiring proof is that $A$ and $B$ are normal. To see this, let $u=ab^{-1}c\in F_P$ and note that by 3.1

$$ue_xf_yu^{-1} = e_{acx/b\wedge cx}f_{(by/y\wedge c)/(a\wedge[by/(y\wedge c)])}.$$

From this we see that $A$ and $B$ are normal. Q.E.D.

3.3. LEMMA. *Suppose $\omega$ is a normal congruence on $E_P$, and suppose $x_0, y_0, t_0\in P$ with $t_0\neq 1$. Let $I$ denote the ideal $I_{x_0y_0}=\bigcup\{D_t \mid t\geqq x_0y_0\}$ of $F_P$. Then*

(i) *if $e_{x_0}f_{y_0}\ \omega\ e_{x_0}f_{y_0t_0}$, then $IA\subseteq\omega$,*

(ii) *if $e_{x_0}f_{y_0}\ \omega\ e_{x_0t_0}f_{y_0}$, then $IB\subseteq\omega$.*

**Proof.** (i) Suppose $x, y, t\in P$ with $xy\geqq x_0y_0$. We wish to show that $e_xf_y\ \omega\ e_xf_{yt}$. Note that $e_xf_y=xf_{xy}x^{-1}$ and $e_xf_{yt}=xf_{xyt}x^{-1}$; hence the result follows if $f_{xy}\ \omega\ f_{xyt}$. To see this, first note that $f_{x_0y_0}=x_0^{-1}e_{x_0}f_{y_0}x_0\ \omega\ x_0^{-1}e_{x_0}f_{y_0t_0}x_0=f_{x_0y_0t_0}$. Hence $f_{x_0y_0t_0}=t_0^{-1}f_{x_0y_0}t_0\ \omega\ t_0^{-1}f_{x_0y_0t_0}t_0=f_{x_0y_0t_0^2}$, and so $f_{x_0y_0}\ \omega\ f_{x_0y_0t_0^2}$. Inductively, we have that $f_{x_0y_0}\ \omega\ f_{x_0y_0t_0^n}$ for each positive integer $n$. Now choose $n$ so large that $x_0y_0t_0^n\geqq xyt\geqq xy$. Then since $\omega$ is a congruence on $E_P$,

$$f_{xy} = f_{xy}\cdot f_{x_0y_0}\ \omega\ f_{xy}\cdot f_{x_0y_0t_0^n} = f_{x_0y_0t_0^n}$$

and

$$f_{xyt} = f_{xyt}\cdot f_{x_0y_0}\ \omega\ f_{xyt}\cdot f_{x_0y_0t_0^n} = f_{x_0y_0t_0^n}.$$

Hence $f_{xy}\ \omega\ f_{xyt}$ and the proof of (i) is complete. The proof of (ii) is analogous.

3.4. THEOREM. *Let $\omega$ be a nonzero normal congruence on $E_P$. Then there is an ideal $I$ of $F_P$ such that $\omega$ is one of the congruences $IA$, $IB$, or $IE_P^2$. Consequently the lattice shown in 3.2 is the lattice of all normal congruences on $E_P$.*

**Proof.** Since $\omega\neq\Delta E_P^2$, there exist $x, y, r, s\in P$ with $x\neq r$ or $y\neq s$ such that $e_xf_y\ \omega\ e_rf_s$. Suppose $x\neq r$; say $x<r$. Then since $e_xf_{y\vee s}=e_xf_y(f_{y\vee s})\ \omega\ e_rf_s(f_{y\vee s})=e_rf_{y\vee s}$, we have by 3.3 that $I_{x(s\vee y)}B\subseteq\omega$. Similarly, if $y<s$, then $I_{(x\vee r)y}A\subseteq\omega$. In any event, at least one of the sets $L=\{t\in P : I_tA\subseteq\omega\}$ and $R=\{T\in P : I_tB\subseteq\omega\}$ is nonvoid.

Suppose $R=\varnothing$ and $L\neq\varnothing$. Let $I_L=\bigcup\{I_t : t\in L\}$ and note that $I_LA=\bigcup\{I_tA : t\in L\}\subseteq\omega$. So let $e_xf_y\ \omega\ e_rf_s$; $x=r$, otherwise $R\neq\varnothing$. Assume $y<s$. Then $(e_xf_y, e_rf_s)\in I_{xy}A$. But by 3.3, $I_{xy}A\subseteq\omega$ so $xy\in L$; hence $I_{xy}A\subseteq I_LA$. Therefore $\omega=I_LA$. By an analogous argument we conclude that if $L=\varnothing$, then $R\neq\varnothing$, so $I_RB=\omega$ where $I_R=\bigcup\{I_t : t\in R\}$.

If neither $L$ nor $R$ is void, then we claim $L=R$ and $\omega=I_LE_P^2$. To see that $L=R$, let $t\in L$. Choose any $t_0\in R$. Then $(e_tf_1, e_tf_{t_0})\in I_tA\subseteq\omega$ as $t\in L$; also $(e_tf_{t_0}, e_{tt_0}f_{t_0})\in I_{t_0}B\subseteq\omega$ and $(e_{tt_0}f_{t_0}, e_{tt_0}f_1)\in I_tA\subseteq\omega$. So $(e_tf_1, e_{tt_0}f_1)\in\omega$. By 3.3 we conclude that $I_tB\subseteq\omega$; i.e. $t\in R$. Thus $L\subseteq R$. Similarly $R\subseteq L$. So $L=R$.

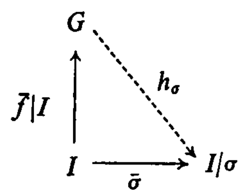
Note that $I_L E_P^2 \subseteq \omega$ since $I_L A \subseteq \omega$ and $I_R B \subseteq \omega$, and $I_L A \vee I_L B = I_L E_P^2$. Now suppose $e_x f_y \, \omega \, e_r f_s$. If $x=r$ and $y=s$, then $(e_x f_y, e_r f_s) \in \Delta E_P^2 \subseteq I_L E_P^2$. Without loss of generality assume $x \neq r$, say $x > r$. If $y=s$, then $(e_x f_y, e_r f_y) \in \omega$, so $I_{ry} B \subseteq \omega$. Thus $I_{ry} B \subseteq I_L E_P^2$, so $(e_x f_y, e_r f_s) = (e_x f_y, e_r f_y) \in I_L E_P^2$. Similarly for the case $x < r$. A similar argument shows if $x=r$ and $y \neq s$, then $(e_x f_y, e_r f_s) \in I_L E_P^2$. Now if $x \neq r$ and $y \neq s$, w.l.o.g. assume $x > r$. Then $e_x f_y \, \omega \, e_x f_s$, and hence $e_x f_s \, \omega \, e_r f_s$. By 3.3, this implies $I_{x(y \wedge s)} A \subseteq \omega$ and $I_{rs} B \subseteq \omega$, so $I_x(y \wedge s) A$ and $I_{rs} B \subseteq I_L E_P^2$. Therefore, $e_x f_y (I_L E_P^2) e_x f_s (I_L E_P^2) e_r f_s$, so $(e_x f_y, e_r f_s) \in I_L E_P^2$, and $\omega \subseteq I_L E_P^2$. This completes the proof.

Now that we have determined the lattice of normal congruences on $E_P$ (and hence the lattice $\Lambda(F_P)/\theta$), we concentrate on determining each $\theta$-class of $\Lambda(F_P)$. If $\omega$ is a normal congruence on $E_P$ then the $\theta$-class belonging to $\omega$ is the set of all congruences $\sigma \in \Lambda(F_P)$ such that $\sigma \cap E_P^2 = \omega$.

Let $I$ be an arbitrary ideal of $F_P$. In the next three theorems we shall determine the $\theta$-class belonging to $IE_P^2$. Let $f$ denote the inclusion map of $P$ into $G$ and let $\bar{f}$ denote the extension of $f$ to $F_P$. Note that $\bar{f}(xy^{-1}z) = xz/y$, and that $\bar{f}|I$ is onto $G$.

3.5. Theorem. *A congruence $\sigma$ on $I$ is a group congruence if and only if there is a subgroup $N$ of $G$ such that for each $u, v \in I$ (with canonical representations $u = xy^{-1}z$, $v = rs^{-1}t$), $u \, \sigma \, v$ if and only if $xzs/rty \in N$.*

**Proof.** Let $\sigma$ be a group congruence on $I$, and consider the following diagram:




$h_\sigma \bar{f} = \bar{\sigma}$

In order to check that the homomorphism $h_\sigma$ exists, we note that if $\bar{f}|I(xy^{-1}z) = \bar{f}|I(rs^{-1}t)$, then $xzs = rty$. Hence $\bar{\sigma}(xy^{-1}z) = \bar{\sigma}(rs^{-1}t)$. Since $\bar{f}|I$ is onto, there is a unique homomorphism induced which we call $h_\sigma$. Now let $N = \ker h_\sigma$ and note that $xy^{-1}z \, \sigma \, rs^{-1}t$ if and only if $\bar{\sigma}(xy^{-1}z) = \bar{\sigma}(rs^{-1}t)$ if and only if $h_\sigma \bar{f}(xy^{-1}z) = h_\sigma \bar{f}(rs^{-1}t)$ if and only if $h_\sigma(xz/y) = h_\sigma(rt/s)$ if and only if $xz/y \div rt/s = xzs/rty \in \ker h_\sigma = N$.

Conversely suppose $N$ is a subgroup of $G$. Let $\sigma_N$ be the relation on $I$ defined by $xy^{-1}z \, \sigma_N \, rs^{-1}t$ if and only if $xzs/rty \in N$, where $y \geqq x, z$ and $s \geqq r, t$ and $xy^{-1}z$, $rs^{-1}t \in I$. It is readily checked that $\sigma_N$ is a congruence on $I$ using the fact that $N$ is a group.

To see that $\sigma_N$ is a group congruence we need only show $I/\sigma_N$ has only one idempotent. So let $e, f$ be idempotents in $I$. Then by 2.10, $e = x(xz)^{-1}z$ and $f = r(rt)^{-1}t$ for some $x, z, r$, and $t$ in $P$. Since $xz(rt)/rt(xz) = 1 \in N$ we have that $e \, \sigma_N \, f$. Thus $I/\sigma_N$ is a group.

3.6. THEOREM. *The correspondences* $\sigma \to \ker h_\sigma$ *and* $N \to \sigma_N$ *described in* 3.1 *between the lattice of group congruences on I and the lattice of subgroups of G are mutually inversive lattice isomorphisms.*

**Proof.** Let $\sigma$ be a group congruence on $I$, and let $\delta = \sigma_{\ker h_\sigma}$. Now as in 3.5 $xy^{-1}z\,\sigma\, rs^{-1}t$ if and only if $xzs/rty \in \ker h_\sigma$. But from the definition of $\delta$, $xy^{-1}z\,\delta\, rs^{-1}t$ if and only if $xzs/rty \in \ker h_\sigma$. Hence $\sigma_{\ker h_\sigma} = \sigma$. On the other hand, let $N$ be a subgroup of $G$. Let $u, v \in I$ with canonical representations $u = xy^{-1}z$ and $v = rs^{-1}t$. Now $u\,\sigma_N\, v$ if and only if $xzs/rty \in N$. Also using the induced homomorphism $h_{\sigma_N}$, $u\,\sigma_N\, v$ if and only if $xzs/rty \in \ker h_{\sigma_N}$. Hence $N = \ker h_{\sigma_N}$. Hence the correspondences are mutually inversive functions. To complete the proof we need only show that the correspondence $N \to \sigma_N$ is a lattice homomorphism.

Let $N$ and $M$ be subgroups of $G$. It will suffice to show that $N \subseteq M$ if and only if $\sigma_N \subseteq \sigma_M$. Now it is clear that $N \subseteq M$ implies $\sigma_N \subseteq \sigma_M$. Conversely if $\sigma_N \subseteq \sigma_M$ let $x$ be in $N$ with $x = y/z$ such that $y, z \in P$. Then $(1, y, 1)\,\sigma_N\,(1, z, 1)$ implies $(1, y, 1)\,\sigma_M\,(1, z, 1)$. Thus $x \in M$ and $N \subseteq M$. This completes the proof of 3.6.

3.7. THEOREM. *The* $\theta$*-class belonging to the normal congruence* $IE_P^2$ *is isomorphic with the lattice of subgroups of G under the correspondence* $N \to \sigma_N \cup \Delta F_P^2$.

**Proof.** Let $\Gamma$ denote the $\theta$-class belonging to $IE_P^2$, $\Omega$ the lattice of subgroups of $G$, and $\Delta$ the lattice of group congruences on $I$. By 3.6 the function from $\Omega$ onto $\Delta$ taking $N$ to $\sigma_N$ is a lattice isomorphism. Hence we only need show that the function from $\Delta$ to $\Gamma$ taking $\delta$ to $\delta \cup \Delta F_P^2$ is a 1-1 onto lattice isomorphism.

To see that this function is 1-1 and onto, let $\delta \cup \Delta F_P^2 = \delta'$ for $\delta \in \Delta$ and $\rho \cap I^2 = \rho^*$ for $\rho \in \Gamma$. Clearly $\delta' \in \Gamma$ and $\rho^* \in \Delta$. Also one sees without difficulty that $(\delta')^* = \delta$, for $\delta \in \Delta$. On the other hand if $\rho \in \Gamma$, then to show that $(\rho^*)' = \rho$ we need only show that whenever $u, v \in F_P$ with $u \neq v$ and $u\,\rho\, v$ then $u, v \in I$. We consider two cases: (1) If $u \notin I$, $v \in I$, then $uu^{-1} \notin I$ and $vv^{-1} \in I$. Also $uu^{-1}\,\rho\, vv^{-1}$. However this is impossible since $\rho \cap E_P^2 = IE_P^2$. (2) If $u \notin I$, $v \notin I$, then $uu^{-1}, vv^{-1}, u^{-1}u, v^{-1}v \notin I$; but $uu^{-1}\,\rho\, vv^{-1}$, so $uu^{-1} = vv^{-1}$ since $\rho \cap E_P^2 = IE_P^2$. Similarly $u^{-1}u = v^{-1}v$. However this implies that $u$ and $v$ are $\mathscr{H}$ related and so by 2.14 we conclude that $u = v$, a contradiction. This shows that $(\rho^*)' = \rho$. Hence the functions $\delta \to \delta'$ and $\rho \to \rho^*$ are mutually inversive functions; and thus $\sigma_N \to \sigma_N \cup \Delta F_P^2$ is a 1-1 onto function.

To see that it is a lattice isomorphism, let $\delta, \sigma \in \Delta$. Then $\delta \vee \sigma = \delta \circ \sigma$, since $\delta \circ \sigma = \sigma \circ \delta$. Also $\delta' \vee \sigma' = \delta' \circ \sigma'$ according to [4]. So $(\delta \vee \sigma)' = (\delta \circ \sigma) \cup \Delta F_P^2$, and $\delta' \vee \sigma' = (\delta \cup \Delta F_P^2) \circ (\sigma \cup \Delta F_P^2)$. From this it follows that $(\delta \vee \sigma)' = \delta' \vee \sigma'$; hence $\sigma_N \to \sigma_N \cup \Delta F_P^2$ preserves $\vee$. Since the inverse of this function clearly preserves $\wedge$, we conclude that $\sigma_N \to \sigma_N \cup \Delta F_P^2$ is a lattice isomorphism.

3.8. COROLLARY. *For each subgroup N of G, let* $\sigma^N$ *denote the relation on* $F_P$ *defined by* $u\,\sigma^N\, v$ *if and only if* $u = v$, *or* $u, v \in I$ *and* $xzs/rty \in N$, *where* $xy^{-1}z$ *and* $rs^{-1}t$ *are the canonical representations of u and v respectively. Then* $\sigma^N$ *is a member* *of the $\theta$-class belonging to $IE_P^2$. Furthermore if $M$ is a subgroup of $G$ then* $\sigma^N \vee \sigma^M = \sigma^{NM}$ *and* $\sigma^N \cap \sigma^M = \sigma^{N \cap M}$.

Now we shall determine the $\theta$-class belonging to $IA$ and $IB$. It turns out that they are both degenerate. Let $g, h: P \to B_P$ be the homomorphisms given by $g(x) = (x, 1)$ and $h(x) = (1, x)$. Let $\bar{g}, \bar{h}: F_P \to B_P$ denote the extensions of $g$ and $h$, and let $\alpha, \beta$ be the congruences on $F_P$ determined by $\bar{g}, \bar{h}$ respectively. Note that $u \; \alpha \; v (u \; \beta \; v)$ if and only if $x = r$ and $yt = sz$ ($z = t$ and $yr = sx$) where $xy^{-1}z$ and $rs^{-1}t$ are the canonical representations of $u$ and $v$. Let $I\alpha = (\alpha \cap I^2) \cup \Delta F_P^2$ ($I\beta = (\beta \cap I^2) \cup \Delta F_P^2$). It is readily checked that $I\alpha$ ($I\beta$) is a congruence on $F_P$ lying in the $\theta$-class belonging to $IA$ ($IB$).

3.9. Theorem. *The $\theta$-class belonging to $IA$ ($IB$) has $I\alpha$ ($I\beta$) as its only member.*

**Proof.** Let $\Gamma$ denote the $\theta$-class belonging to $IA$, and let $\rho$ and $\sigma$ denote the largest and smallest elements of $\Gamma$ respectively. It follows from Theorem 4.2 of [4] that for $u, v \in F_P$ with canonical representations $xy^{-1}z$ and $rs^{-1}t$ respectively that $u \; \sigma \; v$ if and only if $uu^{-1} \, (IA) \, vv^{-1}$ and $eu = ev$ for some $e \in E_P$ such that $e \; IA \; uu^{-1}$. To prove the theorem we need only show that $u \; \rho \; v$ implies $u \; \sigma \; v$. So suppose $u \; \rho \; v$. Then $u^{-1} \; \rho \; v^{-1}$ so $uu^{-1} \; \rho \; vv^{-1}$. Thus $e_x f_{y/x} = uu^{-1} \, (IA) \, vv^{-1} = e_r f_{s/r}$ and so $x = r$. Also $e_{y/z} f_z = u^{-1}u \, (IA) \, v^{-1}v = e_{s/t} f_t$ and so $yt = sz$. Now let $e = e_x f_{sy}$ and note that $eu = ev$ and $e \; IA \; uu^{-1}$. Hence $u \; \sigma \; v$, and we conclude that $\sigma = \rho = I\alpha$. The proof that the $\theta$-class belonging to $IB$ contains only $I\beta$ is analogous.

The following corollary sums up the information contained in 3.7 and 3.9. For an arbitrary ideal $I$ of $F_P$ and an arbitrary congruence $\sigma$ on $F_P$, let $I\sigma$ denote the congruence $(\sigma \cap I^2) \cup \Delta F_P^2$ on $F_P$. The *top* of $\Lambda(F_P)$, $T$, is the set of group congruences on $F_P$ together with the two congruences $\alpha$ and $\beta$.

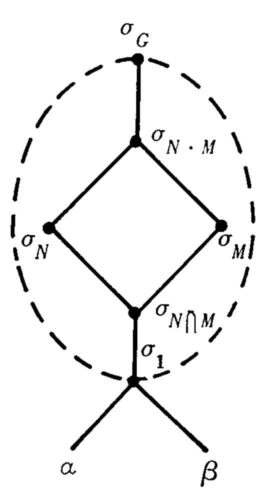



Figure 4

3.10. COROLLARY. *Every nonzero congruence $\sigma$ on $F_P$ can be written uniquely in the form $I\delta$ for some $\delta \in T$ and some ideal $I$ of $F_P$. Furthermore for ideals $I$ and $J$ of $F_P$ and $\gamma$ and $\delta$ in $T$, $I\gamma \subset J\delta$ if and only if $I \subset J$ and $\gamma \subset \delta$.*

3.11. REMARK. If we consider $F_P$ with the topology described in 2.10, then it is natural to ask what the closed congruences on $F_P$ are. It is not hard to see that 1, 0, $\alpha$ and $\beta$ are closed. Also the group congruence $\sigma_N$ is closed if and only if $N$ is cyclic, and if $I$ is an ideal of $F_P$ and $\sigma \in T$ then $I\sigma$ is closed if and only if $I$ is closed and $\sigma$ is closed.

Several additional pieces of information can be obtained from the preceding theorems. We state them below.

3.12. COROLLARY. *$\Lambda(F_P)$ is a nonmodular lattice.*

3.13. COROLLARY. *All one-parameter inverse semigroups except those of the form $F_P$ have a kernel (i.e. minimal ideal). In particular, if $I$ is an ideal of $F_P$ then $F_P/I\alpha$ and $F_P/I\beta$ have a kernel isomorphic with $B_P$ and $F_P/I\sigma_N$ has a kernel isomorphic with $G/N$.*

3.14. COROLLARY. *The lattice of congruences on $F_{P_0}$ is isomorphic with the complement of the top of $\Lambda(F_P)$ under the mapping $\sigma \to \sigma \cup \{(1, 1)\}$.*

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