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One-Parameter Lorentzian Dual Spherical Movements and Invariants of the Axodes

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Abstract: E. Study map is one of the most basic and powerful mathematical tools to study lines in line geometry, it has symmetry property. In this paper, based on the E. Study map, clear expressions were developed for the differential properties of one-parameter Lorentzian dual spherical movements that are coordinate systems independent. This eliminates the requirement of demanding coordinates transformations necessary in the determination of the canonical systems. With the proposed technique, new proofs for Euler–Savary, and Disteli's formulae were derived.

Keywords: axodes; Euler–Savary equation; Disteli-axis

MSC: 53A04; 53A05; 53A17

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Citation: Li, Y.; Alluhaibi, N.; Abdel-Baky, R.A. One-Parameter Lorentzian Dual Spherical Movements and Invariants of the Axodes. *Symmetry* **2022**, *14*, 1930. https://doi.org/10.3390/sym14091930

Academic Editors: Abraham A. Ungar and Alexander Zaslavski

Received: 23 July 2022
Accepted: 10 September 2022
Published: 15 September 2022

1. Introduction

Line families in 3-dim space are investigated widely in line geometry studies. The whole space could be either Euclidean or non-Euclidean. The interesting part of differential geometry is how extensively it has been applied in mechanism design and robot kinematics due to its direct relation to spatial motion (kinematics) [1–5]. However, in spatial kinematics, it is motivated to address the intrinsic ownerships of the line trajectory in view of ruled surfaces. In addition, it is famed that the instantaneous screw axis ($\mathbb{ISA}$) of a moving body traces a pair of ruled surfaces, fixed axodes and named the moving, with $\mathbb{ISA}$ as their ruling in the fixed space and the moving space, respectively. Through the movement, the axodes slide and roll with respect to each other in a way that the tangential contact between them is constantly preserved on the entire length of the two matting rulings (one being in each axode) that together locate the $\mathbb{ISA}$ at any instant. It is considered that a particular movement gives rise to a unique set of axodes and the other way around applies as well. Consider the axodes of any movement are known, then this yields that the specific movement can be rebuilt without knowing the physical elements of the mechanism, their arrangement, particular dimensions, or the method by which they are joined. In the process of synthesis, the characteristic of axodes has become evident in the realization that the axodes are a go-between in the middle of the physical mechanism and the actual movement of its members (see References [1–5]).

One of the most convenient manners to discuss the movement of line space is to link such space and dual numbers. W. Clifford considered dual numbers after them E. Study. He utilized it as an intermediary for their realization of kinematics and differential line geometry. He gave particular notice to the parametrization of directed lines by dual unit vectors and definition of E. Study map; The set of all directed lines in Euclidean three-space $\mathbb{E}^3$ is represented set of points on the dual unit sphere in the dual three-space $\mathbb{D}^3$. The fundamental idea and philosophy of E. Study map related to symmetry. Because E. Study map gives the symmetry property between the directed lines in Euclidean three-space are the corresponding points on the dual unit sphere in the dual three-space. More details on screw calculus and E. Study map can be found in [1–5].

Inspired by such works, if we consider $\mathbb{E}_1^3$ (the Minkowski 3-space $\mathbb{E}_1^3$) instead of $\mathbb{E}^3$ then the study would be much more motivated utilizing the property that the distance function $<,>$ can be zero, negative or positive, whilst the distance function in the Euclidean case is only positive. In such a case, we need to disconnect directed lines on account of whether the distance function is zero, negative, or positive. Directed lines with $<,>= 0$ are named null lines, and directed lines with $<,>< 0$ (resp $<,>> 0$) are named timelike (resp. spacelike) directed lines. In consideration of its link to the physical science of Minkowski space and engineering, many engineers and geometers have addressed curves and straight surfaces and different surfaces and have observed many different aspects (see [6–20]).

In this study, we used the E. Study map in order to give a direct manner for handling the kinematic-geometry of one parameter Lorentzian dual spherical movements by using the aspects of axodes with analogizing to Lorentzian spherical kinematic. Then, new proofs of Disteli and Euler–Savary formulae are given which demonstrate the efficiency and elegance of the E. Study map in Lorentzian spatial kinematics. Furthermore, interdisciplinary research can provide valuable new insights, but synthesizing articles across disciplines with highly varied standards, formats, terminology, and methods required an adapted approach. Recently, many interesting papers related to symmetry, Bertrand curves, submanifold theory, singularity theory and harmonic quasiconformal mappings etc. [21–54]. In future work, we plan to study the one-parameter Lorentzian dual spherical movements and invariants of the axodes for different queries and further improve the results in this paper combine with the technics and results in [21–54]. We intend to do the implementation of those results in our following papers.

2. Preliminaries

In this section, we provide a brief introduction to the theory of dual numbers and dual Lorentzian vectors [1–5,16–20]. Let x, and x^* be real numbers, then the expression: $\widehat{x} = x + \varepsilon x^*$ is named a dual number, such that $\varepsilon \neq 0$, and $\varepsilon^2 = 0$. This is in fact very comparable to the idea of a complex number, the major distinction being than in a complex number $\varepsilon^2 = -1$. Then the set

$$\mathbb{D}^3 = \{\widehat{\mathbf{x}} := \mathbf{x} + \varepsilon \mathbf{x}^* = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)\},$$

together with the Lorentzian inner product

$$< \widehat{\mathbf{x}}, \widehat{\mathbf{y}} >= \widehat{x}_1\widehat{y}_1 - \widehat{x}_2\widehat{y}_2 + \widehat{x}_3\widehat{y}_3,$$

is referred to as dual Lorentzian 3-space $\mathbb{D}_1^3$. Hence,

$$\begin{gathered} < \widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_1 >= - < \widehat{\mathbf{f}}_2, \widehat{\mathbf{f}}_2 >=< \widehat{\mathbf{f}}_3, \widehat{\mathbf{f}}_3 >= 1, \\ \widehat{\mathbf{f}}_1 \times \widehat{\mathbf{f}}_2 = \widehat{\mathbf{f}}_3, \widehat{\mathbf{f}}_2 \times \widehat{\mathbf{f}}_3 = \widehat{\mathbf{f}}_1, \widehat{\mathbf{f}}_3 \times \widehat{\mathbf{f}}_1 = -\widehat{\mathbf{f}}_2. \end{gathered}$$

Here, $\widehat{\mathbf{f}}_1$, $\widehat{\mathbf{f}}_2$, and $\widehat{\mathbf{f}}_3$, are the dual base at the origin point $\widehat{\mathbf{o}}(0,0,0)$ of the dual Lorentzian 3-space $\mathbb{D}_1^3$. Therefore, $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)^t$ is a point whose dual coordinates are $\widehat{x}_i = (x_i + \varepsilon x_i^*) \in \mathbb{D}$. If $\mathbf{x} \neq \mathbf{0}$ the norm of $\widehat{x}$ is given as

$$\|\widehat{\mathbf{x}}\| = \sqrt{|< \widehat{\mathbf{x}}, \widehat{\mathbf{x}} >|} = \|\mathbf{x}\|(1+\varepsilon \frac{< \mathbf{x}, \mathbf{x}^* >}{\|\mathbf{x}\|^2}).$$

Thus, $\widehat{\mathbf{x}}$ is named a timelike (resp. spacelike) dual unit vector if $\|\widehat{\mathbf{x}}\|^2 = -1$ (resp. $\|\widehat{\mathbf{x}}\|^2 = 1$). It is evident that

$$\|\widehat{\mathbf{x}}\|^2 = \pm 1 \Longleftrightarrow \|\mathbf{x}\|^2 = \pm 1, < \mathbf{x}, \mathbf{x}^* >= 0.$$

The Lorentzian (de Sitter space) and hyperbolic dual unit spheres, respectively, are:

$$\mathbb{S}_1^2 = \{\widehat{x} \in \mathbb{D}_1^3 \mid \widehat{x}_1^2 - \widehat{x}_2^2 + \widehat{x}_3^2 = 1\},$$

and

$$\mathbb{H}_+^2 = \{\widehat{\mathbf{x}} \in \mathbb{D}_1^3 \mid \widehat{x}_1^2 - \widehat{x}_2^2 + \widehat{x}_3^2 = -1\},$$

respectively. Then the E. Study map can be stated as follows: The ring shaped hyperboloid correspondence with the set of spacelike lines, the common (shared) asymptotic cone correspondence with the set of null-lines, and the oval shaped hyperboloid correspondence with the set of timelike lines (see Figure 1).

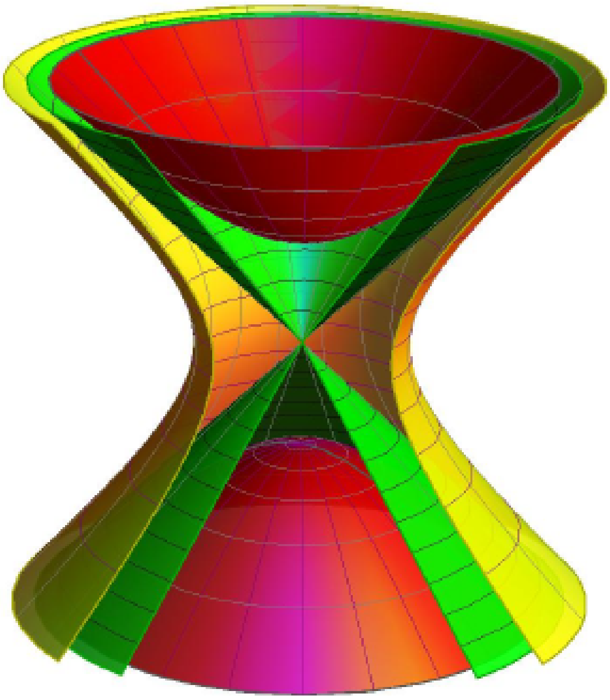

Figure 1. The dual Lorentzian and dual hyperbolic unit spheres.

In view of the E. Study map, a regular curve on the dual unit sphere $\mathbb{H}_+^2$ (or $\mathbb{S}_1^2$) can locally be null (lightlike), spacelike or timelike if all of its velocity vectors are null (lightlike), spacelike or, timelike, respectively. The differentiable curve $t \in \mathbb{R} \mapsto \widehat{x}(t) \in \mathbb{H}_+^2$(or $\mathbb{S}_1^2$), corresponds to a timelike ruled surface $(\widehat{x})$ in Minkowski 3-space $\mathbb{E}_1^3$. In a similar way, the dual curve on $\mathbb{S}_1^2$ corresponds to a spacelike or timelike ruled surface in $\mathbb{E}_1^3$. $\widehat{x}(t)$ are determined by the rulings of the surface and from now on we do not characterize between ruled surface and its explaining dual curve.

Definition 1. *Let $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ be non-null dual vectors in $\mathbb{D}_1^3$. Then [55],*

(i) If $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ are spacelike dual vectors, then

- *In case they span a spacelike dual plane, there exists a unique dual number $\widehat{\varphi} = \varphi + \varepsilon\varphi^*$; $0 \leq \varphi \leq \pi$, and $\varphi^* \in \mathbb{R}$ in which $< \widehat{\mathbf{x}}, \widehat{\mathbf{y}} >= \|\widehat{\mathbf{x}}\| \|\widehat{\mathbf{y}}\| \cos \widehat{\varphi}$. Such number is named the spacelike dual angle between $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$.*
- *In case they span a timelike dual plane, then there exists a unique dual number $\widehat{\varphi} = \varphi + \varepsilon\varphi^* \geq 0$ in which $< \widehat{\mathbf{x}}, \widehat{\mathbf{y}} >= \epsilon \|\widehat{\mathbf{x}}\| \|\widehat{\mathbf{y}}\| \cosh \widehat{\varphi}$, such that $\epsilon = +1$ or $\epsilon = -1$ based to $sign(\widehat{\mathbf{x}}_2) = sign(\widehat{\mathbf{y}}_2)$ or $sign(\widehat{\mathbf{x}}_2) \neq sign(\widehat{\mathbf{y}}_2)$, respectively. Such number is named the central dual angle between $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$.*

(ii) If $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ are timelike dual vectors, then there exists a unique dual number $\widehat{\varphi} = \varphi + \varepsilon\varphi^* \geq 0$ in which $< \widehat{\mathbf{x}}, \widehat{\mathbf{y}} >= \epsilon \|\widehat{\mathbf{x}}\| \|\widehat{\mathbf{y}}\| \cosh \widehat{\varphi}$, such that $\epsilon = -1$ or $\epsilon = +1$ based on $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ having

the same time-orientation or different time-orientation, respectively. This dual number is named the Lorentzian timelike dual angle between $\widehat{\mathbf{x}}$ *and* $\widehat{\mathbf{y}}$*.*

(iii) *Let* $\widehat{\mathbf{x}}$ *be spacelike dual, and* $\widehat{\mathbf{y}}$ *be timelike dual, then there exists a unique* $\widehat{\varphi} = \varphi + \varepsilon\varphi^* \geq 0$ *in which* $< \widehat{\mathbf{x}}, \widehat{\mathbf{y}} >= \epsilon\|\widehat{x}\|\|\widehat{\mathbf{y}}\| \sinh \widehat{\varphi}$*, such that* $\epsilon = +1$ *or* $\epsilon = -1$ *based on whether* $sign(\widehat{\mathbf{x}}_2) = sign(\widehat{y}_1)$ *or* $sign(\widehat{\mathbf{x}}_2) \neq sign(\widehat{\mathbf{y}}_1)$*. Such number is named the Lorentzian timelike dual angle between* $\widehat{\mathbf{x}}$ *and* $\widehat{\mathbf{y}}$*.*

3. One-Parameter Lorentzian Dual Spherical Movements

Consider $\mathbb{S}^2_{1m}$ and $\mathbb{S}^2_{1f}$ as Lorentzian dual unit spheres with $\widehat{\mathbf{o}}$ as a common center in $\mathbb{D}^3_1$. We suppose that $\{\widehat{\mathbf{o}};\ \widehat{\mathbf{e}}_1,\ \widehat{\mathbf{e}}_2(\text{timelike}),\ \widehat{\mathbf{e}}_3\}$ and $\{\widehat{\mathbf{o}};\ \widehat{\mathbf{f}}_1,\ \widehat{\mathbf{f}}_2(\text{timelike}),\ \widehat{\mathbf{f}}_3\}$ be two orthonormal dual bases rigidly associated with $\mathbb{S}^2_{1m}$ and $\mathbb{S}^2_{1f}$, respectively. In case that we let $\{\widehat{\mathbf{o}};\widehat{\mathbf{f}}_1,\ \widehat{\mathbf{f}}_2,\ \widehat{\mathbf{f}}_3\}$ fixed, whereas the components of the set $\{\widehat{\mathbf{o}};\ \widehat{\mathbf{e}}_1,\ \widehat{\mathbf{e}}_2,\ \widehat{\mathbf{e}}_3\}$ are functions of a real parameter $t \in \mathbb{R}$ (say the time). Here, we say that $\mathbb{S}^2_{1m}$ moves according to $\mathbb{S}^2_{1f}$. Such movement is named a one-parameter Lorentzian dual spherical movement which is denoted by $\mathbb{S}^2_{1m}/\mathbb{S}^2_{1f}$. If the Lorentzian dual unit spheres $\mathbb{S}^2_{1m}$ and $\mathbb{S}^2_{1f}$ correspond to the line spaces $\mathbb{L}_m$ and $\mathbb{L}_f$, respectively, then $\mathbb{S}^2_{1m}/\mathbb{S}^2_{1f}$ represents the one-parameter Lorentzian spatial movement $\mathbb{L}_m/\mathbb{L}_f$. Therefore, $\mathbb{L}_m$ is the moving Lorentzian space with respect to the Lorentzian fixed space $\mathbb{L}_f$.

Since each of these orthonormal dual frames has the same orientation, one frame is obtained by using another when rotated about **O**. Putting $< \widehat{\mathbf{e}}_i, \widehat{\mathbf{f}}_j >= \widehat{A}_{ij} = A_{ij} + \varepsilon A^*_{ij}$ and introducing the dual matrix $\widehat{A}(t) = (\widehat{A}_{ij})$. It then follows that the signature matrix ϵ, describing the inner product of dual Lorentzian 3-space $\mathbb{D}^3_1$, is given by

$$\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the dual matrix $\widehat{A}(t) = (A_{ij}(t) + \varepsilon A^*_{ij}(t))$ satisfy the equations $\widehat{A}^T = \epsilon\widehat{A}^{-1}\epsilon$, and $\widehat{A}^{-1} = \epsilon\widehat{A}^T\epsilon\widehat{A}$. Hence, we have

$$\widehat{A}\widehat{A}^{-1} = \widehat{A}\epsilon\widehat{A}^T\epsilon = \widehat{A}^{-1}\widehat{A} = \epsilon\widehat{A}^T\epsilon\widehat{A} = I, \quad (1)$$

which means it is a Lorentzian orthogonal matrix. Such a finding tells that if a one-parameter Lorentzian spatial movement is given in $\mathbb{E}^3_1$, we can find the associated dual Lorentzian orthogonal 3×3 matrix $\widehat{A}(t) = (\widehat{A}_{ij})$, in which $(\widehat{A}_{ij})$ are dual functions of one variable $t \in \mathbb{R}$. As the set of real Lorentzian orthogonal matrices, the set $\mathbb{O}(\mathbb{D}^{3\times3}_1)$ of Lorentzian dual orthogonal 3×3 matrices form a group with matrix multiplication (real Lorentzian orthogonal matrices are a subgroup of Lorentzian dual orthogonal matrices). The identity element of $\mathbb{O}(\mathbb{D}^{3\times3}_1)$ is the 3×3 unit matrix. The transformation group in $\mathbb{D}^3_1$ (the image of Lorentzian movements in the Minkowski 3-space $\mathbb{E}^3_1$) does not contain any translations since the center of the Lorentzian dual unit sphere in $\mathbb{D}^3_1$ should stay fixed. Therefore, in order to perform the Lorentzian movements in $\mathbb{D}^3_1$, we can state the next theorem:

Theorem 1. *The set of all one-parameter Lorentzian spatial movements in* $\mathbb{E}^3_1$*-space is in one-to-one correspondence with the set of Lorentzian dual orthogonal matrices* $\mathbb{O}(\mathbb{D}^{3\times3}_1)$ *in* $\mathbb{D}^3_1$*-space.*

To derive the form of an element of the dual Lie algebra $L(\mathbb{O}_{\mathbb{D}^{3\times3}_1})$ of the dual group $\mathbb{O}(\mathbb{D}^{3\times3}_1)$, we take a dual curve of such dual matrices $\widehat{A}(t)$ such that $\widehat{A}(0)$ is the identity. By taking the derivative of Equation (1) with respect to t, we obtain:

$$\widehat{A}'\widehat{A}^{-1} + \widehat{A}\left(\widehat{A}^{-1}\right)' = 0;\ \text{0 is zero } 3 \times 3 \text{ matrix.}$$

By choosing $\widehat{\psi}(t) = \widehat{A}'\widehat{A}^{-1}$, we see that $\widehat{\psi}^T + \epsilon\widehat{\psi}\epsilon = 0$, that is, the matrix $\widehat{\psi}$ is a skew-adjoint matrix. Thus, according to Theorem 1, the dual Lie algebra $L(\mathbb{O}_{\mathbb{D}_1^{3\times 3}})$ of the dual Lorentzian group $\mathbb{O}(\mathbb{D}_1^{3\times 3})$ is the dual algebra of skew-adjoint 3×3 dual matrices

$$\widehat{\psi}(t) = \begin{pmatrix} 0 & \widehat{\psi}_3 & \widehat{\psi}_2 \\ \widehat{\psi}_3 & 0 & \widehat{\psi}_1 \\ \widehat{\psi}_2 & -\widehat{\psi}_1 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \\ \widehat{\psi}_3 \end{pmatrix} = \widehat{\psi} \in L(O_{\mathbb{D}_1^{3\times 3}}).$$

Suppose that $\widehat{\mathbf{x}}_f$ and $\widehat{\mathbf{x}}_m$ represent a spacelike dual unit vector expressed in $\mathbb{S}^2_{1m}$ and $\mathbb{S}^2_{1f}$, respectively, then we may express their relationship as:

$$\widehat{\mathbf{x}}_f = \widehat{A}\widehat{\mathbf{x}}_m. \tag{2}$$

The inverse transformation is

$$\widehat{\mathbf{x}}_m = \epsilon\widehat{A}^T\epsilon\widehat{\mathbf{x}}_f. \tag{3}$$

Equation (2), by differentiation with respect to t, yields

$$\widehat{\mathbf{x}}_f' = \widehat{A}_m'\widehat{\mathbf{x}}.$$

Then, we have

$$\widehat{\mathbf{x}}_f' = \widehat{\psi}_f\widehat{\mathbf{x}}_f, \text{ and } \widehat{\mathbf{x}}' = \widehat{\psi}_m\widehat{\mathbf{x}},$$

where $\widehat{\mathbf{x}}_f'$, and $\widehat{\mathbf{x}}'$ represent the absolute time derivatives of $\widehat{\mathbf{x}}$ in $\mathbb{S}^2_{1m}$ and $\mathbb{S}^2_{1f}$, respectively. Furthermore, we obtain $\widehat{\psi}_f(t)$, and $\widehat{\psi}_m(t)$ as below:

$$\widehat{\psi}_f(t) := \widehat{A}'\epsilon\widehat{A}^T\epsilon = \begin{pmatrix} 0 & \widehat{\psi}_{3f} & \widehat{\psi}_{2f} \\ \widehat{\psi}_{3f} & 0 & \widehat{\psi}_{1f} \\ -\widehat{\psi}_{2f} & \widehat{\psi}_{1f} & 0 \end{pmatrix} = \begin{pmatrix} \widehat{\psi}_{1f} \\ \widehat{\psi}_{2f} \\ \widehat{\psi}_{3f} \end{pmatrix}, \tag{4}$$

and

$$\widehat{\psi}_m(t) := \epsilon\widehat{A}^T\epsilon\widehat{A}' = \begin{pmatrix} 0 & \widehat{\psi}_{3m} & \widehat{\psi}_{2m} \\ \widehat{\psi}_{3m} & 0 & \widehat{\psi}_{1m} \\ -\widehat{\psi}_{2m} & \widehat{\psi}_{1m} & 0 \end{pmatrix} = \begin{pmatrix} \widehat{\psi}_{1m} \\ \widehat{\psi}_{2m} \\ \widehat{\psi}_{3m} \end{pmatrix}. \tag{5}$$

From the above two equations, we have

$$\widehat{\psi}_f(t) = \widehat{A}(t)\widehat{\psi}_m(t)\widehat{A}^{-1}(t), \text{ and } \left\|\widehat{\boldsymbol{\psi}}_f\right\| = \|\widehat{\boldsymbol{\psi}}_m\|.$$

3.1. Lorentzian Spatial Kinematics and Invariants of the Axodes

In this subsection, we consider the Lorentzian spatial kinematics and invariants of the axodes by bearing in mind the E. Study map. Therefore, the dual curve (polode) $\widehat{\mathbf{r}}_m(t) = \widehat{\boldsymbol{\psi}}_m(t)\|\widehat{\boldsymbol{\psi}}_m(t)\|^{-1}$ represents the locus of the instantaneous screw axis ($\mathbb{ISA}$) on $\mathbb{S}^2_{1m}$. This locus is timelike or spacelike moving axode in $\mathbb{L}_m$-space. This moving axode is the locus of the $\mathbb{ISA}$ as viewed from the moving space $\mathbb{L}_m$. It will be assumed a spacelike ruled surface in our study, and let us denote this surface by π_m. Similarly, the $\mathbb{ISA}$ on $\mathbb{S}^2_{1f}$ is also a dual curve (polode) $\widehat{\mathbf{r}}_f(t) = \widehat{\boldsymbol{\psi}}_f(t)\left\|\widehat{\boldsymbol{\psi}}_f(t)\right\|^{-1}$. This curve like-wise corresponds to the fixed axode π_f. This fixed axode is made up of those lines in the fixed space $\mathbb{L}_f$ which at some instant coincides with a line in the moving space $\mathbb{L}_m$ having zero dual velocity. It is important to note that $\|\widehat{\boldsymbol{\psi}}_m\| = \left\|\widehat{\boldsymbol{\psi}}_f\right\| = \widehat{\psi} = \psi + \varepsilon\psi^*$ is the dual angular speed of the movement $\mathbb{S}^2_{1m}/\mathbb{S}^2_{1f}$. ψ and ψ^* correspond, respectively, to the rotation movements and the translation movements of the movement $\mathbb{L}_m/\mathbb{L}_f$. Hence, the one-parameter Lorentzian spatial movement $\mathbb{L}_m/\mathbb{L}_f$ can be described by rotation ψ about and translation ψ^* along the $\mathbb{ISA}$. Thus, the pitch of the movement $\mathbb{L}_m/\mathbb{L}_f$ is defined by

$$h(t) = \frac{<\boldsymbol{\psi}, \boldsymbol{\psi}^*>}{\|\boldsymbol{\psi}\|^2} = \frac{\psi_1\psi_1^* - \psi_2\psi_2^* + \psi_3\psi_3^*}{\psi_1^2 - \psi_2^2 + \psi_3^2} = \frac{\psi^*}{\psi}. \tag{6}$$

Corollary 1. *During a one-parameter Lorentzian dual spherical movement $\mathbb{S}^2_{1m}/\mathbb{S}^2_{1f}$, the tangent vectors of the fixed and moving timelike polodes are joined as:*

$$\pi_f : \widehat{\mathbf{r}}'_f = \widehat{A}\widehat{\mathbf{r}}'_m\widehat{A}^{-1}. \tag{7}$$

Proof. Without loss of generality, we take the parameter $t \in \mathbb{R}$ as the canonical parameter of the movement $\mathbb{S}^2_{1m}/\mathbb{S}^2_{1f}$, that is, $\left\|\widehat{\boldsymbol{\psi}}_f(t)\right\| = \|\widehat{\boldsymbol{\psi}}_m(t)\| = 1$. Then:

$$\pi_m : \widehat{\mathbf{r}}_m = \widehat{A}^{-1}\widehat{A}', \;\; \pi_f : \widehat{\mathbf{r}}_f = \widehat{A}'\widehat{A}^{-1}.$$

Furthermore, we have:

$$\widehat{\mathbf{r}}'_m = \widehat{A}'^{-1}\widehat{A}' + \widehat{A}^{-1}\widehat{A}'',$$

and

$$\left.\begin{aligned} \widehat{A}\widehat{\mathbf{r}}'_m\widehat{A}^{-1} &= \widehat{A}(\widehat{A}'^{-1}\widehat{A}' + \widehat{A}^{-1}\widehat{A}'')\widehat{A}^{-1} \\ &= \widehat{A}\widehat{A}'^{-1}\widehat{A}'\widehat{A}^{-1} + \widehat{A}\widehat{A}^{-1}\widehat{A}''\widehat{A}^{-1}. \end{aligned}\right\}$$

Furthermore, we find

$$\widehat{\mathbf{r}}'_f = \widehat{A}''\widehat{A}^{-1} + \widehat{A}'\widehat{A}'^{-1}.$$

From the relation $\widehat{A}\widehat{A}^{-1} = I$, we obtain $\widehat{A}'\widehat{A}^{-1} + \widehat{A}\widehat{A}'^{-1} = 0$. Substituting into the expression for $\widehat{A}\widehat{\mathbf{r}}'_m\widehat{A}^{-1}$, we obtain:

$$\left.\begin{aligned} \widehat{A}\widehat{\mathbf{r}}'_m\widehat{A}^{-1} &= -\widehat{A}\widehat{A}'^{-1}\widehat{A}\widehat{A}'^{-1} + \widehat{A}''\widehat{A}^{-1} \\ &= \widehat{A}'\widehat{A}^{-1}\widehat{A}\widehat{A}'^{-1} + \widehat{A}''\widehat{A}^{-1} \\ &= \widehat{A}'\widehat{A}'^{-1} + \widehat{A}''\widehat{A}^{-1} = \widehat{\mathbf{r}}'_f. \end{aligned}\right\}$$

This completes the proof. □

Equation (7) contains only first order derivatives of $\pi_f(\pi_m)$, it is a first-order property of the axodes, in particular is its dual unit speed. More explicitly, we have that

$$\left\|\widehat{\mathbf{r}}'_f\right\| = \sqrt{<\widehat{\mathbf{r}}'_f, \widehat{\mathbf{r}}'_f>} = \sqrt{<\widehat{A}\widehat{\mathbf{r}}'_m\widehat{A}^{-1}, \widehat{A}\widehat{\mathbf{r}}'_m\widehat{A}^{-1}>} = \left\|\widehat{\mathbf{r}}'_m\right\|. \tag{8}$$

The above relation has only first-order derivatives of π_f (π_m); it is a first-order property of the fixed (moving) axode, particularly its dual unit speed. It follows from Equation (8) that $\left\|\widehat{\mathbf{r}}'_f\right\| = \left\|\widehat{\mathbf{r}}'_m\right\|$ and can replaced them by $\widehat{p} = p + \varepsilon p^*$ for short hereinafter. Thus, we can take the dual arc-length parameter $d\widehat{s} := ds + \varepsilon ds^* = \widehat{p}dt$ to replace the arbitrary parameter, t.

Hence, the common distribution parameter of the axodes is

$$\mu(t) := \frac{ds^*}{ds} = \frac{p^*}{p}. \tag{9}$$

Therefore, tangent conditions between the axodes π_m and π_f can be given by the following corollary:

Corollary 2. *During a one-parameter Lorentzian spatial movement $\mathbb{L}_m/\mathbb{L}_f$, the spacelike axodes contact each other along and rolls around the $\mathbb{ISA}$ in the first order (Figure 2).*

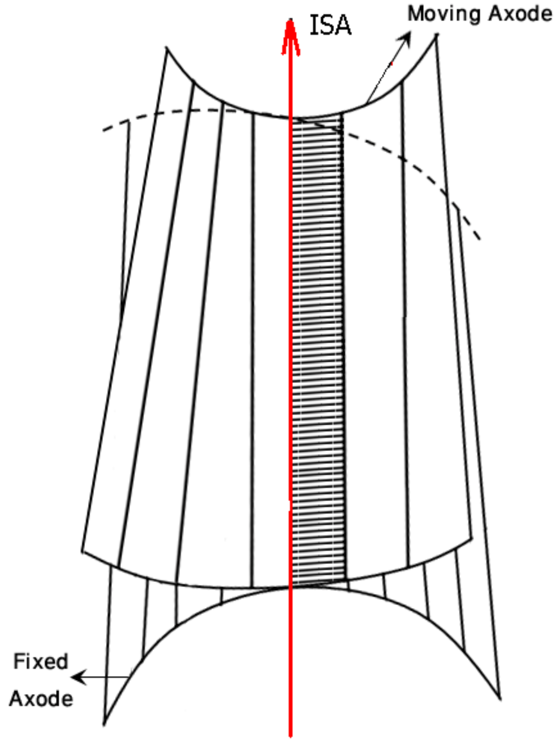

Figure 2. Typical portions of spacelike axodes.

3.2. Euler–Savary Formula for the Spacelike Axodes

We now shall deduce the Euler–Savary and Disteli formulae for the spacelike axodes by using the apparatus just derived above. Thus, as in spherical geometry, we introduce a Lorentzian Blaschke frame

$$\{\widehat{\mathbf{o}};\widehat{\mathbf{r}}_i{=}\widehat{\mathbf{r}}_i(t),\ \widehat{\mathbf{t}}_i(t)=\mathbf{r}_i^{'}\left\|\widehat{\mathbf{r}}_i^{'}\right\|^{-1},\ \widehat{\mathbf{g}}_i(t)=\widehat{\mathbf{r}}_i\times\widehat{\mathbf{t}}_i\}$$

of the polode $\widehat{\mathbf{r}}_i(t)$ on $\mathbb{S}^2_{1i}(i=m,\ f)$. Such that

$$\begin{array}{c}<\widehat{\mathbf{r}}_i,\widehat{\mathbf{r}}_i>=-<\widehat{\mathbf{t}}_i,\widehat{\mathbf{t}}_i>=<\widehat{\mathbf{g}}_i,\widehat{\mathbf{g}}_i>=1,\\ \widehat{\mathbf{r}}_i\times\widehat{\mathbf{t}}_i=\widehat{\mathbf{g}}_i,\widehat{\mathbf{t}}_i\times\widehat{\mathbf{g}}_i=\widehat{\mathbf{r}}_i,\widehat{\mathbf{g}}_i\times\widehat{\mathbf{r}}_i=-\widehat{\mathbf{t}}_i.\end{array} \tag{10}$$

$\widehat{\mathbf{r}}_i$, $\widehat{\mathbf{t}}_i$, and $\widehat{\mathbf{g}}_i$ correspond to three concurrent mutually orthogonal lines, and their point of intersection is the central point $\mathbf{c}_i$ on the the ruling $\widehat{\mathbf{r}}_i$. $\widehat{\mathbf{g}}_i$ is the limit position of mutual orthogonal to $\widehat{\mathbf{r}}_i(t)$ and $\widehat{\mathbf{r}}_i(t+dt)$ and is named the central tangent of the spacelike axode $\widehat{\mathbf{r}}_i(t)$ at the central point and $\widehat{\mathbf{t}}_i$ is named its central normal. By construction, the Blaschke formula is (11)–(14):

$$\begin{pmatrix}\widehat{\mathbf{r}}_i^{'}\\ \widehat{\mathbf{t}}_i^{'}\\ \widehat{\mathbf{g}}_i^{'}\end{pmatrix}=\begin{pmatrix}0&\widehat{p}&0\\ \widehat{p}&0&\widehat{q}_i\\ 0&\widehat{q}_i&0\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{r}}_i\\ \widehat{\mathbf{t}}_i\\ \widehat{\mathbf{g}}_i\end{pmatrix}=\widehat{\omega}_i\times\begin{pmatrix}\widehat{\mathbf{r}}_i\\ \widehat{\mathbf{t}}_i\\ \widehat{\mathbf{g}}_i\end{pmatrix}, \tag{11}$$

where $\widehat{\omega}_i=\widehat{q}_i\widehat{r}_i-\widehat{p}\widehat{\mathbf{g}}_i$, and

$$\widehat{p}(t)=p+\varepsilon p^*=\left\|\widehat{\mathbf{r}}_i^{'}\right\|,\ \widehat{q}_i(t):=q_i+\varepsilon q_i^*=\det(\widehat{\mathbf{r}}_i,\widehat{\mathbf{r}}_i^{'},\widehat{\mathbf{r}}_i^{''})\left\|\widehat{\mathbf{r}}_i^{'}\right\|^{-2}, \tag{12}$$

are invariants of the movement $\mathbb{L}_m/\mathbb{L}_f$. We shall neglect the pure translational movements, that is, $\widehat{q}_m\neq\widehat{q}_f$. Furthermore, we eliminate zero divisors, that is, $q_m^*\neq q_f^*$. The evolute of the spacelike axode π_i is obtained as

$$\widehat{\mathbf{b}}_i(t) := \mathbf{b_i}+\varepsilon\mathbf{b_i^*} = \frac{\widehat{\mathbf{r}}_i' \times \widehat{\mathbf{r}}_i''}{\|\widehat{\mathbf{r}}_i' \times \widehat{\mathbf{r}}_i''\|} = \frac{\widehat{q}_i\widehat{\mathbf{r}}_i - \widehat{p}\widehat{\mathbf{g}}_i}{\sqrt{\widehat{q}_i^2 + \widehat{p}^2}}. \tag{13}$$

It is obvious that $\widehat{\mathbf{b}}_i$ is the Disteli-axis (curvature axis or striction axis) of the spacelike axode π_i. If $\widehat{\varphi}_i = \varphi_i + \varepsilon\varphi_i^*$; $0 \le \varphi_i \le \pi$, and $\varphi_i^* \in \mathbb{R}$ is the radius of curvature between $\widehat{r}_i$ and $\widehat{\mathbf{b}}_i$, then we obtain

$$\widehat{\mathbf{b}}_i = \cos\widehat{\varphi}_i\widehat{\mathbf{r}}_i - \sin\widehat{\varphi}_i\widehat{\mathbf{g}}_i. \tag{14}$$

Here,

$$\widehat{\gamma}_i(t) = \gamma_i + \varepsilon\gamma_i^* = \frac{\widehat{q}_i}{\widehat{p}} = \cot\widehat{\varphi}_i, \text{ with } \widehat{p} \neq 0, \tag{15}$$

is the dual geodesic curvature of the spacelike axode π_i. In view of Equation (15), we conclude the equality:

$$\widehat{\gamma}_f - \widehat{\gamma}_m = \cot\widehat{\varphi}_f - \cot\widehat{\varphi}_m = \frac{\widehat{\omega}}{\widehat{p}}, \text{ with } \widehat{\omega} = \widehat{q}_f - \widehat{q}_m. \tag{16}$$

This is a dual Lorentzian formula of Euler–Savary formula for the polodes of real spherical movement (compare with [1–3]). Equation (16) shows a relationship among the two spacelike axodes in direct contact and the kinematic-geometry representing the instantaneous invariants of the movement $\mathbb{L}_m/\mathbb{L}_f$. By separating the real and dual parts of Equation (16), respectively, we find:

$$\cot\varphi_f - \cot\varphi_m = \frac{\omega}{p}, \tag{17}$$

and

$$\frac{\varphi_m^*}{\sin^2\varphi_m} - \frac{\varphi_f^*}{\sin^2\varphi_f} = \frac{\omega}{p}(\mu - h). \tag{18}$$

The Eqsuations (17) and (18) are new Disteli's formulae for the spacelike axodes. Observe that the scalars ω, ω^* and h are invariants of the choice of the reference point.

Velocity and Acceleration for a Spacelike Line Trajectory

Through the movement $\mathbb{L}_m/\mathbb{L}_f$, each fixed spacelike line $\widehat{\mathbf{x}}$ adjoint with the moving spacelike axode, generally, will trace a timelike or spacelike ruled surface (X) in the fixed space $\mathbb{L}_f$. It will be assumed a spacelike ruled surface in our study. In kinematics, this spacelike ruled surface is referred to as spacelike line trajectory. The vector equation of (X) is represented by

$$\left.\begin{array}{l}\widehat{\mathbf{x}}(\widehat{s}) = \widehat{x}_1(\widehat{s})\widehat{\mathbf{r}}_i(\widehat{s}) + \widehat{x}_2(\widehat{s})\widehat{\mathbf{t}}_i(\widehat{s}) + \widehat{x}_3(\widehat{s})\widehat{\mathbf{g}}_i(\widehat{s}),\\ \widehat{x}_1^2 - \widehat{x}_2^2 + \widehat{x}_3^2 = 1.\end{array}\right\} \tag{19}$$

Here $\widehat{x}_1(\widehat{s})$, $\widehat{x}_2(\widehat{s})$, $\widehat{x}_3(\widehat{s})$ are its dual coordinates with respect to the axode π_i. By Equation (11), the first derivative of $\widehat{\mathbf{x}}$ with respect to $\widehat{s}$ is

$$\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}\,|_i = (\frac{d\widehat{x}_1}{d\widehat{s}} + \widehat{x}_2)\widehat{\mathbf{r}}_i + (\frac{d\widehat{x}_2}{d\widehat{s}} + \widehat{x}_1 + \widehat{x}_3\widehat{\gamma}_i)\widehat{\mathbf{t}}_i + (\frac{d\widehat{x}_3}{d\widehat{s}} + \widehat{x}_2\widehat{\gamma}_i)\widehat{\mathbf{g}}_i. \tag{20}$$

In particular, if $\widehat{\mathbf{x}}$ is a fixed spacelike line in $\mathbb{L}_m$, we have $\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}\,|_m = 0$, which implies

$$\frac{d\widehat{x}_1}{d\widehat{s}} + \widehat{x}_2 = 0,\ \frac{d\widehat{x}_2}{d\widehat{s}} + \widehat{x}_1 + \widehat{x}_3\widehat{\gamma}_m = 0,\ \frac{d\widehat{x}_3}{d\widehat{s}} + \widehat{x}_2\widehat{\gamma}_m = 0. \tag{21}$$

Substituting Equation (21) into Equation (20) and simplifying it, we rewrite Equation (20) as

$$\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}=\widehat{\gamma}(\widehat{x}_3\widehat{\mathbf{t}}_f+\widehat{x}_2\widehat{\mathbf{g}}_f), \tag{22}$$

where $\widehat{\gamma}=\gamma+\varepsilon\gamma^*=\widehat{\gamma}_f-\widehat{\gamma}_m$ is the relative dual geodesic curvature. In order to show the kinematic meaning of $\widehat{\gamma}$, another expressions of $\widehat{x}^{'}$ can be derived by Equation (11), which is

$$\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}=\widehat{\mathbf{x}}^{\prime}\frac{dt}{d\widehat{s}}=\widehat{\omega}(\widehat{x}_3\widehat{\mathbf{t}}_f+\widehat{x}_2\widehat{\mathbf{g}}_f)\frac{dt}{d\widehat{s}}. \tag{23}$$

From Equations (22) and (23), it follows that:

$$\widehat{\omega}dt=\widehat{\gamma}(1+\varepsilon\mu)ds, \tag{24}$$

which reveals the kinematic meaning of the relative curvature $\widehat{\gamma}$. By differentiating Equation (22) with respect to $\widehat{s}$ again and simplifying it, we get:

$$\frac{d^2\widehat{\mathbf{x}}}{d\widehat{s}^2}=\widehat{x}_3\widehat{\gamma}\widehat{\mathbf{r}}_f+(\widehat{x}_2\widehat{\gamma}^2+\widehat{x}_3\frac{d\widehat{\gamma}}{d\widehat{s}})\widehat{\mathbf{t}}_f+(\widehat{x}_2\frac{d\widehat{\gamma}}{d\widehat{s}}-\widehat{x}_1\widehat{\gamma}+\widehat{x}_3\widehat{\gamma}^2)\widehat{\mathbf{g}}_f. \tag{25}$$

3.3. Disteli Formulae for a Spacelike Line Trajectory

In planar kinematics, at each point of a regular curve, there exist only one curvature circle of the curve. The radius and center of this circle can be determined by the famous Euler–Savary formula; if the position of the point is given in the moving plane. Although the Disteli formulae of a line trajectory had been introduced for various types of geometry, we find our motivation in this fact.

Now, we are searching the Disteli's axis $\widehat{\mathbf{b}}$ of the spacelike ruled surface (X) by using the tools just derived. The significance of Disteli's axis is that it is the axis of osculating helicoidal surface is obtained by the helical motion of a line, that is, the generating line is always at the same angle with the Disteli's axis and always at the same distance from it. Therefore, we consider a timelike circle on the fixed Lorentzian dual unit sphere $\mathbb{S}^2_{1f}$ given by the equation

$$<\widehat{\mathbf{x}},\widehat{\mathbf{b}}>=\cos\widehat{\varphi},$$

where $\widehat{\varphi}=\varphi+\varepsilon\varphi^*$ is a given dual spherical radius of curvature and $\widehat{\mathbf{b}}$ is a fixed spacelike dual unit vector which determines the circle's center. According to E. Study's map, this equation represents a set of all spacelike oriented lines $\widehat{\mathbf{x}}$ which forms the given spacelike dual angle with the Disteli's axis $\widehat{\mathbf{b}}$. Such a set of spacelike lines depend on two parameters and are named spacelike line congruence (it is specified by two linear equations of the Pl̀ucker coordinates) [1–3]. Thus, the timelike osculating circle of this curve is specified by the equations:

$$<\widehat{\mathbf{x}},\widehat{\mathbf{b}}>=\cos\widehat{\varphi},\ <\frac{d\widehat{\mathbf{x}}}{d\widehat{s}},\widehat{\mathbf{b}},>=0,\ <\frac{d^2\widehat{\mathbf{x}}}{d\widehat{s}^2},\widehat{\mathbf{b}}>=0, \tag{26}$$

which are obtained from the condition that the osculating circle should have contact of at least second order with the curve.

Let us define the Blaschke frame as follows:

$$\begin{gathered}\widehat{\mathbf{x}}=\widehat{\mathbf{x}}(\widehat{s}),\ \widehat{\mathbf{t}}(\widehat{s})=\tfrac{d\widehat{\mathbf{x}}}{d\widehat{s}}\left\|\tfrac{d\widehat{\mathbf{x}}}{d\widehat{s}}\right\|^{-1},\ \widehat{\mathbf{g}}(\widehat{s})=\widehat{\mathbf{x}}\times\widehat{\mathbf{t}},\\ \widehat{\mathbf{x}}\times\widehat{\mathbf{t}}=\widehat{\mathbf{g}},\widehat{\mathbf{t}}\times\widehat{\mathbf{g}}=\widehat{\mathbf{x}},\widehat{\mathbf{g}}\times\widehat{\mathbf{x}}=-\widehat{\mathbf{t}},\\ <\widehat{\mathbf{x}},\widehat{\mathbf{x}}>=-<\widehat{\mathbf{t}},\widehat{\mathbf{t}}>=<\widehat{\mathbf{g}},\widehat{\mathbf{g}}>=1.\end{gathered} \tag{27}$$

At any instant, it is seen from Equations (26) and (27) that:

$$<\widehat{\mathbf{t}},\widehat{\mathbf{r}}_f>=<\widehat{\mathbf{t}},\widehat{\mathbf{x}}>=<\widehat{\mathbf{t}},\widehat{\mathbf{b}}>=0. \tag{28}$$

Then, all these three spacelike oriented lines $\widehat{\mathbf{r}}_f$, $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{b}}$ constitute a spacelike normal net, that is, they belong to a spacelike line congruence whose focus line is the timelike line $\widehat{\mathbf{t}}$. We locate $\widehat{\mathbf{t}}$ relative to the set $\{\widehat{\mathbf{r}}_f, \ \widehat{\mathbf{t}}_f, \ \widehat{\mathbf{g}}_f\}$ by its intercept distance ψ^*, measured along the $\mathbb{ISA}$ and the angle ψ, measured with respect to $\widehat{\mathbf{g}}_f$. We introduce the dual angles $\widehat{\vartheta} = \vartheta + \varepsilon\vartheta^*$and $\widehat{\alpha} = \alpha + \varepsilon\alpha^*$ which define the positions of $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{b}}$, along $\widehat{\mathbf{t}}$. These dual angles are all measured relative to the $\mathbb{ISA}$ (see Figure 3). The following convention governs the signs: (ϑ, ϑ^*) and (α, α^*) are according to the right-hand Lorentzian screw movement with the thumb pointing along $\widehat{\mathbf{t}}$; the sense of $\widehat{\mathbf{t}}$ is such that $\psi + \varepsilon\psi^* \geq 0$ are defined with the thumb in the direction of the $\mathbb{ISA}$. Since $\widehat{\mathbf{x}}$ is a spacelike dual unit vector, we can write out the components of $\widehat{\mathbf{x}}$ in the following form:

$$\widehat{\mathbf{x}} = \cos\widehat{\vartheta}\widehat{\mathbf{r}}_f + \sin\widehat{\vartheta}\widehat{\mathbf{m}}; \quad \widehat{\mathbf{m}} = \sinh\widehat{\psi}\widehat{\mathbf{t}}_f + \cosh\widehat{\psi}\widehat{\mathbf{g}}_f. \tag{29}$$

Furthermore, a set of coordinates may be used to match the Disteli's axis by the equation

$$\widehat{\mathbf{b}} = \cos\widehat{\alpha}\widehat{\mathbf{r}}_f + \sin\widehat{\alpha}\widehat{\mathbf{m}}. \tag{30}$$

Substituting from Equations (25) and (30) into Equation (26), yields

$$\widehat{\gamma}\widehat{x}_3\cot\widehat{\alpha} - (\widehat{x}_2\widehat{\gamma}^2 + \widehat{x}_3\frac{d\widehat{\gamma}}{d\widehat{s}})\sinh\widehat{\psi} + (-\widehat{x}_1\widehat{\gamma} + \widehat{x}_2\frac{d\widehat{\gamma}}{d\widehat{s}} + \widehat{x}_3\widehat{\gamma}^2)\cosh\widehat{\psi} = 0. \tag{31}$$

Into Equation (31) we substitute from Equation (29) to obtain

$$\left(\cot\widehat{\vartheta} - \cot\widehat{\alpha}\right)\cosh\widehat{\psi} = \widehat{\gamma}. \tag{32}$$

This is precisely a new dual extension of the Euler–Savary formula from ordinary spherical kinematics (compare with [1–3]). By separating the real and the dual parts, respectively, we get:

$$(\cot\vartheta - \cot\alpha)\cosh\psi = \gamma, \tag{33}$$

and

$$\psi^*(\cot\vartheta - \cot\alpha)\sinh\psi - (\frac{\vartheta^*}{\sin^2\vartheta} - \frac{\alpha^*}{\sin^2\alpha})\cosh\psi = \gamma^*. \tag{34}$$

The Euler–Savary formula in Equation (33) together with Equation (34) are new the Disteli formulae of Lorentzian spatial kinematics. Once the angles α and ϑ are known, Equation (34) gives the correspondence between α^* and ϑ^* in terms of ψ and $\psi^*$and the second order invariant γ^*. According to Figure 3, the sign of α^* ($+$ or $-$) in Equation (34) indicates that the positions of the Disteli's axis $\widehat{\mathbf{b}}$ are located on the positive or negative direction along the common central normal $\widehat{\mathbf{t}}$.

Once again, we can derive (32) as follows: From Equations (23), (27) and (29), we have:

$$\begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix} = \begin{pmatrix} \cos\widehat{\vartheta} & \sin\widehat{\vartheta}\sinh\widehat{\psi} & \sin\widehat{\vartheta}\cosh\widehat{\psi} \\ 0 & \cosh\widehat{\psi} & \sinh\widehat{\psi} \\ -\sin\widehat{\vartheta} & \cos\widehat{\vartheta}\sinh\widehat{\psi} & \cos\widehat{\vartheta}\cosh\widehat{\psi} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{r}}_f \\ \widehat{\mathbf{t}}_f \\ \widehat{\mathbf{g}}_f \end{pmatrix}. \tag{35}$$

By construction, the Blaschke formula is:

$$\frac{d}{d\widehat{s}}\begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix} = \begin{pmatrix} 0 & \widehat{p}_x & 0 \\ \widehat{p}_x & 0 & \widehat{q}_x \\ 0 & \widehat{q}_x & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{t}} \\ \widehat{\mathbf{g}} \end{pmatrix}, \tag{36}$$

where

$$\left.\begin{array}{c} \widehat{p}_x := p_x + \varepsilon p_x^* = \widehat{\gamma}\sin\widehat{\vartheta}, \\ \widehat{q}_x = q_x + \varepsilon q_x^* = \det\left(\widehat{\mathbf{x}}, \frac{d\widehat{\mathbf{x}}}{d\widehat{s}}, \frac{d^2\widehat{\mathbf{x}}}{d\widehat{s}^2}\right)\left\|\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}\right\|^{-2} = \widehat{\gamma}\cos\widehat{\vartheta} - \frac{\cosh\widehat{\psi}}{\sin\widehat{\vartheta}}. \end{array}\right\} \tag{37}$$

By Equations (30), and (35), moreover, we have

$$\left.\begin{array}{l}\widehat{\mathbf{r}}_f = \cos\widehat{\vartheta}\widehat{\mathbf{x}} - \sin\widehat{\vartheta}\widehat{\mathbf{g}},\\ \widehat{\mathbf{b}} = \cos\widehat{\varphi}\widehat{\mathbf{x}} - \sin\widehat{\varphi}\widehat{\mathbf{g}}.\end{array}\right\} \tag{38}$$

Note that $\widehat{\varphi} := \varphi + \varepsilon\varphi^* = \widehat{\vartheta} - \widehat{\alpha} = \cos^{-1}(<\widehat{\mathbf{x}}, \widehat{\mathbf{b}}>)$ is the dual radius of curvature of (X). It follows from the differentiations of Equations (38) that:

$$\left.\begin{array}{l}\frac{d\widehat{\mathbf{r}}_f}{d\widehat{s}} = -(\sin\widehat{\vartheta}\widehat{\mathbf{x}} + \cos\widehat{\vartheta}\widehat{\mathbf{g}})\frac{d\widehat{\vartheta}}{d\widehat{s}} + (\widehat{p}_x\cos\widehat{\vartheta} - \widehat{q}_x\sin\widehat{\vartheta})\widehat{\mathbf{t}},\\ \frac{d\widehat{\mathbf{b}}}{d\widehat{s}} = -(\sin\widehat{\varphi}\widehat{\mathbf{x}} + \cos\widehat{\varphi}\widehat{\mathbf{g}})\frac{d\widehat{\alpha}}{d\widehat{s}} + (\widehat{p}_x\cos\widehat{\varphi} - \widehat{q}_x\sin\widehat{\varphi})\widehat{\mathbf{t}}.\end{array}\right\} \tag{39}$$

The spacelike dual vector $\widehat{\mathbf{b}}$ is also called the evolute of (X). Therefore, the following condition should be satisfied

$$< \frac{d\widehat{\mathbf{b}}}{d\widehat{s}}, \widehat{\mathbf{t}} >= 0 \Leftrightarrow 0 = \widehat{p}_x\cos\widehat{\varphi} - \widehat{q}_x\sin\widehat{\varphi},$$

which leads to

$$\widehat{\gamma}_x := \gamma_x + \varepsilon\gamma_x^* = \frac{\widehat{q}_x}{\widehat{p}_x} = \cot\widehat{\varphi}. \tag{40}$$

This equation gives the linkage among the dual spherical curvature $\widehat{\gamma}_x$ and the dual radius of curvature $\widehat{\varphi}$. By Equation (39), we also obtain

$$- < \frac{d\widehat{\mathbf{r}}_f}{d\widehat{s}}, \widehat{\mathbf{t}} >= \widehat{p}_x\cos\widehat{\vartheta} - \widehat{q}_x\sin\widehat{\vartheta}. \tag{41}$$

Furthermore, from Equations (37), (40) and (41), a simple calculation shows that:

$$\begin{aligned} - < \frac{d\widehat{\mathbf{r}}_f}{d\widehat{s}}, \widehat{\mathbf{t}} > &= (\widehat{\gamma}\sin\widehat{\vartheta}\cos\widehat{\vartheta} - \widehat{\gamma}\sin\widehat{\vartheta}\cot\widehat{\varphi}\sin\widehat{\vartheta})\\ &= \widehat{\gamma}\sin\widehat{\vartheta}(\cos\widehat{\vartheta} - \cot\widehat{\varphi}\sin\widehat{\vartheta})\\ &= \frac{\widehat{\gamma}\sin\widehat{\vartheta}}{\sin\widehat{\varphi}}(\sin\widehat{\varphi}\cos\widehat{\vartheta} - \cos\widehat{\varphi}\sin\widehat{\vartheta})\\ &= \frac{\widehat{\gamma}\sin\widehat{\vartheta}\sin(\widehat{\varphi}-\widehat{\vartheta})}{\sin\widehat{\varphi}}\\ &= \frac{\widehat{\gamma}\sin\widehat{\vartheta}\sin(\widehat{\varphi}-\widehat{\vartheta})}{\sin(\widehat{\varphi}-\widehat{\vartheta})\cos\widehat{\vartheta} + \cos(\widehat{\varphi}-\widehat{\vartheta})\sin\widehat{\vartheta}}\\ &= \frac{\widehat{\gamma}}{\cot\widehat{\vartheta} + \cot(\widehat{\varphi}-\widehat{\vartheta})}\\ &= \frac{\widehat{\gamma}}{\cot\widehat{\vartheta} - \cot\widehat{\alpha}}. \end{aligned} \tag{42}$$

On the other hand, we have

$$\begin{aligned} \frac{d\widehat{\mathbf{r}}_f}{d\widehat{s}} &= \widehat{\omega}_f \times \widehat{\mathbf{r}}_f,\\ &= \widehat{\omega}_f \times (\cos\widehat{\vartheta}\widehat{\mathbf{x}} - \sin\widehat{\vartheta}\widehat{\mathbf{g}}),\\ &= \sin\widehat{\vartheta}\sinh\widehat{\psi}\widehat{\mathbf{x}} + \cosh\widehat{\psi}\widehat{\mathbf{t}} + \cos\widehat{\vartheta}\sinh\widehat{\psi}\widehat{\mathbf{g}}. \end{aligned}$$

Thus, we get

$$< \frac{d\widehat{\mathbf{r}}_f}{d\widehat{s}}, \widehat{\mathbf{t}} >= -\cosh\widehat{\psi}. \tag{43}$$

Upon substitution of Equation (43) into Equation (42), we have Equation (32), as claimed.

On the other hand, we can derive another version of the dual Euler–Savary formula in Equation (32) as follows: From Equations (37) and (40) one finds easily

$$\cot\widehat{\vartheta} - \cot\widehat{\varphi} = \frac{\cosh\widehat{\psi}}{\widehat{\gamma}\sin^2\widehat{\vartheta}}.$$

This is a new Lorentzian dual spherical Euler–Savary formula for the timelike dual curve $\widehat{\mathbf{x}}$ which corresponds to a spacelike ruled surface and its osculating circle in terms of the dual angle $\widehat{\psi}$ as well as the second order invariant $\widehat{\gamma}$. By separating the real and the dual parts, respectively, we get:

$$\cot\vartheta - \cot\varphi = \frac{\cosh\psi}{\gamma\sin^2\vartheta},$$

and

$$\varphi^* = \frac{1}{\gamma}\frac{\sin^2\varphi}{\sin^2\vartheta}\{(\gamma - 2\cot\vartheta\cosh\psi)\vartheta^* + \frac{\gamma^*}{\gamma}\cosh\psi + \psi^*\sinh\psi\}. \tag{44}$$

The above equations are new Disteli's formulae of a spacelike line trajectory; According to Figure 3, the sign of φ^* ($+$ or $-$) in the above equation indicates that the position of the Disteli's axis $\widehat{\mathbf{b}}$ is located in the positive or negative direction of the central normal $\widehat{\mathbf{t}}$ at the central point $\mathbf{c}(\vartheta^*, \psi^*)$. Since the central points of (X) are on the normal plane, the Disteli's formulae can be displayed in the timelike plane $Sp\{\widehat{\mathbf{r}}_f, \widehat{\mathbf{t}}\}$ (or $Sp\{\widehat{\mathbf{r}}_m, \widehat{\mathbf{t}}\}$). Hence, any arbitrary point $\mathbf{c}(\vartheta^*, \psi^*)$ on the timelike plane $Sp\{\widehat{\mathbf{r}}_f, \widehat{\mathbf{t}}\}$ is defined as central point of the spacelike line trajectory whose generator is the spacelike oriented line $\widehat{\mathbf{x}}$ and the radius can be calculated by Equation (34); its length which is the line segment from $\mathbf{s}$ to the point $\mathbf{c}$ on the plane $Sp\{\widehat{\mathbf{r}}_f, \widehat{\mathbf{t}}\}$. Meanwhile, the central point $\mathbf{c}$ is in the direction of $\widehat{\mathbf{t}}$ if $\alpha^* > 0$ and in the opposite direction of $\widehat{\mathbf{t}}$ if $\alpha^* < 0$. The central point $\mathbf{c}(\vartheta^*, \psi^*)$ can be on the $\mathbb{ISA}$ if $\vartheta^* = 0$ ($\varphi^* = -\alpha^*$) and on the Disteli's axis $\widehat{\mathbf{b}}$ if $\alpha^* = 0 \Leftrightarrow \varphi^* = \vartheta^*$. In the latter case the central point $\mathbf{c}(\vartheta^*, \psi^*)$ can be located by letting $\varphi^* = 0$ in Equation (44) which is simplified as a linear equation

$$L:\ \psi^* = (-\gamma + 2\cot\vartheta\coth\psi)\vartheta^* - \frac{\gamma^*}{\gamma}\coth\psi. \tag{45}$$

Equation (45) is linear in the position coordinates $\psi^*$and ϑ^* of the spacelike oriented line $\widehat{\mathbf{x}}$. Hence, for $\mathbb{L}_m/\mathbb{L}_f$ the spacelike lines in a given direction fixed in $\mathbb{L}_m$-space lies on a timelike plane. The line L would change its position if ϑ has different value, but ψ =constant. Nevertheless, a family of line envelope a spacelike curve on the timelike plane $Sp\{\widehat{\mathbf{r}}_f, \widehat{\mathbf{t}}\}$. Meanwhile, if the parameter ψ of a spacelike line has a different value but ϑ =consent, the position of the timelike plane is different. Hence, the set of all spacelike lines L given in Equation (45) is a spacelike line congruence for all values of (ϑ^*, ψ^*).

At the end of this section, we want to derive another Lorentzian dual Euler–Savary formula for the spacelike axodes as follows: Substituting $\widehat{\vartheta} = \widehat{\varphi}_f$, $\widehat{\alpha} = \widehat{\varphi}_f$, and $\psi^* = \psi = 0$ into Equation (32), we attain, after some calculations, that

$$\cot\widehat{\varphi}_f - \cot\widehat{\varphi}_f = \widehat{\gamma}, \tag{46}$$

which is the associated dual Euler–Savary formula. By separating the last equation into real and dual parts we obtain

$$\cot\varphi_f - \cot\varphi_f = \gamma, \tag{47}$$

and

$$\frac{\varphi_f^*}{\sin^2\varphi_f} - \frac{\varphi_m^*}{\sin^2\varphi_f} = \gamma^*. \tag{48}$$

The above two equations give new Lorentzian Disteli formulae for the axodes $\mathbb{L}_m/\mathbb{L}_f$.

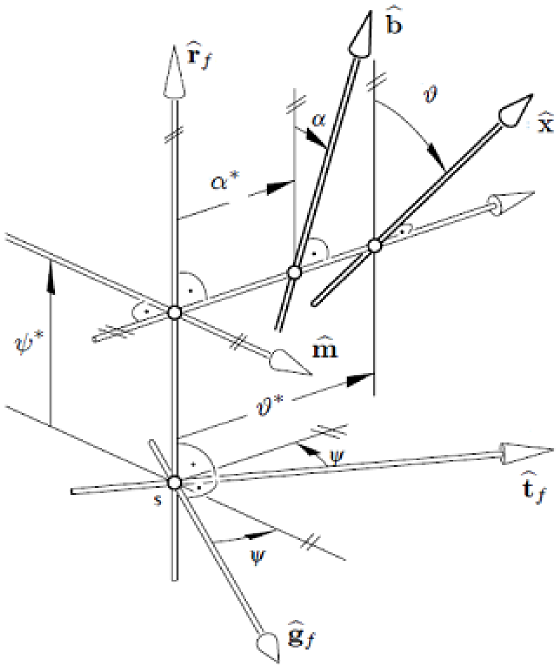

Figure 3. Illustrations of $\widehat{\mathbf{x}}$ and the Disteli-axis $\widehat{\mathbf{b}}$.

4. Conclusions

The main advantage of the work is to consider one-parameter Lorentzian spatial movements in terms of the E. Study map. With the proposed technique, new proofs for Disteli's, and Euler–Savary formulae were derived. In addition, new metric aspects are defined for the Disteli axis of a spacelike trajectory ruled surface under a one-parameter Lorentzian spatial movement of a body in Minkowski 3-space.

Hopefully, our work may give a productive contribution to the application of four-bar mechanisms, dual Lorentzian spherical motions, theory of mechanism synthesis for higher order approximations, spatial mechanisms in engineering design and gear theory.

Author Contributions: Conceptualization, Y.L., N.A. and R.A.A.-B.; methodology, Y.L., N.A. and R.A.A.-B.; investigation, Y.L., N.A. and R.A.A.-B.; writing—original draft preparation, Y.L., N.A. and R.A.A.-B.; writing—review and editing, Y.L., N.A. and R.A.A.-B.; All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China (Grant No. 12101168) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We gratefully acknowledge the constructive comments from the editor and the anonymous referees.

Conflicts of Interest: The authors declare no conflict of interest.

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