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ONE-PARAMETER SEMIGROUPS HOLOMORPHIC AWAY FROM ZERO

BY

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ABSTRACT. Suppose T is a one-parameter semigroup of bounded linear operators on a Banach space, strongly continuous on $[0, \infty)$. It is known that $\limsup_{x\to 0} |T(x) - I| < 2$ implies T is holomorphic on $(0, \infty)$. Theorem I is a generalization of this as follows: Suppose $M > 0$, $0 < r < s$, and ρ is in $(1,2)$. If $|(T(h) - I)^n| \le M\rho^n$ whenever nh is in $[r, s]$, $n = 1, 2, \cdots$, $h > 0$, then there exists $b > 0$ such that T is holomorphic on $[b, \infty)$. Theorem II shows that, in some sense, $b \to 0$ as $r \to 0$. Theorem I is an application of Theorem III: Suppose $M > 0$, $0 < r < s$, ρ is in $(1,2)$, and f is continuous on $[-4s, 4s]$. If $|\Sigma_{q=0}^{n}\binom{n}{q}(-1)^{n-q}f(t+qh)| \le M\rho^n$ whenever nh is in $[r, s]$, $n = 1, 2, \cdots$, $h > 0$, $[t, t+nh] \subset [-4s, 4s]$, then f has an analytic extension to an ellipse with center zero. Theorem III is a generalization of a theorem of Beurling in which the inequality on the differences is assumed for all nh. An example is given to show the hypothesis of Theorem I does not imply T holomorphic on $(0, \infty)$.

1. **Introduction.** Suppose T is a one-parameter semigroup of bounded linear operators on a Banach space. Recent work of A. Beurling [3] gives the following:

Theorem A. *Suppose T is weakly measurable on $(0, \infty)$. Then if $\limsup_{x\to 0} |T(x) - I| < 2$, T is holomorphic on $(0, \infty)$.*

This is a generalization of a theorem due to J. Neuberger [8]:

Theorem B. *Suppose T is strongly continuous on $[0, \infty)$. Then if $\limsup_{x\to 0} |T(x) - I| < 2$, $AT(x)$ is bounded for all $x > 0$, A being the infinitesimal generator of T.*

Under the assumption of strong continuity on $[0, \infty)$, Theorem A also follows from a theorem of Kato [5].

Theorem I of this note presents a generalization of Theorems A and B as follows: Suppose T is strongly continuous on $[0, \infty)$. Suppose $M > 0$, $0 < r < s$, and ρ is in $(1,2)$. If $|(T(h) - I)^n| \le M\rho^n$ whenever nh is in $[r, s]$, $n = 1, 2, \cdots$, $h > 0$, then there exists $b > 0$ such that T is holomorphic on $[b, \infty)$. An example,

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due to Neuberger [9], is presented in §4 to show that the hypothesis of Theorem I does not imply T holomorphic on $(0, \infty)$. However, Theorem II says that, in some sense, $h \to 0$ as $r \to 0$.

Theorems A and B trace their beginnings, at least in part, to some earlier work of Neuberger having to do with quasianalytic classes of functions determined by conditions on finite differences. In [7] Neuberger proved the following:

Theorem C. *Suppose ρ and M are positive numbers, $1 \le \rho < 2$, and suppose G is a collection of continuous real-valued functions f on $(0, 1)$ such that if u and v are in $(0, 1)$ then*

$$\left|\sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i} f(u + i(v-u)/n)\right| \le M\rho^n, \qquad n = 1, 2, \cdots.$$

Then G is a quasianalytic collection in the sense that no two members of G agree on an open subinterval of $(0, 1)$.

The question was raised in [7], and also by D. G. Kendall in [6] in the context of Markov semigroups, of whether G could contain a nonanalytic member. Beurling has proved the following theorem which answers this question negatively.

Theorem D. *Suppose f is a function continuous on $[-4, 4]$ and for some $M > 0$ and ρ in $[3/2, 2)$,*

$$\left|\sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i} f(u + i(v-u)/n)\right| \le M\rho^n, \qquad n = 1, 2, \cdots,$$

u, v in $[-4, 4]$. Then f can be extended analytically to the rhombus with vertices at ± 4, $\pm 4ik\alpha^2$ where $\alpha = (2-\rho)/4$.

Theorems III and IV of this note generalize Theorem D in that analyticity of the function f (in some open set centered at zero) is still deduced even though the inequality on the differences is assumed only for $|v - u|$ in some interval $[r, s]$, $0 < r < s$. Theorem III is stated for purposes of comparison with Theorem D. Theorem IV is a more detailed statement and includes Theorem III; consequently, a separate proof of Theorem III is not given. Theorem D is contained in [3] as a special case. The proof here of Theorem IV parallels Beurling's proof in [3] and uses, as does his proof, some techniques described in [2]. Theorem D is applied by Neuberger in [8] to prove Theorem B. Theorem I is proved from Theorem IV using some of the same techniques.

2. **Definitions and statement of theorems.** Suppose X is a (complex) Banach space and T is a one-parameter semigroup of bounded linear transformations from X to X, strongly continuous on $[0, \infty)$. For p in X and f in X^*, denote by $z_{p,f}$ the function on $[0, \infty)$ defined by $z_{p,f}(x) = f(T(x)p)$.

An additive abelian semigroup (in the complex plane, in this note) will be called a semimodule. An angular semimodule is a semimodule which is an open set and which has zero as a limit point. A spinal semimodule is a semimodule which includes a ray from the origin and an open set intersected by this ray. These definitions are given in [4, pp. 256–269].

The statement that U is an extension of T to a domain S in the complex plane means that (1) S is a semimodule; (2) for η in S, $U(\eta)$ is a bounded linear transformation from X to X; (3) if λ, η in S, then $U(\lambda)U(\eta) = U(\lambda + \eta)$; (4) $S \cap [0, \infty)$ is not empty and if x is in $S \cap [0, \infty)$ then $U(x) = T(x)$. If U is an extension of T to S, then the functions $z_{p,f}$ have an obvious extension $\widetilde{z_{p,f}}$ to S: $\widetilde{z_{p,f}}(\lambda) = f(U(\lambda)p)$, f in X^*, p in X, λ in S.

U is said to be a holomorphic (analytic) extension of T to S if, for p in X, f in X^*, $\widetilde{z_{p,f}}$ is holomorphic in S. Seemingly this is a definition of weak differentiability, but if U is holomorphic in S by the definition just given, then U is continuous and differentiable in S in the uniform operator topology, uniformly on compact subsets of S. For a proof and discussion see [4, pp. 92–94].

Theorem I. *Suppose r, s, ρ are positive numbers with $1 < \rho < 2$, $r < s$, and suppose there exists $M > 0$ such that if n is a nonnegative integer, $h > 0$,*

$$|(T(h) - I)^n| \leq M\rho^n \quad \textit{whenever } n = 0 \textit{ or } nh \textit{ is in } [r, s].$$

Then there exists $b > 0$ such that T has a holomorphic extension to a domain which includes $[b, \infty)$.

Theorem II. *Suppose $\rho > 0$ and for each positive integer j, r_j and s_j are numbers such that* (i) $0 < r_j < s_j$; (ii) *$r_j \to 0$ as $j \to \infty$;* (iii) *$\{r_j/s_j\}_{j=1}^{\infty}$ is bounded away from* 1. *Suppose that for each positive integer j, T_j is a strongly continuous semigroup on $[0, \infty)$ and there exists $M_j > 0$ such that $|(T_j(h) - I)^n| \leq M_j\rho^n$ whenever $n = 0$ or nh is in $[r_j, s_j]$. Then there is a sequence $b_1, b_2, \cdots$ of positive numbers converging to* 0 *such that T has a holomorphic extension to a domain which includes $[b_j, \infty)$.*

Some additional notation and definitions are given before the next theorems are stated. If $\beta, \theta > 0$ and t_0 is a real number, then $E_{\beta,\theta}(t_0)$ denotes the ellipse with foci at $t_0 - \beta$, $t_0 + \beta$ and with sum of semiaxes equal to β/θ. $E_{\beta,\theta}(0)$ will be denoted simply $E_{\beta,\theta}$. Also $\sum_{v=0}^{n}\binom{n}{v}(-1)^{n-v}f(t + vh)$, for f a function on $[t, t + nh]$, will be denoted by $\Delta_h^n f(t)$.

The statement that f has an analytic extension to $E_{\beta,\theta}(t_0)$ means that there is a function $\tilde{f}$, analytic at every point within and on $E_{\beta,\theta}(t_0)$, such that if x is in $[t_0 - \beta, t_0 + \beta]$, $\tilde{f}(x) = f(x)$.

Theorem III. *Suppose r, s are positive numbers, $r < s$, f is a function continuous on $[-4s, 4s]$, and for some $M > 0$, ρ in $(1,2)$,*

$$\left| \sum_{q=0}^{n} \binom{n}{q} (-1)^{n-q} f(u + q(v-u)/n) \right| \leq M\rho^n$$

if u, v are in $[-4s, 4s]$, $|v - u|$ is in $[r, s]$, $n = 1, 2, \cdots$. Then if σ is in $(\rho, 2)$ there exists a number β, $0 < \beta < \sigma(s - r)/8$, such that f can be extended analytically to the ellipse $E_{\beta,\sigma/2}$.

Theorem IV. *Suppose r, s, ρ are positive numbers with $1 < \rho < 2$, $r < s$. Then there are positive numbers D, β, σ such that the following is true: Suppose $K > 0$ and denote by G_K a collection of functions f such that for some real number t_0,*

(1) f is continuous on $[t_0 - D, t_0 + D]$, and

(2) $|\Delta_h^n f(t)| \leq K\rho^n$, whenever $n = 0$ or nh is in $[r, s]$ and $[t, t + nh] \subset [t_0 - D, t_0 + D]$.

Then there exists $\tilde{K} > 0$ such that if f is in G_K, f continuous on $[t_0 - D, t_0 + D]$, then f has an analytic extension $\tilde{f}$ to $E_{\beta,\sigma/2}(t_0)$ and $\tilde{f}$ is bounded by $\tilde{K}$ in $E_{\beta,\sigma/2}(t_0)$.

3. **Proofs.** The proof of Theorem IV is given first. It depends upon the following theorem of S. Bernstein [1, p. 112]:

Theorem E. *Suppose f is a function continuous on $[-\beta, \beta]$ and there exist polynomials P_n of degree n, θ_0 in $(0, 1)$, and $M > 0$ such that*

$$(3) \qquad \int_{-\beta}^{\beta} |f(t) - P_n(t)|^2\, dt < M\theta_0^{2n}, \qquad n = 0, 1, 2, \cdots.$$

Then if θ is in $(\theta_0, 1)$, f has an analytic extension $\tilde{f}$ to $E_{\beta,\theta}$. Furthermore, if M, $\beta > 0$ and $0 < \theta_0 < \theta < 1$, there exists $\tilde{M}$ such that for any continuous function f for which there exist polynomials P_n of degree n such that (3) holds, the extension $\tilde{f}$ is bounded by $\tilde{M}$ in $E_{\beta,\theta}$.

Lemma. *If r_0, $\delta_0 > 0$, n is a positive integer, and $|x| \geq 4\pi n/\delta_0$, then*

$$\int_{r_0/n}^{(r_0+\delta_0)/n} \sin^{2n}(hx/2)\, dh \geq \delta_0/4n^2.$$

Proof of lemma. Suppose n is a positive integer and $x \geq 4\pi n/\delta_0$. Then there is a positive integer $K \geq 1$ such that x is in $[4\pi nK/\delta_0, 4\pi n(K + 1)/\delta_0]$. It is easy to verify that $\int_0^{2K\pi} \sin^{2n} u\, du \geq K\pi/n$. Hence, one has

$$\int_{r_0/n}^{(r_0+\delta_0)/n} \sin^{2n}(bx/2)\,db = 2/x \int_{r_0x/2n}^{(r_0+\delta_0)x/2n} \sin^{2n} u\,du$$

$$\geq 2/x \int_0^{2K\pi} \sin^{2n} u\,du \geq 2K\pi/xn,$$

using that $x \geq 4nK\pi/\delta_0$ and hence $\delta_0 x/2n \geq 2K\pi$. But also $x \leq 4\pi n(K+1)/\delta_0$ and hence $2K\pi/xn \geq \delta_0 K/2n^2(K+1) \geq \delta_0/4n^2$.

Proof of Theorem IV. Suppose r, s, ρ are positive numbers with $1 < \rho < 2$, $r < s$.

Choose σ such that $\rho < \sigma < 2$.

The choice of D and β is more complicated but an explicit procedure follows.

Choose σ_0 such that $\rho < \sigma_0 < \sigma$; choose α such that

(i) $0 < \alpha < 1/2$,

(ii) $\alpha < 1 - (r/s)$, and

(iii) for all positive integers n,

$$\binom{n}{[\alpha n]}\rho^n < \sigma_0^n;$$

choose $B > 0$ such that $(s/B)^\alpha < \sigma_0/4$. Denote $r/(1-\alpha)$ by r_0 and denote $s - r_0$ by δ_0. Then let $D = 3B + s$ and $\beta = \sigma_0\delta_0/8e\pi$.

Suppose $K > 0$ and denote by G_K a collection of functions as described in the statement of the theorem. The selection of $\tilde{K}$ is made as follows: Denote by σ_1 a number such that $\sigma_0 < \sigma_1 < \sigma$, by K_0 a number such that

$$K_0 \geq \max\{24\sqrt{2\pi}BK/\delta_0,\ 16K^2(6B+s)\},$$

and by K_1 a number such that

$$2\beta(K_0)^2(\sigma_0/2)^{2n} + n^3K_0(\sigma_0/2)^{2n} < K_1(\sigma_1/2)^{2n}, \qquad n = 1, 2, \cdots.$$

Choose $\tilde{K}$ to be a number such that if f is continuous on $[-\beta, \beta]$ and, for some polynomials P_n, (3) holds with M replaced by K_1 and θ_0 replaced by $\sigma_1/2$, then f has an analytic extension $\tilde{f}$ to $E_{\beta,\sigma/2}$ and $\tilde{f}$ is bounded by $\tilde{K}$ on $E_{\beta,\sigma/2}$. The theorem of Bernstein quoted above says this is possible.

Suppose now that f is a member of G_K. Then f is continuous on $[t_0 - D, t_0 + D]$ for some real number t_0. It can be assumed that $t_0 = 0$. The essence of the proof is the construction of polynomials P_n which approximate f on $[-\beta, \beta]$ in such a way that Bernstein's theorem can be invoked.

The first step is to replace f by functions f_n which coincide with f on $[-B, B]$ and vanish off $[-3B, 3B]$.

The norms of $L^1(-\infty, \infty)$, $L^2(-\infty, \infty)$, and $L^\infty(-\infty, \infty)$ will be denoted $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$, respectively. Also, if g is a function, n a nonnegative integer, and t a number, then $g^{(n)}(t)$ denotes the nth derivative of g at t.

For n a positive integer, define $Q_{n,B}$ by

$$Q_{n,B}(t) = \begin{cases} B^{-2n}(B^2 - t^2)^n, & \text{if } |t| \le B; \\ 0, & \text{if } |t| > B. \end{cases}$$

An important property of $Q_{n,B}$ is that

(4) $$|Q_{n,B}^{(m)}(t)| \le (2n/B)^n, \qquad n = 0, 1, \cdots, n-1, \text{ all real } t.$$

To verify this suppose t is in $(-B, B)$. Then, for any positive integer n, $|Q_{n,B}^{(m)}(t)| \le (2n/B)^m$ for *all* nonnegative integers m: use induction on n and the fact that if f, g each possess m derivatives at t then

$$(fg)^{(m)}(t) = \sum_{v=0}^{m} \binom{m}{v} f^{(v)}(t) g^{(m-v)}(t).$$

If $|t| > B$ then $Q_n^{(m)}(t) = 0$ for all nonnegative integers m. Finally, if $t = \pm B$, an elementary argument gives $Q_{n,B}^{(m)}(t) = 0$ for $m = 0, 1, 2, \cdots, n-1$.

Denote $\int_{-\infty}^{\infty} Q_{n,B}(t)\,dt$ by $\gamma_{n,B}$.

Then $\gamma_{n,B} = 2B(2 \cdot 4 \cdot 6 \cdots (2n))/(3 \cdot 5 \cdot 7 \cdots (2n+1))$ [2, p. 4] and $\gamma_{n,B} > B/\sqrt{n}$ for all positive integers n.

As in [2, p. 4] and [3, p. 392], multiplier functions are now defined.

Define $k_{n,B}(t) = \gamma_n^{-1} \int_{-\infty}^{t} (Q_{n,B}(u + 2B) - Q_{n,B}(u - 2B))\,du$.

Using (4) one has

(5) $$|k_{n,B}^{(v)}(t)| \le \gamma_n^{-1}(2n/B)^{v-1} \le (\sqrt{n}/B)(2n/B)^{v-1},$$

$v = 1, 2, \cdots, n$, t any real number.

If a function g has v continuous derivatives on $[t, t + vh]$, then there exists a number c in $[t, t + vh]$ such that $(\Delta_h^v g)(t) = h^v g^{(v)}(c)$.

This together with (5) gives the following:

$$|\Delta_h^v k_{n,B}(t)| < h^v(\sqrt{n}/B)(2n/B)^{v-1} = (\sqrt{n}/B)(2hn/B)^{v-1}h, \qquad v = 1, 2, \cdots, n;$$

hence,

(6) $$|\Delta_h^v k_{n,B}(t)| \le (2hn/B)^v, \qquad v = 0, 1, 2, \cdots, n.$$

Now for each positive integer n, define f_n by

$$f_n(t) = \begin{cases} (fk_{n,B})(t), & \text{if } |t| \leq D; \\ 0, & \text{if } |t| > D. \end{cases}$$

Then for each positive integer n,

(i) f_n is continuous on $(-\infty, \infty)$;

(ii) f_n has its support in $[-3B, 3B]$ and agrees with f on $[-B, B]$;

(iii) $\|f_n\|_1 < 6BK$.

The next step is to define polynomials P_n, of degree n, such that (3) holds with M replaced by K_1 and θ_0 replaced by $\sigma_1/2$.

Denote by $\hat{f}_n$ the Fourier transform of f_n. For $n = 1, 2, \cdots$, define

$$P_n(t) = \frac{1}{\sqrt{2\pi}} \int_{|x|<4\pi n/\delta_0} \left(\hat{f}_n(x) \sum_{v=0}^{n} (itx)^v/v!\right) dx. \tag{7}$$

If $\hat{f}_n$ denotes the Fourier transform of f_n, then the Fourier transform of $\Delta_b^n f_n$ at x is $\hat{f}_n(x)(e^{ibx} - 1)^n$. By Parseval's relation

$$\begin{aligned} &\int_{-\infty}^{\infty} |\hat{f}_n(x)|^2 |e^{ibx} - 1|^{2n} dx \\ &\qquad = \int_{-\infty}^{\infty} |\Delta_b^n f_n(x)|^2 dx, \qquad n = 1, 2, \cdots, b > 0. \end{aligned} \tag{8}$$

In what follows, a bound on $|\Delta_b^n f_n|$ is deduced for nb in $[r/(1-\alpha), s] = [r_0, s]$. Suppose t is a real number, n is a positive integer, $b > 0$, and g_1, g_2 are functions each of whose domain includes $[t, t + nb]$. Then

$$(\Delta_b^n g_1 g_2)(t) = \sum_{v=0}^{n} \binom{n}{v} (\Delta_b^{n-v} g_1)(t)(\Delta_b^v g_2)(t + (n-v)b). \tag{9}$$

From (9),

$$\begin{aligned} |\Delta_b^n f_n(t)| < &\sum_{v=0}^{[\alpha n]} \binom{n}{v} |\Delta_b^{n-v} f(t)||\Delta_b^v k_n(t + (n-v)b)| \\ &+ \sum_{v=[\alpha n]+1}^{n} \binom{n}{v} |\Delta_b^{n-v} f(t)||\Delta_b^v k_n(t + (n-v)b)|. \end{aligned}$$

Suppose nb is in $[r_0, s]$. Then if t is outside $[-3B - s, 3B]$, $\Delta_b^n f_n(t) = 0$. Suppose t is in $[-3B - s, 3B]$. Since nb is in $[r_0, s]$, if $v \leq [\alpha n] \leq \alpha n$, then $(n - v)b$ is in $[r, s]$ and hence $|\Delta_b^{n-v} f(t)| \leq K\rho^{n-v}$. Also $2bn/B < 1$, since $(s/B)^\alpha < \sigma_0/4 < 1/2$. Hence,

$$\sum_{v=0}^{[\alpha n]} \binom{n}{v} |\Delta_b^{n-v} f(t)| |\Delta_b^v k_n(t+(n-v)b)|$$

(10)
$$\leq \sum_{v=0}^{[\alpha n]} \binom{n}{v} K\rho^{n-v}(2bn/B)^v \leq nK \binom{n}{[\alpha n]} \rho^n$$

$$< nK\sigma_0^n, \quad \text{using also that } \alpha < \tfrac{1}{2} \text{ and } \rho > 1.$$

Any function g bounded by M on $[t, t+nb]$ satisfies $|\Delta_b^n g(t)| \leq M2^n$, $n = 1, 2, \cdots, b > 0$. Hence for nb in $[r_0, s]$ and t a real number,

$$\sum_{v=[\alpha n]+1}^{n} \binom{n}{v} |\Delta_b^{n-v} f(t)| |\Delta_b^v k_n(t+(n-v)b)|$$

(11)
$$\leq \sum_{v=[\alpha n]+1}^{n} \binom{n}{v} K2^{n-v}(2bn/B)^v$$

$$\leq 2^n K \sum_{v=[\alpha n]+1}^{n} \binom{n}{v} (s/B)^v < 4^n K(s/B)^{\alpha n} < K\sigma_0^n.$$

Thus if nb is in $[r_0, s]$, (10) and (11) give $|\Delta_b^n f_n| \leq 2Kn(\sigma_0)^n$, $n = 1, 2, \cdots$. Since $\Delta_b^n f_n$ is zero outside $[-3B-s, 3B]$, using (8) one gets

$$\int_{-\infty}^{\infty} |\hat{f}_n(x)|^2 2^{2n} \sin^{2n}(bx/2)\,dx = \int_{-\infty}^{\infty} |\hat{f}_n(x)|^2 |e^{ibx}-1|^{2n}\,dx$$
$$< (6B+s)(2Kn(\sigma_0)^n)^2, \quad \text{if } nb \text{ in } [r_0, s].$$

Hence

$$\int_{[r_0/n, s/n]} \left(\int_{|x| \geq 4\pi n/\delta_0} |\hat{f}_n(x)|^2 \sin^{2n}(bx/2)\,dx \right) db$$
$$< n\delta_0(6B+s)4K^2(\sigma_0/2)^{2n}, \qquad n = 1, 2, \cdots.$$

Using the lemma stated above and reversing the order of integration one has

$$\int_{|x| \geq 4\pi n/\delta_0} |\hat{f}_n(x)|^2\,dx < 4n^3(6B+s)4K^2(\sigma_0/2)^{2n}$$
$$< K_0 n^3(\sigma_0/2)^{2n}, \qquad n = 1, 2, \cdots.$$

Define $g_n(t) = 1/\sqrt{2\pi} \int_{|x|<4\pi n/\delta_0} (\hat{f}_n(x)e^{itx})dx$, $n = 1, 2, \cdots$. If for n a positive integer,

$$h_n(t) = \begin{cases} \hat{f}_n(t), & |t| < 4\pi n/\delta_0, \\ 0, & |t| \geq 4\pi n/\delta_0, \end{cases}$$

then $g_n(t) = \int_{-\infty}^{\infty} h_n(x)e^{itx}dx$, so $h_n = \hat{g}_n$ in $L^2(-\infty, \infty)$. Hence

$$\|\hat{f}_n - \hat{g}_n\|_2 = \|\hat{f}_n - h_n\|_2 = \int_{|x| \geq 4\pi n/\delta_0} |\hat{f}_n(x)|^2\, dx < K_0 n^3(\sigma_0/2)^{2n};$$

also

$$\int_{-\beta}^{\beta} |f(t) - g_n(t)|^2\, dt < \int_{-B}^{B} |f(t) - g_n(t)|^2\, dt$$

$$= \int_{-B}^{B} |f_n(t) - g_n(t)|^2\, dt \leq \|f_n - g_n\|_2 = \|\hat{f}_n - \hat{g}_n\|_2$$

which yields

$$\int_{-\beta}^{\beta} |f(t) - g_n(t)|^2\, dt < K_0 n^3(\sigma_0/2)^{2n}, \qquad n = 1, 2, \cdots. \tag{12}$$

Suppose t is a real number and n is a positive integer. Then

$$|g_n(t) - P_n(t)| \leq \frac{1}{\sqrt{2\pi}} \int_{|x| < 4\pi n/\delta_0} |\hat{f}_n(x)|\,|tx|^{n+1}/(n+1)!\, dx$$

$$\leq \|\hat{f}_n\|_\infty |t|^{n+1}/\sqrt{2\pi}(n+1)! \int_{|x| < 4\pi n/\delta_0} |x|^{n+1}\, dx$$

$$\leq (4\pi n/\delta_0)^{n+2}(2\|f_n\|_1 |t|^{n+1}/\sqrt{2\pi}(n+1)!(n+2)),$$

using that $|e^{itx} - \sum_{\nu=0}^{n} (itx)^\nu/\nu!| \leq |tx|^{n+1}/(n+1)!$ and that $n! > (n/e)^n$.

If $|t| \leq \beta = \sigma_0\delta_0/8e\pi$, then

$$|g_n(t) - P_n(t)| < (48BK\pi/\sqrt{2\pi}\delta_0)(n/(n+2))(n/(n+1))^{n+1}(\sigma_0/2)^{n+1}$$

$$< K_0(\sigma_0/2)^n, \qquad n = 1, 2, \cdots.$$

Hence,

$$\int_{-\beta}^{\beta} |g_n(t) - P_n(t)|^2\, dt < 2\beta K_0^2(\sigma_0/2)^{2n}. \tag{13}$$

Recalling that K_1 is a number such that $2\beta(K_0)^2(\sigma_0/2)^{2n} + n^3 K_0(\sigma_0/2)^{2n} < K_1(\sigma_1/2)^{2n}$, $n = 1, 2, \cdots$, one gets, from (12) and (13),

$$\int_{-\beta}^{\beta} |f(t) - P_n(t)|^2\, dt < K_1(\sigma_1/2)^{2n}, \qquad n = 1, 2, \cdots,$$

and the desired analytic extension of f follows from Bernstein's theorem.

Proof of Theorem I. Suppose r, s, ρ are positive numbers with $1 < \rho < 2$, $r < s$, and suppose M is a number such that if n is a nonnegative integer, $h > 0$, and nh is in $[r, s]$, then $|(T(h) - I)^n| < M\rho^n$. Suppose D, β, σ are positive numbers such that Theorem IV holds for $r, s, \rho, D, \beta, \sigma$. The claim is that the conclusion of Theorem I holds for $b = D - \beta$.

Suppose $t \geq 0$. It is easy to verify that for $h > 0$, n a nonnegative integer, p in X, and f in X^*, $|\Delta_h^n z_{p,f}(t)| \leq |f| \, \|p\| \, |T(t)| \, |(T(h) - I)^n|$. Denote by M_0 a number such that $|T(t)| \leq M_0$ if t is in $[0, 2D]$. If nh is in $[r, s]$, $\|p\| \leq 1$, $|f| \leq 1$, then $|\Delta_h^n z_{p,f}(t)| \leq M_0 M \rho^n$ if $[t, t + nh] \subset [0, 2D]$. Hence, by Theorem IV, there exists $\widetilde{M}$ such that if $\|p\| \leq 1$, $|f| \leq 1$, $z_{p,f}$ has an analytic extension $\widetilde{z_{p,f}}$ to $E_{\beta,\sigma/2}(D)$ and $\widetilde{z_{p,f}}$ is bounded by $\widetilde{M}$ in $E_{\beta,\sigma/2}(D)$.

Denote by $B(x; \epsilon)$ the ball in the complex plane with center at x and radius ϵ, x a real number, $\epsilon > 0$.

The ellipse $E_{\beta,\sigma/2}(D)$ has its foci at $D - \beta$, $D + \beta$. Hence there exists $\delta > 0$ such that $B(b; 2\delta) = B(D - \beta; 2\delta)$ is contained in $E_{\beta,\sigma/2}(D)$. Since $\widetilde{z_{p,f}}$ is bounded by $\widetilde{M}$ in $E_{\beta,\sigma/2}(D)$, $\|p\| \leq 1$, $|f| \leq 1$, then if λ is in $B(b; \delta)$,

$$|\widetilde{z_{p,f}^{(n)}}(\lambda)| \leq n! \widetilde{M} \delta^{-n}, \qquad n = 0, 1, 2, \cdots.$$

The claim is that if t is in $B(b;\delta)$, then $A^n T(t)$ is a bounded operator on X, $\|A^n T(t)\| \leq n! \widetilde{M} \delta^{-n}$, $n = 0, 1, 2, \cdots$. The argument verifying this, by induction on n, is presented below; for the case $n = 1$, it is found in [8] of Neuberger.

If $n = 0$, the claim is obviously true since $|\widetilde{z_{p,f}}(t)| \leq \widetilde{M}$, $\|p\| \leq 1$, $|f| \leq 1$, t in $B(b; \delta)$, implies $\|T(t)\| \leq \widetilde{M}$, t in $B(b; \delta)$.

Suppose K is a positive integer and suppose $A^{K-1}T(t)$ is a bounded operator on X, t in $B(b; \delta)$. It will be shown that $A^K T(t)$ is a bounded operator on X, for t in $B(b; \delta)$, and that $\|A^K T(t)\| \leq K! \widetilde{M} \delta^{-K}$.

Suppose t is in $B(b; \delta)$ and p is in the domain of A. Then $A^K T(t)p = \lim_{x \to t^+} (x - t)^{-1}(T(x) - I)A^{K-1}T(t)p$, if this limit exists. By assumption, $A^{K-1}T(x)p$ exists for all x in $B(b; \delta)$. Then

$$\begin{aligned}
&\lim_{x \to t^+} (x - t)^{-1}(T(x - t) - I)A^{K-1}T(t)p \\
&= \lim_{x \to t^+} (x - t)^{-1}(T(x - t)A^{K-1}T(t)p - A^{K-1}T(t)p) \\
&= \lim_{x \to t^+} (x - t)^{-1}(A^{K-1}T(x)p - A^{K-1}T(t)p) \\
&= \lim_{x \to t^+} A^{K-1}T(t)((x - t)^{-1}(T(x - t) - I))p \\
&= A^{K-1}T(t)\left(\lim_{x \to t^+} (x - t)^{-1}(T(x - t) - I)p\right) = A^{K-1}T(t)Ap,
\end{aligned}$$

and thus $A^K T(t)p$ exists. The above equalities also show that if p is any point of X and $\lim_{x \to t^+} (x - t)^{-1}(A^{K-1}T(x)p - A^{K-1}T(t)p)$ exists, then this limit is $A^K T(t)p$.

Suppose f is in X^*, p is in X, $\|f\| \leq 1$, $\|p\| \leq 1$. Then

$$|f((x-t)^{-1}(A^{K-1}T(x) - A^{K-1}T(t))p)|$$
$$= |(x-t)^{-1}(z_{p,f}^{(K-1)}(x) - z_{p,f}^{(K-1)}(t))| = |z_{p,f}^{(K)}(x_0)|$$

for some x_0 in $[x, t]$, and $|z_{p,f}^{(K)}(x_0)| \leq \delta^{-K}K!\tilde{M}$ if $[x, t]$ is in $B(b; \delta)$.

Hence if $[x, t]$ is in $B(b; \delta)$,

$$\|(x-t)^{-1}(A^{K-1}T(x) - A^{K-1}T(t))\| \leq \delta^{-K}K!\tilde{M}.$$

Thus if t is in $B(b; \delta)$, $\lim_{x\to t^+}(x-t)^{-1}(A^{K-1}T(x)p - A^{K-1}T(t)p)$ exists for p in a dense set (the domain of A), and also $\|(x-t)^{-1}(A^{K-1}T(x) - A^{K-1}T(t))\| \leq \delta^{-K}K!\tilde{M}$ when $[x, t]$ is in $B(b; \delta)$.

Hence for any p in X, t in $B(b; \delta)$, $\lim_{x\to t^+}(x-t)^{-1}(A^{K-1}T(x)p - A^{K-1}T(t)p)$ exists, this limit is $A^KT(t)p$, and $\|A^KT(t)\| \leq \delta^{-K}K!\tilde{M}$.

Suppose λ is in $B(b; \delta/2)$. Then $W(\lambda)p = \sum_{n=0}^{\infty}((\lambda-b)^n/n!)A^nT(b)p$ defines $W(\lambda)$ as a bounded linear transformation on X. Furthermore, W is holomorphic at each λ in $B(b; \delta/2)$ since if f is in X^*, p in X, then

$$f(W(\lambda)p) = \sum_{n=0}^{\infty}((\lambda-b)^n/n!)f(A^nT(b)p)$$

$$= \sum_{n=0}^{\infty}((\lambda-b)^n/n!)z_{p,f}^{(n)}(b) = \widetilde{z_{p,f}}(\lambda)$$

and $\widetilde{z_{p,f}}$ is holomorphic at λ.

Thus there is a function W from $B(b; \delta/2)$ to the set of bounded linear transformations on X, W is holomorphic at each λ in $B(b; \delta/2)$, and if x is in $(b-(\delta/2), b+(\delta/2))$, then $W(x) = T(x)$. By a theorem of Hille [4, p. 477] T has an analytic extension to the interior of a spinal semimodule which includes $[b, \infty)$.

Proof of Theorem II. Suppose ρ is a number, $1 < \rho < 2$, and $\{[r_j, s_j]\}_{j=1}^{\infty}$ is a sequence of intervals such that $r_j \to 0$ as $j \to \infty$ and such that there exists $\epsilon > 0$ such that $r_j/s_j < 1-\epsilon$, $j = 1, 2, \cdots$. Suppose that for each j, T_j is a strongly continuous semigroup on $[0, \infty)$ and there exists $M_j > 0$ such that if n is a nonnegative integer, $b > 0$, and $n = 0$ or nb is in $[r_j, s_j]$, then $|(T_j(b) - I)^n| \leq M_j\rho^n$.

Denote $r_j/(1-\epsilon)$ by s_j'. Then $r_j < s_j' < s_j$, for all j, and $s_j' \to 0$ as $j \to \infty$.

Suppose σ, σ_0 are numbers such that $\rho < \sigma_0 < \sigma < 2$, and suppose α is a number such that $\alpha < \epsilon$, α is in $(0, 1/2)$, and

$$\binom{n}{[\alpha n]}\rho^n < \sigma_0^n,$$

$n = 1, 2, \cdots$. Then $\alpha < 1 - (r_j/s_j') = \epsilon$, $j = 1, 2, \cdots$.

Since $s_j' \to 0$ as $j \to \infty$, there exists a sequence of positive numbers $\{B_j\}_{j=1}^{\infty}$ such that $B_j \to 0$ as $j \to \infty$ and such that $(s_j'/B_j)^{\alpha} < \sigma_0/4$, $j = 1, 2, \cdots$. Denote $r_j/(1-\alpha)$ by $r_{0,j}$ and denote $s_j' - r_{0,j}$ by $\delta_{0,j}$. Let $D_j = 3B_j + s_j'$ and let $\beta_j = \delta_{0,j}\sigma_0/8e\pi$. Then Theorem IV holds for r_j, s_j', ρ, D_j, β_j, σ, and hence for r_j, s_j, ρ, D_j, β_j, σ since $[r_j, s_j'] \subset [r_j, s_j]$. Let $b_j = D_j - \beta_j$, $j = 1, 2, \cdots$. Then for each j, Theorem I holds for T_j, r_j, s_j, ρ, and b_j. Clearly $D_j \to 0$ as $j \to \infty$. Hence $b_j \to 0$ as $j \to \infty$.

4. **Example.** The following example is due to Neuberger [9]. Suppose $X = C_{[0,1];0}$, the space of all functions h continuous on $[0, 1]$, with $h(0) = 0$, and with $\|h\| = \sup_{x \in [0,1]}\{|h(x)|\}$.

For each $\lambda \geq 0$, define

$$(T(\lambda)h)(x) = \begin{cases} 0 & \text{if } \lambda - x \geq 0, \\ h(x-\lambda) & \text{if } x - \lambda \geq 0, \end{cases}$$

x in $[0, 1]$, h in $C_{[0,1];0}$.

Then T is a one-parameter semigroup of operators on $C_{[0,1];0}$. T is strongly continuous at $\lambda > 1$ since $T(\lambda) = 0$ for all $\lambda > 1$; T is strongly continuous at $\lambda < 1$ since each element of $C_{[0,1];0}$ is uniformly continuous on $[0, 1]$; and T is strongly continuous at $\lambda = 1$ since each element h of $C_{[0,1];0}$ is continuous at 0 and $h(0) = 0$.

Suppose α is a number such that $0 < \alpha < 1/2$. Then there exists $M > 0$ and ρ in $(1,2)$ such that $\sum_{v=0}^{[\alpha n]} \binom{n}{v} < M\rho^n$, $n = 0, 1, 2, \cdots$. Denote $1/\alpha$ by r, and suppose s is any number $> r$. Then if $n\lambda$ is in $[r, s]$, $|(T(\lambda) - I)^n| < M\rho^n$, $n = 0, 1, 2, \cdots$, since

$$\begin{aligned} \|(T(\lambda) - I)^n h\| &= \left\| \sum_{v=0}^{n} \binom{n}{v} (-1)^{n-v} T(v\lambda)h \right\| \\ &\leq \left\| \sum_{v=0}^{[\alpha n]} \binom{n}{v} (-1)^{n-v} T(v\lambda)h \right\| \\ &\quad + \left\| \sum_{v=[\alpha n]+1}^{n} \binom{n}{v} (-1)^{n-v} T(v\lambda)h \right\| \\ &\leq \|h\| \sum_{v=0}^{\alpha n} \binom{n}{v} \leq M\rho^n \|h\|, \end{aligned}$$

using that $v\lambda \geq \alpha n\lambda > 1$ implies $\sum_{v=[\alpha n]+1}^{[\alpha n]} \binom{n}{v} (-1)^{n-v} T(v\lambda)h = 0$.

However, T does not have an analytic extension to an open set which has zero as a limit point. Suppose t_0 is in $(0, 1)$, suppose $g(x) = x$, x in $[0, 1]$, and suppose $f_{t_0}(h) = h(t_0)$, h in $C_{[0,1];0}$. Then the function $z_{g,f_{t_0}}$, where $z_{g,f_{t_0}}(\lambda) = f_{t_0}(T(\lambda)g)$, is not analytic at t_0 since

$$z_{g,f_{t_0}}(x) = \begin{cases} t_0 - x & \text{if } t_0 - x \geq 0, \\ 0 & \text{if } x - t_0 \geq 0. \end{cases}$$

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