

ONE-SIDED CONGRUENCES ON INVERSE SEMIGROUPS

BY

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ABSTRACT. By the kernel of a one-sided (left or right) congruence ρ on an inverse semigroup S , we mean the set of ρ -classes which contain idempotents of S . We provide a set of independent axioms characterizing the kernel of a one-sided congruence on an inverse semigroup and show how to reconstruct the one-sided congruence from its kernel. Next we show how to characterize those partitions of the idempotents of an inverse semigroup S which are induced by a one-sided congruence on S and provide a characterization of the maximum and minimum one-sided congruences on S inducing a given such partition. The final two sections are devoted to a study of idempotent-separating one-sided congruences and a characterization of all inverse semigroups with only trivial full inverse subsemigroups. A Green-Lagrange-type theorem for finite inverse semigroups is discussed in the fourth section.

1. Basic notions, terminology. We adhere throughout to the notation and terminology of A. H. Clifford and G. B. Preston [1]. Throughout the paper, S will always denote an inverse semigroup (i.e., for each $a \in S$, there exists a unique element $a^{-1} \in S$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$) and E_S will denote the set of idempotents of S . The elementary properties of inverse semigroups may be found in [1]. In particular, we shall liberally use, without comment, the fact that E_S is a semilattice (a commutative semigroup of idempotents) and that $a^{-1}E_S a \subseteq E_S \quad \forall a \in S$. We shall also use the fact that if S is an inverse semigroup then the Green's relations L and R on S are given by

$$L = \{(a, b) \in S \times S: a^{-1}a = b^{-1}b\}$$

and

$$R = \{(a, b) \in S \times S: aa^{-1} = bb^{-1}\}.$$

We denote the lattice of left congruences on S by $L(S)$ and the lattice of right congruences on S by $R(S)$. For each $\rho \in L(S) \cup R(S)$ we set

$$\rho_{-1} = \{(a, b) \in S \times S: (a^{-1}, b^{-1}) \in \rho\}.$$

It is obvious that ρ_{-1} is a left (right) congruence on S iff ρ is a right (left) congruence on S , and that $(\rho_{-1})_{-1} = \rho$.

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PROPOSITION 1.1. *The lattice of left congruences on an inverse semigroup S is isomorphic to the lattice of right congruences on S .*

PROOF. The mapping $\phi: \rho \rightarrow \rho_{-1} \forall \rho \in L(S)$ is easily seen to be a lattice isomorphism of $L(S)$ onto $R(S)$.

By virtue of this observation we feel at liberty to restrict our attention to left congruences on inverse semigroups throughout the remainder of the paper. Any result concerning left congruences on S has an obvious dual result concerning right congruences on S .

We remark that it is clear that $R_{-1} = L$ and $L_{-1} = R$, where R and L denote the usual Green's relations on S : note also that if $\rho \in R(S) \cup L(S)$, then $\rho_{-1} = \rho$ iff ρ is a congruence on S . (This last remark follows from the fact that if ρ is a congruence on an inverse semigroup then $(a, b) \in \rho$ iff $(a^{-1}, b^{-1}) \in \rho$; see, for example, J. M. Howie [3].)

2. The kernel of a left congruence on an inverse semigroup. If ρ is a left congruence on an inverse semigroup S then the kernel of ρ is the set of ρ -classes which contain idempotents of S , i.e.,

$$\text{Ker } \rho = \{e\rho: e \in E_S\}.$$

It is well known that if ρ and σ are two left congruences on an inverse semigroup, then $\rho = \sigma$ iff $\text{Ker } \rho = \text{Ker } \sigma$. (See [1, Theorem 7.39].) This result naturally suggests the following question: How do we characterize the kernel of a left congruence on an inverse semigroup and how do we reconstruct the left congruence from its kernel? In Theorem 2.1 we provide an answer to this question. The corresponding problem for two-sided congruences on inverse semigroups has been solved by G. B. Preston [4].

We introduce the following notation. If $A = \{A_i: i \in I\}$ is a nonempty set of disjoint subsets of S then we shall consistently use the notation $A(A) = \bigcup_{i \in I} A_i$ (or more simply $A(A) = A$ if no confusion can arise) and if $a \in A$, then $A(a)$ will denote the element A_i of A to which a belongs, i.e. $A_i = A(a)$ iff $a \in A_i$.

DEFINITION 2.1. A set $A = \{A_i: i \in I\}$ of disjoint subsets of the inverse semigroup S is called a *left kernel system* of S if it satisfies the conditions:

- (L1) $E_S \subseteq A$;
- (L2) for each $i \in I$, $E_S \cap A_i \neq \emptyset$;
- (L3) for each $a \in A$ and $j \in I$, $\exists k \in I$ such that $aA_j \subseteq A_k$;
- (L4) for each $a \in A$, $a^{-1}A(a) \subseteq A(a^{-1}a)$;
- (L5) if $a^{-1}b \in A(a^{-1}a)$ for some $b \in A$, then $a \in A$.

(We remark that L3 implies in particular that A is a subsemigroup of S .)

LEMMA 2.1. *If ρ is a left congruence on S with $\text{Ker } \rho = \{A_i; i \in I\}$, then $\text{Ker } \rho$ is a left kernel system of S .*

PROOF. $\text{Ker } \rho$ clearly satisfies L1, L2 and L4. Let $a \in A, j \in I$ and $f \in A_j \cap E_S, e \in A(a) \cap E_S$. Then for each $b \in A_j, ab \rho af = aa^{-1}af = (afa^{-1})a \rho (afa^{-1})e \in E_S$.

Thus $aA_j \subseteq A((afa^{-1})e)$, and so $\text{ker } \rho$ satisfies L3. Finally, if $a^{-1}b \in A(a^{-1}a)$ for some $b \in A$ and if $g \in A(b) \cap E_S$, then

$$a = a(a^{-1}a) \rho a(a^{-1}b) = (aa^{-1})b \rho (aa^{-1})g \in E_S,$$

and so $a \in A$.

We make two small remarks about Lemma 2.1 and Definition 2.1. The lemma implies that even though each element $e\rho$ of $\text{Ker } \rho$ is not necessarily a subsemigroup of S (for example an R -class of S is not necessarily a subsemigroup of S but R is a left congruence on S), the union $\bigcup_{e \in E_S} e\rho$ of the elements of $\text{Ker } \rho$ is always a subsemigroup of S . We also remark that, in the special case in which S is a group with identity e , $\text{Ker } \rho = \{A(e)\}$ and conditions L1–L5 reduce simply to the condition that $A(e)$ is a subgroup of S .

The next theorem tells us that conditions L1–L5 provide the desired characterization of the kernel of a left congruence on an inverse semigroup.

THEOREM 2.1. *If $A = \{A_i; i \in I\}$ is a left kernel system of S then the relation*

$$(*) \quad \rho_A = \{(a, b) \in S \times S: a^{-1}b \in A(a^{-1}a) \text{ and } b^{-1}a \in A(b^{-1}b)\}$$

is a left congruence on S with kernel A . If ρ is a left congruence on S with kernel A then A is a left kernel system of S and $\rho = \rho_A$.

PROOF. In view of Lemma 2.1 and the uniqueness theorem of Clifford and Preston it suffices to show that if A is a left kernel system of S then ρ_A is a left congruence on S with kernel A . Let A satisfy L1–L5 and introduce the following notation: $a \sim b$ iff a and b are in the same element A_i of A , i.e., $a \sim b$ iff $a, b \in A_i$ for some $i \in I$. Note that \sim is a partial equivalence relation on S .

It is clear that ρ_A is reflexive and symmetric: To prove that ρ_A is transitive let $(a, b) \in \rho_A$ and $(b, c) \in \rho_A$. Then $a^{-1}b \sim a^{-1}a$, $b^{-1}a \sim b^{-1}b$, $b^{-1}c \sim b^{-1}b$ and $c^{-1}b \sim c^{-1}c$. Hence, by L3 and L1,

$$c^{-1}aa^{-1}c = (c^{-1}aa^{-1}c)c^{-1}c \sim (c^{-1}aa^{-1}c)c^{-1}b = c^{-1}aa^{-1}b.$$

Hence $(a^{-1}c)^{-1}(a^{-1}b) \in A((a^{-1}c)^{-1}(a^{-1}c))$ and $a^{-1}b \in A$ and it follows by L5 that $a^{-1}c \in A$. Thus by many applications of L3 we have

$$\begin{aligned}
a^{-1}c &= a^{-1}cc^{-1}c \sim a^{-1}cc^{-1}b = a^{-1}cc^{-1}bb^{-1}b \sim a^{-1}cc^{-1}bb^{-1}a \\
&= a^{-1}cc^{-1}bb^{-1}aa^{-1}a \sim a^{-1}cc^{-1}bb^{-1}aa^{-1}b \\
&= a^{-1}bb^{-1}cc^{-1}b \sim a^{-1}bb^{-1}cc^{-1}c = a^{-1}bb^{-1}c \sim a^{-1}bb^{-1}b \\
&= a^{-1}b \sim a^{-1}a.
\end{aligned}$$

Hence $a^{-1}c \in A(a^{-1}a)$, and a similar argument shows that $c^{-1}a \in A(c^{-1}c)$. Thus $(a, c) \in \rho_A$ and so ρ_A is transitive.

Now suppose that $(a, b) \in \rho_A$ and $c \in S$. Then $a^{-1}b \sim a^{-1}a$, $b^{-1}a \sim b^{-1}b$, and, by L3,

$$\begin{aligned}
(ca)^{-1}ca &= a^{-1}c^{-1}ca = (a^{-1}c^{-1}ca)a^{-1}a \sim a^{-1}c^{-1}caa^{-1}b \\
&= a^{-1}c^{-1}cb = (ca)^{-1}cb,
\end{aligned}$$

and similarly $(cb)^{-1}cb \sim (cb)^{-1}ca$. It follows that ρ_A is a left congruence on S .

Now let $K = \{K_j; j \in J\}$ be the kernel of ρ_A . We let $K_j \in K$ and show first that $K_j \subseteq A_i$ for some $i \in I$. Choose $e \in K_j \cap E_S$ and let $k \in K_j$. Then $(k, e) \in \rho_A$ and so $k^{-1}k \sim k^{-1}e$ and $ek \sim e$. Since $k^{-1}k \sim k^{-1}e$ it follows that $k \in A$ by L5. Hence, by L3, $k = kk^{-1}k \sim kk^{-1}e \sim kk^{-1}(ek) = ek \sim e$, so $k \in A(e)$. It follows that $K_j \subseteq A(e)$.

Conversely, let $A_i \in A$ and suppose that $e \in A_i \cap E_S$. Then $e \in K_j$ for some $j \in J$. Let $a \in A_i$. Then $a \sim e$ and so $ea \sim e$ by L3. Also, $a^{-1}e \in a^{-1}A(a) \subseteq A(a^{-1}a)$ by L4. It follows that $a^{-1}e \sim a^{-1}a$ and this, together with $ea \sim e$, implies that $(a, e) \in \rho_A$, i.e., $a \in K_j$, and so $A_i \subseteq K_j$. Thus $K = A$ is the kernel of ρ_A . This completes the proof of the theorem.

We now provide examples which show that the conditions L1–L5 are mutually independent. Let I_2 be the symmetric inverse semigroup on two letters $\{1, 2\}$. (I_2 is the set of all one-one mappings from subsets of $\{1, 2\}$ onto subsets of $\{1, 2\}$ (including the empty mapping) with composition defined in the usual way—see [1] for details.) We use the notation $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ for the mapping which takes x to z and y to w , $\begin{pmatrix} x \\ y \end{pmatrix}$ for the mapping which takes x to y , and 0 for the empty mapping. Note that

$$E_{I_2} = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, 0 \right\}.$$

EXAMPLE 1. Let

$$A = \left\{ \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ 0, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \right\}.$$

One may readily check that A satisfies conditions L1–L4 but not L5.

EXAMPLE 2. Let

$$A = \left\{ \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \left\{ 0, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \right\}.$$

Conditions L1, L2, L3, and L5 are satisfied but L4 is not.

EXAMPLE 3. If

$$A = \left\{ \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, 0, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\} \right\},$$

then A satisfies L1, L2, L4 and L5 but not L3.

EXAMPLE 4. Let $A = \{0\}$, A satisfies L2–L5 but not L1.

EXAMPLE 5. If

$$A = \left\{ \{0\}, \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \right\} \right\},$$

then conditions L1, L3, L4 and L5 are satisfied but L2 is not satisfied.

Thus no four of L1–L5 imply the fifth and so L1–L5 are mutually independent. We make the remark that while it is true that if ρ is a left congruence on S then $A = \bigcup_{e \in E_S} e\rho$ is a subsemigroup of S , it is not true that A is necessarily an *inverse* subsemigroup. For example the partition

$$\left\{ \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, 0, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \right\}$$

defines a left congruence on I_2 and here

$$A = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, 0, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

is not an inverse subsemigroup of I_2 .

3. Idempotent-equivalent one-sided congruences. The set of left congruences on S which induce the same fixed partition $E = \{E_i; i \in I\}$ of E_S constitutes a complete sublattice of $L(S)$ (with both 0 and 1 elements). Two left congruences on S are called *idempotent-equivalent* if they induce the same partition of E_S . In view of these remarks it is natural to ask which partitions of E_S are induced by left congruences on S and to seek a characterization of the maximum and minimum left congruences on S which induce a given partition of E_S . We provide the solution to this problem in the following theorem. Reilly and Scheiblich have answered the corresponding set of questions for two-sided congruences on inverse semigroups in [5].

THEOREM 3.1. *A partition $E = \{E_i; i \in I\}$ of E_S into disjoint subsets is induced by some left congruence on S iff E satisfies:*

$$(N) \quad \forall i, j \in I, \exists k = k(i, j) \in I \text{ such that } E_i E_j \subseteq E_k.$$

The minimum left congruence on S inducing such a partition of E_S is

$$\xi = \{(a, b) \in S \times S : \exists i \in I \text{ such that } a^{-1}a, b^{-1}b \in E_i \text{ and } ae = be \text{ for some } e \in E_i\}.$$

The maximum left congruence on S inducing such a partition of E_S is

$$\zeta = \{(a, b) \in S \times S : a^{-1}bb^{-1}a \in E(a^{-1}a), b^{-1}aa^{-1}b \in E(b^{-1}b) \text{ and for each } k \in I, \exists l, m \in I \text{ such that } a^{-1}bE_k b^{-1}a \subseteq E_l \text{ and } b^{-1}aE_k a^{-1}b \subseteq E_m\}.$$

PROOF. Let ρ be a left congruence on S and let $E = \{E_i : i \in I\}$ be the partition of E_S induced by ρ . It is an easy matter to see that E satisfies the condition (N), for if $e_1, e_2 \in E_i$ and $f_1, f_2 \in E_j$ it follows that $e_1 f_1 \rho e_1 f_2 = f_2 e_1 \rho f_2 e_2 = e_2 f_2$ and so $E_i E_j \subseteq E_k$ for some $k \in I$.

Suppose now that E is a partition of E_S which satisfies (N). We show that there is some left congruence on S which induces this partition E of E_S by showing directly that both ξ and ζ have this property. We first prove that ξ is a left congruence on S with the properties stated in the theorem.

Evidently, ξ is reflexive and symmetric. Suppose now that $(a, b) \in \xi$ and $(b, c) \in \xi$: Then there exists $i \in I$ such that $a^{-1}a, b^{-1}b, c^{-1}c \in E_i$ and $ae = be, bf = cf$ for some $e, f \in E_i$. Then $a(ef) = bef = bfe = cfe = c(ef)$, and $ef \in E_i$, so $(a, c) \in \xi$. Now let c be any element of S and suppose that $a^{-1}c^{-1}ca \in E_j$. Then $(a^{-1}a)(a^{-1}c^{-1}ca)(a^{-1}a) \in E_j$ and it follows by (N) that

$$e(b^{-1}c^{-1}cb)e = (eb^{-1})(c^{-1}c)(be) = e(a^{-1}c^{-1}ca)e \in E_j,$$

and hence that $b^{-1}c^{-1}cb = (b^{-1}b)(b^{-1}c^{-1}cb)(b^{-1}b) \in E_j$. Furthermore, if we let $f = ea^{-1}c^{-1}ca = ea^{-1}c^{-1}cae = eb^{-1}c^{-1}cbe = b^{-1}c^{-1}cbe$, we easily see that $f \in E_j$ and that

$$caf = caa^{-1}c^{-1}cae = cae = cbe = cbb^{-1}c^{-1}cbe = cbf,$$

and hence $(ca, cb) \in \xi$. Thus ξ is a left congruence on S .

It is almost immediate from the definition of ξ that if $e, f \in E_S$, then $(e, f) \in \xi$ iff $e, f \in E_i$ for some $i \in I$. Thus ξ induces the partition E of E_S and it remains to show that if ρ is any left congruence on S which induces this partition E of E_S , then $\xi \subseteq \rho$. Let $(a, b) \in \xi$: There exists $i \in I$ such that $a^{-1}a, b^{-1}b \in E_i$ and $ae = be$ for some $e \in E_i$. Since ρ is a left congruence on S which induces the partition E of E_S it follows that $aE_i \subseteq a(e\rho) \subseteq x\rho$ for some $x \in S$. But $a = a(a^{-1}a) \in aE_i$ and so $a \in x\rho$. By a similar argument, $bE_i \subseteq y\rho$ and $b \in y\rho$ for some $y \in S$, but since $ae = be \in aE_i \cap bE_i$, it follows that $x\rho = y\rho$ and that $a, b \in x\rho$. Thus $(a, b) \in \rho$ and so $\xi \subseteq \rho$. This completes the proof of the claims made about ξ .

Now let ζ be the relation defined in the statement of the theorem. That ζ is reflexive follows from (N), and that ζ is symmetric is immediately obvious. Suppose now that $(a, b) \in \zeta$ and $(b, c) \in \zeta$. Then there are elements $i, j, k \in I$ such that $a^{-1}a, a^{-1}bb^{-1}a \in E_p, b^{-1}b, b^{-1}aa^{-1}b, b^{-1}cc^{-1}b \in E_j, c^{-1}c, c^{-1}bb^{-1}c \in E_k$, and for each $l \in I$, there exist $m, n, p, q \in I$ such that $a^{-1}bE_l b^{-1}a \subseteq E_m, b^{-1}aE_l a^{-1}b \subseteq E_n, b^{-1}cE_l c^{-1}b \subseteq E_p$ and $c^{-1}bE_l b^{-1}c \subseteq E_q$. To prove the transitivity of ζ , we need to prove that $a^{-1}cc^{-1}a \in E_i$ and $c^{-1}aa^{-1}c \in E_k$, and that if $l \in I$, there exist $r, s \in I$ such that $a^{-1}cE_l c^{-1}a \subseteq E_r$ and $c^{-1}aE_l a^{-1}c \subseteq E_s$. Now $a^{-1}cc^{-1}a = (a^{-1}cc^{-1}a)(a^{-1}a)$, so by (N), $a^{-1}cc^{-1}a$ is in the same element of E as $(a^{-1}cc^{-1}a)(a^{-1}bb^{-1}a)$. But

$$(a^{-1}cc^{-1}a)(a^{-1}bb^{-1}a) = a^{-1}b(b^{-1}cc^{-1}b)b^{-1}a \in a^{-1}bE_j b^{-1}a,$$

and $a^{-1}bb^{-1}a = a^{-1}b(b^{-1}b)b^{-1}a \in a^{-1}bE_j b^{-1}a$. Hence $a^{-1}bE_j b^{-1}a \cap E_i \neq \emptyset$, and it follows that $a^{-1}bE_j b^{-1}a \subseteq E_i$. Hence $a^{-1}cc^{-1}a \in E_i$ and similarly $c^{-1}aa^{-1}c \in E_k$. Now choose $l \in I$ arbitrarily and let $e \in E_l$. Then, by (N), $a^{-1}cec^{-1}a$ is in the same element of E as $(a^{-1}cec^{-1}a)(a^{-1}bb^{-1}a)$, since $a^{-1}cec^{-1}a = (a^{-1}cec^{-1}a)(a^{-1}a)$. But

$$(a^{-1}cec^{-1}a)(a^{-1}bb^{-1}a) = a^{-1}bb^{-1}cec^{-1}bb^{-1}a \in a^{-1}b(b^{-1}cE_l c^{-1}b)b^{-1}a,$$

and $a^{-1}b(b^{-1}cE_l c^{-1}b)b^{-1}a \subseteq E_r$ for some $r \in I$, since there exist $p, r \in I$ for which $b^{-1}cE_l c^{-1}b \subseteq E_p$ and $a^{-1}bE_p b^{-1}a \subseteq E_r$. Hence $a^{-1}cec^{-1}a \in E_r$ for each $e \in E_l$ and it follows that $a^{-1}cE_l c^{-1}a \subseteq E_r$. Similarly, $c^{-1}aE_l a^{-1}c \subseteq E_s$, some $s \in I$, and so ζ is an equivalence relation on S . Now let c be any element of S and let $(a, b) \in \zeta$ as above, and suppose that $a^{-1}c^{-1}ca \in E_p$. Then since $a^{-1}c^{-1}ca = (a^{-1}c^{-1}ca)(a^{-1}a)$, we have that $E_p E_i \subseteq E_p$. It follows that $a^{-1}c^{-1}cb \cdot b^{-1}c^{-1}ca = (a^{-1}c^{-1}ca)(a^{-1}bb^{-1}a) \in E_p$, and the proof that $b^{-1}c^{-1}cb$ and $b^{-1}c^{-1}ca \cdot a^{-1}c^{-1}cb$ are in the same element of E is similar. Further, if $l \in I$ and $e \in E_l$, then

$$\begin{aligned} a^{-1}c^{-1}cb e b^{-1}c^{-1}ca &= (a^{-1}c^{-1}ca)(a^{-1}b e b^{-1}a) \in (a^{-1}c^{-1}ca)(a^{-1}bE_l b^{-1}a) \\ &\subseteq E_p E_m \subseteq E_q, \text{ for some } q \in I, \end{aligned}$$

and it follows that $a^{-1}c^{-1}cb E_l b^{-1}c^{-1}ca \subseteq E_q$. Similarly $b^{-1}c^{-1}ca E_l a^{-1}c^{-1}cb \subseteq E_r$ for some $r \in I$, and so ζ is a left congruence on S .

Again, it follows fairly easily that ζ induces the partition E of E_S : Let ρ be any left congruence on S which induces the partition E of E_S , and let $(a, b) \in \rho$. Then $(a^{-1}a, a^{-1}b) \in \rho$ and $(b^{-1}a, b^{-1}b) \in \rho$ and also $(b^{-1}a, b^{-1}aa^{-1}b) = (b^{-1}aa^{-1}a, b^{-1}aa^{-1}b) \in \rho$. Since $(b^{-1}a, b^{-1}b) \in \rho$ it follows that $(b^{-1}b, b^{-1}aa^{-1}b) \in \rho$ and hence that $b^{-1}b, b^{-1}aa^{-1}b \in E_i$ for some $i \in I$. Similarly $a^{-1}a, a^{-1}bb^{-1}a \in E_j$ for some $j \in I$. Now let $l \in I$ and $e \in E_l$. Then

$(ea^{-1}b, ea^{-1}a) \in \rho$, i.e., $(ea^{-1}b, a^{-1}ae) \in \rho$, and hence $(b^{-1}aea^{-1}b, b^{-1}aa^{-1}ae) \in \rho$, i.e., $(b^{-1}aea^{-1}b, b^{-1}ae) \in \rho$. Let $A = \{A_i; i \in I\}$ be the kernel of ρ . Then $b^{-1}ae \in b^{-1}aE_i \subseteq b^{-1}aA_j$, and since $b^{-1}aea^{-1}b \in A_j$ for some $j \in I$, we have $b^{-1}aA_i \subseteq A_j$ by the left congruence property of ρ . In particular, $b^{-1}af \in A_j$ for all $f \in E_i$. Hence $b^{-1}afa^{-1}b \in A_j$ for all $f \in E_i$, and it follows that $b^{-1}aE_i a^{-1}b \subseteq A_j$. Since $b^{-1}aE_j a^{-1}b \subseteq E_S$, we evidently have that $b^{-1}aE_i a^{-1}b \subseteq E_j$. Similarly, $a^{-1}bE_i b^{-1}a \subseteq E_k$, for some $k \in I$. Hence $(a, b) \in \xi$ and so $\rho \subseteq \xi$. This completes the proof of the theorem.

We mention two corollaries of the theorem which are of independent interest and may be proved directly.

COROLLARY 3.1. *A partition $E = \{E_i; i \in I\}$ of E_S is induced by a left congruence on S iff E is induced by a right congruence on S .*

(This follows from the left-right symmetry of condition (N).)

COROLLARY 3.2. *Any left (right, two-sided) congruence on E_S may be extended to a left or right congruence on S whose restriction to E_S coincides with the original left congruence on E_S .*

(This follows since condition (N) is of course just the condition which insures that the partition E defines a congruence on the semilattice E_S .)

4. Idempotent-separating one-sided congruences. A one-sided congruence ρ on S is called *idempotent-separating* if distinct idempotents of S are in distinct equivalence classes of ρ . Idempotent-separating left congruences on S are clearly idempotent-equivalent in the sense of the previous section. For idempotent-separating left congruences on S we are able to obtain a Green-Lagrange type theorem (Theorem 4.3): We first show how the results of the previous two sections may be simplified in the idempotent-separating case.

THEOREM 4.1. *The maximum idempotent-separating left (right) congruence on S is R (L).*

PROOF. This may be proved by noting that the relation ξ of Theorem 3.1 reduces to R in the special case in which $E = \{\{e\}; e \in E_S\}$. For with this partition E , the relation ξ clearly reduces to the relation

$$\xi = \{(a, b) \in S \times S: a^{-1}a = a^{-1}bb^{-1}a \text{ and } b^{-1}b = b^{-1}aa^{-1}b\}.$$

If $(a, b) \in \xi$, then

$$\begin{aligned} aa^{-1} &= a(a^{-1}a)a^{-1} = a(a^{-1}bb^{-1}a)a^{-1} = (aa^{-1})(bb^{-1})(aa^{-1}) \\ &= (bb^{-1})(aa^{-1})(bb^{-1}) = b(b^{-1}aa^{-1}b)b^{-1} = b(b^{-1}b)b^{-1} = bb^{-1}, \end{aligned}$$

so $(a, b) \in \mathcal{R}$, i.e., $\xi \subseteq \mathcal{R}$. Since \mathcal{R} is an idempotent-separating left congruence on S , $\mathcal{R} = \xi$ as required. One may of course provide a direct proof of this result without using Theorem 3.1.

Thus the study of idempotent-separating left congruences on S reduces to the study of those left congruences on S which are contained in \mathcal{R} . Recall that T is called a *full inverse subsemigroup* of S if T is an inverse subsemigroup of S which contains all idempotents of S .

THEOREM 4.2. *The lattice of idempotent-separating left congruences on an inverse semigroup S is isomorphic to the lattice of full inverse subsemigroups of S .*

PROOF. Let ρ be an idempotent-separating left congruence on S with kernel $\text{Ker } \rho = \{A(e) : e \in E_S\}$. As usual, set $A = \bigcup_{e \in E_S} A(e)$. Then A is a subsemigroup of S which contains E_S . In addition, if $a \in A$, then $a \in A(aa^{-1})$ because $\rho \subseteq \mathcal{R}$, and so $a^{-1} = a^{-1}(aa^{-1}) \rho a^{-1}a \in E_S$, i.e., $a^{-1} \in A$. Hence A is a full inverse subsemigroup of S .

Conversely, if A is a full inverse subsemigroup of S , let $\mathcal{A} = \{A \cap R_e : e \in E_S\}$. We claim that \mathcal{A} is a left kernel system of S . \mathcal{A} clearly satisfies L1 and L2. If $a \in A$, then $\forall e \in E_S$, $aA(e) \subseteq A$ since A is a subsemigroup of S , and also $aA(e) \subseteq aR_e \subseteq R_{ae}$ since \mathcal{R} is a left congruence on S . Hence $aA(e) \subseteq R_{ae} \cap A = A(ae)$, so L3 is satisfied. Furthermore, $a^{-1} \in A$ so $a^{-1}A(a) \subseteq A(a^{-1}a)$ by L3 so L4 is satisfied. Finally, if $a^{-1}b \in A(a^{-1}a)$ for some $b \in A$, then $a^{-1}b \mathcal{R} a^{-1}a$, so $a^{-1}bb^{-1}a = a^{-1}a$: Hence $a = a(a^{-1}a) = aa^{-1}bb^{-1}a = (aa^{-1})b(b^{-1}a) \in A$ (since aa^{-1} , b , $b^{-1}a \in A$), and so L5 is satisfied. One can easily check (by using Theorem 2.1) that this establishes an isomorphism between the lattice of idempotent-separating left congruences on S and the lattice of full inverse subsemigroups of S .

LEMMA 4.1. *If ρ is an idempotent-separating left congruence on S then ρ_{-1} is an idempotent-separating right congruence on S and $\rho \circ \rho_{-1} = \rho_{-1} \circ \rho$ is the smallest equivalence relation containing both ρ and ρ_{-1} .*

PROOF. That ρ_{-1} is an idempotent-separating right congruence on S is obvious. The rest of the proof follows from Lemma 1.4 of [1] and Lallement's result [6] that any left congruence on a regular semigroup which is contained in Green's relation \mathcal{R} commutes with any right congruence which is contained in \mathcal{L} .

The following theorem describes how a \mathcal{D} -class of S is further partitioned by an idempotent-separating left congruence on S . It may be thought of as a generalization of Green's lemma [1, Lemma 2.2] and provides an analogue of Lagrange's theorem for finite groups in the case where S is finite.

THEOREM 4.3 (A GREEN-LAGRANGE THEOREM). *Let ρ be an idempotent-separating left congruence on S with kernel $A = \{A(e) : e \in E_S\}$. For each $a \in S$ define*

$$\begin{aligned} R_\rho(a) &= \bigcup \{H_x : x \in a\rho\}, \\ L_\rho(a) &= \bigcup \{H_x : x \in a\rho_{-1}\}, \\ D_\rho(a) &= \bigcup \{H_x : x \in a(\rho \circ \rho_{-1})\}. \end{aligned}$$

Then

- (1) $\forall a \in S, a\rho = aA(a^{-1}a)$.
- (2) *If $a \perp b$ then for some $x, y \in S, xa = b$ and $yb = a$: the mappings $\phi: u \rightarrow xu \forall u \in a\rho$ and $\psi: v \rightarrow yv \forall v \in b\rho$ are mutually inverse one-one L -class preserving mappings of $a\rho$ onto $b\rho$ and $b\rho$ onto $a\rho$ respectively.*
- (3) *For each $e \in E_S, e(\rho \cap \rho_{-1}) = e\rho \cap H_e = e\rho_{-1} \cap H_e$ is a subgroup of H_e : If $e, f \in E_S$ and $f \in D_\rho(e)$ then $e(\rho \cap \rho_{-1}) \cong f(\rho \cap \rho_{-1})$.*
- (4) *If S is finite, then for each $a \in S,$*
 - (i) $|b\rho| = |a\rho|$ and $|b\rho_{-1}| = |a\rho_{-1}| \forall b \in D_\rho(a)$; $|a\rho \cap H_a| = |b\rho \cap H_b|$ and $|a\rho_{-1} \cap H_a| = |b\rho_{-1} \cap H_b| \forall b \in D_\rho(a)$;
 - (ii) $|a\rho| |R_\rho(a)| |D_\rho(a)|$ and $|a\rho_{-1}| |L_\rho(a)| |D_\rho(a)|$;
 - (iii) $|a\rho|, |a\rho_{-1}| |a(\rho \circ \rho_{-1})|$;
 - (iv) $|a\rho| = |a\rho_{-1}|$ if $D_\rho(a) \cap E_S \neq \emptyset$.

PROOF. (1) Let $ax \in aA(a^{-1}a)$. Then $ax \rho a(a^{-1}a) = a$, so $ax \in a\rho$. Conversely if $x \rho a$ then $xx^{-1} = aa^{-1}$ and so $x = (xx^{-1})x = aa^{-1}x = a(a^{-1}x)$, and $a^{-1}x \rho a^{-1}a$, so $a^{-1}x \in A(a^{-1}a)$. Hence $x \in A(a^{-1}a)$.

(2) This is straightforward and is omitted.

(3) Let $e \in E_S$: Then if $a \in e\rho \cap H_e, (a, e) \in \rho$ and so $a^{-1} = a^{-1}e \rho a^{-1}a = e$, i.e., $a^{-1} \in e\rho \cap H_e$, and so $a \in e\rho_{-1} \cap H_e$. It follows easily that $e\rho \cap H_e = e\rho_{-1} \cap H_e = e(\rho \cap \rho_{-1})$. Clearly $e(\rho \cap \rho_{-1})$ is a subgroup of H_e since ρ induces a left congruence on the group H_e .

Suppose now that $e, f \in E_S$ and that $e \in D_\rho(f)$. There are elements $a, b \in S$ such that $b \in H_f, e\rho_{-1}a$ and $a\rho b$, and it follows that $aa^{-1} = f, a^{-1}a = e, bb^{-1} = b^{-1}b = f, e\rho a^{-1}$ and $a^{-1}\rho_{-1}b^{-1}$. It is straightforward to check that the mapping $\chi: e(\rho \cap \rho_{-1}) \rightarrow f(\rho \cap \rho_{-1})$ defined by $\chi(u) = b^{-1}a u a^{-1}b \forall u \in e(\rho \cap \rho_{-1})$ is an isomorphism from $e(\rho \cap \rho_{-1})$ onto $f(\rho \cap \rho_{-1})$.

(4) Suppose now that S is finite.

(i) First note that by part (2) of the theorem, if $a \perp b$ then $|a\rho| = |b\rho|$ and $|a\rho \cap H_a| = |b\rho \cap H_b|$. Let $|a\rho \cap H_a| = k$: Since $H_a = \bigcup_{x \in H_a} (x\rho \cap H_x)$

it follows that $|H_a| = lk$ for some $l \in Z^+$. We claim that if $c \in a\rho$ then $|a\rho \cap H_a| = |c\rho \cap H_c| = k$. Let $A = \{x\rho \cap H_a : x \in H_a\}$ and $B = \{x\rho \cap H_c : x \in H_c\}$ and define $\nu : A \rightarrow B$ by $\nu(x\rho \cap H_a) = x\rho \cap H_c \forall x \in H_a$. One checks that ν is well defined and is a one-one mapping of A onto B , and so $|A| = |B|$. But $|H_a| = |A| |a\rho \cap H_a|$ and $|H_c| = |B| |c\rho \cap H_c|$, and since $|H_a| = |H_c|$ it follows that $|a\rho \cap H_a| = |c\rho \cap H_c|$. The remainder of (i) follows easily.

(ii) We have $k = |b\rho \cap H_b| \forall b \in D_\rho(a)$ and $|H_b| = lk$. Let n be the number of L -classes which $a\rho$ intersects and let m be the number of R -classes which $a\rho_{-1}$ intersects. Then since $R_\rho(a) = \bigcup_{x \in a\rho} H_x$, $|R_\rho(a)| = nlk$, $|a\rho| = nk$ and also $D_\rho(a) = \bigcup_{x \in a\rho_{-1}} R_\rho(a)$ so $|D_\rho(a)| = mnlk$. Thus $|a\rho| |R_\rho(a)| |D_\rho(a)|$ and similarly $|a\rho_{-1}| |L_\rho(a)| |D_\rho(a)|$.

(iii) We need only note that $a(\rho \circ \rho_{-1}) = \bigcup_{x \in a\rho_{-1}} x\rho = \bigcup_{x \in a\rho} x\rho_{-1}$, and so $|a\rho|, |a\rho_{-1}| |a(\rho \circ \rho_{-1})|$.

(iv) Note that if $e \in E_S$ then the mapping $a \rightarrow a^{-1} \forall a \in e\rho$ is a bijection between $e\rho$ and $e\rho_{-1}$: The result follows easily.

5. Inverse semigroups with only trivial full inverse subsemigroups. We now turn to the question of characterizing inverse semigroups S which have no idempotent-separating left congruences between i_S and R . In view of Theorem 4.2 this is equivalent to the problem of characterizing those inverse semigroups S which have no full inverse subsemigroups other than E_S and S . We shall refer to E_S and S as the trivial full inverse subsemigroups of S .

We use the notation $\langle W \rangle$ to denote the subsemigroup generated by a subset W of S . It is clear that if S is an inverse semigroup with only trivial full inverse subsemigroups, then either $S = E_S$ is a semilattice or $S = \langle a, a^{-1}, E_S \rangle \forall a \in S \setminus E_S$. This comment will be used without further reference.

LEMMA 5.1. *If $S = \langle a, a^{-1}, E_S \rangle$ for some $a \in S \setminus E_S$ then $S = E_S \cup E_S \langle a, a^{-1} \rangle$.*

PROOF. Let s be an element of $S \setminus E_S$. Then

$$\begin{aligned} s &= e_1 w_1 e_2 w_2 \cdots e_n w_n e_{n+1}, \text{ some } e_i \in E_S, w_i \in \langle a, a^{-1} \rangle, \\ &= e_1 w_1 e_2 w_2 \cdots e_{n-1} w_{n-1} e_n w_n (w_n^{-1} w_n) e_{n+1} \\ &= e_1 w_1 e_2 w_2 \cdots e_{n-1} w_{n-1} e_n (w_n e_{n+1} w_n^{-1}) w_n \\ &= e_1 w_1 e_2 w_2 \cdots e_{n-1} w_{n-1} f_n w_n \text{ where } f_n = e_n w_n e_{n+1} w_n^{-1} \in E_S. \end{aligned}$$

If we proceed inductively in this manner it is easy to see that $s = f_1 w_1 \cdots w_n$, some $f_1 \in E_S$, and so $s \in E_S \langle a, a^{-1} \rangle$.

LEMMA 5.2. *If $S = E_S \cup E_S A$ for some inverse subsemigroup A of S and if S has only trivial full inverse subsemigroups, then A has only trivial full inverse subsemigroups.*

PROOF. Suppose that B is a proper full inverse subsemigroup of A and set $B_1 = E_S \cup E_S B$. A standard argument shows that B_1 is a full inverse subsemigroup of S . Furthermore, since $B \subseteq E_S B \subseteq B_1$ and since B contains non-idempotents, $B_1 \neq E_S$ and so $B_1 = S$. Now let x be any element of A : Then $x \in S$ and so $x \in B_1$. If $x \in E_S$, $x \in A \cap E_S \subseteq B$ and so $x \in B$. If $x \notin E_S$, then $x = fb$ for some $f \in E_S$, $b \in B$. Thus $xx^{-1} = fbb^{-1} \in E_S \cap A \subseteq B$, and so $x = fb = (fbb^{-1})b \in B$. This yields $A \subseteq B$ and so $A = B$ which contradicts the fact that B is proper. Hence A has no proper full inverse subsemigroups.

We recall that an inverse semigroup S is called an *elementary inverse semigroup* if S is generated by an element and its inverse, i.e., $S = \langle a, a^{-1} \rangle$ for some $a \in S$. We make use of the following lemma concerning elementary inverse semigroups due to L. M. Gluskin [3].

LEMMA 5.3 (GLUSKIN [2]). *If S is an elementary inverse semigroup generated by a and $b (= a^{-1})$ then every element $w \in S$ is expressible in the form*

$$w = a^i b^j a^k, \quad 0 \leq i, k \leq j, \quad j \geq 1.$$

Furthermore, if $l \geq i$ and $n \geq k$, then $a^l b^m a^n = a^i b^j a^k$ implies that $b^m = b^{j+(n-k)+(i-i)}$; if $l \leq i$ and $n \geq k$, then $a^l b^m a^n = a^i b^j a^k$ implies that $b^{m+(i-l)} = b^{j+(n-k)}$.

We deduce a number of technical corollaries and lemmas concerning elementary inverse semigroups. The arguments involved in the first two of these are quite routine, involving only the commuting of idempotents, the expansion $x = xx^{-1}x$ and Lemma 5.3, so we omit these details and provide statements of the results without proof.

COROLLARY 5.1. *If S is an elementary inverse semigroup generated by a and $b (= a^{-1})$ and if a is of infinite order then*

- (a) $a^l b^m a^n = a^i b^j a^k$ implies $m - l - n = j - i - k$;
- (b) $a^i b^j a^k \in E_S$ iff $i + k = j$.

LEMMA 5.4. *If S is an elementary inverse semigroup generated by a and $b (= a^{-1})$ then multiplication in S is defined by*

$$(a^i b^j a^k)(a^l b^m a^n) = \begin{cases} a^i b^{j+m-k-l} a^n & \text{if } k+l \leq j, m, \\ a^i b^j a^{k+l-m+n} & \text{if } m \leq k+l \leq j, \\ a^{k+l-j+i} b^m a^n & \text{if } j \leq k+l \leq m, \\ a^{k+l-j+i} b^{k+l} a^{k+l-m+n} & \text{if } k+l \geq j, m. \end{cases}$$

From these two results we obtain the following

COROLLARY 5.2. *If S is an elementary inverse semigroup generated by a and b and if S has only trivial full inverse subsemigroups, then a is of finite order.*

PROOF. Suppose that a is of infinite order. As in Lemma 5.1, $S = \langle E_S, a^2, b^2 \rangle$ and thus $a \in \langle E_S, a^2, b^2 \rangle$. Hence $a = ew$ for some $e \in E_S$, $w \in \langle a^2, b^2 \rangle$ and by the preceding lemmas, $e = a^i b^{i+j} a^j$ and $w = a^{2k} b^{2l} a^{2m}$ for some integers i, j, k, l and m such that $i, j \geq 0, 0 \leq k, m \leq l$. Thus by the previous lemma

$$a = \begin{cases} a^i b^{i+j+2l-j-k} a^{2m} & \text{if } j+2k \leq i+j, 2l, \\ a^i b^{i+j} a^{j+2k-2l+2m} & \text{if } 2l \leq j+2k \leq i+j, \\ a^{j+2k-i-j+i} b^{2l} a^{2m} & \text{if } i+j \leq j+2k \leq 2l, \\ a^{j+2k-i-j+i} b^{j+2k} a^{j+2k-2l+2m} & \text{if } j+2k \geq i+j, 2l. \end{cases}$$

Then since a is of infinite order, Corollary 5.1 yields $2l - 2k - 2m = -1$, which is a contradiction since $2l - 2k - 2m$ is even. Hence a is of finite order.

LEMMA 5.5. *If S is an elementary inverse semigroup with only trivial full inverse subsemigroups, then either S is a group of prime order or S is the five-element Brandt semigroup.*

PROOF. Let S be generated by a and $b (= a^{-1})$. By Corollary 5.2, $\exists m, n \in \mathbb{Z}^+$ such that $m > n$ and $a^m = a^n$. With m and n chosen minimally it follows easily (using Gluskin's Lemma 5.3) that $G = \{a^n, a^{n+1}, \dots, a^{m-1}\} = \{b^n, \dots, b^{m-1}\}$ is a cyclic group and an ideal of S .

Now set $A = E_S \cup G$: since G is an ideal of S , A is a full inverse subsemigroup of S and hence $A = S$ or $A = E_S$. If $A = S$, then $a \in E_S$ or $a \in G$, and if $a \in E_S$ then $a = a^{-1}$ and $S = \{a\}$: If $a \in G$ then $S = G$ is a group which has no proper subgroups and so is a cyclic group of prime order.

Suppose now that $A = E_S$. Then $G = \{0\}$ because G is an ideal of S and is a group consisting of idempotents: We claim now that this implies that

$a^2 = b^2 = 0$. Suppose that $a^2 \neq 0$: Then $a^n = 0$ for some $n > 2$ and $\{a, a^2, \dots, a^{n-1}, a^n = 0\}$ and $\{b, b^2, \dots, b^{n-1}, b^n = 0\}$ are sets of n distinct elements with $a^i \notin E_S$ and $b^i \notin E_S$ for $i < n$. Now set $B = E_S \cup E_S \langle a^2, b^2 \rangle$. B is clearly a full inverse subsemigroup of S and $B \neq E_S$ since $a^2 \in B$; we produce a contradiction by showing that $a \notin B$, and so $B \neq S$. First note that if e is an idempotent of S other than 0 then $e = a^i b^{i+j} a^j$ for some $i, j \in \mathbb{Z}^+ \cup \{0\}$ such that $0 < i + j < n$. (This follows by a routine argument using the fact that the elements b, b^2, \dots, b^{n-1} are all distinct.) That $a \notin E_S \langle a^2, b^2 \rangle$ may be proved by an argument along the same lines as the argument used in Corollary 5.2. Thus $a^2 = b^2 = 0$ and so $S = \{a, b, ab, ba, 0\}$. One checks easily that the elements a, b, ab and ba are all distinct, and thus S is the five-element Brandt semigroup (with nonzero idempotents ab and ba).

LEMMA 5.6. *Let $S = \langle E_S, a, a^{-1} \rangle$ be an inverse semigroup with only trivial full inverse subsemigroups and suppose that $\langle a, a^{-1} \rangle = G$ is a cyclic group of prime order with identity $e_0 \in E_S$. Then $S = E_S \cup G$ with multiplication between E_S and G defined by*

$$\begin{aligned}
 & eg = ge = ee_0 \quad \text{if } e \in E_S, g \in G \text{ and } ee_0 \neq e_0, \\
 (*) \quad & eg = ge = g \quad \text{if } e \in E_S, g \in G \text{ and } ee_0 = e_0.
 \end{aligned}$$

PROOF. From Lemma 5.2 we know that $S = E_S \cup E_S G$. Before embarking on the proof we recall that the natural partial ordering on E_S is defined by $e \leq f$ iff $ef = fe = e$: We make liberal use of this partial ordering in the proof. Note that for each $e, f \in E_S$, $ef \leq e$ and $ef \leq f$.

We first claim that for each $e \in E_S$, $R_{ee_0} = \{ea^i : i = 1, \dots, p\}$, $L_{ee_0} = \{a^i e : i = 1, \dots, p\}$, and $D_{ee_0} = \{a^i e a^j : i, j = 1, \dots, p\}$. Clearly $\{ea^i : i = 1, \dots, p\} \subseteq R_{ee_0}$. Suppose now that $s \in R_{ee_0}$: If $s \in E_S$ then $s = ee_0$; if $s \notin E_S$ then $s = fa^i$, some $f \in E_S$, $i \in \{1, \dots, p\}$, and so $ee_0 = ss^{-1} = fa^i a^{-i} f = fe_0 f = fe_0$, so $s = fa^i = fe_0 a^i = ee_0 a^i = ea^i$. A similar argument shows that $L_{ee_0} = \{a^i e : i = 1, \dots, p\}$. It follows that the set of idempotents of D_{ee_0} is $\{a^i e a^{-i} : i = 1, \dots, p\}$, and if $x R a^i e a^{-i}$ for some i then $x = a^i e a^{-i} a^j = a^i e a^k$, some k . Thus $D_{ee_0} = \{a^i e a^j : i, j = 1, \dots, p\}$.

We now claim that in fact $D_{ee_0} = \{ee_0\}$ if $ee_0 \neq e_0$ (i.e., if $ee_0 < e_0$). Suppose that $D_{f_0 e_0} \neq \{f_0 e_0\}$ for some $f_0 \in E_S$ for which $f_0 e_0 \neq e_0$, and consider $B = \bigcup_{e \leq f_0 e_0} D_e$ and let $A = E_S \cup B$: We claim that A is a full inverse subsemigroup of S . It clearly suffices to show that A is a subsemigroup of S since if $x \in D_e$ then $x^{-1} \in D_e$. Let $a^i e f_0 e_0 a^j, a^k f f_0 e_0 a^l \in B$. Then

$$a^i e f_0 e_0 a^j a^k f f_0 e_0 a^l = a^{i+j+k} (a^{-j-k} e f_0 e_0 a^{j+k} f f_0 e_0) a^l \in B.$$

(Note that $a^{-i-k}ef_0e_0a^{i+k}ff_0e_0 \in E_S$ and $a^{-i-k}ef_0e_0a^{i+k}ff_0e_0 \leq f_0e_0$.) Also, if $f \in E_S$ and $a^ief_0e_0a^i \in B$, then $f(a^ief_0e_0a^i) = a^i(a^{-i}fa^i)ef_0e_0a^i \in B$ since $a^{-i}fa^ief_0e_0 \leq f_0e_0$; similarly $(a^ief_0e_0a^i)f \in B$, and thus A is a full inverse subsemigroup of S . Now $D_{f_0e_0} \neq \{f_0e_0\}$ and $D_{f_0e_0} \subseteq A$, so $A \neq E_S$. On the other hand $a \in D_{e_0}$ and $e_0 \not\leq e_0f_0$, so $a \notin A$, so $A \neq S$. This contradicts the fact that S has no full inverse subsemigroups other than E_S and S . Hence $D_{ee_0} = \{ee_0\}$ if $ee_0 \neq e_0$. Thus $a^iee_0 = a^ie = ee_0a^i = ea^i = ee_0 \forall i = 1, \dots, p$ provided $ee_0 \neq e_0$. Clearly, if $ee_0 = e_0$, then $a^ie = a^ie_0e = a^ie_0 = a^i = ea^i$. This proves the lemma.

LEMMA 5.7. Let $S = \langle E_S, a, a^{-1} \rangle$ be an inverse semigroup with only trivial full inverse subsemigroups and suppose that $\langle a, a^{-1} \rangle = T$ is the five-element Brandt semigroup $T = \{a, b, e_1, e_2, e_0\}$ (where $b = a^{-1}$, $e_1 = aa^{-1}$, $e_2 = a^{-1}a$, e_0 is the zero). Then $S = E_S \cup T$ with multiplication between E_S and T defined by

$$\begin{aligned}
 &ea = ee_1 \quad \text{if } ee_1 \neq e_1, & ae = ee_2 \quad \text{if } ee_2 \neq e_2, \\
 &= a \quad \text{if } ee_1 = e_1; &= a \quad \text{if } ee_2 = e_2; \\
 (**) & & \\
 &eb = ee_2 \quad \text{if } ee_2 \neq e_2, & be = ee_1 \quad \text{if } ee_1 \neq e_1, \\
 &= b \quad \text{if } ee_2 = e_2; &= b \quad \text{if } ee_1 = e_1.
 \end{aligned}$$

PROOF. This is similar in spirit to the proof of Lemma 5.6 so we provide only an outline of the proof. One checks first that for each $e \in E_S$, $R_{ee_1} = \{ee_1, ea\}$, $R_{ee_2} = \{ee_2, eb\}$, $L_{ee_1} = \{be, ee_1\}$, $L_{ee_2} = \{ee_2, ae\}$ and so $D_{ee_1} = \{ee_1, ea, be, bea\}$ and $D_{ee_2} = \{ee_2, eb, ae, aeb\}$. Since $a^2 = b^2 = e_0$ and since $S = E_S \cup E_S T$, it follows that $S = E_S \cup (\bigcup_{e < e_1} D_e) \cup (\bigcup_{e < e_2} D_e)$. One then proves, as in the proof of Lemma 5.6, that $D_{ee_1} = \{ee_1\}$ if $ee_1 \neq e_1$ and that $D_{ee_2} = \{ee_2\}$ if $ee_2 \neq e_2$. The result $S = E_S \cup T$ and the multiplication (**) then follow easily.

Motivated by the results of Lemmas 5.6 and 5.7 we introduce two semigroups $F(E, e_0, G)$ and $F(E, e_0, e_1, e_2, a, b)$ as follows:

(1) Let E be a semilattice, e_0 a fixed element of E and G a group with identity e_0 : We define $F(E_1, e_0, G) = E \cup G$ with multiplication inherited by that in E and G and multiplication between elements of E and elements of G defined by (*) (in Lemma 5.6).

(2) Let E be a semilattice, e_0, e_1 and e_2 distinct fixed elements of E for which $e_0 \leq e_1$, $e_0 \leq e_2$ and let a, b be elements not in E : let $T = \{a, b, e_0, e_1, e_2\}$ be the five-element Brandt semigroup with $b = a^{-1}$, $e_1 = ab$, $e_2 = ba$ and e_0 as zero. Now define $F(E, e_0, e_1, e_2, a, b) = E \cup T$ with

multiplication inherited by that in E and T and multiplication between elements of E and elements of T defined by (**) (in Lemma 5.7).

Combining all of the results of this section we have the following theorem.

THEOREM 5.1. *An inverse semigroup S has no proper full inverse subsemigroups (other than E_S and S) iff either*

(i) $S \cong F(E, e_0, e_1, e_2, a, b)$ for some semilattice E and elements $e_0, e_1, e_2 \in E$ and $a, b \notin E$; or

(ii) $S \cong F(E, e_0, G)$ for some semilattice E , some $e_0 \in E$ and some cyclic group G of prime order.

REMARKS. (1) Semilattices and cyclic groups of prime order are special cases of the semigroup $F(E, e_0, G)$.

(2) Theorem 5.1 also characterizes inverse semigroups S which have no idempotent-separating left (right) congruences between i_S and $R(L)$.

(3) Theorem 5.1 suggests that a general study of the connection between the lattice of full inverse subsemigroups of an inverse semigroup and the structure of the semigroup might prove rewarding.

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