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One-Sided Cumulative Sum (CUSUM) Control Charts for the Erlang-Truncated Exponential Distribution

G. Srinivasa Rao

Department of Statistics, School of Mathematical Sciences, University of Dodoma
Dodoma, Tanzania, PO Box:259
E-mail: gaddesrao@yahoo.com

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Abstract: In this article, we construct one-sided cumulative sum (CUSUM) control charts for controlling the parameters of a random variable with erlang-truncated exponential distribution. The rejection of the Wald's sequential probability ratio test (SPRT) is viewed as the decision lines of a CUSUM control chart for which the variate is a quality characteristic. Parameters of the CUSUM chart, e.g. lead distance and mask angle, are presented. The results show that the Average Run Length (ARL) of the resulting control charts changes substantially for a slight shift in the parameters of the distribution.

Key words: Sequential Probability Ratio Test (SPRT), Cumulative Sum (CUSUM) Control Chart, Average Run Length (ARL), erlang-truncated exponential distribution

I. INTRODUCTION

In statistical quality control the cumulative sum control charts (CUSUM Charts) have found importance as a parallel process control technique to the well-known Shewhart control charts. An alternative method for testing statistical hypothesis parallel to Neyman's theory is the popular sequential probability ratio test (SPRT) due to Wald (1942). Page (1954, 1961) suggested the cumulative sum charts which are more effective than Shewhart control charts in detecting small and moderate size departures from a simple acceptable quality level [Montgomery (2001)]. In this article we develop one-sided CUSUM control charts for erlang-truncated exponential distribution to detect the shift of the process parameters. We have also examined how the parameters of the V-mask are influenced by the probability of defectives departure from its target value and the average run length (ARL) of the CUSUM scheme. Johnson (1961) introduced this method of construction of the CUSUM chart. Johnson and Leone (1962) made use of simultaneous applications of SPRT to test a simple H_0 against two separate simple alternative hypotheses on either side of the null hypothesis. The concerned decision lines for both cases of alternative hypotheses to come to the respective rejection of the null hypothesis are taken to construct a cumulative sum control chart for a process variate assumed to follow a normal distribution. Johnson (1966) extended the same procedure to a CUSUM chart for the Weibull process variate. Nabar and Bilgi (1994) extended the CUSUM chart procedure to the case of inverse Gaussian distribution. Kantam and Rao (2006) studied the cumulative sum control chart for log-logistic distribution. Chakraborty and Khurshid (2011) constructed one-sided cumulative sum (CUSUM) control charts the zero-truncated binomial distribution.

The rest of the paper is organized as follows: the CUSUM chart for control of parameter λ when ν is known along with ARL is given in Section 2. The CUSUM chart for control of parameter ν when λ is known along with ARL is shown in Section 3, and conclusions are listed in Section 4.

II. THE CUSUM CHART FOR CONTROL OF PARAMETER λ WHEN ν IS KNOWN

The Erlang-truncated exponential (ETE) distribution was introduced and studied by El-Alosey (2007), Mohsin (2009) and Mohsin et al (2010). The erlang-truncated exponential distribution has the following density function

$$f(x;\nu,\lambda)=\nu(1-e^{-\lambda})\,\exp\left[-\nu x(1-e^{-\lambda})\right];\quad \text{for } x>0 \tag{1}$$

and the distribution function

$$F(x;\nu,\lambda)=1-\exp\left[-\nu x(1-e^{-\lambda})\right];\quad \text{for } x>0. \tag{2}$$

Here $\nu>0$ and $\lambda>0$ are the shape and scale parameters, respectively. Now onwards ETE distribution with the shape parameter ν and scale parameter λ will be denoted by ETE(ν,λ).
The mean and variance of ETE (ν,λ) are $\left[\nu(1-e^{-\lambda})\right]^{-1}$ and $\left[\nu(1-e^{-\lambda})\right]^{-2}$, respectively.
If we assume that $X_1, X_2, ...X_m$ be i.i.d. random variables taken from ETE distribution with the probability density function (1). The likelihood ratio to test the null hypothesis $H_0:\lambda=\lambda_0$ against the alternative hypothesis $H_1:\lambda=\lambda_1\,(>\lambda_0)$ with known ν is given by

$$\frac{L_1}{L_0}=\frac{f_{X_1,X_2,...X_m}(x_1,x_2,...x_m;\lambda_1,\nu)}{f_{X_1,X_2,...X_m}(x_1,x_2,...x_m;\lambda_0,\nu)}= =\left(\frac{1-e^{-\lambda_1}}{1-e^{-\lambda_0}}\right)^m \exp\left[\nu\left(e^{-\lambda_1}-e^{-\lambda_0}\right)\sum_{i=1}^m x_i\right] \tag{3}$$

The continuation region of the sequential probability ratio test (SPRT) discriminating between the two hypotheses is given by

$$\ln\left(\frac{\beta}{1-\alpha}\right)<m\ln\left(\frac{1-e^{-\lambda_1}}{1-e^{-\lambda_0}}\right) +\nu\left(e^{-\lambda_1}-e^{-\lambda_0}\right)\sum_{i=1}^m x_i<\ln\left(\frac{1-\beta}{\alpha}\right) \tag{4}$$

where α and β are the probability of Type I and Type II errors, respectively.
If $\beta=0$ then the right side of inequality in (4) becomes

$$\sum_{i=1}^m x_i<\frac{\ln\alpha+m\ln\left(\frac{1-e^{-\lambda_1}}{1-e^{-\lambda_0}}\right)}{\nu\left(e^{-\lambda_0}-e^{-\lambda_1}\right)} \tag{5}$$

The cumulative sum (CUSUM) control chart (as shown in Fig. 1) is constructed as follows: The CUSUM control chart is formed by plotting the sum $S_m=\sum_{i=1}^m x_i$ against the number of observations m. A visual procedure with aid of the V-Mask is sometimes used to determine whether the process is under control or out of control (see Fig. 1). A V-Mask is an overlay shape in the form of a V on its side that is superimposed on the graph of the cumulative sums. The origin point of the V-Mask (see Fig. 1) is placed on top of the latest cumulative sum point and past points are examined to see if there was any fall above or below the sides of the V. As long as all the previous points lie between the sides of the V, the process is in control. Otherwise (even if one point lies outside) the process is suspected of being out of control. Suppose here O is the last plotted point, P is the vertex of the mask and the point Q is obtained by drawing a perpendicular to the line OP. A small shift in the value of λ from λ_0 to λ_1 is indicated if any plotted point falls below the line PQ. Lucas (1982) proposed a control chart which consists of control lines rather than a V-mask, in our study; we prefer to use the original V-mask procedure.

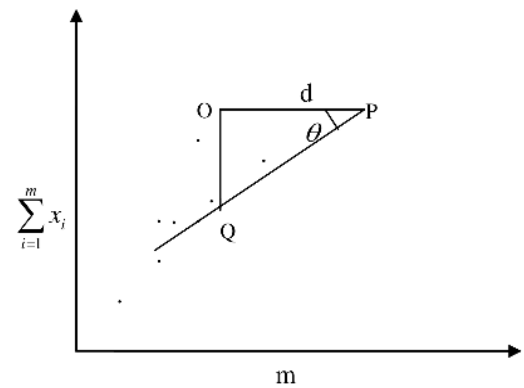

Fig. 1. Cumulative sum control chart

The parameters d and θ of V-mask from Fig. 1 are given by

$$d=\frac{-\ln\alpha}{\ln\left(\frac{1-e^{-\lambda_1}}{1-e^{-\lambda_0}}\right)} \tag{6}$$

and

$$\theta=\tan^{-1}\left[\frac{\ln\left(\frac{1-e^{-\lambda_1}}{1-e^{-\lambda_0}}\right)}{\nu\left(e^{-\lambda_0}-e^{-\lambda_1}\right)}\right] \tag{7}$$

II. 1. Average Run Length (ARL)

In general, the average number of trials required for detecting a shift in the process average for the first time is called the average run length (ARL). If α is the producer's risk then the approximate formula for ARL for the CUSUM control chart [see also Johnson (1961), Johnson and Leone (1962)] detecting a shift of the parameter from λ_0 to λ_1 is given by

$$\text{ARL}=\frac{-\log\alpha}{E\left(\log Z\right)_{\lambda=\lambda_1}},$$

where $Z=\dfrac{f(x;\lambda_1)}{f(x;\lambda_0)}$. Thus, we get

$$\text{ARL}=\frac{-\log\alpha}{\ln\left(\frac{1-e^{-\lambda_1}}{1-e^{-\lambda_0}}\right)-\left(\frac{e^{-\lambda_0}-e^{-\lambda_1}}{1-e^{-\lambda_1}}\right)} \tag{8}$$

The parameters of the V-mask of the one-sided CUSUM chart, such as the lead distance d and the angle θ, are calculated for a number of combinations of the values of λ, ν and α for controlling parameters λ when ν is known. The obtained values of d, θ and ARL for controlling parameter λ when ν is known are displayed in the Tab. 1, 2 and 3 respectively.

III. THE CUSUM CHART FOR CONTROL OF THE PARAMETER ν WHEN λ IS KNOWN

The CUSUM chart to control parameter ν when λ is known is constructed in the following procedure. The approximate likelihood ratio for detecting a shift in the value of ν from ν_0 to ν_1 $(> \nu_0)$is given by

$$\frac{L_1}{L_0} = \frac{f_{X_1,X_2,...X_m}(x_1, x_2, ...x_m; \nu_1, \lambda)}{f_{X_1,X_2,...X_m}(x_1, x_2, ...x_m; \nu_0, \lambda)} = = \left(\frac{\nu_1}{\nu_0}\right)^m \exp\left[-(\nu_1 - \nu_0)\left(1 - e^{-\lambda}\right)\sum_{i=1}^{m} x_i\right] \quad (9)$$

Proceeding like in Section 2, we get the inequality below hold true for ν

$$\sum_{i=1}^{m} x_i < \frac{\ln\alpha + m\ln(\nu_1/\nu_0)}{(\nu_1 - \nu_0)(1 - e^{-\lambda})} \quad (10)$$

The CUSUM chart is constructed in the similar manner as we explained in Section 2 for change of ν from ν_0 to ν_1 $(> \nu_0)$. The parameters of the V-mask (the lead distance d and the angle θ) are given by

$$d = \frac{-\ln\alpha}{\ln(\nu_1/\nu_0)} \quad (11)$$

and

$$\theta = \tan^{-1}\left[\frac{\ln(\nu_1/\nu_0)}{(\nu_1 - \nu_0)(1 - e^{-\lambda})}\right] \quad (12)$$

III. 1. Average Run Length (ARL)

The ARL for detecting a change in ν from ν_0 to ν_1 $(> \nu_0)$ is approximately given by

$$\text{ARL} = \frac{-\log\alpha}{E(\log Z)_{\nu=\nu_1}},$$

where $Z = \frac{f(x;\lambda,\nu_1)}{f(x;\lambda,\nu_0)}$. Thus, we get

$$\text{ARL} = \frac{-\log\alpha}{\ln(\nu_1/\nu_0) + \frac{\nu_0}{\nu_1} - 1}. \quad (13)$$

The parameters of the V-mask of the one-sided CUSUM chart, such as lead distance d and angle θ are calculated for a number of combinations of the values of λ, ν and α for controlling the parameters ν when λ is known. The obtained values of d, θ and ARL for controlling the parameter ν when λ is known are displayed in the Tab. 4, 5 and 6 respectively.

IV. CONCLUSIONS

It is noticeable from Tab. 1 for all the combinations of λ and fixed α that the values of d are independent from shape parameter ν and decrease as the ratio λ_1/λ_0 increases, whereas for constant ratio λ_1/λ_0, the values of d increases as α increases for controlling parameter λ. Tab. 2 indicates that angle θ of the V-mask decreases as the ratio λ_1/λ_0 increases and for constant ratio λ_1/λ_0 the angle decreases as ν increases for controlling parameter λ.

Tab. 1. Values of d for controlling parameter λ when ν is known

λ_0	λ_1	α				
		0.05	0.025	0.01	0.005	0.001
0.50	0.55	41.33	50.89	63.53	73.09	95.30
0.50	0.60	21.89	26.95	33.64	38.71	50.47
0.50	0.65	15.40	18.96	23.68	27.24	35.51
0.50	0.70	12.16	14.97	18.69	21.50	28.03
0.50	0.75	10.21	12.57	15.70	18.06	23.54
0.50	0.80	8.91	10.97	13.70	15.76	20.55
0.50	0.85	7.99	9.83	12.28	14.12	18.41
0.50	0.90	7.29	8.98	11.21	12.89	16.81
0.50	0.95	6.75	8.31	10.38	11.94	15.57
0.50	1.00	6.32	7.78	9.71	11.18	14.57

Tab. 3 depicts the values of ARL (an average number of observations required to detect the shift of the process parameter) for different combinations of α, ν, λ_0 and λ_1. Here it is interesting to note that ARL is independent of shape parameter ν, similar as in the case of lead distance d. It seems to be evident from Tab. 3 that for fixed α and λ_0 the ARL decreases as λ_1 increases (or the ratio increases), and for fixed values λ_0 and λ_1 the ARL increases as α decreases for controlling parameter λ.

Tab. 4 shows that for all the combinations of ν and fixed α, the values of d are independent of scale parameter λ and decrease as the ratio ν_1/ν_0 increases, whereas for constant ratio ν_1/ν_0, the values of d increases as α increases for controlling parameter ν. Tab. 5 indicates that angle θ of the V-mask decreases as the ratio ν_1/ν_0 increases and for constant ratio ν_1/ν_0 the angle decreases as λ increases for controlling parameter ν.

Tab. 6 describes the values of ARL (an average number of observations required to detect the shift of the process parameter) for different combinations of α, λ, ν_0 and ν_1. Here also ARL is independent of scale parameter λ, similar as in the case of lead distance d. It is clearly evident from Tab. 6 that for fixed α and ν_0 the ARL decreases as ν_1 increases (or the ratio increases), and for fixed values ν_0 and ν_1 the ARL increases as α decreases for controlling parameter ν.

Moreover, in comparison to angle θ, the ARL and the values of d for controlling λ differ from those for controlling ν.

From Tables 1 and 4, we have observed that the values of d for controlling the parameter λ, when ν is known, and we have concluded that d follows the same trends as in the case of controlling ν when λ is known but that the values of d for controlling ν are smaller than for controlling λ. The similar trend follows for angle θ and also the ARL values. Among these two CUSUM control charts the values of parameters d, θ and ARL for controlling ν are uniformly smaller than for controlling λ.

Tab. 2. Values of θ for controlling parameter λ when ν is known

λ_0	λ_1	ν								
		0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.50	0.55	76.24	75.14	74.06	72.98	71.92	70.87	69.83	68.81	67.80
0.50	0.60	75.80	74.67	73.56	72.45	71.36	70.28	69.22	68.17	67.14
0.50	0.65	75.39	74.23	73.09	71.96	70.84	69.74	68.65	67.58	66.52
0.50	0.70	75.01	73.83	72.65	71.50	70.36	69.23	68.12	67.03	65.95
0.50	0.75	74.66	73.45	72.25	71.07	69.91	68.76	67.63	66.52	65.43
0.50	0.80	74.33	73.09	71.87	70.67	69.49	68.32	67.17	66.04	64.94
0.50	0.85	74.01	72.76	71.52	70.30	69.10	67.91	66.75	65.60	64.48
0.50	0.90	73.72	72.45	71.19	69.95	68.73	67.53	66.35	65.19	64.05
0.50	0.95	73.45	72.16	70.88	69.62	68.39	67.17	65.98	64.80	63.65
0.50	1.00	73.19	71.88	70.59	69.32	68.06	66.83	65.63	64.44	63.28

Tab. 3. Values of ARL for controlling parameter λ when ν is known

λ_0	λ_1	α				
		0.05	0.025	0.01	0.005	0.001
0.50	0.55	1167.98	1438.22	1795.47	2065.71	2693.20
0.50	0.60	334.53	411.93	514.25	591.65	771.38
0.50	0.65	168.79	207.84	259.47	298.53	389.21
0.50	0.70	106.94	131.69	164.40	189.14	246.60
0.50	0.75	76.57	94.29	117.71	135.43	176.56
0.50	0.80	59.13	72.81	90.90	104.58	136.35
0.50	0.85	48.06	59.17	73.87	84.99	110.81
0.50	0.90	40.51	49.88	62.27	71.64	93.40
0.50	0.95	35.08	43.20	53.93	62.05	80.90
0.50	1.00	31.03	38.21	47.70	54.88	71.56

Tab. 4. Values of d for controlling parameter ν when λ is known

ν_0	ν_1	α				
		0.05	0.025	0.01	0.005	0.001
0.60	0.65	37.43	46.09	57.53	66.19	86.30
0.60	0.70	19.43	23.93	29.87	34.37	44.81
0.60	0.75	13.43	16.53	20.64	23.74	30.96
0.60	0.80	10.41	12.82	16.01	18.42	24.01
0.60	0.85	8.60	10.59	13.22	15.21	19.83
0.60	0.90	7.39	9.10	11.36	13.07	17.04
0.60	0.95	6.52	8.03	10.02	11.53	15.03
0.60	1.00	5.86	7.22	9.02	10.37	13.52

Tab. 5. Values of θ for controlling parameter ν when λ is known

ν_0	ν_1	λ										
		0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.60	0.65	76.19	75.20	74.26	73.38	72.54	71.76	71.02	70.32	69.66	69.04	68.45
0.60	0.70	75.68	74.65	73.69	72.77	71.91	71.10	70.34	69.62	68.94	68.31	67.70
0.60	0.75	75.18	74.13	73.13	72.19	71.30	70.47	69.69	68.95	68.25	67.60	66.98
0.60	0.80	74.70	73.61	72.58	71.62	70.71	69.86	69.05	68.29	67.58	66.91	66.28
0.60	0.85	74.23	73.11	72.06	71.07	70.13	69.26	68.43	67.66	66.93	66.24	65.60
0.60	0.90	73.77	72.62	71.54	70.52	69.57	68.67	67.83	67.04	66.29	65.59	64.93
0.60	0.95	73.32	72.14	71.03	70.00	69.02	68.11	67.25	66.44	65.68	64.96	64.29
0.60	1.00	72.88	71.67	70.54	69.48	68.49	67.55	66.67	65.85	65.08	64.35	63.17

Tab. 6. Values of ARL for controlling parameter ν when λ is known

ν_0	ν_1	α				
		0.05	0.025	0.01	0.005	0.001
0.60	0.65	960.28	1182.47	1476.19	1698.38	2214.29
0.60	0.70	265.26	326.64	407.77	469.15	611.66
0.60	0.75	129.44	159.39	198.98	228.93	298.47
0.60	0.80	79.50	97.89	122.21	140.61	183.32
0.60	0.85	55.28	68.07	84.98	97.77	127.48
0.60	0.90	41.53	51.14	63.84	73.45	95.77
0.60	0.95	32.88	40.49	50.54	58.15	75.82
0.60	1.00	27.03	33.29	41.55	47.81	62.33

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Gadde Srinivasa Rao was born on 25th August, 1965 in Guntur, AP, India. He received the BSc (1985), MSc (1988), M.Phil. (1994) and PhD (2002) in Statistics from the Acharya Nagarjuna University, Guntur, India. He is presently working as Professor of Statistics at the Department of Statistics, University of Dodoma, Tanzania. His research interests include statistical inference, statistical process control, acceptance sampling plans and reliability estimation. He boasts more than 50 publications in different peer-reviewed journals.

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