One-sided Markov additive processes and related exit problems

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Lévy processes, that is, processes with stationary and independent increments, have become a classical model in applied probability. Many real-world problems, however, exhibit non-stationary behavior in longer time intervals. One may think about seasonality of prices, recurring patterns of activity, burst arrivals, occurrence of events in phases and so on. This motivates the interest in regime-switching models, where the process under consideration is modulated by an exogenous background process. Markov Additive Processes (MAPs) form a natural generalization of Lévy processes to regime-switching models. The focus of this book is on the path properties of MAPs. Both MAPs and their reflections at constant boundaries are considered. We address the basic exit problems, such as first passage over a level and first exit from an interval. We restrict ourselves to the one-sided case, where all the jumps have the same sign; nevertheless arbitrary phase-type jumps can be easily incorporated into the model. Most of the results appear in the form of Laplace transforms.

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Faculteit der Natuurwetenschappen, Wiskunde en Informatica

‘Experience is not what happens to you.
It is what you do with what happens to you.’
Aldous Huxley

‘Consistency is contrary to nature, contrary to life.
The only completely consistent people are the dead.’
Aldous Huxley

Acknowledgments

This book resulted from the research carried out during the four years of my PhD studies. It will always remind me of this amazing period of my life, great experiences, and, certainly, people I met on my way. I would like to express my gratitude to all of you who were there, who counseled, supported and kept me motivated, who shared my free time, my joy and adventures, and to those who loved. At this very point I feel fortunate, content and happy. Thank you all for that!

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Thank you all,
Jevgenij
15 July 2011

Contents

Chapter 1

Introduction

The Compound Poisson Process (CPP) is one of the most basic and popular models in applied probability. It can represent a workload arrival process, where customers (or requests) arrive in a memoryless manner and bring independent and identically distributed (i.i.d.) amounts of work. In risk theory the jumps of a CPP can be interpreted as, for example, claims arriving at an insurance company. A CPP is one of the simplest examples of a Lévy process, that is, a process with stationary and independent increments. Brownian Motion is another well-known example. Lévy processes allow for greater flexibility in modeling real-life phenomena compared to CPPs, and at the same time often the associated problems turn out to remain tractable. Our main textbook references concerning Lévy processes are Bertoin [1996] and Kyprianou [2006].

The stationarity property of the Lévy process is somewhat restrictive, because many real-world problems exhibit non-stationary behavior in longer time intervals. One may think about seasonality of prices, recurring patterns of activity, burst arrivals, occurrence of events in phases and so on. This motivates so-called *regime-switching* models, where the process of interest X is *modulated* by an exogenous background process J, which represents different modes or phases of activity, see Figure 1.1 for an illustration. It is common to assume that J is a continuous-time Markov chain with a finite state space; we also make this assumption throughout this work.

Markov Additive Processes (MAPs) form a natural generalization of Lévy processes to regime-switching models. A MAP is a bivariate Markov process (X, J), such that the *additive component* X has stationary and independent increments

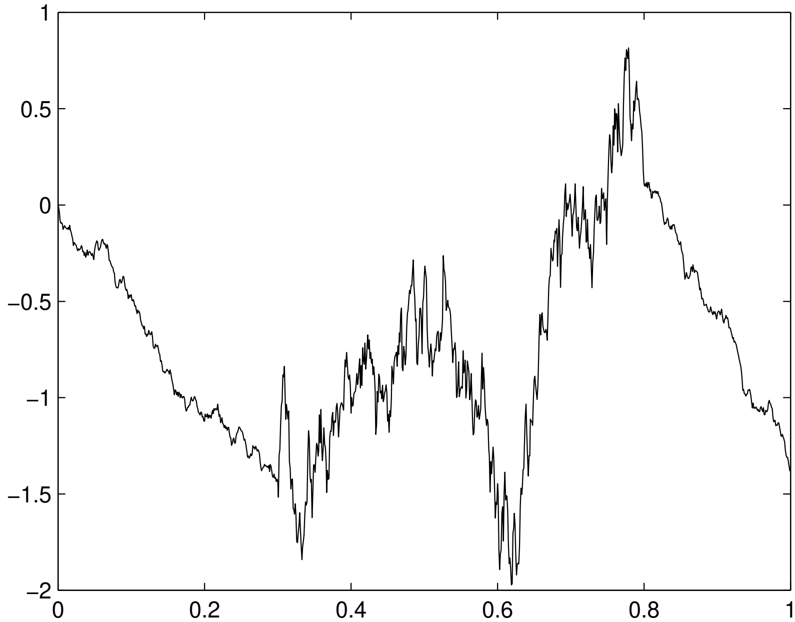

Figure 1.1 Illustration of regime-switching (J jumps at 0.3 and 0.8).

given the state of the background process J. From another point of view, as will be shown later, a MAP is a Markov-modulated Lévy process. That is, X evolves as a Lévy process for which the parameters change in time according to the background process J. In addition, phase changes (jumps of J) may induce jumps of X. The use of MAPs is widespread, making it a classical model in applied probability with a variety of application areas, such as queues, insurance risk, inventories, data communication, finance, environmental problems and so forth. The focus of this thesis is on the path properties of MAPs. We believe that our results can be best explained and demonstrated using the terminology of queueing theory and risk theory. A textbook introduction to MAPs can be found in Asmussen [2003, Ch. XI]; see also Prabhu [1998, Ch. 7].

In many applications it is natural to assume that all the jumps of the process X have the same sign. That is, X has either positive jumps (arrivals of work in a queue), or negative jumps (arrivals of claims to an insurance company). Throughout this work we assume that there are *no positive jumps*; the case of no negative jumps can be dealt with in a symmetric fashion. To avoid trivialities we exclude

MAPs whose additive component X has almost surely (a.s.) non-increasing paths. Processes satisfying these assumptions are said to be *spectrally negative*. They form an important special case, which leads to tractable exit problems and rather explicit solutions to those. It is noted that textbooks on Lévy processes often contain a chapter treating the spectrally negative case, see Bertoin [1996, Ch. VII] and Kyprianou [2006, Ch. 8]. The present thesis generalizes some of the results contained in these chapters to the MAP setting.

In this work we consider various exit problems for spectrally negative MAPs. The fundamental one, which underlies the other problems, is to characterize the *first passage time* of X over a positive level. Another quantity of interest is the first passage time of X over a negative level jointly with the corresponding *overshoot*, that is, the amount by which the level is exceeded when crossed by a jump. The two-sided exit, such as *exit from an interval over the upper boundary*, is yet another important problem considered in this book. It concerns the probability of exiting over the upper boundary and the time of such an exit given that it occurs. This problem is complemented by the two-sided exit over a lower boundary, where additionally the corresponding overshoot is of interest. The so-called *scale matrix*, which is the matrix analogue of the scale function of a Lévy process, plays an important role in the study of these problems.

The concept of reflection at a certain boundary is important in applications. For example, $-X$ reflected at 0 can serve as a model of workload evolution in a queue; here we use $-X$ to get the spectrally positive case, which is natural in the queueing setting. By adding another reflecting barrier at a level $B > 0$ one can model a queue with a finite buffer of size B. The first passage time over the level B can be interpreted as the first time of buffer overflow in this queue. Another interesting interpretation comes from risk theory. Consider a risk process subject to dividend payments. It is assumed that a classical barrier strategy is used, where dividends are paid continuously at a certain barrier B. Paying dividends essentially amounts to reflecting the risk process X at the level B. It is then important to characterize the time of bankruptcy (first passage over 0), the debt (overshoot over 0) and the amount of dividends paid up to the bankruptcy time. It is noted that X reflected at B can be seen as $-X$ reflected at 0 by flipping the picture upside down. The problems of the first passage of X and $-X$ reflected at 0 are also addressed in this work. Most of the results appear in the form of Laplace transforms.

So-called *PHase-type* (PH) distributions fit naturally into the framework of MAPs. A random variable is said to be of phase-type if it is distributed as the life time of some transient finite-state continuous-time Markov chain, see Asmussen [2003, Ch. III]. The main examples are the exponential, Erlang and hyperexponential distributions. Importantly, a MAP with positive PH jumps (these can be jumps of underlying Lévy processes or jumps at phase changes) can be reduced to a MAP with no positive jumps without losing any information. This procedure is called *fluid embedding*. Informally, it involves enlarging the state space of the background process J and replacing the jumps of X by linear stretches of unit slope. Hence the results of this thesis can be used to analyze MAPs with upward jumps of phase-type and arbitrary downward jumps. Furthermore, if the jumps in both directions are PH then the analysis can be reduced to MAPs with continuous sample paths. It is noted that a MAP has continuous sample paths if and only if it is a *Markov-Modulated Brownian Motion* (MMBM). Roughly speaking, MMBM is a process with piecewise Brownian paths with drift and variance parameters determined by the background process J. The variance parameters are allowed to be zero, in which case the corresponding pieces are linear. An MMBM is also called a *second-order fluid model* in the literature. It is noted that in the case of MMBM many results simplify considerably and become more explicit.

In Section 1.1 we present an outline of this thesis. The main results of this work are summarized in Section 1.2.

1.1 Outline

To appreciate the results of this monograph, it is helpful to have a working knowledge of Lévy processes and their path properties. In addition, experience with problems coming from queueing theory or risk theory may facilitate understanding of some basic concepts and techniques. Deep knowledge of these subjects is, however, not essential. In order to make this work self-contained and accessible to a wider audience, we start by presenting some basic theory in Chapter 2. We define a MAP and associate to it a matrix exponent, which characterizes the law of the process. In addition, we discuss the concepts of killing, time-reversal, and reflection, define the first passage problem, and elaborate on the phase-type method.

The following two chapters are devoted to the fundamental problem of the first passage time over a positive level. In Chapter 3 we consider an important special case when the MAP is *time-reversible*. This special case is substantially easier to analyze. The ideas and results, however, provide a good introduction to the general case treated in Chapter 4. The latter chapter can be seen as a foundation for the rest of the thesis. It relies on the theory of analytic matrix functions, which is summarized in the beginning of the chapter. The main results and their proofs are followed by applications, which concern Markov-modulated queues. So, for example, we present a generalization of the celebrated Pollaczek-Khintchine formula to the MAP setting.

Chapter 5 and Chapter 6 are devoted to the special case of an MMBM. In other words, the additive component of the MAP is assumed to have continuous sample paths. In Chapter 5 we present some preliminary results concerning MMBM by applying the theory of Chapter 4 to both the original process and its negative, which is possible as both are spectrally negative. These results are then used to study the workload process of an MMBM-driven queue with a finite buffer. We identify the stationary distribution of the workload process and the so-called *loss vectors*, which can be interpreted as expected overflow and unused capacity in a unit of time in stationarity. In addition, we solve the two-sided exit problem. Chapter 6 presents some further properties of an MMBM reflected to stay in a strip. We fully characterize this model at inverse local times, which for example yields identities for the first passage problem of a reflected MMBM.

Chapter 7 and Chapter 8 take the theory of two-sided exit and reflection to the next level by considering general spectrally negative MAPs. A fundamental role in this theory is played by scale matrices, which generalize scale functions associated with Lévy processes. Chapter 7 is entirely devoted to the construction of the scale matrix and the identification of its first properties. An important role in this construction is played by so-called *occupation densities*. Using scale matrices, in Chapter 8 we solve the problems of first passage over a negative level as well as two-sided exit. We characterize the first passage process killed upon arrival of an excursion exceeding a certain height. We extend the results of Chapter 6 and derive identities for the first passage of the reflected process.

Throughout this book we extensively use matrices, functions of matrices, Jordan forms, analytic functions and Laplace transforms. For completeness and convenience we gather the corresponding theory in the Appendix. The notation of this book may look puzzling at first glance despite all the efforts of the author to make it as transparent and convenient to use as possible. Getting acquainted with notation and conventions is facilitated by the 'List of Symbols' and 'Index'. Finally, the references to related work are given in the individual chapters. I would like to apologize to all those authors, whose work should have been mentioned in this thesis, but has remained unknown to me at the completion of this manuscript.

1.2 Contribution

A MAP is a generalization of a Lévy process with many analogous properties and characteristics. Some of the results on MAPs can be obtained mechanically by repeating the proofs for the case of a Lévy process. Scalars and functions are replaced by their multidimensional analogs such as matrices and matrix-valued functions. This is, however, not always the case. Various new mathematical objects appear in the theory of MAPs, posing new challenges. Often some details which go unnoticed in the case of a Lévy process, because of their triviality, become important issues, which require novel ideas and lead to better understanding of the problem at hand. Consider, for example, the first passage process. In the Lévy case it is a killed Lévy process with the Laplace exponent $-\Phi(q)$. In the MAP case it is a MAP with matrix exponent $\Lambda(q)$. The function $\Phi(q)$ is a non-negative increasing function, whereas $\Lambda(q)$ is the transition rate matrix of a certain Markov chain for each $q \geq 0$. Moreover, the case when $\Phi(0) = 0$ corresponds to $\Lambda(0)$ being recurrent. This is a very simple example of the above statement. The reader will encounter many others while reading the book.

This book is based on a number of research papers: Ivanovs and Mandjes [2010], Ivanovs et al. [2010], D'Auria et al. [2010], Ivanovs [2010], D'Auria et al. [2012], Ivanovs [2011], and Ivanovs and Palmowski [2011]. The most important contributions of this work are the following.

- We show in Chapter 4 that the theory of *analytic matrix functions* plays a fundamental role in explaining the relation between the first passage process and the original process. This results in an explicit construction of the matrix exponent $\Lambda(q)$ of the first passage process, and leads in a simple and direct way to the celebrated matrix integral equation. It turns out that the theory of analytic matrix functions is exactly 'that missing component' in many previous works which did not allow to treat various problems about MAPs in their general form.
    - We provide a number of alternative approaches to the first passage problem.
    - We analyze the (closely related) generalized Cramér-Lundberg equation. We determine the number of roots of this equation using entirely analytic arguments. This extends and unifies various partial results found in the literature.
    - An alternative simple analysis of the time-reversible case is given in

Chapter 3. It leads to important additional properties of $\Lambda(q)$ in this special case.

- We derive a number of results concerning MMBM in Chapter 5.
  - We identify the so-called loss vectors corresponding to the two-sided reflection of a general MMBM, which generalizes the results of Kella and Stadje [2004].
  - We provide a simple approach to the two-sided exit problem for an MMBM, see also Jiang and Pistorius [2008]. Additionally, we solve the delicate case when the asymptotic drift is zero.
  - Using a simple probabilistic argument we identify the stationary distribution of an MMBM reflected at two barriers. We also present relations between different approaches to this problem.
- In Chapter 7 we construct a scale matrix $W(x)$ for a general spectrally negative MAP. This result is based on a number of novel ideas. So-called occupation densities (also called local times) play a fundamental role in this chapter. We show that $e^{\Lambda x}W(x)$ can be interpreted as the expected local time at zero up to the first passage over level x. This observation is essential in order to establish various properties of $W(x)$.
- We characterize the two-sided reflection of a spectrally negative MAP at inverse local times at the upper boundary. This method, in its simpler form, is first used in the case of an MMBM in Chapter 6. In addition, in Chapter 8 we analyze the first passage process killed upon arrival of an excursion exceeding a certain height. These results are then combined to obtain identities for the first passage of reflected processes.

Chapter 2

Basic theory

This chapter provides some introduction to the theory of MAPs. We set up the notation and present some related concepts. We start with a definition and some basic properties of a Lévy process, and then proceed to MAPs. We present such fundamental notions as the matrix exponent of a MAP, its Perron-Frobenius eigenvalue, the asymptotic drift, and the first passage process. We discuss the concepts of killing, time reversal, reflection, and fluid embedding. Our basic reference book with respect to MAPs and queueing theory is Asmussen [2003], with Asmussen and Kella [2000] being an important supplement to it.

Throughout this work we use bold symbols to denote column vectors unless otherwise specified. In particular, $\mathbf{1}$ and $\mathbf{0}$ are the vectors of 1's and 0's respectively. A coordinate vector with i-th element being 1 and all others being 0 is denoted through $\boldsymbol{e}_i$. The symbols $\mathbb{I}$ and $\mathbb{O}$ denote the identity matrix and the matrix of 0s of appropriate dimensions. The typeface $\mathbb{P}$ and $\mathbb{E}$ are used for probability and expectation. The expression $\mathbb{E}[X; A]$ means $\mathbb{E}X1_A$, where 1_A is the indicator of an event A (if say $A = \{X > 0\}$, we write $\mathbb{E}[X; X > 0]$). The Laplace transform of a random variable X means $\mathbb{E}e^{-\alpha X}$. The distribution of X is allowed to have atoms; we do not use the term Laplace-Stieltjes transform to distinguish this case. We often deal with negative random variables, e.g. jumps of a MAP, in which case it is convenient to write $\mathbb{E}e^{\alpha X}$. The meaning should be clear from the context. We provide a 'List of Symbols' for further convenience.

2.1 Lévy processes

Our main reference book concerning Lévy processes is Bertoin [1996]. A good introduction for a less experienced reader is Kyprianou [2006]. The books Williams [1991] and Kallenberg [2002] serve as a general reference to probability theory.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a right-continuous complete filtration $(\mathcal{F}_t)_{t\geq 0}$. Consider an adapted real-valued stochastic process $X(t), t \geq 0$ with càdlàg (right-continuous with left limits) sample paths.

Definition 2.1. We say that X is a Lévy process if for every $s, t \geq 0$ the increment $X(t+s) - X(t)$ is independent of $\mathcal{F}_t$ and has the same law as $X(s) - X(0)$.

It is usually assumed that $X(0) = 0$. We often write $\mathbb{P}_x$ to denote the law of X when $X(0) = x$. Even though the paths of a Lévy process are not a.s. continuous in general, it can be shown that for any fixed $t > 0$ a Lévy process X is a.s. continuous at t. Every Lévy process satisfies the so-called *strong Markov property.*

Proposition 2.2. *Let T be a stopping time with $\mathbb{P}(T < \infty) > 0$. Then conditionally on $\{T < \infty\}$, the process $X(T + t) - X(T), t \geq 0$ is independent of $\mathcal{F}_T$ and has the same law as the original process X.*

Throughout this work it will be assumed that X has no positive jumps. In this case X is uniquely characterized (in the sense of *finite-dimensional distributions*) by its *Laplace exponent* $\psi(\alpha), \alpha \geq 0$, which in particular satisfies

$$\mathbb{E}e^{\alpha X(t)} = e^{\psi(\alpha)t}, t \geq 0.$$

Every such Laplace exponent is given by the famous Lévy-Khintchine formula

$$\psi(\alpha) = a\alpha + \frac{1}{2}\sigma^2\alpha^2 + \int_{-\infty}^{0} (e^{\alpha x} - 1 - \alpha x 1_{\{x>-1\}})\nu(\mathrm{d}x), \tag{2.1}$$

where $a \in \mathbb{R}, \sigma \geq 0$ and ν is a measure on $(-\infty, 0)$ with $\int_{-\infty}^{0}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty$, see Theorem I.1 and Chapter VII of Bertoin [1996]. An important special case arises when $\int_{-1}^{0} |x|\nu(\mathrm{d}x) < \infty$. In this case (2.1) can be rewritten as

$$\psi(\alpha) = d\alpha + \frac{1}{2}\sigma^2\alpha^2 + \int_{-\infty}^{0} (e^{\alpha x} - 1)\nu(\mathrm{d}x), \qquad d = a - \int_{-1}^{0} x\nu(\mathrm{d}x). \tag{2.2}$$

Such a process can be interpreted as an independent sum of a purely deterministic drift, a Brownian motion and a non-increasing jump process. The constant d in (2.2) is referred to as the *drift*. Moreover, X has paths of *bounded variation* on compacts a.s. if and only if $\sigma = 0$ and $\int_{-1}^{0} |x|\nu(\mathrm{d}x) < \infty$, that is, (2.2) holds with

$\sigma = 0$. Finally, a bounded variation Lévy process is a CPP if the Lévy measure ν is finite and $d = 0$. In this case $\nu(-\infty, 0)$ is the intensity and $\nu(\mathrm{d}x)/\nu(-\infty, 0)$ is the distribution of jumps of a CPP.

Often it is convenient to add an isolated point ∂ to the value set of X. This point will serve as a 'cemetery'. More concretely, let e_q be an exponential random variable of rate $q \geq 0$ ($e_0 = \infty$ a.s. by convention), independent of X. If we redefine X to be ∂ for all $t \geq e_q$, then such a process is called a Lévy process 'killed' at e_q. Observe that

$$\mathbb{E}e^{\alpha X(t)}1_{\{t<e_q\}} = e^{(\psi(\alpha)-q)t},$$

hence $\psi(\alpha) - q$ can be interpreted as the Laplace exponent of X killed at e_q.

It is known that $\psi(\alpha)$ is analytic in $\mathbb{C}^{\mathrm{Re}>0}$ and continuous in $\mathbb{C}^{\mathrm{Re}\geq 0}$, see also Appendix A.6. Moreover, the real-valued right derivative $\psi'(0+)$ equals $\mathbb{E}X(1) \in \mathbb{R} \cup \{-\infty\}$, where $\mathbb{E}X(1) = -\infty$ corresponds to large jumps of X being non-integrable. If X has non-increasing paths, then $\psi(\alpha) \leq 0$; otherwise $\mathbb{P}(X(1) > 0) > 0$, which implies that $\lim_{\alpha\to\infty} \psi(\alpha) = \infty$. Moreover, $\psi(\alpha), \alpha \geq 0$ is convex by Hölder's inequality.

In the study of Lévy processes without positive jumps it is usual to exclude the trivial case when the paths are non-increasing a.s. The remaining processes are called *spectrally negative* Lévy processes. Let X be a spectrally negative Lévy process. Then for every $q > 0$ there is a unique positive solution to $\psi(\alpha) = q$, which we denote through $\Phi(q)$. In addition, $\Phi(0) = \lim_{q\downarrow 0} \Phi(q)$, which is 0 if $\psi'(0+) \geq 0$, and is the positive solution to $\psi(\alpha) = 0$ otherwise, see Figure 2.1. Hence we say that Φ is a *right inverse* of ψ. The equation $\psi(\alpha) = q$ is sometimes called the Cramér-Lundberg (C-L) equation. The solution $\Phi(q)$ of the C-L equation characterizes the associated first passage process, see Section 2.6. It plays a fundamental role in the study of path properties of Lévy processes.

2.2 Markov additive processes (MAPs)

Consider a real-valued càdlàg process $(X(t))_{t\geq 0}$ and a right-continuous jump process $(J(t))_{t\geq 0}$ with a finite state space E, such that (X, J) is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Definition 2.3. We say that $(X(t), J(t))$ is a Markov Additive Process (MAP) if, given $\{J(t) = i\}$, the pair $(X(t+s) - X(t), J(t+s))$ is independent of $\mathcal{F}_t$ and has the same law as $(X(s) - X(0), J(s))$ given $\{J(0) = i\}$ for all $s, t \geq 0$ and $i \in E$.

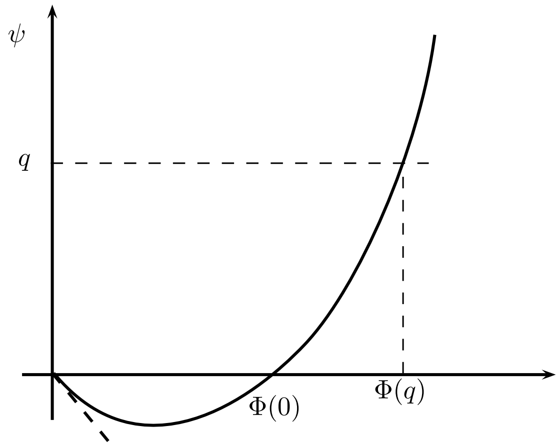

Figure 2.1 The Laplace exponent ψ and its right inverse Φ.

It is usual to say that X is an additive component and J is a background process representing the environment. Note that MAP is a generalization of a Lévy process. The increments of the additive component are not stationary in general; their distributions depend on the states of J.

It should be noted that in some other works MAP stands for a wider class of processes. Namely, any Markov process with values in an arbitrary measurable space can serve as the background process J, see Çinlar [1972] for the basic definitions. Unlike the case of a finite E, where the structure of a MAP is completely understood, the case of an infinite E is much more complicated in general, as it is noted and demonstrated in Asmussen [2003, Ch. XI]. Some of the results contained in this work can be extended to the case of a countable E. A model belonging to the latter class can be found in Prabhu [1998, Ch. 7], see also Virtamo and Norros [1994].

In the following we use $\mathbb{P}_i$ in the context of MAPs to denote the law of (X, J) given $\{X(0) = 0, J(0) = i\}$. Moreover, we write $\mathbb{P}(A, J)$ for an event A to denote the $N \times N$ matrix with entries $\mathbb{P}_i(A, J = j)$. Similarly, $E[Z; J]$ is used to denote the matrix with entries $\mathbb{E}_i[Z; J = j] = \mathbb{E}_i[Z1_{\{J=j\}}]$. Sometimes it is important to consider a shifted additive component with $X(0) = x_0$, in which case with slight abuse of notation we write $\mathbb{P}_{x_0}(A, J)$ and $\mathbb{E}_{x_0}[Z, J]$ to denote the above matrices.

Next we assert the strong Markov property for MAPs, which can be proven by following the same steps as in the case of Lévy processes, see Bertoin [1996, Proposition I.6].

Proposition 2.4. *Let T be a stopping time with $\mathbb{P}(J(T)=i)>0$. Then conditionally on $\{J(T)=i\}$, the process $(X(T+t)-X(T), J(T+t)), t\geq 0$ is independent of $\mathcal{F}_T$ and has law $\mathbb{P}_i$.*

Importantly, a MAP has a very special structure, which we reveal in the following. It is immediate from the definition that J is a Markov chain. For each $i,j\in E$ let U_{ij}^n be a sequence of i.i.d. random variables and X_i^n be a sequence of i.i.d. Lévy processes. All the objects J, X_i^n, U_{ij}^n are assumed to be independent. Letting $T_0=0, T_1, \ldots$ be a sequence of the successive jump epochs of J we define the process X for $t\in[T_n, T_{n+1})$ recursively through

$$X(t) = 1_{\{n>0\}}\left(X(T_n-)+U_{ij}^n\right)+X_j^n(t-T_n), \tag{2.3}$$

where $i=J(T_n-)$ and $j=J(T_n)$. In words, X evolves as the Lévy process $X_j=X_j^0$ during the intervals when J is in the state j, and jumps according to $U_{jk}=U_{jk}^0$, whenever J jumps from j to k. It is straightforward to check that (X,J) is a MAP in its own filtration (completed to satisfy the usual conditions). More interestingly, the converse is also true.

Proposition 2.5. *Every MAP has Representation (2.3).*

Proof. Assume that $J(0)=i$ for some $i\in E$. Define $X_i(t)$ to be $X(t)$ if $t\in[0,T_1)$ and put $X_i(t)=\partial$ otherwise. Note that X_i is a Lévy process killed at independent exponential time T_1. To see this, pick an arbitrary $A\in\mathcal{F}$ and $B\in\mathcal{F}_t$, and write, using the Markov property applied at time t,

$$\begin{aligned}
&\mathbb{P}_i(\{X(t+s)-X(t)\in A\}\cap B\cap\{t+s<T_1\})\\
&=\mathbb{P}_i(\{X(t+s)-X(t)\in A\}\cap B\cap\{t+s<T_1\}|J(t)=i)\mathbb{P}_i(J(t)=i)\\
&=\mathbb{P}_i(\{X(s)\in A\}\cap\{s<T_1\})\mathbb{P}_i(B\cap\{t<T_1\}).
\end{aligned}$$

Let $U_{ij}=X(T_1)-X(T_1-)$ given $J(T_1)=j$. By the Markov property U_{ij} is seen to be independent of X_i and T_1. An application of the strong Markov property at the jump epochs of J completes the proof. □

Every MAP evolves as a Lévy process X_i while J is in the state i. This explains the other commonly used name for a MAP - 'Markov-modulated Lévy process'. Sometimes it is stressed that a MAP is a Markov-modulated Lévy process with additional jumps at transition epochs of J. Observe that a MAP has a.s. continuous sample paths if and only if every $U_{ij}=0$ and every X_i is a Brownian motion (possibly a deterministic drift). Such a MAP is called Markov-modulated

Brownian motion (MMBM). Another important special case arises when every X_i is a CPP. Such a MAP is called Markov-modulated CPP.

Let us fix the basic setup of this thesis. Let (X, J) be a MAP without positive jumps. The trivial case when the paths of X are a.s. non-increasing is excluded from consideration. This results in a class of *spectrally negative* MAPs. Note that some underlying Lévy processes X_i of a spectrally negative MAP are allowed to be non-increasing. Essentially without loss of generality it is assumed that J is *irreducible*. The following list presents some related notation.

- $E = \{1, 2, \dots, N\}$ is the state space of J,
- Q is the *irreducible transition probability matrix* of J and $\boldsymbol{\pi}$ is its unique *stationary distribution*,
- $\psi_i(\alpha)$ is the Laplace exponent of the Lévy process X_i,
- $G_{ij}(\alpha) = \mathbb{E}e^{\alpha U_{ij}}$ is the Laplace transform of the jump when J has a transition from i to j (if $q_{ij} = 0$ we put $U_{ij} = 0$; by convention we also put $U_{ii} = 0$).

Some basic facts concerning the Laplace transform are given in A.7. In addition, we mention that the right derivative $G'_{ij}(0+)$ equals $\mathbb{E}U_{ij}$, which may not be finite.

The state space E of J is partitioned into E_+ and $E_\downarrow$, where $i \in E_\downarrow$ if the Lévy process X_i has a.s. non-increasing paths. In other words, $i \in E_+$ if and only if $\mathbb{P}(X_i(1) > 0) > 0$. The cardinalities of E_+ and $E_\downarrow$ are denoted through N_+ and $N_\downarrow$ respectively. For convenience we often assume that indices in E_+ are smaller than indices in $E_\downarrow$. Note that according to our basic assumptions stated above it is ensured that $N_+ \geq 1$.

Definition 2.6. Let M be a matrix with N rows. Then M_+ denotes the matrix obtained from M by dropping all the rows except those indexed by E_+. We say that M_+ is the *restriction* of M to the rows in E_+. Similarly we define $M_\downarrow$.

Finally, we use the symbol $\mathbb{I}^+$ to denote the $N_+ \times N_+$ identity matrix.

2.3 Matrix exponent of a MAP

Any Lévy process (without positive jumps) is characterized by the associated Laplace exponent. In the case of a MAP this exponent becomes a matrix-valued function

$$F(\alpha) = \mathrm{diag}(\psi_1(\alpha), \dots, \psi_N(\alpha)) + Q \circ G(\alpha), \tag{2.4}$$

where $G(\alpha) = (G_{ij}(\alpha))$ and $A \circ B$ denotes entrywise (Hadamard) matrix multiplication. The matrix-valued function $F(\alpha)$ is referred to as the *matrix exponent* of a MAP. Recall that $\mathbb{E}[e^{\alpha X(t)}; J(t)]$ denotes the matrix with entries $\mathbb{E}_i[e^{\alpha X(t)}; J(t) = j]$.

Proposition 2.7. *It holds for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ that*

$$\mathbb{E}[e^{\alpha X(t)}; J(t)] = e^{F(\alpha)t}.$$

Proof. Up to $o(h)$ terms,

$$\mathbb{E}_i[e^{\alpha X(h)}; J(h) = j] = 1_{\{i=j\}}(1 + q_{ii}h)\mathbb{E}e^{\alpha X_i(h)} + 1_{\{i\neq j\}}q_{ij}h\mathbb{E}e^{\alpha U_{ij}},$$

because J jumps from i to j in an interval of length h with probability $q_{ij}h + o(h)$, and $\mathbb{E}e^{\alpha X_k(h)} = 1 + o(1)$ for any $k \in E$. But $(1 + q_{ii}h)\mathbb{E}e^{\alpha X_i(h)} = 1 + q_{ii}h + \psi_i(\alpha)h + o(h)$, hence we obtain

$$\mathbb{E}[e^{\alpha X(h)}; J(h)] = \mathbb{I} + F(\alpha)h$$

up to $o(h)$ terms. The Markov property states that

$$\mathbb{E}[e^{\alpha X(t+h)}; J(t+h)] = \mathbb{E}[e^{\alpha X(t)}; J(t)]\mathbb{E}[e^{\alpha X(h)}; J(h)],$$

and so

$$\frac{\partial}{\partial t}\mathbb{E}[e^{\alpha X(t)}; J(t)] = \mathbb{E}[e^{\alpha X(t)}; J(t)]F(\alpha).$$

Finally, note that $\mathbb{E}[e^{\alpha X(0)}; J(0)] = \mathbb{I}$, which implies the result according to the standard solution formula for systems of linear differential equations. □

Let $(\lambda(\alpha), \boldsymbol{h}(\alpha))$ be an eigenvalue-eigenvector pair of $F(\alpha), \alpha \geq 0$, that is, $F(\alpha)\boldsymbol{h}(\alpha) = \lambda(\alpha)\boldsymbol{h}(\alpha)$. Then

$$\mathbb{E}_i e^{\alpha X(t)} h_{J(t)}(\alpha) = \mathbb{E}_i[e^{\alpha X(t)}; J(t)]\boldsymbol{h}(\alpha) = e^{\lambda(\alpha)t}h_i(\alpha). \tag{2.5}$$

This leads to the following important result.

Proposition 2.8. $e^{\alpha X(t)-\lambda(\alpha)t}h_{J(t)}(\alpha)$ *is an $\mathcal{F}_t$-martingale under $\mathbb{P}_i$ for any $i \in E$.*

Proof. Using the Markov property we write

$$\mathbb{E}_i[e^{\alpha X(t+s)-\lambda(\alpha)(t+s)}h_{J(t+s)}(\alpha)|\mathcal{F}_t] = \mathbb{E}_i[e^{\alpha X(t)-\lambda(\alpha)t}; J(t)]\mathbb{E}[e^{\alpha X(s)-\lambda(\alpha)s}h_{J(s)}(\alpha)].$$

But according to (2.5) the last term in the right hand side is $\boldsymbol{h}(\alpha)$. □

In the final part of this section we present another important martingale of Asmussen and Kella [2000].

Theorem 2.9. *Let $Y(t)$ be an adapted continuous process having finite variation on compact intervals. Set $Z(t) = X(t)+Y(t)$ and pick $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. Then for every initial distribution of (X, J)*

$$M(t) = \int_0^t e^{\alpha Z(s)} \boldsymbol{e}_{J(s)}^\top \mathrm{d}s F(\alpha) + e^{\alpha Z(0)} \boldsymbol{e}_{J(0)}^\top - e^{\alpha Z(t)} \boldsymbol{e}_{J(t)}^\top + \alpha \int_0^t e^{\alpha Z(s)} \boldsymbol{e}_{J(s)}^\top \mathrm{d}Y(s)$$

is a vector-valued local martingale.

Remark 2.10. Even though Theorem 2.9 is stated in Asmussen and Kella [2000] for the case of no killing, it holds in a more general setting. Namely, the transition rate matrix Q of J can be transient, see Section 2.5. The proof does not require any changes.

The local martingale $M(t)$ becomes a martingale if certain extra conditions are met.

Corollary 2.11. *Assume that $Y(t)$ has finite expected variation on compact intervals and $Z(t)$ is bounded from above on the interval $[0, T]$, where T is a stopping time. Then $M(t \wedge T)$ is a martingale.*

Proof. Letting τ_n be the localizing sequence of stopping times, we have by Doob's optional stopping theorem Kallenberg [2002, Thm. 7.12] that $\mathbb{E}[M(t\wedge T\wedge\tau_n)|\mathcal{F}_s] = M(s\wedge T\wedge\tau_n)$. Under the extra conditions of the corollary we have for every $t > 0$ that $\mathbb{E}\sup_{0\leq s\leq t}|M(s\wedge T)| < \infty$. The dominated convergence theorem then shows that $\mathbb{E}[M(t\wedge T)|\mathcal{F}_s] = M(s\wedge T)$. □

We remark that for many applications it is sufficient to pick $T = \infty$ in Corollary 2.11.

2.4 Perron-Frobenius eigenvalue

For a fixed $\alpha \geq 0$, one of the eigenvalues of $F(\alpha)$ will play a special role.

Proposition 2.12. *For $\alpha \geq 0$ the matrix $F(\alpha)$ has a real simple eigenvalue $k(\alpha)$, which is larger than the real part of any other eigenvalue. The corresponding left-eigenvector $\boldsymbol{v}(\alpha)$ and right-eigenvector $\boldsymbol{h}(\alpha)$ can be chosen so that $v_i(\alpha) > 0$ and $h_i(\alpha) > 0$ for all $i \in E$. The normalization requirement*

$$\boldsymbol{\pi}\boldsymbol{h}(\alpha) = 1, \qquad \boldsymbol{v}(\alpha)\boldsymbol{h}(\alpha) = 1$$

results in a unique choice of $\boldsymbol{v}(\alpha)$ and $\boldsymbol{h}(\alpha)$.

Proof. Observe that the off-diagonal elements of $F = F(\alpha)$ are non-negative. Hence we can find a non-negative matrix M and some $m \in \mathbb{R}$, such that $F = M - m\mathbb{I}$. Moreover, irreducibility of Q implies irreducibility of M. The Perron-Frobenius theory, see Horn and Johnson [1985, Thm. 8.4.4], states that the spectral radius λ of M is a simple eigenvalue of M with the corresponding eigenvector $\boldsymbol{h}$ satisfying $h_i > 0$ for all $i \in E$. Clearly, $(k, \boldsymbol{h})$, where $k = \lambda - m$, is an eigenvalue-eigenvector pair of F. The maximality property of k is immediate.

The uniqueness of $\boldsymbol{h}(\alpha)$ follows from the facts that the matrix $(F(\alpha) - k(\alpha)\mathbb{I})$ has rank $N - 1$, and the vector $\boldsymbol{\pi}$ is not in the row space of this matrix. If the latter is false then $\boldsymbol{u}^\top(F(\alpha) - k(\alpha)\mathbb{I}) = \boldsymbol{\pi}$ for some vector $\boldsymbol{u}$. But multiplication of this equality by any $\boldsymbol{h}(\alpha)$ with positive elements results in $0 = \boldsymbol{\pi}\boldsymbol{h}(\alpha) > 0$. Similarly, $\boldsymbol{h}(\alpha)$ is not in the column space of $(F(\alpha) - k(\alpha)\mathbb{I})$, which completes the proof. □

It is noted that the eigenvalue $k(\alpha)$ plays in many respects the same role as the Laplace exponent of a Lévy process. In the following we establish some important properties of the function $k(\alpha)$. Let us first discuss smoothness of $k(\alpha)$.

Proposition 2.13. *The functions $k(\alpha), \boldsymbol{v}(\alpha), \boldsymbol{h}(\alpha)$ are infinitely many times differentiable for $\alpha > 0$. It holds that,*

$$k'(\alpha) = \boldsymbol{v}(\alpha)F'(\alpha)\boldsymbol{h}(\alpha), \qquad \alpha \geq 0,$$

where $k'(0) \in \mathbb{R} \cup \{-\infty\}$ is interpreted as the right derivative. Moreover, existence of $k^{(n)}(0)$ implies existence of the right derivatives $\boldsymbol{v}^{(n)}(0)$ and $\boldsymbol{h}^{(n)}(0)$.

Proof. Firstly, $k(\alpha)$ is a continuous function according to Hurwitz's theorem, see Appendix A.5. Denoting $M(\alpha) = F(\alpha) - k(\alpha)\mathbb{I}$, we have that $M(\alpha)\boldsymbol{h}(\alpha + \epsilon) \to \mathbf{0}$ as $\epsilon \to 0$, because the entries of $\boldsymbol{h}(\alpha + \epsilon)$ are bounded as seen from the normalization requirement. Recall that $\boldsymbol{\pi}$ is not in the row space of $M(\alpha)$, and apply Proposition A.10 to deduce continuity of $\boldsymbol{h}(\alpha)$ for $\alpha \geq 0$. Continuity and positivity of $\boldsymbol{h}(\alpha)$ imply that the entries of $\boldsymbol{v}(\alpha)$ are bounded in some neighborhood of α, hence $\boldsymbol{v}(\alpha + \epsilon)\boldsymbol{h}(\alpha) \to 1$ and $\boldsymbol{v}(\alpha + \epsilon)M(\alpha) \to \mathbf{0}$. Use Proposition A.10 with $\boldsymbol{u} = \boldsymbol{h}(\alpha)$ to show continuity of $\boldsymbol{v}(\alpha)$.

Theorem 6.3.12 in Horn and Johnson [1985] shows that $k(\alpha)$ is differentiable and the identity in the display holds. Some care should be taken when $\alpha = 0$ (observe that the entries of $F'(0)$ either converge or approach $-\infty$). We also have, as $\epsilon \downarrow 0$,

$$M(\alpha)\frac{1}{\epsilon}(\boldsymbol{h}(\alpha + \epsilon) - \boldsymbol{h}(\alpha)) + M'(\alpha)\boldsymbol{h}(\alpha) \to \mathbf{0}.$$

Observe that $\boldsymbol{\pi}(\boldsymbol{h}(\alpha+\epsilon)-\boldsymbol{h}(\alpha))/\epsilon = 0$, so we can apply Proposition A.10 to claim differentiability of $\boldsymbol{h}(\alpha)$. Moreover,

$$\frac{1}{\epsilon}(\boldsymbol{v}(\alpha+\epsilon)-\boldsymbol{v}(\alpha))\boldsymbol{h}(\alpha) = \frac{1}{\epsilon}\boldsymbol{v}(\alpha+\epsilon)(\boldsymbol{h}(\alpha)-\boldsymbol{h}(\alpha+\epsilon)) \to -\boldsymbol{v}(\alpha)\boldsymbol{h}'(\alpha),$$

which allows us to deduce differentiability of $\boldsymbol{v}(\alpha)$ using Proposition A.10.

Next we can differentiate the identity in the display to show that $k''(\alpha)$ exists. Then we use similar steps as above to prove that $\boldsymbol{v}(\alpha)$ and $\boldsymbol{h}(\alpha)$ are twice differentiable. These steps can be repeated arbitrarily many times. □

Observe that $F(0) = Q$, which has non-positive eigenvalues, so $k(0) = 0$, $\boldsymbol{h}(0) = \mathbf{1}$ and $\boldsymbol{v} = \boldsymbol{\pi}$. Proposition 2.13 then implies

$$\begin{aligned} k'(0) &= \sum_{i\in E} \pi_i \left(\psi_i'(0) + \sum_{j\in E} q_{ij} G_{ij}'(0)\right) \\ &= \sum_{i\in E} \pi_i \left(\mathbb{E}X_i(1) + \sum_{j\in E} q_{ij}\mathbb{E}U_{ij}\right) = \mathbb{E}_{\boldsymbol{\pi}} X(1). \end{aligned} \tag{2.6}$$

The last equality follows from the fact that J, started with its stationary distribution $\boldsymbol{\pi}$, will on average spend a fraction π_i of the time in the state i, and will make $\pi_i q_{ij}$ jumps $i \to j$ in the time interval $[0,1]$. In particular, we see that $k'(0)$ is finite if and only if all the first moments $\mathbb{E}X_i(1)$ and $\mathbb{E}U_{ij}$ exist. Otherwise $k'(0) = -\infty$, which indicates the presence of non-integrable jumps.

In order to simplify notation we often write κ for $k'(0)$. It is common to refer to κ as to the *stationary drift* of $X(t)$. Alternatively, the term *asymptotic drift* is used, which is supported by the following lemma.

Lemma 2.14. *It holds $\mathbb{P}_i$-a.s. that $\lim_{t\to\infty} X(t)/t = \kappa$ for all $i \in E$.*

Proof. Let $T_i(t) = \int_0^t 1_{\{J(s)=i\}}\mathrm{d}s$ be the amount of time spent by J in the state i in the interval $[0,t]$. Let also $N_{ij}(t)$ be the number of transitions of J from i to j in the interval $[0,t]$. According to Proposition 2.5 there exist Lévy processes X_i and sequences of i.i.d. random variables U_{ij}^n, such that

$$X(t) = \sum_{i\in E} X_i(T_i(t)) + \sum_{i,j\in E, i\neq j}\ \sum_{n=1}^{N_{ij}(t)} U_{ij}^n$$

$\mathbb{P}_i$-a.s. Finally, observe that $\mathbb{P}_i$-a.s. it holds that $T_i(t)/t \to \pi_i$, $N_{ij}(t)/t \to \pi_i q_{ij}$ and, moreover, $X_i(t)/t \to \mathbb{E}X_i(1)$ as $t \to \infty$. Identity (2.6) completes the proof. □

Proposition 2.15. *It holds* $\mathbb{P}_i$*-a.s. that (a)* $X(t) \to -\infty$*, (b)* $X(t) \to \infty$*, (c)* $\liminf X(t) = -\infty, \limsup X(t) = \infty$ *according to (a)* $\kappa < 0$*, (b)* $\kappa > 0$ *and (c)* $\kappa = 0$.

Proof. The cases (a) and (b) are immediate from Lemma 2.14. The case (c) follows from random walk theory, see Theorem 2.4 in Asmussen [2003, Ch. VIII]. The random walk is defined through $S_n = X(T_i^n)$, where T_i^n is the time of n-th entrance of J into the state i. □

2.5 Killing and time reversal

In this section we discuss two important concepts. Firstly, the state space of a MAP (X, J) can be appended with an absorbing state (∂_X, ∂_J). This extends the class of MAPs by allowing the background Markov chain J to be transient. As it is common in the theory of Markov chains, we do not include the additional state ∂_J in the descriptor of the process. In other words, the matrix exponent $F(\alpha)$ is assumed to have the same form (2.4) as before, but the transition rate matrix Q is allowed to be transient. Equivalently, such a process can be seen as a 'regular' MAP with state-dependent killing. That is, every Lévy process X_i is 'killed' with some rate $q_i \geq 0$. This kind of process appears naturally when one considers first passage times, as is shown in Section 2.6.

In the rest of this section we discuss time reversion. Assume that $J(0)$ is distributed according to $\boldsymbol{\pi}$. Fix $T > 0$ and define a process $(\hat{X}, \hat{J})$ on $[0, T)$ through

$$\hat{J}(t) = J((T-t)-), \qquad \hat{X}(t) = X(T) - X((T-t)-).$$

The left limits are taken to guarantee that $\hat{X}$ and $\hat{J}$ have càdlàg paths. It is well-known that $\hat{J}$ is an irreducible Markov chain with the same stationary distribution $\boldsymbol{\pi}$, see Asmussen [2003, Prop. II.5.2]. The transition rate matrix of $\hat{J}$ is given by $\hat{Q} = \Delta_{\boldsymbol{\pi}}^{-1} Q^\top \Delta_{\boldsymbol{\pi}}$, where $\Delta_{\boldsymbol{\pi}}$ is a diagonal matrix with $\boldsymbol{\pi}$ on the diagonal. Moreover, time reversion of a Lévy process results in a process with an identical law, see Bertoin [1996, Lem. II.2]; this remains true if $T > 0$ is an independent random variable. Considering Representation (2.3), we see that $(\hat{X}, \hat{J})$ is a MAP with $\hat{Q}$ defined above, $\hat{X}_i$ having the law of X_i, and $\hat{U}_{ij}$ distributed as U_{ji}. In other words, the matrix exponent of a time-reversed process is given by

$$\hat{F}(\alpha) = \Delta_{\boldsymbol{\pi}}^{-1} F(\alpha)^\top \Delta_{\boldsymbol{\pi}}. \tag{2.7}$$

The process (X, J) is called time-reversible if it has the same law as $(\hat{X}, \hat{J})$. That is, $\hat{F}(\alpha) = F(\alpha)$, which according to (2.7) is true if and only if $\Delta_{\boldsymbol{\pi}} F(\alpha)$ is a symmetric matrix. Furthermore, this is equivalent to J being time-reversible and U_{ij} having the same law as U_{ji} for all $i \neq j$.

Finally, we note that the eigenvalues of $F(\alpha)$ and $\hat{F}(\alpha)$ coincide. This in particular implies that the asymptotic drifts coincide too.

2.6 First passage process

Let $\overline{X}(t)$ be the *supremum* of the additive component X up to time t, that is, $\overline{X}(t) = \sup_{0 \le s \le t}\{X(s)\}$. We often write $\overline{X}$ to denote $\overline{X}(\infty)$. Similarly, one defines the *infimum* process $\underline{X}(t)$. The first passage time over level $x \ge 0$ is defined through

$$\tau_x = \inf\{t \ge 0 : X(t) > x\}. \tag{2.8}$$

It is known that τ_x is a stopping time, see Bertoin [1996, Corollary I.8] for an idea of the proof. Observe that $X(\tau_x) = x$, because of the absence of positive jumps. Moreover, the events $\{\tau_x < t\}$ and $\{\overline{X}(t) > x\}$ coincide. The strong Markov property then implies that $(\tau_x, J(\tau_x))_{x \ge 0}$ is a MAP. This is a killed MAP, where the killing time is given by $x = \overline{X}$.

Alternatively, we can consider the process (X, J) killed with rate $q \ge 0$. Again $J(\tau_x)_{x \ge 0}$ is a Markov chain. Letting $\Lambda(q)$ be its $N_+ \times N_+$ transition rate matrix we have for $i, j \in E_+$ the following identity

$$(e^{\Lambda(q)x})_{ij} = \mathbb{P}(J(\tau_x) = j, \tau_x < e_q \mid J(\tau_0) = i) = \mathbb{E}_{J(\tau_0)=i}[e^{-q\tau_x}; J(\tau_x) = j]. \tag{2.9}$$

Hence $\Lambda(q)$ is also the matrix exponent of the MAP $(-\tau_x, J(\tau_x))_{x \ge 0}$. Finally, let $\Pi(q)$ denote the matrix with initial distributions of this new MAP. More formally, $\Pi(q)$ is a $N \times N_+$ matrix with elements $\mathbb{P}_i(J(\tau_0) = j, \tau_0 < e_q)$, where $i \in E$ and $j \in E_+$, with the obvious ordering. The strong Markov property implies

$$\mathbb{E}[e^{-q\tau_x}; J(\tau_x)] = \mathbb{E}[e^{-q\tau_0}; J(\tau_0)]\mathbb{E}_{J(\tau_0)}[e^{-q\tau_x}; J(\tau_x)] = \Pi(q)e^{\Lambda(q)x}.$$

Moreover, the matrices $\Pi(q)$ and $\Lambda(q)$ fully characterize the first passage process $(\tau_x, J(\tau_x))_{x \ge 0}$ under $\mathbb{P}_i, i \in E$ in terms of its finite-dimensional distributions. The first part of this thesis is devoted to the study of the matrices $\Pi(q)$ and $\Lambda(q)$ and their relation to the matrix exponent $F(\alpha)$.

Let us present some basic properties of the matrices $\Pi(q)$ and $\Lambda(q)$. Firstly, $\tau_0 = 0$ $\mathbb{P}_i$-a.s. for any $i \in E_+$, see e.g. Kyprianou [2006, Thm. 6.5]. Therefore

$\Pi(q)_+$ is the identity matrix, see also Definition 2.6. Secondly, it is easy to see that irreducibility of Q implies irreducibility of $\Lambda(0)$. Moreover, if $\kappa \geq 0$ then $\mathbb{P}_i(\tau_x < \infty) = 1$ for all $x \geq 0$ and $i \in E$ by Proposition 2.15. This shows that $J(\tau_x)$ is an irreducible recurrent Markov chain. We denote its stationary distribution through $\boldsymbol{\pi}_\Lambda$. If, however, $\kappa < 0$ or $q \neq 0$ then $\Lambda(q)$ is transient. We summarize these properties in the following proposition.

Proposition 2.16. *It holds that* $\Pi(q)_+ = \mathbb{I}^+$ *for all* $q \geq 0$*. Moreover,* $\Lambda(q)$ *is an irreducible recurrent transition rate matrix if* $q = 0$ *and* $\kappa \geq 0$*, and is transient otherwise.*

Remark 2.17. Let $\rho(q)$ denote the Perron-Frobenius eigenvalue of $\Lambda(q)$. If $\kappa \geq 0$, then $\rho(0) = 0$, that is, $\Lambda(0)$ has a simple eigenvalue at 0 (with corresponding eigenvector $\mathbf{1}$) and all others in $\mathbb{C}^{\mathrm{Re}<0}$. Otherwise $\rho(q) < 0$, that is, all the eigenvalues of $\Lambda(q)$ are in $\mathbb{C}^{\mathrm{Re}<0}$, see Appendix A.4.

It is instructive to consider a MAP on a single state, which is just a Lévy process. Then $\Lambda(q) = -\Phi(q)$, where $\Phi(q)$ is the right-inverse of $\psi(\alpha), \alpha \geq 0$, see Bertoin [1996, Thm. VII.1]. Note also that $\Phi(q)$ is 0 if $q = 0$ and $\kappa \geq 0$ and is strictly positive otherwise, see Section 2.1.

In general, identification of $\Lambda(q)$, as well as its relation to the C-L equation, is more complicated. These questions are addressed in Chapter 3 in the case when (X, J) is time-reversible. This case is substantially simpler, but it illustrates some of the main ideas and concepts well. The general theory is presented in Chapter 4.

Remark 2.18. The concept of killing plays a pivotal role in this book. It is often easier to analyze a killed MAP. Then the analysis can be extended to the case of no killing by a limiting argument. For notational convenience we often suppress the killing rate $q \geq 0$ and simply assume that the transition rate matrix Q is transient. In this context we write Λ and Π to denote $\Lambda(q)$ and $\Pi(q)$. One should be careful when making q explicit. When doing so, one must substitute Q with $Q - q\mathbb{I}$, and hence $F(\alpha)$ with $F(\alpha) - q\mathbb{I}$. In fact, this setup can be generalized. That is, Q need not satisfy $Q\mathbf{1} = -q\mathbf{1}$; it can be any transient transition rate matrix, see Section 2.5. Finally, it is always assumed that there is no killing when talking about stationary distributions and stationary (asymptotic) drift.

2.7 Phase-type distributions and MAPs

Let J be a continuous-time Markov chain on $m < \infty$ states. Assume that $\boldsymbol{\alpha}$ is its initial distribution and T is its $m \times m$ transition rate matrix. In addition, assume that J is transient, that is $\boldsymbol{t} = -T\mathbf{1} \geq 0$ has a positive entry. It is convenient to add an absorbing state ∂ to the state space of J. Finally, the *life time* of J is denoted through $\zeta = \inf\{t > 0 : J(t) = \partial\}$, which is known to be finite a.s.

Definition 2.19. The distribution of ζ is called phase-type with parameters $(m, \boldsymbol{\alpha}, T)$.

It is not difficult to prove the following, see Asmussen [2003, Ch. III.4].

Proposition 2.20. *For $x \geq 0$, the cumulative distribution function of ζ is $F(x) = 1 - \boldsymbol{\alpha} e^{Tx}\mathbf{1}$ and the density is $f(x) = \boldsymbol{\alpha} e^{Tx}\boldsymbol{t}$. The Laplace transform of ζ is*

$$\mathbb{E}e^{-s\zeta} = \boldsymbol{\alpha}(s\mathbb{I} - T)^{-1}\boldsymbol{t}.$$

It is mentioned in the introduction that phase-type distributions fit naturally in the framework of MAPs. Consider a MAP with transition rate matrix T and $X_i(t) = t$ for all i. The matrix exponent of such MAP is $T + \alpha\mathbb{I}$. Given that J is started with distribution $\boldsymbol{\alpha}$, the value of X right before entrance into the absorbing state has phase-type distribution with parameters $(m, \boldsymbol{\alpha}, T)$. Roughly speaking, by slanting a phase-type jump we get a simple MAP, whose additive component evolves linearly. This idea can be taken one step further. Namely, phase-type jumps of an arbitrary MAP can be replaced by linear stretches of unit slope. This procedure requires adding supplementary states to the background process (as many as there are phases), and is called *fluid embedding*. It allows to apply the results of this thesis to a general MAP with all upward jumps of phase-type by constructing an auxiliary spectrally negative MAP with an enlarged state space.

Let us describe fluid embedding in more detail. Consider a general MAP (X, J). First we address jumps of X at switching epochs of J. Suppose the distribution of the jump U_{ij} is phase-type with parameters $(m, \boldsymbol{\alpha}, T)$. The corresponding entry of $F(\alpha)$ is $q_{ij}G_{ij}(\alpha)$. Let us use fluid embedding with respect to U_{ij} to construct an auxiliary MAP. Figure 2.2 presents the matrix exponent of the auxiliary MAP. It is a $(N + m)$-dimensional matrix with upper left block being $F(\alpha)$, except that the (i, j)-th entry is 0. The lower right block is $T + \alpha\mathbb{I}$, representing the jump. Suppose next that a Lévy process X_i is an independent sum of some Lévy process with Laplace exponent $\tilde{\psi}_i(\alpha)$ and a CPP, where the latter models arrivals of jumps with intensity λ. We assume that these jumps are of phase-type with parameters $(m, \boldsymbol{\alpha}, T)$. Let us use fluid embedding to 'eliminate' these jumps. The resulting matrix exponent has a very similar form to the one depicted in Figure 2.2. In this case $j = i$, the (i, i)-th entry is $q_{ii} - \lambda + \tilde{\psi}_i(\alpha)$, and the last part of the i-th row reads $\lambda\boldsymbol{\alpha}$.

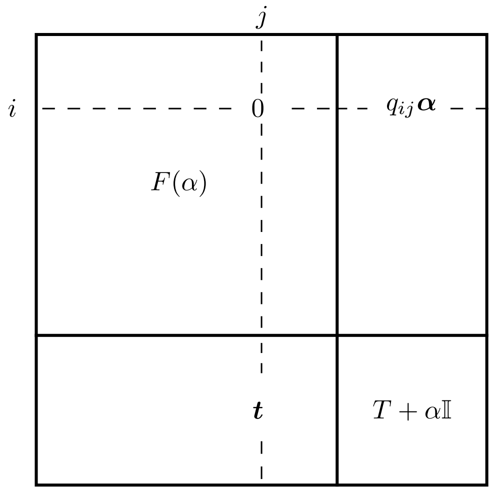

Figure 2.2 An example of fluid embedding.

Let us conclude the discussion about fluid embedding with some comments. Note that if the distribution of the jump U_{ij} is a mixture of phase-type and some other distribution then we can still use fluid embedding to eliminate the phase-type component. Moreover, negative jumps of phase-type can be handled in exactly the same way by introducing linear stretches of slope -1. It is noted that instead of enlarging the state space of J, it can be sometimes useful to reduce it by eliminating certain states, see Section 7.1 for further discussion.

Phase-type distributions arise when one considers all-time supremum $\overline{X}$ of a spectrally negative MAP (X, J). This quantity turns out to be important for queueing theory, see Section 2.8. Observe that $\overline{X}$ is the life time of $J(\tau_x)$. The latter is transient if Q is transient, or Q is recurrent and $\kappa < 0$. Assuming either of these conditions, we see that $(\overline{X}|J(0) = i)$ is phase-type with initial distribution given by Π_i and transition rate matrix Λ. It is important to note that $(\overline{X}|J(0) = i)$ may have a mass at 0. This mass is $(1 - \Pi_i\mathbf{1})$, which is always 0 if $i \in E_+$.

2.8 Reflection

In the first part of this section we consider an arbitrary sample path of a stochastic process forgetting about the probability space. That is, we assume that $X(t), t \geq 0$ is a real càdlàg function. In the following we define the two-sided reflection of X with respect to the strip $[0, B]$, where $B \in [0, \infty]$. It is noted that the one-sided reflection at 0 is a special case with $B = \infty$. It will be assumed that the initial value of X belongs to the strip, that is, $X(0) \in [0, B]$.

Definition 2.21. The two-sided reflection $R(t)$ of $X(t)$, with respect to the strip $[0, B]$, is defined through

$$R(t) = X(t) + L(t) - U(t), \tag{2.10}$$

where $R(t), L(t), U(t)$ are real càdlàg functions which satisfy the following conditions:

- $L(t)$ and $U(t)$ are non-decreasing with $L(0) = U(0) = 0$,
- $R(t) \in [0, B]$ for all $t \geq 0$,
- $R(s) = 0$ if $\forall t > s : L(s-) < L(t)$, and $R(s) = B$ if $\forall t > s : U(s-) < U(t)$.

It is known that such a triplet of functions exists and is unique, see Kella [2006] and Kruk et al. [2007], and is called the solution of the *two-sided Skorokhod problem*. The functions $L(t)$ and $U(t)$ are called *regulators* at the lower and upper barriers respectively, that is at 0 and at B. The last condition of the above definition states that the *points of increase* of L and U are contained in $\{t \geq 0 : R(t) = 0\}$ and $\{t \geq 0 : R(t) = B\}$ respectively. It can be alternatively written as

$$\int_0^\infty R(t)\mathrm{d}L(t) = 0, \qquad \int_0^\infty (B - R(t))\mathrm{d}U(t) = 0.$$

Let us consider the one-sided reflection, that is, we set $B = \infty$. In this case the solution to the Skorokhod problem has a simple explicit form: $U(t) = 0, L(t) = -(\underline{X}(t) \wedge 0)$ and $R(t) = X(t) - (\underline{X}(t) \wedge 0)$, where $X(0) \in [0, \infty)$.

Theorem 2.22. *Let (X, J) be a MAP with negative asymptotic drift, that is, $\kappa < 0$. Then the one-sided reflection $(R(t), J(t))$ has a stationary version. The corresponding stationary distribution coincides with the distribution of $(\overline{\hat{X}}(\infty), \hat{J}(0))$, where $(\hat{X}, \hat{J})$ is the time-reversed MAP.*

Proof. Assume that $X(0) = 0$ and $J(0)$ is distributed according to $\boldsymbol{\pi}$. In this case $R(t) = X(t) - \underline{X}(t)$. Observe that for any time horizon $T > 0$ it holds that

$$\overline{\hat{X}}(T) = X(T) - \inf_{0 \leq t \leq T}\{X((T-t)-)\} = X(T) - \underline{X}(T-)$$

by the construction presented in Section 2.5, see also Figure 2.3. Note that neither X, nor J, jumps at T with probability 1. Hence $(R(T), J(T))$ has the same distribution as $(\overline{\hat{X}}(T), \hat{J}(0))$.

Recall that the time-reversed process has the same asymptotic drift as the original process, hence $\hat{X}(T) \to -\infty$ as $T \to \infty$ by Proposition 2.15, and so $\overline{\hat{X}} < \infty$ a.s. This completes the proof for the case when $X(0) = 0$. Finally, if $X(0) > 0$ then X hits $(-\infty, 0]$ in finite time a.s. The strong Markov property allows us to reduce the problem to the case when $X(0) = 0$. □

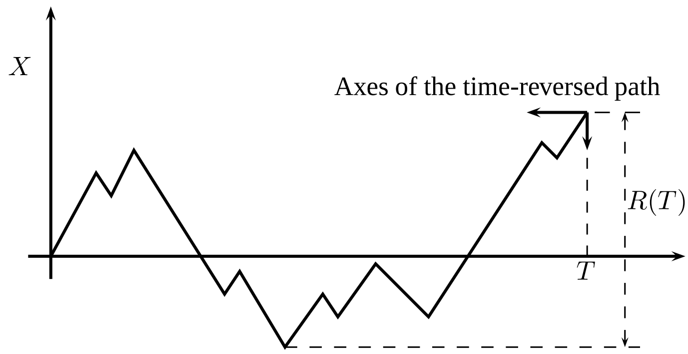

Figure 2.3 Reflection and time reversal.

Consider the stationary version of the reflected process $(R(t), J(t))$, and let a random pair (R^*, J^*) have this stationary distribution. The following corollary is an immediate consequence of Theorem 2.22 and the observations made in Section 2.7.

Corollary 2.23. *The distribution of* $(R^* | J^* = i)$ *is phase-type with initial distribution* $\hat{\Pi}_i$ *and transition rate matrix* $\hat{\Lambda}$*. It has mass* $(1 - \hat{\Pi}_i \mathbf{1})$ *at 0, which is 0 if* $i \in E_+$*.*

Chapter 3

First passage: time-reversible case

Consider a spectrally negative MAP, and the matrices Π and Λ characterizing its first passage process, see Section 2.6. In this chapter we identify Π and Λ in the special case when the MAP is time-reversible. This special case is substantially easier to analyze. In particular, there will be no need to go beyond classical linear algebra. The ideas and results, however, provide a good introduction to the general case which will be dealt with later in this monograph. Moreover, the results in this important special case are of a particularly neat and simple form, which can be used to greatly reduce the computational efforts required to obtain numerical output, as demonstrated in Section 3.2. Hence the purpose of this chapter is twofold. Firstly, we illustrate the main ideas and concepts needed to identify Λ, and secondly, we establish some additional important properties of Λ for this important special case.

3.1 Main results

Throughout this chapter it is assumed that (X, J) is a time-reversible spectrally negative MAP. In other words, J is a time-reversible Markov chain, and the laws of U_{ij} and U_{ji} coincide for all $i \neq j$, see also Section 2.5. We will prove that the transition rate matrix Λ is *similar* to some real diagonal matrix Γ, in the sense that $\Lambda = V\Gamma V^{-1}$ for some invertible matrix V, see also Appendix A.2. Moreover, we provide a procedure to construct the matrices V and Γ given the matrix exponent $F(\alpha)$. For simplicity it is assumed that there is no killing: $q = 0$.

Let $\lambda_1, \ldots, \lambda_n$ be all the distinct zeros of $\det(F(\alpha))$ in $(0, \infty)$. Let also p_i denote the dimension of the null space of $F(\lambda_i)$ (geometric multiplicity of the null-eigenvalue). Define Γ_i to be a $p_i \times p_i$ diagonal matrix with λ_i on the diagonal, and V_i to be an $N \times p_i$ matrix formed from a basis of the (right) null-space of $F(\lambda_i)$. Finally, we construct matrices Γ and V as follows:

$$\begin{aligned} &\Gamma = \mathrm{diag}(\Gamma_1, \dots, \Gamma_n) \text{ and } V = [V_1, \dots, V_n], && \text{if } \kappa < 0, \\ &\Gamma = \mathrm{diag}(0, \Gamma_1, \dots, \Gamma_n) \text{ and } V = [\mathbf{1}, V_1, \dots, V_n], && \text{if } \kappa \geq 0, \end{aligned} \tag{3.1}$$

where κ is the asymptotic drift of X. It will be shown in the following that $\sum_{i=1}^n p_i = N_+ - 1_{\{\kappa \geq 0\}}$ and hence V is an $N \times N_+$ matrix. Let us now formulate the main theorem of this chapter.

Theorem 3.1. *If (X, J) is a time-reversible MAP, then Γ and V_+ are $N_+ \times N_+$-dimensional matrices, V_+ is invertible, and*

$$\Lambda = -V_+ \Gamma (V_+)^{-1}, \qquad \Pi = V(V_+)^{-1}.$$

Let us start by establishing a lemma, which can be considered as a weak analog of the above theorem.

Lemma 3.2. *If $\lambda > 0$ and $\boldsymbol{v}$ are such that $F(\lambda)\boldsymbol{v} = \mathbf{0}$, then*

$$\Lambda \boldsymbol{v}_+ = -\lambda \boldsymbol{v}_+ \text{ and } \boldsymbol{v} = \Pi \boldsymbol{v}_+.$$

Proof. According to Proposition 2.8, $e^{\lambda X(t)} v_{J(t)}$ is a martingale under $\mathbb{P}_i$ for any $i \in E$. By Doob's optional stopping theorem, see e.g. Kallenberg [2002, Theorem 7.12], we have

$$v_i = \mathbb{E}_i[e^{\lambda X(\tau_x \wedge t)} v_{J(\tau_x \wedge t)}]$$

for any $x \geq 0$ and $t > 0$. Proposition 2.15 implies that either $\tau_x < \infty$ or $X(t) \to -\infty$ as $t \to \infty$ $\mathbb{P}_i$-a.s., where the latter case corresponds to $\kappa < 0$. Moreover, $X(t) \leq x$ on $[0, \tau_x]$, hence using the dominated convergence theorem we obtain

$$v_i = \mathbb{E}_i[1_{\{\tau_x < \infty\}} e^{\lambda x} v_{J(\tau_x)}] = e^{\lambda x} \mathbb{P}_i(J(\tau_x)) \boldsymbol{v}.$$

Plugging in $x = 0$ we get $\boldsymbol{v} = \Pi \boldsymbol{v}_+$; and $(\Lambda + \lambda \mathbb{I}) \boldsymbol{v}_+ = \mathbf{0}$ is obtained by differentiating $\boldsymbol{v}_+ = e^{\lambda x} e^{\Lambda x} \boldsymbol{v}_+$ at $x = 0$. □

This lemma shows that $\Lambda V_+ = -\Gamma V_+$ and $V = \Pi V_+$, see also Proposition 2.16 and Remark 2.17. Moreover, the matrix V_+ is composed from eigenvectors of Λ, where the eigenvectors corresponding to the same eigenvalue are linearly independent (the restriction to E_+ can not ruin independence, because $V = \Pi V_+$). It is a standard fact from linear algebra that all these vectors are linearly independent. Hence the proof of Theorem 3.1 will be complete if we establish that

$$\sum_{i=1}^{n} p_i \geq N_+ - 1_{\{\kappa \geq 0\}}, \tag{3.2}$$

because the number of columns in V_+ cannot exceed N_+. We address this question in the rest of this section. It is exactly this part of the proof, where time-reversibility is essential.

The following lemma is an important starting point of the analysis.

Lemma 3.3. *The eigenvalues of* $F(\alpha), \alpha \geq 0$ *are real with algebraic and geometric multiplicities being the same.*

Proof. Time-reversibility is equivalent to the requirement that $\Delta_{\boldsymbol{\pi}} F(\alpha)$ is a symmetric matrix for all $\alpha \geq 0$, see Section 2.5. Then $\Delta_{\boldsymbol{\pi}}^{1/2} F(\alpha) \Delta_{\boldsymbol{\pi}}^{-1/2}$ is a real symmetric matrix too. Theorem A.7 completes the proof. □

Let $g_i(\alpha)$ be the i-th largest eigenvalue of $F(\alpha), \alpha \geq 0$ (so that $g_1(\alpha) = k(\alpha)$, the Perron-Frobenius eigenvalue defined earlier). It is well-known that the eigenvalues of a matrix, which is continuous in its parameter, trace continuous curves in the complex plane. This is an immediate consequence of Hurwitz's theorem, see Appendix A.5. Therefore, $g_i : [0, \infty) \mapsto \mathbb{R}$ are continuous functions. The next lemma presents some additional properties of these functions.

Lemma 3.4. *It holds that*

- $g_1(0) = 0$ *and* $g_i(0) < 0$ *for* $i = 2, \ldots, N$,
- $g_i(\alpha) \to \infty$ *as* $\alpha \to \infty$ *for* $i = 1, \ldots, N_+$.

Proof. The first statement is immediate from $F(0) = Q$. Recall that $\psi_i(\alpha) \to \infty$ as $\alpha \to \infty$ for $i \in E_+$. Moreover, the off-diagonal elements of $F(\alpha)$ are bounded for $\alpha \geq 0$. Gershgorin's theorem, see Appendix A.1, completes the proof. □

Proof of Theorem 3.1. Recall that we are left to prove (3.2). Lemma 3.4 shows that the functions $g_2(\alpha), \ldots, g_{N_+}(\alpha)$, and in addition $g_1(\alpha)$ provided that $\kappa = k'(0) < 0$, hit 0 in the interval $(0, \infty)$ at least once (recall that $k(\alpha) = g_1(\alpha)$) by continuity. But the algebraic and geometric multiplicities of the null-eigenvalue of $F(\alpha)$ coincide according to Lemma 3.3. These multiplicities are given by p_i, so (3.2) holds. □

Remark 3.5. Recall that inequality (3.2) is in fact an equality. Hence the functions $g_2(\alpha), \ldots, g_{N_+}(\alpha)$, and in addition $g_1(\alpha)$ provided that $\kappa < 0$, hit 0 exactly once for positive α. Moreover, these are the only functions $g_i(\alpha)$ hitting 0 for positive α.

Let us summarize the results. Firstly, note that there is a one-to-one correspondence between the positive zeros of $\det(F(\alpha))$ and non-zero eigenvalues of Λ. Secondly, the (geometric) multiplicity of any such eigenvalue coincides with the dimension of the corresponding null space of $F(\alpha)$. This results in the following corollary, which is a counterpart of Lemma 3.2.

Corollary 3.6. *If $\lambda \neq 0$ and $\boldsymbol{h}$ are such that $\Lambda \boldsymbol{h} = \lambda \boldsymbol{h}$ then $F(-\lambda)\boldsymbol{v} = \boldsymbol{0}$, where $\boldsymbol{v} = \Pi \boldsymbol{h}$.*

Proof. Let $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^n$ be a basis of the null space of $F(-\lambda)$. Then $\boldsymbol{v}^i_+, i = 1, \ldots, n$ is a basis of the eigenspace of Λ corresponding to the eigenvalue λ. Hence $\boldsymbol{h}$ can be written as a linear combination of these vectors. But $\boldsymbol{v}^i = \Pi \boldsymbol{v}^i_+$, and hence $\Pi \boldsymbol{h}$ is in the null space of $F(-\lambda)$. □

3.2 Computational aspects

In this section we consider the problem of finding the positive zeros of $\det(F(\alpha))$, which are then used to construct matrices Γ and V. Time-reversibility greatly reduces the computational efforts required to construct these matrices, as compared to the general case discussed in the following chapters. Firstly, we have restricted ourselves to the domain of positive reals, whereas in general the right half of the complex plane is to be considered. Secondly, it turns out that the functions $g_i(\alpha)/\alpha, \alpha > 0$ are strictly increasing (which furthermore implies that $g_i(\alpha)$ are strictly increasing after they hit 0), see Lemma 3.8. Hence a simple root finding procedure can be employed to find the zeros of functions $g_i(\alpha)$, which are exactly the zeros of $\det(F(\alpha))$. Finally, this idea can be extended further for a MAP, which can be represented as a superposition of multiple time-reversible MAPs. In this case one can reduce the computational burden by several orders of magnitude.

Let us first present a useful lemma.

Lemma 3.7. *The positive zeros of* $\det(F(\alpha))$ *are bounded by*

$$C = \max\{\Phi_i(2q_i) : i \in E_+\}, \text{ where } q_i = -q_{ii}.$$

Proof. We only need to show that $F(\alpha)$ is strictly diagonally dominant and hence non-singular for all $\alpha > C$, see Appendix A.1. Consider the i-th row of $F(\alpha)$ and note that the off-diagonal elements are non-negative; their sum is bounded by q_i. If $i \in E_+$ then the diagonal element is $-q_i + \psi_i(\alpha) > q_i$, otherwise it is smaller than $-q_i$, because $\psi_i(\alpha) < 0$. □

Define $h_i(\alpha) = g_i(\alpha)/\alpha$ for $\alpha > 0$ and let d_j be the deterministic drift of the Lévy process X_j if this process has paths of bounded variation, and ∞ otherwise, see Section 2.1 and in particular Identity (2.2).

Lemma 3.8. *The functions $h_i(\alpha)$ are strictly increasing with*

(i) $\lim_{\alpha\downarrow 0} h_1(\alpha) = \kappa$ *and* $\lim_{\alpha\downarrow 0} h_i(\alpha) = -\infty$ *for* $i > 1$,

(ii) $\lim_{\alpha\to\infty} h_i(\alpha) = c_i$, *where c_i is the i-th largest number among the d_i-s.*

Proof. Fix a $c \in \mathbb{R}$, and consider a time-reversible MAP $(X(t) - ct, J(t))_{t\geq 0}$. Its matrix exponent is $\tilde{F}(\alpha) = F(\alpha) - c\alpha$. Trivially $\tilde{g}_i(\alpha) = g_i(\alpha) - c\alpha$ and $\tilde{h}_i(\alpha) = h_i(\alpha) - c$. But the functions $\tilde{g}_i(\alpha)$, and hence also $\tilde{h}_i(\alpha)$, hit 0 for positive α at most once. This shows that $h_i(\alpha)$ are strictly increasing, because c is arbitrary.

Claim (i) follows immediately from Lemma 3.4. Finally, note that $\tilde{N}_+$ (in self-evident notation) is non-increasing in c. More precisely, $\tilde{N}_+$ decreases when $c = d_j$ for some j, because then $X_j(t) - ct$ becomes a process with non-increasing paths. This means that one of the functions $h_i(\alpha)$ approaches d_j but does not hit it, which proves the second claim. Some care should be taken when $\tilde{N}_+ = 0$. In this case $\det(\tilde{F}(\alpha))$ has no positive zeros as follows from the familiar diagonal dominance argument. □

We now consider the situation in which the MAP (X, J) is a superposition of multiple independent MAPs $(X^{(1)}, J^{(1)}), \ldots, (X^{(M)}, J^{(M)})$, see also Kosten [1984], Stern and Elwalid [1991]. Then $F(\alpha)$ can be written as $F^{(1)}(\alpha) \oplus \ldots \oplus F^{(M)}(\alpha)$, with $F^{(1)}(\alpha), \ldots, F^{(M)}(\alpha)$ being matrix exponents, and $\oplus$ and $\otimes$ denoting the Kronecker sum and product respectively, see Bellman [1960]. If the underlying MAPs are time-reversible, then so is (X, J). Following the procedure outlined above, one can identify Γ and V by finding the zeros of the eigenvalue functions $g_i(\alpha)$ of $F(\alpha)$. If N_m is the dimension of the matrix exponent $F^{(m)}(\alpha)$, this would require working with eigenvalues of the $\prod_{m=1}^{M} N_m$ dimensional matrix $F(\alpha)$. It is, however, known that these eigenvalues and eigenvectors are given by

$$\sum_{m=1}^{M} g_{i(m)}^{(m)}(\alpha) \quad \text{and} \quad \bigotimes_{m=1}^{M} \boldsymbol{v}_{i(m)}^{(m)}(\alpha), \quad i(m) \in \{1, \ldots, N_m\}.$$

This essentially entails that the bulk of the computations can be performed at the level of individual MAPs $(X^{(m)}, J^{(m)})$. This procedure may lead to reducing the computational burden by several orders of magnitude.

Chapter 4

First passage: general theory

In the previous chapter we presented results on the first passage problem under the assumption of time-reversibility. In particular, we identified the matrix Λ, which turned out to be similar to a real diagonal matrix. It was shown that the non-zero eigenvalues of $-\Lambda$ coincide with the positive zeros of $\det(F(\alpha))$. Moreover, the corresponding eigenspaces and null spaces have a very close relation. There is no reason to expect that all the eigenvalues of Λ are real in general. In the following we will consider the zeros of $\det(F(\alpha))$ in its region of analyticity, that is, in $\mathbb{C}^{\mathrm{Re}>0}$. The main difficulty stems from the fact that Λ may not be similar to a diagonal matrix. In other words, some eigenvalues may have geometric multiplicity strictly smaller then their algebraic one. In this case the eigenvectors do not provide enough information, and Jordan chains are to be considered.

We have seen that null spaces of $F(\alpha)$ correspond to eigenspaces of Λ. But then what kind of object associated with $F(\alpha)$ corresponds to an arbitrary Jordan chain of Λ? This question will be answered using the theory of analytic matrix functions. It turns out that the concepts of eigenvalues and Jordan chains can be extended to analytic matrix functions, where in the classical case this matrix function is a monic matrix polynomial of degree 1. So the zeros of $\det(F(\alpha))$ and the corresponding null spaces can be called eigenvalues and eigenspaces in that generalized sense.

Before we proceed to the theory of analytic matrix functions, let us illustrate the above mentioned problems with a simple example.

Example 4.1. Let (X, J) be a MAP with the following matrix exponent

$$F(\alpha) = \begin{pmatrix} -1+\alpha & 1 & 0 \\ 0 & -1+\alpha+\alpha^2 & 1 \\ 1 & 0 & -1+\frac{2}{5}\alpha \end{pmatrix}.$$

Observe that $N_+ = 3$. Moreover, $\det(F(\alpha))$ is a fourth order polynomial with the following zeros: $-3/2, 0, 2, 2$. So the only zero in $\mathbb{C}^{\text{Re}>0}$ is $\alpha = 2$, which has multiplicity 2. But the algebraic and geometric multiplicities of the null eigenvalue of $F(2)$ are both 1. The construction (3.1) requires *two* vectors, and hence cannot be satisfactory.

This chapter is organized as follows. First, we present some fundamental definitions and results from the theory of analytic matrix functions. This theory builds upon the theory of matrix polynomials. Most of the basic concepts are well illustrated by the latter, see Gohberg et al. [1982]. Having the necessary tools at hand, we proceed in Section 4.2 to the main results concerning the relation between $F(\alpha)$ and matrices Π and Λ. These results appeared in D'Auria et al. [2010]. The present proof is, however, new in one direction, see Lemma 4.13. Next, using our main results, we derive in Section 4.3 a matrix integral equation. This equation can be considered as a commonly used tool for identification of Π and Λ, and appears in a number of papers in the literature under different assumptions. Moreover, we discuss the relationship between the results, and different methods of proving them. In Section 4.4 we present an alternative approach based on entirely analytic arguments. These arguments are taken from Ivanovs et al. [2010], and served as a basis for D'Auria et al. [2010]. The material of this section, which is rather technical, is not required in the rest of this thesis.

In the final two sections of this chapter we discuss the applicability of our results. Importantly, our results serve as a basis for developing a *technique*, which allows to derive various further identities. This technique is an extension of the approach known as 'martingale calculations for MAPs', see Asmussen [2003, Ch. XI, 4a], to its final and general form. That is, no assumptions on simplicity and the number of zeros are required. The discussion of the technique is followed by applications. In particular, we consider a Markov-modulated queue and derive a generalization of the famous Pollaczek-Khintchine formula. Moreover, we compute the transforms of the supremum and infimum of X up to an independent exponentially distributed time. Further applications of the technique can be found in Chapter 5.

4.1 Generalized Jordan chains

The concept of a Jordan chain associated with an analytic matrix function plays a fundamental role our theory. In this section we review some basic facts from analytic matrix function theory. These facts can be found, for example, in Gohberg and Rodman [1981], Hryniv and Lancaster [1999].

Let $A(z)$ be an analytic matrix-valued function of a complex variable z taking values in some domain $D \subset \mathbb{C}$. It is assumed that $A(z)$ is $n \times n$ matrix, and $\det(A(z))$ is not identically zero. For any $\lambda \in D$ and z in a close enough neighborhood of λ we can write

$$A(z) = \sum_{i=0}^{\infty} \frac{1}{i!} A^{(i)}(\lambda)(z-\lambda)^i,$$

where $A^{(i)}(\lambda)$ denotes the i-th derivative of $A(z)$ at λ, see Appendix A.5. A complex number $\lambda \in D$ is an *eigenvalue* of $A(z)$ if $\det(A(\lambda)) = 0$.

Definition 4.2. We say that vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1} \in \mathbb{C}^n$ with $\boldsymbol{v}_0 \neq \mathbf{0}$ form a *Jordan chain* of $A(z)$ corresponding to the eigenvalue λ if

$$\sum_{i=0}^{j} \frac{1}{i!} A^{(i)}(\lambda)\boldsymbol{v}_{j-i} = \mathbf{0}, \qquad j = 0, \ldots, r-1. \tag{4.1}$$

Remark 4.3. A classical Jordan chain of a square matrix M, see Appendix A.2, is obtained by considering a first order monic polynomial $A(z) = z\mathbb{I} - M$. Indeed, in this case equations (4.1) reduce to

$$M\boldsymbol{v}_0 = \lambda\boldsymbol{v}_0, \quad M\boldsymbol{v}_1 = \lambda\boldsymbol{v}_1 + \boldsymbol{v}_0, \quad \ldots, \quad M\boldsymbol{v}_{r-1} = \lambda\boldsymbol{v}_{r-1} + \boldsymbol{v}_{r-2}.$$

Let m be the multiplicity of λ as a zero of $\det(A(z))$ and p be the dimension of the null space of $A(\lambda)$. Then there exists a *canonical system of Jordan chains*

$$\boldsymbol{v}_0^{(k)}, \boldsymbol{v}_1^{(k)}, \ldots, \boldsymbol{v}_{r_k-1}^{(k)}, \qquad k = 1, \ldots, p$$

corresponding to the eigenvalue λ, such that

- the vectors $\boldsymbol{v}_0^{(1)}, \ldots, \boldsymbol{v}_0^{(p)}$ form the basis of the null space of $A(\lambda)$,
- $\sum_{i=1}^{p} r_i = m$.

Such a canonical system of Jordan chains is specified by the matrices:

$$\begin{aligned} V &= [\boldsymbol{v}_0^{(1)}, \boldsymbol{v}_1^{(1)}, \ldots, \boldsymbol{v}_{r_1-1}^{(1)}, \ldots, \boldsymbol{v}_0^{(p)}, \boldsymbol{v}_1^{(p)}, \ldots, \boldsymbol{v}_{r_p-1}^{(p)}], \\ \Gamma &= \operatorname{diag}[\Gamma_{r_1}(\lambda), \ldots, \Gamma_{r_p}(\lambda)], \end{aligned} \tag{4.2}$$

where $\Gamma_k(\lambda)$ is the Jordan block of size $k \times k$ corresponding to the eigenvalue λ, see Appendix A.2.

Definition 4.4. A pair of matrices (V, Γ) given by (4.2) is called a *Jordan pair* of $A(z)$ corresponding to the eigenvalue λ.

It is noted that a Jordan pair is unique up to similarity. That is, if (V, Γ) and (V', Γ') are two Jordan pairs corresponding to the same eigenvalue, then $V' = VS$ and $\Gamma' = S^{-1}\Gamma S$ for some invertible matrix S. Unlike the classical case, the vectors forming a Jordan chain are *not* necessarily linearly independent; furthermore, a Jordan chain may contain a null vector.

Assume for a moment that $A(z)$ is a matrix polynomial of degree l, that is $A(z) = \sum_{i=0}^{l} A_i z^i$. Then (4.1) is equivalent to the following matrix equation

$$\sum_{i=0}^{l} A_i V \Gamma^i = \mathbb{O}, \tag{4.3}$$

see also Proposition 1.10 and the comments thereafter in Gohberg et al. [1982]. It is instructive to prove this equivalence using the Identity (A.1) in Appendix A.3. In the classical case we obtain a familiar equation $MV = V\Gamma$. The following result is well known and is an immediate consequence of (4.1).

Proposition 4.5. *Let $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ be a Jordan chain of $A(z)$ corresponding to the eigenvalue λ, and let $C(z)$ be an $m \times n$ dimensional matrix. If $B(z) = C(z)A(z)$ is $r-1$ times differentiable at λ, then*

$$\sum_{i=0}^{j} \frac{1}{i!} B^{(i)}(\lambda) \boldsymbol{v}_{j-i} = \boldsymbol{0}, \qquad j = 0, \ldots, r-1.$$

Note that if $B(z)$ is a square matrix then $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ is a Jordan chain of $B(z)$ corresponding to the eigenvalue λ. It is, however, not required that $C(z)$ and $B(z)$ are square matrices. Finally we state the following corollary.

Corollary 4.6. *Let (V, Γ) be a Jordan pair of $A(z)$. Assume that*

$$\boldsymbol{c}(z) A(z) = \sum_{i=1}^{K} f_i(z) \boldsymbol{u}_i,$$

where $\boldsymbol{c}(z), \boldsymbol{u}_i \in \mathbb{C}^n$ and $f_i(z)$ are entire functions. Then it holds that

$$\sum_{i=1}^{K} \boldsymbol{u}_i \, V \, f_i(\Gamma) = \boldsymbol{0}. \tag{4.4}$$

Proof. It is enough to show that Identity (4.4) holds for an arbitrary Jordan chain $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{n-1}$ of $A(z)$ corresponding to some eigenvalue λ. Letting $\boldsymbol{b}(z) = \sum_{i=1}^{K} f_i(z)\boldsymbol{u}_i$, we have according to Proposition 4.5 that

$$\sum_{j=0}^{k} \frac{1}{j!}\boldsymbol{b}^{(j)}(\lambda)\,\boldsymbol{v}_{k-j} = \boldsymbol{0}, \quad k = 0, \ldots, n-1.$$

It only remains to observe that the columns of $Vf_i(\Gamma)$ are given by

$$\sum_{j=0}^{k} \frac{1}{j!} f_i^{(j)}(\lambda)\boldsymbol{v}_{k-j}, \quad k = 0, \ldots, n-1,$$

according to (A.1) in Appendix A.3. □

4.2 Main results

Consider a (possibly killed) spectrally negative MAP $(X(t), J(t))$ with matrix exponent $F(\alpha)$. The killing rate $q \geq 0$ is implicit here and in the following. If $q = 0$ then we define the asymptotic drift κ as in Section 2.4. Observe that $F(\alpha)$ is an analytic matrix function on $\mathbb{C}^{\mathrm{Re}>0}$, because $\psi_i(\alpha)$ and $G_{ij}(\alpha)$ are analytic on this domain for any i, j, see Section 2.1 and Section A.7. Hence we can talk about (generalized) Jordan pairs of $F(\alpha), \alpha \in \mathbb{C}^{\mathrm{Re}>0}$. Let us immediately formulate the main result of this chapter.

Theorem 4.7. *A pair (V, Γ) is a Jordan pair of $F(\alpha)$ corresponding to an eigenvalue $\lambda \in \mathbb{C}^{\mathrm{Re}>0}$ if and only if (V_+, Γ) is a Jordan pair of $\alpha\mathbb{I} + \Lambda$ corresponding to the eigenvalue $\lambda \neq 0$ and $V = \Pi V_+$.*

This theorem follows immediately from Lemma 4.11 and Lemma 4.13 given in the final part of this section. As a consequence of Theorem 4.7, the zeros of $\det(F(\alpha))$ and $\det(\alpha\mathbb{I} + \Lambda)$ in $\mathbb{C}^{\mathrm{Re}>0}$ coincide. This leads to the following result, see also Remark 2.17.

Theorem 4.8. *The number of zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ is equal to*

(i) N_+ if Q is transient,

(ii) $N_+ - 1_{\{\kappa \geq 0\}}$ if Q is recurrent.

The equation $\det(F(\alpha)) = 0$ can be seen as a generalization of the Cramér-Lundberg equation in the theory of Lévy processes: $\psi(\alpha) - q = 0$. It is noted that a number of special cases of Theorem 4.8 are scattered over the literature, see e.g. Asmussen et al. [2004], Karandikar and Kulkarni [1995], Mandjes and Scheinhardt [2008], Regterschot and de Smit [1986], Sonneveld [2004], Tzenova et al. [2005].

Furthermore, Theorem 4.7 allows to obtain a complete characterization of the matrices Λ and Π in terms of the Jordan pairs of $F(\alpha)$. Namely, let $\lambda_1, \ldots, \lambda_n$ be all the distinct zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$. Let also (V_i, Γ_i) be a Jordan pair of $F(\alpha)$ corresponding to λ_i, and put $(V_0, \Gamma_0) = (\mathbf{1}, 0)$. Then we define

$$V = [V_i] \text{ and } \Gamma = \operatorname{diag}(\Gamma_i), \text{ where } i = \begin{cases} 0, \ldots, n & \text{if } q = 0, \kappa \geq 0, \\ 1,\ldots,\mathrm{n} & \text{otherwise.} \end{cases}$$

In other words, we use (V_0, Γ_0) only in the case of no killing and non-negative asymptotic drift.

Corollary 4.9. *The matrices V and Γ have N_+ columns, V_+ is invertible, and the following holds*

$$\Lambda = -V_+\Gamma(V_+)^{-1} \textit{ and } \Pi = V(V_+)^{-1}.$$

Proof. Use Theorem 4.7 to show that $-\Lambda V_+ = V_+\Gamma$ and $\Pi V_+ = V$, see also (4.3). Observe that V_+ is composed from the *classical* Jordan chains of $-\Lambda$. So the invertibility of V_+ is a standard fact from linear algebra. □

The matrices V and Γ will play a central role in the remainder of this thesis. Hence we introduce the following definition.

Definition 4.10. A pair (V, Γ) is called (right) spectral pair of $F(\alpha)$.

The word 'right' in the above definition indicates that the pair (V, Γ) is constructed from Jordan pairs corresponding to the eigenvalues in $\mathbb{C}^{\mathrm{Re}>0}$. A MAP without negative jumps similarly leads to a left spectral pair, where the eigenvalues in $\mathbb{C}^{\mathrm{Re}<0}$ are of interest. Moreover, if MAP has no jumps then $F(\alpha)$ is analytic on the whole of $\mathbb{C}$. The latter case gives rise to a full spectral pair, see Section 5.1 for an in-depth discussion.

Let us present a proof of Theorem 4.7, which is split into two parts. First, we prove a generalization of Lemma 3.2.

Lemma 4.11. *Let $\boldsymbol{v}^0, \ldots, \boldsymbol{v}^r$ be a Jordan chain of $F(\alpha)$ corresponding to the eigenvalue $\lambda \in \mathbb{C}^{\mathrm{Re}>0}$. Then $\boldsymbol{v}^0_+, \ldots, \boldsymbol{v}^r_+$ is a Jordan chain of $\alpha\mathbb{I} + \Lambda$ corresponding to the eigenvalue λ, and $\Pi\boldsymbol{v}^i_+ = \boldsymbol{v}^i$ for $i = 0, \ldots, r$.*

Proof. Consider the martingale $M(t \wedge T)$ presented in Corollary 2.11 with $Y = 0$ and $T = \tau_x$. It leads to the following equation for any $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ and $k \in E$

$$\mathbb{E}_k \left[\int_0^{t\wedge\tau_x} e^{\alpha X(s)} \boldsymbol{e}_{J(s)}^\top \mathrm{d}s \right] F(\alpha) + \boldsymbol{e}_k^\top - \mathbb{E}_k[e^{\alpha X(t\wedge\tau_x)} \boldsymbol{e}_{J(t\wedge\tau_x)}^\top] = \mathbf{0}^\top.$$

Observe that $\mathbb{E}_k[e^{\alpha X(t\wedge\tau_x)}; J(t\wedge\tau_x)] - \boldsymbol{e}_k^\top$ is infinitely many times differentiable in $\alpha \in \mathbb{C}^{\mathrm{Re}>0}$ in view of Proposition A.15. Moreover, differentiation can be performed under the expectation sign. Apply Proposition 4.5 to see that for all $j = 0, \dots, r$ the following holds true:

$$\sum_{i=0}^{j} \frac{1}{i!} \mathbb{E}_k \left[X^i(t\wedge\tau_x) e^{\lambda X(t\wedge\tau_x)} \boldsymbol{e}_{J(t\wedge\tau_x)}^\top \right] \boldsymbol{v}^{j-i} - \boldsymbol{e}_k^\top \boldsymbol{v}^j = 0.$$

Let $t \to \infty$ and use the dominated convergence theorem and Lemma A.18 to obtain

$$\sum_{i=0}^{j} \frac{1}{i!} x^i e^{\lambda x} \mathbb{P}_k(J(\tau_x)) \boldsymbol{v}^{j-i} - \boldsymbol{e}_k^\top \boldsymbol{v}^j = 0. \tag{4.5}$$

Note that the case of no killing when $\mathbb{P}_k(\tau_x = \infty) > 0$ should be treated with care. In this case $\lim_{t\to\infty} X(t) = -\infty$ a.s. according to Proposition 2.15, so the above equation is still valid.

Considering (4.5) for all $k \in E$ and choosing $x = 0$, we obtain $\Pi \boldsymbol{v}_+^j = \boldsymbol{v}^j$. For $k \in E_+$ we get

$$\sum_{i=0}^{j} \frac{1}{i!} x^i e^{(\lambda\mathbb{I}+\Lambda)x} \boldsymbol{v}_+^{j-i} - \boldsymbol{v}_+^j = \mathbf{0}_+,$$

which results in $(\lambda\mathbb{I}+\Lambda)\boldsymbol{v}_+^j + \boldsymbol{v}_+^{j-1} = \mathbf{0}_+$ after differentiating at $x = 0$. □

In the rest of this section we prove the converse of Lemma 4.11. Before we do so, let us present a martingale, see also Rogers [1994, Section 7]. The proof of the lemma below relies on some basic properties of the conditional expectation, see for example Kallenberg [2002, Theorem 6.1].

Lemma 4.12. *Fix $a > 0$ and a vector $\boldsymbol{h}$, and let*

$$f(j, x) = \mathbb{E}_j[h_{J(\tau_a)} | X(0) = x] = \mathbb{P}_j(J(\tau_{a-x}))\boldsymbol{h},$$

where $f(\partial_J, \partial_X) = 0$ by convention. Then $f(J(t\wedge\tau_a), X(t\wedge\tau_a))$ is an $\mathcal{F}_{t\wedge\tau_a}$-martingale for any initial distribution of J.

Proof. It is sufficient to show that

$$f(J(t_a), X(t_a)) = \mathbb{E}_i[h_{J(\tau_a)} | \mathcal{F}_{t_a}] \quad \text{a.s.},$$

where $t_a = t \wedge \tau_a$. Use the strong Markov property to write $\mathbb{E}_i[h_{J(\tau_a)} 1_B]$, where $B \in \mathcal{F}_{t_a}$, as follows:

$$\sum_j \int_{x \le a} \mathbb{P}_i[B | J(t_a) = j, X(t_a) = x] f(j, x) \mathbb{P}_i(J(t_a) = j, X(t_a) \in \mathrm{d}x)$$
$$= \mathbb{E}_i[f(J(t_a), X(t_a)) 1_{\{B\}}].$$

This completes the proof. □

Let us present the second part of the main result, which can be seen as a generalization of Corollary 3.6.

Lemma 4.13. *Let $\boldsymbol{h}_0, \ldots, \boldsymbol{h}_r$ be a Jordan chain of $\alpha\mathbb{I} + \Lambda$ corresponding to the eigenvalue $\lambda < 0$. Then $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_r$ is a Jordan chain of $F(\alpha)$ corresponding to the eigenvalue λ, where $\boldsymbol{v}_i = \Pi \boldsymbol{h}_i$ for $i = 0, \ldots, r$.*

Proof. Observe that $(\Lambda + \lambda\mathbb{I})\boldsymbol{h}_k = -\boldsymbol{h}_{k-1}$ for $k = 1, \ldots, r$. Write the exponential $e^{(\Lambda+\lambda\mathbb{I})x}$ as a series to obtain

$$e^{\Lambda x}\boldsymbol{h}_k = e^{-\lambda x}(\boldsymbol{h}_k - x\boldsymbol{h}_{k-1} + \ldots + \frac{1}{k!}(-x)^k \boldsymbol{h}_0),$$

where $k \le r$. Letting $\boldsymbol{h} = \boldsymbol{h}_k$ in Lemma 4.12 we get

$$f(j, x) = \boldsymbol{e}_j^\top \Pi e^{\Lambda(a-x)} \boldsymbol{h}_k = \boldsymbol{e}_j^\top \sum_{i=0}^{k} \frac{1}{i!}(x-a)^i e^{\lambda(x-a)} \boldsymbol{v}_{k-i}.$$

From $\mathbb{E}_j f(J(t_a), X(t_a)) = f(j, 0)$ we obtain

$$\sum_{i=0}^{k} \frac{1}{i!} \mathbb{E}[(X(t_a) - a)^i e^{\lambda(X(t_a)-a)}; J(t_a)] \boldsymbol{v}_{k-i} = \sum_{i=0}^{k} \frac{1}{i!}(-a)^i e^{-\lambda a} \boldsymbol{v}_{k-i}.$$

Multiply this equation by $a^{r-k}/(r-k)!$ and sum over $k = 0, \ldots, r$ to obtain

$$\sum_{i=0}^{r} \frac{1}{i!} \mathbb{E}[X^i(t_a) e^{\lambda X(t_a)}; J(t_a)] \boldsymbol{v}_{r-i} = \boldsymbol{v}_r, \tag{4.6}$$

see Lemma A.20. Observe that $e^{\alpha X(t_a)} \le e^{\alpha X(t)} + e^{\alpha a} \le e^{\alpha X(t)} + 1$ for any $\alpha < 0$. Let $a \to \infty$ and use the dominated convergence theorem with Lemma A.18 to see that (4.6) holds with t_a replaced by t. This in turn implies that

$$\sum_{i=0}^{r} \frac{1}{i!} \frac{\partial^i e^{F(\lambda)t}}{\partial^i \lambda} \boldsymbol{v}_{r-i} = \boldsymbol{v}_r.$$

Differentiate at $t = 0$ and exchange the order of differentiation (which is possible according to Clairaut's theorem in calculus) to finish the proof. □

Proof of Theorem 4.7. Combine the results of Lemma 4.11 and Lemma 4.13. □

4.3 The matrix integral equation

Let (X, J) be a MAP, where the associated transition rate matrix Q is allowed to be transient. It is known that the matrices Π and Λ characterizing the corresponding first passage process solve a certain matrix integral equation. Moreover, this pair is a unique solution in a certain domain. Under various specific assumptions, this equation appears in e.g. Asmussen [1995], Breuer [2008], Dieker and Mandjes [2009], Miyazawa [2009], Pistorius [2006], Prabhu and Zhu [1989], Rogers [1994].

Let $\mathcal{M}_+$ be a set of matrices in $\mathbb{R}^{N_+\times N_+}$ with all the eigenvalues in $\mathbb{C}^{\mathrm{Re}>0}$. In addition, let $\mathcal{M}_0$ be a set of matrices in $\mathbb{R}^{N_+\times N_+}$ with a simple eigenvalue at 0 and all others in $\mathbb{C}^{\mathrm{Re}>0}$. Define $\mathcal{M} = \mathcal{M}_+$ if the MAP (X, J) is killed or $\kappa < 0$, and $\mathcal{M} = \mathcal{M}_0$ otherwise. Moreover, we denote through $\mathcal{P}$ the set of $N \times N_+$ real matrices P satisfying $P_+ = \mathbb{I}$. Clearly, $-\Lambda \in \mathcal{M}$ and $\Pi \in \mathcal{P}$.

For any choice of $M \in \mathcal{M}$ and $P \in \mathcal{P}$ we write $F(P, M)$ to denote the following $N \times N_+$ matrix

$$F(P,M) = \Delta_{\boldsymbol{a}} PM + \frac{1}{2}\Delta^2_{\boldsymbol{\sigma}} PM^2 + \int_{-\infty}^{0} \Delta_{\boldsymbol{\nu}}(\mathrm{d}x) P \left(e^{Mx} - \mathbb{I} - Mx1_{\{x>-1\}}\right) + \int_{-\infty}^{0} Q \circ G(\mathrm{d}x) P e^{Mx}, \tag{4.7}$$

where $(a_i, \sigma_i, \nu_i(\mathrm{d}x))$ are the Lévy triplets corresponding to the Lévy processes X_i, see (2.1), and $G_{ij}(\mathrm{d}x)$ is the distribution of U_{ij}. In other words, if we forget for a while that P and M are matrices and substitute $P = 1$ and $M = \alpha$, we get exactly $F(\alpha)$.

Let us comment on the integrals appearing in (4.7), see also Appendix A.3. These integrals converge absolutely for the above choice of M. It is enough to show that

$$\int_{-\infty}^{0} \|e^{Mx} - \mathbb{I} - Mx1_{\{x>-1\}}\|_\infty \nu_i(\mathrm{d}x) < \infty, \qquad \int_{-\infty}^{0} \|e^{Mx}\|_\infty G_{ij}(\mathrm{d}x) < \infty$$

for all i, j. In fact, it is sufficient to show the above with M replaced by its Jordan matrix Γ. Use Identity (A.1) to compute the elements of $e^{\Gamma x} - \mathbb{I} - \Gamma x 1_{\{x>-1\}}$ and $e^{\Gamma x}$. Observe that it is only left to establish absolute convergence of

$$\int_{-\infty}^{0} [\partial^k (e^{\lambda x} - 1 - \lambda x 1_{\{x>-1\}})/\partial^k \lambda] \nu_i(\mathrm{d}x), \qquad \int_{-\infty}^{0} x^k e^{\lambda x} G_{ij}(\mathrm{d}x),$$

where λ is any eigenvalue of M and k does not exceed its multiplicity. Use Lemma A.18, Lemma A.19, and $\int_{-\infty}^{0}(1 \wedge x^2)\nu_i(\mathrm{d}x) < \infty$ to conclude.

Theorem 4.14. *It holds that* $F(\Pi,-\Lambda)=\mathbb{O}$. *Moreover,* $(\Pi,-\Lambda)$ *is the only such pair in* $\mathcal{P}\times\mathcal{M}$.

Proof. Consider a Jordan decomposition (H,Γ) of $M\in\mathcal{M}$, that is, $M=H\Gamma H^{-1}$, and define $V=PH$. For an entire function f, see Appendix A.3, we have that $Pf(M)=PHf(\Gamma)H^{-1}=Vf(\Gamma)H^{-1}$. Pick an arbitrary $n\in\{1,\dots,N_+\}$ and consider the n-th column of Γ. Suppose this column corresponds to the k-th column in some Jordan block with eigenvalue λ. Then according to (A.1) the n-th column of $Vf(\Gamma)$ is

$$\sum_{i=0}^{k-1}\frac{1}{i!}f^{(i)}(\lambda)\boldsymbol{v}_{n-i}.$$

Hence the n-th column of $F(P,M)H$ is given by

$$\sum_{i=0}^{k-1}\frac{1}{i!}\Big[\Delta_{\boldsymbol{a}}\frac{\partial^i\lambda}{\partial^i\lambda}+\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\frac{\partial^i\lambda^2}{\partial^i\lambda}+\int_{-\infty}^{0}\frac{\partial^i(e^{\lambda x}-1-\lambda x1_{\{x>-1\}})}{\partial^i\lambda}\Delta_{\boldsymbol{\nu}}(\mathrm{d}x)$$
$$+Q\circ\int_{-\infty}^{0}\frac{\partial^i e^{\lambda x}}{\partial^i\lambda}G(\mathrm{d}x)\Big]\boldsymbol{v}_{n-i}=\sum_{i=0}^{k-1}\frac{1}{i!}F^{(i)}(\lambda)\boldsymbol{v}_{n-i},$$

where the last equality holds, because the differentiation operators can be taken outside of the integral signs for any $\lambda\in\mathbb{C}^{\mathrm{Re}>0}$. If $\lambda=0$ then $k=1$, and hence there is no differentiation. So $F(P,M)=\mathbb{O}$ if and only if (V,Γ) is composed from Jordan pairs of $F(\alpha)$. Theorem 4.7 shows that $F(\Pi,-\Lambda)=\mathbb{O}$, and Corollary 4.9 establishes uniqueness. □

Observe that if $N_+=N$ then $P=\mathbb{I}$, so that P 'disappears' from the matrix integral equation. In the case of a Markov-modulated Brownian motion (MMBM) we obtain a result of Rogers [1994] and Asmussen [1995].

Corollary 4.15. *If* (X,J) *is an MMBM then* $(\Pi,-\Lambda)$ *is the unique pair* $(P,M)\in\mathcal{P}\times\mathcal{M}$ *such that*

$$\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}PM^2+\Delta_{\boldsymbol{a}}PM+QP=\mathbb{O}.$$

4.4 Alternative approaches: the analytic method

Consider the main result of this chapter, namely Theorem 4.7, which relates the Jordan pairs of $F(\alpha)$ and $\alpha\mathbb{I}+\Lambda$. This result can be equivalently stated in terms of a matrix integral equation, see Theorem 4.14 and its proof. Moreover, the statement that $(\Pi,-\Lambda)$ is a solution is equivalent to Lemma 4.13, whereas uniqueness is equivalent to Lemma 4.11. Hence an alternative approach would be to attack the matrix integral equation right away.

A proof of the fact that $(\Pi, -\Lambda)$ is a solution of the matrix integral equation can be found in Breuer [2008], where the author applies the infinitesimal generator of (X, J) to the function $f(x, j) = \mathbb{E}[e^{\alpha\tau_x}; J(\tau_x) = j]$. It should also be mentioned that an early work Prabhu and Zhu [1989], see also Prabhu [1998, Ch. 7], considers the matrix integral equation for a Markov-modulated compound Poisson process. The theory of infinitesimal generators plays a key role in these works. Finally, a method based on discretization and Wiener-Hopf factorization for Lévy processes can be found in Dieker and Mandjes [2009].

In the rest of this section we consider the generalized C-L equation, and prove that it has a certain number of zeros in $\mathbb{C}^{\mathrm{Re}>0}$, see Theorem 4.8. This proof entirely relies on analytic arguments. This result then can be used with either Lemma 4.11 or Lemma 4.13 to establish a one-to-one relation between the zeros of $\det(F(\alpha))$ and the eigenvalues of $-\Lambda$ in $\mathbb{C}^{\mathrm{Re}>0}$, resulting in an alternative proof of Theorem 4.7.

In the following we prove Theorem 4.8(i), where Q is assumed to be transient, from first principles using analytic arguments. This proof is based on Ivanovs et al. [2010]. Importantly, Theorem 4.8(ii) follows from Theorem 4.8(i) by a limiting argument, which we sketch in the next paragraph.

Note that Theorem 4.8(i), and either Lemma 4.11 or Lemma 4.13, show that there is a one-to-one relation between the zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ and the eigenvalues of $-\Lambda$. Assume that Q is recurrent. Consider a sequence $(Q - q/n\mathbb{I})$ of transient matrices. We use $F_n(\alpha)$ and Λ_n to denote matrices $F(\alpha)$ and Λ corresponding to $(Q - q/n\mathbb{I})$. It is easy to see from (2.9) that $e^{\Lambda_n} \to e^{\Lambda}$. This further implies $\Lambda_n \to \Lambda$, see Proposition A.11. Hence the eigenvalues of Λ_n converge to the eigenvalues of Λ (preserving multiplicities) as $n \to \infty$. All the eigenvalues of Λ are in $\mathbb{C}^{\mathrm{Re}>0}$ except a simple one at 0 if $\kappa \geq 0$. But $F(\alpha)$ is analytic in $\mathbb{C}^{\mathrm{Re}>0}$, which allows to use Hurwitz's theorem, see Appendix A.5, to extend the one-to-one relation between zeros and eigenvalues to the recurrent case.

It is noted that one can also try to approach the recurrent case stated in Theorem 4.8(ii) using analytic arguments. This case, however, presents many additional difficulties, see Ivanovs et al. [2010]. In this work we had to exclude the case when $\kappa \in \{0, -\infty\}$.

Basic idea

The main ingredient of the proof is given by the following lemma, which relies on the argument principle, see Appendix A.5. Let $D \subset \mathbb{C}$ be a bounded domain with boundary γ, which is a simple closed curve.

Lemma 4.16. *Let $M : D \mapsto \mathbb{C}^{N\times N}$ be a matrix-valued function and $f(z) := \det(M(z))$. Assume that*

A1 all $m_{ij}(z)$ are analytic on D and continuous on $D \cup \gamma$,

A2 $\forall i$ and $\forall z \in \gamma : |m_{ii}(z)| \geq \sum_{j\neq i} |m_{ij}(z)| \neq 0$,

A3 $f(z) \neq 0$ for $z \in \gamma$.

Then $f(z)$ and $\prod_{i=1}^{N} m_{ii}(z)$ have the same number of zeros in D.

The main idea of the proof is taken from Gail et al. [1992], where the authors use the following procedure. First they introduce an additional parameter t; the original function is retrieved by taking $t = 1$. For $t = 0$, however, the function has a nice form (that is, it nicely factorizes) making the analysis of the number of zeros easy. Then essentially continuity arguments are used to conclude that the number of zeros, as a function of the new parameter t, is constant. This basic idea used in a related context can be also found in Boudreau et al. [1962] and Sonneveld [2004].

Proof of Lemma 4.16. Define $f(z,t) := \det(M_t(z))$ for $t \in [0,1]$, where $M_t(z)$ is a $N \times N$ matrix obtained from $M(z)$ by multiplying the off-diagonal elements by t. Note that $f(z,0) = \prod_{i=1}^{N} m_{ii}(z)$ and $f(z,1) = f(z)$. Moreover, $f(z,t) \neq 0$ for all $z \in \gamma$. To see this use assumption A3 when $t = 1$ and A2 when $t < 1$. In the second case $M_t(z), z \in \gamma$ is strictly diagonally dominant and thus non-singular, see Appendix A.1. Since $f(z,t)$ is a continuous function on $\overline{D} \times [0,1]$, one can choose $\delta > 0$, such that $f(z,t) \neq 0$ on $E_\delta \times [0,1]$, where $E_\delta := \{z \in D : y \in \gamma, |z-y| < \delta\}$ is a boundary strip of D. This is true, because otherwise there exists a converging sequence of the zeros with a limit (z^*, t^*), such that $z^* \in \gamma$ and $f(z^*, t^*) = 0$.

Let n_t denote the number of zeros (counting multiplicities) of the function $f_t(z) := f(z,t)$ in D. Take some simple closed curve $\gamma' \subset E_\delta$ (which is possible) and write using the argument principle

$$n_t = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma'} \frac{f_t'(z)}{f_t(z)} \mathrm{d}z.$$

Note that n_t is integer-valued and continuous, because $f_t'(z)/f_t(z)$ is continuous in t uniformly in $z \in \gamma'$. This means that n_t is constant. $\square$

Properties of the Laplace exponent of a Lévy process

Next we present two technical lemmas concerning the Laplace exponent $\psi(\alpha)$ of a Lévy process without positive jumps.

Lemma 4.17. *It holds that* $|\psi(\alpha)| \to \infty$ *as* $|\alpha| \to \infty$ *through* $\mathbb{C}^{\mathrm{Re}\geq 0}$ *unless* X *is a pure jump process, that is,* $\psi(\alpha) = \int_{-\infty}^{0}(e^{\alpha x} - 1)\nu(\mathrm{d}x)$.

Proof. Let us start by proving that

$$\lim_{|\alpha|\to\infty, \alpha\in\mathbb{C}^{\mathrm{Re}\geq 0}} \frac{\psi(\alpha)}{\alpha^2} = \frac{1}{2}\sigma^2. \tag{4.8}$$

This limit follows immediately from Representation (2.1) if we can show that $\int_{-1}^{0}(e^{\alpha x} - 1 - \alpha x)\nu(\mathrm{d}x)/\alpha^2 \to 0$. The latter is true, because the dominated convergence theorem applies, see Lemma A.19. Thus the proof of the lemma is complete if $\sigma > 0$. Assume that $\sigma = 0$ and $\int_{-\infty}^{0}(1 \wedge |x|)\nu(\mathrm{d}x) < \infty$. Consider Representation (2.2) and follow the same steps as above to show that $\psi(\alpha)/\alpha \to d$.

It is only left to consider the case when $\int_{-\infty}^{0}(1 \wedge |x|)\nu(\mathrm{d}x) = \infty$ and $\sigma = 0$. Note that $|\int_{-\infty}^{-1}(e^{\alpha x} - 1)\nu(\mathrm{d}x)|$ is bounded for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ and hence we can assume without loss of generality that

$$\psi(\alpha) = a\alpha + \int_{-1}^{0}(e^{\alpha x} - 1 - \alpha x)\nu(\mathrm{d}x).$$

The rest of the proof will be split in two steps.

Step 1. We show that $\mathrm{Im}(\psi(u + \mathrm{i}v))/v \to \infty$ as $|v| \to \infty$ uniformly in $u \geq 0$. Note that

$$\mathrm{Im}(\psi(u + \mathrm{i}v)) = av + \int_{-1}^{0}(e^{ux}\sin(vx) - vx)\nu(\mathrm{d}x)$$

is an odd function in v, thus it is enough to consider the case when $v > 0$. Note also that $y - \sin(y) \geq 0$ for $y \geq 0$, and so $e^{ux}\sin(vx) - vx \geq 0$ for $x < 0$. Thus we have for any $\epsilon > 0$

$$\begin{aligned}\frac{\mathrm{Im}(\psi(u + \mathrm{i}v))}{v} &\geq a + \int_{-1}^{\epsilon}\left(\frac{e^{ux}\sin(vx) - vx}{v}\right)\nu(\mathrm{d}x)\\ &\geq a + \int_{-1}^{\epsilon}(-x)\nu(\mathrm{d}x) - \int_{-1}^{\epsilon}\frac{1}{v}\nu(\mathrm{d}x) \to a + \int_{-1}^{\epsilon}(-x)\nu(\mathrm{d}x) \text{ as } v \to \infty.\end{aligned}$$

Recall that $\int_{-1}^{0}(-x)\nu(\mathrm{d}x) = \infty$ to complete the proof of the first step.

Step 2. We show that given any constants $C > 0$ and $C_v > 0$ one can choose a large $C_u > 0$, so that $\mathrm{Re}(\psi(u + \mathrm{i}v)) > C$ for all u and v such that $|v| \leq C_v$ and

$u > C_u$. Observe that the process under consideration cannot have non-increasing paths a.s., hence $\psi(u) \to \infty$ as $u \to \infty$, see Section 2.1. Next note that

$$\frac{\partial \mathrm{Re}(\psi(u + \mathrm{i}v))}{\partial v} = -\int_{-1}^{0} xe^{ux} \sin(vx)\nu(\mathrm{d}x),$$

because differentiation can be done under the integral sign according to Proposition A.15. But

$$\left|\int_{-1}^{0} xe^{ux} \sin(vx)\nu(\mathrm{d}x)\right| \leq C_v \int_{-1}^{0} x^2\nu(\mathrm{d}x) < \infty$$

for $|v| \leq C_v$. So it is enough to choose C_u such that $\psi(u) > C + C_v^2 \int_{-1}^{0} x^2\nu(\mathrm{d}x)$ for all $u > C_u$.

Now pick any $C > 0$. The result of Step 1 implies that there exists a large enough $C_v > 0$, so that $|\mathrm{Im}(\psi(u + \mathrm{i}v))| > C$ for all $u \geq 0$ and all v satisfying $|v| > C_v$. Combining this with the result of Step 2, we see that there exists $C_u > 0$, such that $|\psi(\alpha)| > C$ when $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ and $|\alpha| > C_u + C_v$, which completes the proof. □

Lemma 4.18. *For a fixed $q > 0$ the equation $\psi(\alpha) = q$ has no solution in $\mathbb{C}^{\mathrm{Re}\geq 0}\backslash\mathbb{R}$.*

Proof. Let $u \geq 0, v \neq 0$ and assume that $\psi(u + \mathrm{i}v) = q$, then

$$au + \frac{1}{2}\sigma^2(u^2 - v^2) + \int_{-\infty}^{0} (e^{ux} \cos(vx) - 1 - ux1_{\{x>-1\}})\nu(\mathrm{d}x) = q,$$
$$av + \sigma^2 uv + \int_{-\infty}^{0} (e^{ux} \sin(vx) - vx1_{\{x>-1\}})\nu(\mathrm{d}x) = 0.$$

Divide the second equation by v, multiply it by u and subtract it from the first equality to obtain

$$\frac{1}{2}\sigma^2(-u^2 - v^2) + \int_{-\infty}^{0} \left(-\frac{u}{v}e^{ux} \sin(vx) - 1 + e^{ux} \cos(vx)\right) \nu(\mathrm{d}x) > 0.$$

This is impossible, because

$$\frac{q}{r}e^{-q} \sin r - 1 + e^{-q} \cos r \leq 0 \text{ for } q \geq 0, r \neq 0,$$

which follows from $\sin r/r \leq 1$ and $e^q - 1 \geq q$. □

Completing the proof

Proof of Theorem 4.8(i). Define a bounded domain

$$D_R = \{\alpha \in \mathbb{C}^{\mathrm{Re}>0} : |\alpha| < R\}$$

for all $R > 0$. Let us first show that there exists $R > 0$ large enough, so that $F(\alpha)$ is irreducibly diagonally dominant in $\mathbb{C}^{\mathrm{Re}\geq 0}\backslash D_R$. Observe that for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ the matrix $F(\alpha)$ is irreducible. Moreover, its (i,j)-th off-diagonal element is bounded in absolute value by q_{ij}. Recall that Q is transient, hence it is enough to establish that $|q_{ii} + \psi_i(\alpha)| \geq |q_{ii}|$. This inequality is satisfied if $|\psi(\alpha)|$ is large, or if $\mathrm{Re}(\psi_i(\alpha)) \leq 0$, because $q_{ii} < 0$. Note that

$$e^{\mathrm{Re}(\psi_i(\mathrm{i}r))} = |e^{\psi_i(\mathrm{i}r)}| = |\mathbb{E}e^{\mathrm{i}rX_i(1)}| \leq 1$$

for any i and $r \in \mathbb{R}$. Hence $\mathrm{Re}(\psi_i(\alpha)) \leq 0$ for all α on the imaginary axis. Moreover, if $\psi_i(\alpha) = \int_{-\infty}^{0}(e^{\alpha x} - 1)\nu(\mathrm{d}x)$ then $\mathrm{Re}(\psi_i(\alpha)) \leq 0$ for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. Finally, use Lemma 4.17 to see that there exists $R > 0$ as claimed above.

An irreducibly diagonal matrix is non-singular, see Appendix A.1. Hence for R large enough $\det(F(\alpha))$ has no zeros in $\mathbb{C}^{\mathrm{Re}\geq 0}\backslash D_R$. Moreover, D_R and $F(\alpha)$ satisfy the assumptions of Lemma 4.16. Thus it remains to show that $\prod(\psi_i(\alpha) + q_{ii})$ has N_+ zeros in $\mathbb{C}^{\mathrm{Re}>0}$. Lemma 4.18 shows that we only need to consider $\alpha > 0$. If X_i has non-increasing paths a.s. then $\psi_i(\alpha) < 0$, otherwise $\psi_i(\alpha) + q_{ii}$ has one simple zero by convexity, see Section 2.1. □

4.5 Applications via martingale calculations

The results presented in this chapter are not only about identification of the matrices Λ and Π characterizing the first passage of a MAP. We prefer to see our contribution rather as the development of a new technique: the theory of analytic matrix functions, combined with the special structure of the Jordan pairs of $F(\alpha)$, and their relation to the matrices Λ and Π, enables the derivation of a set of further identities. In the next section we demonstrate this technique with a simple example. More specifically, we compute the transform of the stationary workload in a Markov-modulated queue. This result can be seen as a generalization of the famous Pollaczek-Khintchine formula to the MAP setting. Further examples of the applicability of our technique can be found in Chapter 5.

Our technique roughly consists of the following steps.

- use a martingale argument to arrive at an initial equation involving the unknown quantities and $F(\alpha)$;

- use the properties of Jordan chains such as stated in Proposition 4.5, Equation (A.1), and Corollary 4.6, to rewrite the initial equation in terms of a spectral pair (V, Γ);

- use the special structure of (V, Γ), such as invertibility of V_+, to simplify the equation;

- rewrite the equation in terms of matrices Λ and Π using Corollary 4.9 to gain some probabilistic insight and claim uniqueness of the solution.

It is noted that this approach can be seen as an extension of the ideas known as 'martingale calculations for MAPs', see Asmussen [2003, Ch. XI, 4a], to its final and general form. It is important that no assumptions about the number and simplicity of the eigenvalues are needed. For some problems certain eigenvalues are inherently non-simple. An example of such a problem is discussed in Chapter 5, where a Markov-modulated Brownian motion (MMBM) is considered. In the case of MMBM, one works with a full spectral pair instead of a right spectral pair. In this setting Corollary 5.7 is essential to prove uniqueness of the solution.

4.6 Queues and extremum processes

In this section we compute the transform of the stationary workload in a Markov-modulated queue. Moreover, we obtain the transform of the workload at an exponential epoch, which then leads to the transforms of extremes of X considered up to this epoch. Let (Y, J) be a MAP without negative jumps. Consider the one-sided reflection of Y at 0 as defined in Section 2.8. Recall that the reflected process R has the form $R(t) = Y(t) + L(t)$, where the regulator L is given by $L(t) = -(\underline{Y}(t) \wedge 0)$. The aim of this section is to characterize the stationary distribution of $(R(t), J(t))$ when it exists. It is noted that the case of no positive jumps is rather simple, see Corollary 2.23.

It is convenient to define the process $X(t) = -Y(t)$, because then (X, J) is a MAP without positive jumps. As from now we forget about the process Y and work exclusively with X, whose matrix exponent is denoted as usual through $F(\alpha)$. Throughout this section it is assumed that Q is *recurrent* (the case of no killing). In this context Λ is to be understood as $\Lambda(0)$. Firstly, we have $R(t) = -X(t)+L(t)$, where $L(t) = \overline{X}(t) \vee 0$. Assume that the asymptotic drift of X is positive, that is $\kappa > 0$. Repeating the arguments of Theorem 2.22, we see that the process $(R(t), J(t))$ has a stationary version. The corresponding distribution coincides with the distribution of $(-\underline{\hat{X}}(\infty), \hat{J}(0))$. As before we let a random pair (R^*, J^*) have this distribution as well.

Observe that $-R(t)$ is bounded from above and $-L(t)$ is continuous. Moreover, $\mathbb{E}\overline{X}(\infty) < \infty$, because $\overline{X}(\infty)$ is of phase type. Hence $-L(t)$ has a finite expected variation on compact intervals. So according to Corollary 2.11 $M(t)$ given by

$$\int_0^t e^{-\alpha R(s)} \boldsymbol{e}_{J(s)}^\top \mathrm{d}s F(\alpha) + e^{-\alpha R(0)} \boldsymbol{e}_{J(0)}^\top - e^{-\alpha R(t)} \boldsymbol{e}_{J(t)}^\top - \alpha \int_0^t e^{-\alpha R(s)} \boldsymbol{e}_{J(s)}^\top \mathrm{d}L(s)$$

is a martingale for any initial distribution of (X, J) and all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. Let $(X(0), J(0))$ be distributed as (R^*, J^*). Then from $\mathbb{E}M(1) = M(0) = \mathbf{0}^\top$ we get

$$\mathbb{E}[e^{-\alpha R^*}; J^*]F(\alpha) = \alpha \mathbb{E} \int_0^1 \boldsymbol{e}_{J(s)}^\top \mathrm{d}L(s) = \alpha \boldsymbol{\ell}.$$

There is no $e^{-\alpha R(s)}$ under the integral sign, because the points of increase of $L(t)$ are contained in $\{t : R(t) = 0\}$. It is only left to determine the vector of non-negative constants $\boldsymbol{\ell}$, which we will do using the technique outlined above.

Before we start let us note that $l_i = 0$ for all $i \in E_\downarrow$. Moreover, $\boldsymbol{\ell}\mathbf{1} = \mathbb{E}L(t)/t$ for any $t > 0$. But $R(t)/t \to 0$ and $X(t)/t \to \kappa$ a.s. as $t \to \infty$, see Lemma 2.14, hence $L(t)/t \to \kappa$ and then $\boldsymbol{\ell}\mathbf{1} = \kappa$.

Let $(V_\lambda, \Gamma_\lambda)$ be a Jordan pair of $F(\alpha)$ corresponding to some eigenvalue $\lambda \in \mathbb{C}^{\mathrm{Re}>0}$ then $\boldsymbol{\ell} V_\lambda \Gamma_\lambda = \mathbf{0}$ according to Corollary 4.6. This implies $\boldsymbol{\ell} V_\lambda = \mathbf{0}$, which can be also derived directly from Proposition 4.5. Hence for a right spectral pair (V, Γ) of $F(\alpha)$, see Definition 4.10, it holds that $\boldsymbol{\ell} V = \kappa \boldsymbol{e}_1^\top$ and so $\boldsymbol{\ell}_+ = \kappa \boldsymbol{e}_1^\top (V_+)^{-1}$. But $\boldsymbol{e}_1^\top (V_+)^{-1} = \boldsymbol{\pi}_\Lambda$, the stationary distribution corresponding to Λ. To see this it is enough to check that $\boldsymbol{e}_1^\top (V_+)^{-1} \Lambda = \mathbf{0}$, which is indeed true in view of Corollary 4.9. This results in the following proposition.

Proposition 4.19. *Let $R(t) \geq 0$ be a reflection of $-X(t)$ at 0. If $\kappa > 0$, then the process $(R(t), J(t))$ has a stationary version. A random pair (R^*, J^*) having this stationary distribution satisfies*

$$\mathbb{E}[e^{-\alpha R^*}; J^*]F(\alpha) = \alpha \boldsymbol{\ell}$$

for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$, where $\boldsymbol{\ell}_\downarrow = \mathbf{0}$ and $\boldsymbol{\ell}_+ = \kappa \boldsymbol{\pi}_\Lambda = \kappa \boldsymbol{e}_1^\top (V_+)^{-1}$.

Using the above outlined technique we essentially showed that

$$\ell_i = \mathbb{E} \int_0^1 1_{\{J(s)=i\}} \mathrm{d}L(s) = \kappa (\pi_\Lambda)_i, \text{ where } i \in E_+.$$

This is a very simple relation which should have a direct probabilistic proof. This is indeed the case. First, observe that $\ell_i = \lim_{t\to\infty} \frac{1}{t} \int_0^t 1_{\{J(s)=i\}} \mathrm{d}L(s)$. Note that

$\mathrm{d}L(t)$ coincides with $\mathrm{d}\overline{X}(t)$ for large t, that is, for $t \geq \inf\{t \geq 0 : R(t) = 0\}$. So ℓ_i represents (up to a common scaling factor κ) the long run proportion of time $J(\tau_x)$ spends in state i, which is exactly $(\pi_\Lambda)_i$. More precisely, the change of variable $x = \overline{X}(s)$ leads to

$$\frac{1}{t}\int_0^t 1_{\{J(s)=i\}}\mathrm{d}\overline{X}(s) = \frac{\overline{X}(t)}{t}\int_0^{\overline{X}(t)} 1_{\{J(\tau_x)=i\}}\mathrm{d}x/\overline{X}(t).$$

But $\overline{X}(t)/t \to \kappa$, which completes the proof.

Finally, it is important to observe that the transform of $(R(t), X(t), J(t))$ at an independent exponentially distributed time e_q can be obtained along the same lines as Proposition 4.19.

Proposition 4.20. *Let R be the reflection of $-X$ at 0, where $R(0) = -X(0) = w_0 \geq 0$. Then for $q > 0$ and $\alpha, \beta \geq 0$ it holds that*

$$-\frac{1}{q}\mathbb{E}[e^{-\alpha R(e_q)-\beta L(e_q)}; J(e_q)](F(\alpha) - q\mathbb{I})$$
$$= e^{-\alpha r_0}\mathbb{I} + (\alpha+\beta)[\Pi(q)e^{\Lambda(q)r_0}(\Lambda(q) - \beta\mathbb{I})^{-1}, \mathbb{O}].$$

Proof. First observe that $-\alpha R - \beta L = \alpha X - (\alpha+\beta)L$. Apply Corollary 2.11 with $Z(t) = \alpha X(t) - (\alpha+\beta)L(t)$. For this, note that $(\alpha X, J)$ is a MAP; its matrix exponent at 1 is given by $F(\alpha)$. In addition, $\mathbb{E}L(e_q) < \infty$, because $\overline{X}(e_q)$ is of phase-type. So $M(t \wedge e_q)$ is a martingale. Note that $\mathbb{E}\int_0^{e_q} e^{-\alpha R(s)-\beta L(s)}\boldsymbol{e}_{J(s)}^\top \mathrm{d}s = \mathbb{E}[e^{-\alpha R(e_q)-\beta L(e_q)}; J(e_q)]/q$. Apply Doob's optional stopping theorem at $t \wedge e_q$ and let $t \to \infty$ to obtain

$$\frac{1}{q}\mathbb{E}[e^{-\alpha R(e_q)-\beta L(e_q)}; J(e_q)](F(\alpha) - q\mathbb{I})$$
$$= (\alpha+\beta)\mathbb{E}\int_0^{e_q} e^{-\alpha R(s)-\beta L(s)}\boldsymbol{e}_{J(s)}^\top \mathrm{d}L(s) - e^{-\alpha r_0}\mathbb{I}.$$

It is left to identify the matrix $\mathbb{E}\int_0^{e_q} e^{-\beta L(s)}\boldsymbol{e}_{J(s)}^\top \mathrm{d}L(s)$. This can be done either using the approach based on the theory of generalized Jordan chains, or the probabilistic argument. The latter written in a succinct form reads

$$\mathbb{E}\int_0^{e_q} e^{-\beta L(s)}\boldsymbol{e}_{J(s)}^\top \mathrm{d}L(s) = \Pi(q)e^{\Lambda(q)r_0}\mathbb{E}_+\int_0^\infty e^{-qs}e^{-\beta\overline{X}(s)}\boldsymbol{e}_{J(s)}^\top \mathrm{d}\overline{X}(s)$$
$$= \Pi(q)e^{\Lambda(q)r_0}\int_0^\infty e^{-\beta y}\mathbb{E}_+[e^{-q\tau_y^+}; J(\tau_y^+)]\mathrm{d}y = -\Pi(q)e^{\Lambda(q)r_0}(\Lambda(q) - \beta\mathbb{I})^{-1},$$

where we restrict $J(s), s > 0$ under the integral signs to the indices in E_+. The restriction of $\mathbb{E}\int_0^{e_q} e^{-\beta L(s)}\boldsymbol{e}_{J(s)}^\top \mathrm{d}L(s)$ to the columns in $E_\downarrow$ is the zero matrix. □

The following corollary identifies the transforms of the infimum and the supremum of X, see also Kyprianou and Palmowski [2008, Thm. 4]. Another representation of this transform is given in Dieker and Mandjes [2009, Thm. 3.2].

Corollary 4.21. *For $q > 0$ and $\alpha, \beta \geq 0$ it holds that*

$$-\frac{1}{q}\mathbb{E}[e^{\alpha X(e_q)-(\alpha+\beta)\overline{X}(e_q)}; J(e_q)](F(\alpha) - q\mathbb{I}) = \mathbb{I} + (\alpha+\beta)[\Pi(q)(\Lambda(q) - \beta\mathbb{I})^{-1}, \mathbb{O}],$$

$$-\frac{1}{q}(F(\alpha) - q\mathbb{I})\mathbb{E}[e^{(\alpha+\beta)\underline{X}(e_q)-\beta X(e_q)}; J(e_q)]$$
$$= \mathbb{I} + (\alpha+\beta)\Delta_{\boldsymbol{\pi}}^{-1}[\hat{\Pi}(q)(\hat{\Lambda}(q) - \beta\mathbb{I})^{-1}, \mathbb{O}]^\top \Delta_{\boldsymbol{\pi}}.$$

Proof. The first equation is a direct consequence of Proposition 4.20 with $r_0 = 0$. Using (2.7) this can be rewritten as

$$-\frac{1}{q}(\hat{F}(\alpha) - q\mathbb{I})\Delta_{\boldsymbol{\pi}}^{-1}\mathbb{E}[e^{\alpha X(e_q)-(\alpha+\beta)\overline{X}(e_q)}; J(e_q)]^\top \Delta_{\boldsymbol{\pi}}$$
$$= \mathbb{I} + (\alpha+\beta)\Delta_{\boldsymbol{\pi}}^{-1}[\Pi(q)(\Lambda(q) - \beta\mathbb{I})^{-1}, \mathbb{O}]^\top \Delta_{\boldsymbol{\pi}}.$$

Finally, a time reversal argument as in the proof of Theorem 2.22 shows that $(\overline{X}(e_q), X(e_q), J(0), J(e_q))$ has the distribution of the following random vector $(X(e_q) - \underline{X}(e_q), X(e_q), J(e_q), J(0))$ under $\hat{\mathbb{P}}$. □

Remark 4.22. It is important to note that the identities of Corollary 4.21 hold true for a wider set of (α, β). Observe that the eigenvalues of $\Lambda(q)$ and $\hat{\Lambda}(q)$ coincide, because so do the zeros of $\det(F(\alpha)-q\mathbb{I})$ and $\det(\hat{F}(\alpha)-q\mathbb{I})$. In particular, the Perron-Frobenius eigenvalues coincide; they are denoted through $\rho(q) < 0$. The identities of Corollary 4.21 are valid for all $\alpha \geq 0$ and $\beta > \rho(q)$. To see this, use analyticity and Proposition A.17.

Chapter 5

Markov-modulated Brownian motion (MMBM) in a strip

MAPs with a.s. continuous sample paths form an important special case of spectrally negative MAPs. Let (X, J) be such a MAP. Then every $U_{ij} = 0$ and every X_i is a Brownian motion with parameters (a_i, σ_i^2). It is noted that σ_i is allowed to be 0, that is, X_i can be a deterministic drift. We call such a process Markov-modulated Brownian motion or MMBM for short. This section is devoted to the study of MMBM reflected to stay in a strip $[0, B]$, see Section 2.8.

The present model is also called in the literature *a second-order fluid model* or *a fluid model with Brownian noise*. It was introduced as a generalization of an extensively studied fluid flow model, where it is assumed that all the variance parameters are 0, making the process piecewise linear. Fluid models were initially proposed for manufacturing and telecommunication systems, where units of work (products or packets) are processed so fast that they can be modelled as fluid instead of discrete units. Since then the use of fluid models has become widespread, making it a classical model in applied probability with a variety of application areas, like the theory of queues and dams, risk processes and insurance, biology problems, etc. The literature on this topic is extensive; we only mention the seminal papers by Kosten [1974/75], Anick et al. [1982], a survey by Kulkarni [1997], and a more recent paper by Ahn et al. [2007] with an extensive list of references.

Second-order fluid models were simultaneously introduced in Asmussen [1995], Karandikar and Kulkarni [1995], and Rogers [1994]. The paper by Rogers [1994] can be considered as one of the most influential papers, not just in the theory of fluid models, but in the much more general theory of fluctuations of MAPs. In that paper the stationary distribution of a reflected MMBM is derived for both a single barrier and two barriers assuming that either all the variance parameters are zero or all are positive, see the comments in Section 5.5. The case of a single barrier, see also Asmussen [1995] and Karandikar and Kulkarni [1995], is a special case of the latter problem with $B = \infty$. In this case the analysis can be extended to MAPs with one-sided jumps, see Section 2.8 and Section 4.5. An important reference in this context is Prabhu and Zhu [1989], where the stationary distribution of an infinite buffer Markov-modulated M/G/1 queue is obtained.

Fluid models play a prominent role in applied probability. The importance comes from the fact that they are flexible enough to model a variety of different phenomena, and at the same time the analysis often remains tractable. Recall that phase-type jumps can be easily incorporated in the model, see also Section 2.7. This shows the importance of an MMBM, where some variance parameters are allowed to be 0. This case is often omitted in the literature, as in Rogers [1994].

This chapter is organized in the following way. First, we present some fundamental preliminary results concerning MMBM. Then we find the transform of the stationary distribution of a MMBM reflected to stay in a strip. A central role here is played by the loss vectors, which are determined using our technique outlined in Section 4.5. Next, we solve the two-sided exit problem; see also Jiang and Pistorius [2008] for an alternative approach. The stationary distribution of an MMBM in a strip turns out to have an explicit form. We provide an easy to understand argument based on time reversal leading to this result. Finally, we discuss alternative approaches to this problem and show how they relate to each other.

5.1 Preliminaries

Start by observing that the matrix exponent $F(\alpha)$ of an MMBM (X, J) is of a particularly simple form:

$$F(\alpha) = \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\alpha^2 + \Delta_{\boldsymbol{a}}\alpha + Q.$$

Thus the asymptotic drift is $\kappa = \boldsymbol{\pi}\boldsymbol{a}$, see (2.6). Importantly, both (X, J) and $(-X, J)$ are MAPs without positive jumps; the matrix exponent of the latter is $F(-\alpha)$. Hence the first passage theory presented in this thesis can be applied to both processes. We denote the corresponding first passage processes by τ_x^+ and

τ_x^- respectively. Consistently with Section 2.6, the transition rate matrix of $J(\tau_x^\pm)$ is denoted through $\Lambda^\pm$, and the matrix with initial distributions through $\Pi^\pm$. Let E_- and $E_\uparrow$ be the analogues of E_+ and $E_\downarrow$ corresponding to the process $(-X, J)$. That is,

$$E_+ = \{j \in E : \sigma_j > 0 \text{ or } a_j > 0\}, \quad E_\downarrow = \{j \in E : \sigma_j = 0 \text{ and } a_j \leq 0\},$$
$$E_- = \{j \in E : \sigma_j > 0 \text{ or } a_j < 0\}, \quad E_\uparrow = \{j \in E : \sigma_j = 0 \text{ and } a_j \geq 0\}.$$

We let N_- and $N_\uparrow$ be the cardinalities of E_- and $E_\uparrow$. So, for example, Λ^- is a $N_- \times N_-$ matrix. Throughout this section it is assumed that X is not a monotone process, that is, $N_+ > 0$ and $N_- > 0$.

In the case of MMBM the matrix exponent $F(\alpha)$ is a second order matrix polynomial. So it is an analytic matrix function on the whole of $\mathbb{C}$. This section is devoted to the study of the eigenvalues of $F(\alpha)$ in $\mathbb{C}$ and the corresponding Jordan pairs, see Section 4.1 for general definitions. Most of the results of this section will follow immediately from Section 4.2.

Recall that there is a one-to-one relationship between the Jordan pairs of $F(\alpha)$ and $\Lambda^+ + \alpha\mathbb{I}^+$ in $\mathbb{C}^{\mathrm{Re}>0}$, see Theorem 4.7. It then follows that the same relationship holds between the Jordan pairs of $F(\alpha)$ and $\Lambda^- - \alpha\mathbb{I}^-$ in $\mathbb{C}^{\mathrm{Re}<0}$. Next we consider Corollary 4.9. Let (V^+, Γ^+) be formed from the Jordan pairs of $F(\alpha), \alpha \in \mathbb{C}^{\mathrm{Re}>0}$ and in addition $(\mathbf{1}, 0)$ (as the first component) if $\kappa \geq 0$. Similarly, let (V^-, Γ^-) be formed from the Jordan pairs of $F(\alpha), \alpha \in \mathbb{C}^{\mathrm{Re}<0}$ and in addition $(\mathbf{1}, 0)$ (as the first component) if $\kappa \leq 0$. It is assumed that $\boldsymbol{\pi}$ is not defined if Q is transient. Then it holds that

$$\begin{array}{ll} \Lambda^+ = -V_+^+\Gamma^+(V_+^+)^{-1} & \Pi^+ = V^+(V_+^+)^{-1} \\ \Lambda^- = \quad V_-^-\Gamma^-(V_-^-)^{-1} & \Pi^- = V^-(V_-^-)^{-1}, \end{array} \tag{5.1}$$

where the second line is true because a Jordan pair (V, Γ) of $\Lambda^- - \alpha\mathbb{I}^-$ solves $\Lambda^- V - V\Gamma = \mathbb{O}^-$ according to (4.3), so that it is a classical Jordan pair of Λ^-.

Observe that $\det(F(\alpha))$ is a polynomial of degree $N_+ + N_-$. If the matrix Q is transient then $\det(F(\alpha))$ has N_+ zeros in $\mathbb{C}^{\mathrm{Re}>0}$ and N_- zeros in $\mathbb{C}^{\mathrm{Re}<0}$. Hence none of the zeros lies on the imaginary axis. If Q is recurrent then these numbers are $N_+ - 1_{\{\kappa \geq 0\}}$ and $N_- - 1_{\{\kappa \leq 0\}}$. Hence if $\kappa \neq 0$ then there is a single zero on the imaginary axis at 0. Finally, in the 0-drift case $\kappa = 0$ there is again a unique zero on the imaginary axis at 0. Its multiplicity, however, is 2. To see this, consider the zeros of $\det(F(\alpha) - q\mathbb{I})$ and let $q \downarrow 0$. These zeros correspond to the eigenvalues of $-\Lambda^+(q)$ and $\Lambda^-(q)$, and hence exactly 2 of them converge to 0. Finally, the null space of $F(0) = Q$ is spanned by the single vector $\mathbf{1}$. This implies that there exists a vector $\boldsymbol{h}$ such that

$$F(0)\boldsymbol{h} + F'(0)\mathbf{1} = Q\boldsymbol{h} + \Delta_{\boldsymbol{a}}\mathbf{1} = \mathbf{0}, \tag{5.2}$$

according to (4.1). Then $(\mathbf{1}, \boldsymbol{h})$ is a Jordan chain corresponding to the null eigenvalue. So if Q is recurrent we can pick a Jordan pair (V_0, Γ_0) corresponding to the eigenvalue 0 of the form:

$$\begin{aligned} &V_0 = \mathbf{1}, \Gamma_0 = 0 && \text{if } \kappa \neq 0, \\ &V_0 = (\mathbf{1}, \boldsymbol{h}), \Gamma_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} && \text{if } \kappa = 0. \end{aligned} \tag{5.3}$$

The Jordan chain $(\mathbf{1}, \boldsymbol{h})$ will play an important role in the analysis of the 0-drift case in the rest of this chapter. Let us present a simple lemma concerning this chain.

Lemma 5.1. *If $\kappa = 0$ and $(\mathbf{1}, \boldsymbol{h})$ is a Jordan chain of $F(\alpha)$ corresponding to the null eigenvalue, then*

$$\boldsymbol{\pi}(\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\mathbf{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h}) \neq 0.$$

Proof. Assume that $\boldsymbol{\pi}(\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\mathbf{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h}) = 0$. Then the vector in brackets should be in the column space of Q, because Q has rank $N - 1$ and $\boldsymbol{\pi}Q = \mathbf{0}$. Hence there exists a vector $\boldsymbol{v}$ such that

$$\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\,\mathbf{1} + \Delta_{\boldsymbol{a}}\,\boldsymbol{h} + Q\boldsymbol{v} = \mathbf{0}.$$

So $(\mathbf{1}, \boldsymbol{h}, \boldsymbol{v})$ is a Jordan chain of $F(\alpha)$ corresponding to the null eigenvalue. This implies then that 0 is a zero of $\det(F(\alpha))$ of multiplicity at least 3, see Gohberg et al. [1982, Theorem 7.1]. We have arrived at a contradiction. □

We conclude this discussion with a definition.

Definition 5.2. A pair of matrices (V, Γ) is called a *spectral pair* if it is composed of Jordan pairs $(V_\lambda, \Gamma_\lambda)$ of $F(\alpha)$: $V = [V_\lambda], \Gamma = \text{diag}(\Gamma_\lambda)$, where λ runs over all eigenvalues of $F(\alpha)$ in $\mathbb{C}$. If 0 is an eigenvalue then it is in addition assumed that (V_0, Γ_0), as defined in (5.3), is used as the first element in constructing (V, Γ).

Observe that Γ is a square $(N_+ + N_-)$-dimensional matrix and V is a $N \times (N_+ + N_-)$ matrix.

5.2 Transform of the stationary distribution and the loss vectors

Consider the two-sided reflection $R(t)$ of $X(t)$ with respect to the strip $[0, B]$, where $B \in (0, \infty]$, as defined in Section 2.8. Recall that $R(t)$ has the representation $R(t) = X(t) + L(t) - U(t)$, where L and U are the regulators at respectively lower and upper barriers, that is, 0 and B. It is well-known that the process $(R(t), J(t))$ has a stationary version (independent of the sign of the asymptotic drift of X). This is a consequence of the fact that the state space of the Markov process (R, J) is compact. Moreover, it is a regenerative process, which implies uniqueness of the stationary distribution. These facts as well as the material presented in this section up to Theorem 5.3 are taken from Asmussen and Kella [2000]. Theorem 5.3 resolves an open problem left in that paper. As usual, a random pair (R^*, J^*) refers to the stationary distribution of $(R(t), J(t))$.

Let us show that

$$M(t) = \int_0^t e^{\alpha R(s)} \boldsymbol{e}_{J(s)}^\top \mathrm{d}s F(\alpha) + e^{\alpha R(0)} \boldsymbol{e}_{J(0)}^\top - e^{\alpha R(t)} \boldsymbol{e}_{J(t)}^\top$$
$$+ \alpha \int_0^t \boldsymbol{e}_{J(s)}^\top \mathrm{d}L(s) - \alpha e^{\alpha B} \int_0^t \boldsymbol{e}_{J(s)}^\top \mathrm{d}U(s)$$

is a martingale for any $\alpha \in \mathbb{C}$ and any initial distribution of (X, J). This is a consequence of Corollary 2.11 with $Y(t) = L(t) - U(t)$ and the fact that the points of increase of L and U are contained in $\{t \geq 0 : R(t) = 0\}$ and $\{t \geq 0 : R(t) = B\}$ respectively. It is only required to show that $\mathbb{E}L(t) < \infty$ (and hence also $\mathbb{E}U(t) < \infty$ by considering the process $(-X, J)$), which can be reduced to the same problem for a Brownian motion instead of MMBM. But the latter fact is well-known, see Harrison [1985]. Finally, the result is true for $\alpha \in \mathbb{C}^{\mathrm{Re}<0}$, because $(-X, J)$ is an MMBM with matrix exponent $F(-\alpha)$.

Assume that the process (X, J) is started with (R^*, J^*) to obtain

$$\mathbb{E}[e^{\alpha R^*}; J^*]F(\alpha) = -\alpha \boldsymbol{\ell} + \alpha e^{\alpha B} \boldsymbol{u}, \tag{5.4}$$

where $\boldsymbol{\ell} = \mathbb{E} \int_0^1 \boldsymbol{e}_{J(s)}^\top \mathrm{d}L(s)$ and $\boldsymbol{u} = \mathbb{E} \int_0^1 \boldsymbol{e}_{J(s)}^\top \mathrm{d}U(s)$. Observe that $\boldsymbol{u}$ and $\boldsymbol{\ell}$ can be interpreted as the expected overflow and unused capacity in a unit of time in stationarity. We refer to them as to the loss vectors. Note that $\boldsymbol{u}_\downarrow = \boldsymbol{0}$ and $\boldsymbol{\ell}_\uparrow = \boldsymbol{0}$. So it is only required to determine $\boldsymbol{u}_+$ and $\boldsymbol{\ell}_-$.

In this section we construct a system of linear equations, which uniquely determines the loss vectors $\boldsymbol{u}$ and $\boldsymbol{\ell}$. These equations are formulated in terms of an arbitrary spectral pair (V, Γ) of $F(\alpha)$.

Theorem 5.3. *The vectors $\boldsymbol{u}_+$ and $\boldsymbol{\ell}_-$ are the unique solutions to the system of linear equations*

$$(\boldsymbol{u}_+, \boldsymbol{\ell}_-) \begin{pmatrix} V_+ \, e^{B\,\Gamma} \\ -V_- \end{pmatrix} = (\boldsymbol{k}, 0, \ldots, 0), \tag{5.5}$$

where

$$\boldsymbol{k} = \boldsymbol{\pi} \left(\Delta_{\boldsymbol{a}} V_0 + \frac{1}{2} \Delta_{\boldsymbol{\sigma}}^2 V_0 \Gamma_0 \right). \tag{5.6}$$

Remark 5.4. Observe that Equation (5.6) can be rewritten as

$$\boldsymbol{k} = \begin{cases} \kappa, & \text{if } \kappa \neq 0 \\ (0, \boldsymbol{\pi}(\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 \mathbf{1} + \Delta_{\boldsymbol{a}} \boldsymbol{h})), & \text{if } \kappa = 0. \end{cases} \tag{5.7}$$

Let us first present a result concerning $\boldsymbol{k}$.

Lemma 5.5. *It holds that $(\boldsymbol{u} - \boldsymbol{\ell})\mathbf{1} = \kappa$. If $\kappa = 0$, then*

$$B\boldsymbol{u}\mathbf{1} + (\boldsymbol{u} - \boldsymbol{\ell})\boldsymbol{h} = \boldsymbol{\pi}(\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 \mathbf{1} + \Delta_{\boldsymbol{a}} \boldsymbol{h}). \tag{5.8}$$

Proof. Differentiate Equation (5.4) at 0 and right multiply by $\mathbf{1}$ to get $(\boldsymbol{u} - \boldsymbol{\ell})\mathbf{1} = \boldsymbol{\pi a} = \kappa$. If, however, we right multiply the result of differentiation by $\boldsymbol{h}$, we obtain the identity

$$(\boldsymbol{u} - \boldsymbol{\ell})\boldsymbol{h} = \mathbb{E}[R^*; J^*]Q\boldsymbol{h} + \mathbb{P}(J^*)\Delta_{\boldsymbol{a}}\boldsymbol{h} = -\mathbb{E}[R^*; J^*]\Delta_{\boldsymbol{a}}\mathbf{1} + \mathbb{P}(J^*)\Delta_{\boldsymbol{a}}\boldsymbol{h},$$

where the second equality follows from (5.2). Differentiating Equation (5.4) twice at 0 and multiplying by $\mathbf{1}$, we find

$$B\,\boldsymbol{u}\mathbf{1} = \mathbb{E}[R^*; J^*]\Delta_{\boldsymbol{a}}\mathbf{1} + \mathbb{P}(J^*)\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\mathbf{1}.$$

Sum up the above two equations to complete the proof. □

Proof of Theorem 5.3. We split the proof into two steps. First we show that $(\boldsymbol{u}_+, \boldsymbol{\ell}_-)$ solves (5.5), and then we show that the solution is unique.
Step 1. Apply Corollary 4.6 to (5.4) to obtain

$$\boldsymbol{u}Ve^{B\Gamma}\Gamma - \boldsymbol{\ell}V\Gamma = \mathbf{0}. \tag{5.9}$$

Let $\hat{\Gamma}$ be the matrix Γ with Jordan block Γ_0 replaced with $\mathbb{I}$. Suppose first that $\kappa \neq 0$. Then $(V_0, \Gamma_0) = (\mathbf{1}, 0)$ and so (5.9) can be rewritten as

$$\boldsymbol{u}Ve^{B\Gamma}\hat{\Gamma} - \boldsymbol{\ell}V\hat{\Gamma} = ((\boldsymbol{u} - \boldsymbol{\ell})\mathbf{1}, 0, \ldots, 0).$$

Multiply by $\hat{\Gamma}^{-1}$ from the right to obtain (5.5).

Now suppose that $\kappa = 0$. Addition of

$$\hat{\boldsymbol{k}} = \boldsymbol{u} V_0 e^{B\Gamma_0} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - \boldsymbol{\ell} V_0 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

to the first two elements of the vectors appearing on both sides of (5.9), leads to

$$\boldsymbol{u} V e^{B\Gamma} \hat{\Gamma} - \boldsymbol{\ell} V \hat{\Gamma} = (\hat{\boldsymbol{k}}, 0, \ldots, 0).$$

To complete Step 1 it is enough to show that $\hat{\boldsymbol{k}} = \boldsymbol{k}$. A simple computation reveals that $\hat{\boldsymbol{k}} = \boldsymbol{u}(\mathbf{1}, (B-1)\mathbf{1} + \boldsymbol{h}) - \boldsymbol{\ell}(\mathbf{1}, -\mathbf{1} + \boldsymbol{h}) = (0, B\boldsymbol{u}\mathbf{1} + (\boldsymbol{u} - \boldsymbol{\ell})\boldsymbol{h})$, where we used that $(\boldsymbol{u} - \boldsymbol{\ell})\mathbf{1} = \kappa = 0$. Use Lemma 5.5 and (5.7) to see that $\hat{\boldsymbol{k}} = \boldsymbol{k}$.
Step 2. Without loss of generality we assume that $\kappa \geq 0$. Consider the pairs $(V^{\pm}, \Gamma^{\pm})$, see Section 5.1, constructed from the Jordan pairs appearing in the spectral pair (V, Γ). It is easy to see that $(\boldsymbol{u}_+, \boldsymbol{\ell}_-)$ solves (5.5) if and only if

$$(\boldsymbol{u}_+, -\boldsymbol{\ell}_-) \begin{pmatrix} V_+^+ e^{B\Gamma^+} & V_+^- e^{B\Gamma^-} \\ V_-^+ & V_-^- \end{pmatrix} = \kappa \boldsymbol{e}_1^\top$$

and, in addition, Equation (5.8) is satisfied if $\kappa = 0$. The above display can be rewritten as

$$(\boldsymbol{u}_+, -\boldsymbol{\ell}_-) \begin{pmatrix} V_+^+ & V_+^- e^{B\Gamma^-} \\ V_-^+ e^{-B\Gamma^+} & V_-^- \end{pmatrix} = \kappa \boldsymbol{e}_1^\top \begin{pmatrix} e^{-B\Gamma^+} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}^- \end{pmatrix} = \kappa \boldsymbol{e}_1^\top, \tag{5.10}$$

because of the form of Γ^+. According to (5.1) we have

$$V_-^+ e^{-B\Gamma^+} = \Pi_-^+ e^{B\Lambda^+} V_+^+ \text{ and } V_+^- e^{B\Gamma^-} = \Pi_+^- e^{B\Lambda^-} V_-^-.$$

So we obtain

$$\begin{pmatrix} V_+^+ & V_+^- e^{B\Gamma^-} \\ V_-^+ e^{-B\Gamma^+} & V_-^- \end{pmatrix} = \begin{pmatrix} \mathbb{I}^+ & \Pi_+^- e^{B\Lambda^-} \\ \Pi_-^+ e^{B\Lambda^+} & \mathbb{I}^- \end{pmatrix} \begin{pmatrix} V_+^+ & \mathbb{O} \\ \mathbb{O} & V_-^- \end{pmatrix}. \tag{5.11}$$

Observe that $\Pi_-^+ e^{B\Lambda^+} \Pi_+^- e^{B\Lambda^-}$ and $\Pi_+^- e^{B\Lambda^-} \Pi_-^+ e^{B\Lambda^+}$ are irreducible transition probability matrices. It is then not difficult to see that the first matrix on the right hand side of (5.11), call it M, is also irreducible. If $\kappa > 0$ then M is irreducibly diagonally dominant and hence non-singular, because $\Pi_+^- e^{B\Lambda^-}$ is transient. If $\kappa = 0$, then M has a simple eigenvalue at 0 by Perron-Frobenius theory. So

$$(\boldsymbol{u}_+, -\boldsymbol{\ell}_-) \begin{pmatrix} \mathbb{I}^+ & \Pi_+^- e^{B\Lambda^-} \\ \Pi_-^+ e^{B\Lambda^+} & \mathbb{I}^- \end{pmatrix} = \mathbf{0}$$

determines the vector $(\boldsymbol{u}_+, -\boldsymbol{\ell}_-)$ up to a scalar. This scalar is then identified using (5.8):

$$(\boldsymbol{u}_+, -\boldsymbol{\ell}_-)\begin{pmatrix} B\mathbf{1}_+ + \boldsymbol{h}_+ \\ \boldsymbol{h}_- \end{pmatrix} = \boldsymbol{\pi}(\Delta_{\boldsymbol{a}}\boldsymbol{h} + \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\mathbf{1}),$$

which is non-zero by Lemma 5.1. □

We finish this section with two corollaries.

Corollary 5.6. *It holds that*

$$(\boldsymbol{u}_+, -\boldsymbol{\ell}_-)\begin{pmatrix} \mathbb{I}^+ & \Pi_+^- e^{B\Lambda^-} \\ \Pi_-^+ e^{B\Lambda^+} & \mathbb{I}^- \end{pmatrix} = \kappa \begin{cases} (\boldsymbol{\pi}_{\Lambda^+}, \mathbf{0}_-), & \text{if } \kappa > 0 \\ (\mathbf{0}_+, \mathbf{0}_-), & \text{if } \kappa = 0 \\ (\mathbf{0}_+, \boldsymbol{\pi}_{\Lambda^-}), & \text{if } \kappa < 0 \end{cases},$$

where $\boldsymbol{\pi}_{\Lambda^+}$ is the unique stationary distribution of Λ^+, which is well-defined if $\kappa > 0$. Similarly, if $\kappa < 0$, then $\boldsymbol{\pi}_{\Lambda^-}$ denotes the stationary distribution of Λ^-.

Proof. We only consider the case when $\kappa > 0$. Observe that $\boldsymbol{e}_1^\top (V_+^+)^{-1} = c\boldsymbol{\pi}_{\Lambda^+}$ for some constant c, because $\boldsymbol{e}_1^\top (V_+^+)^{-1}\Lambda^+ = \mathbf{0}$ according to (5.1). Moreover, $1 = c\boldsymbol{\pi}_{\Lambda^+}\mathbf{1}_+ = c$, because the first column of V^+ is $\mathbf{1}$. The result then follows from (5.10) and (5.11). □

Corollary 5.7. *For any $B > 0$ the matrix*

$$\begin{pmatrix} \mathbb{I}^+ & \Pi_+^- e^{B\Lambda^-} \\ \Pi_-^+ e^{B\Lambda^+} & \mathbb{I}^- \end{pmatrix}$$

is invertible if Q is transient, or Q is recurrent and $\kappa \neq 0$. If $\kappa = 0$ then it has rank $N_+ + N_- - 1$, and the vector $\begin{pmatrix} B\mathbf{1}_+ + \boldsymbol{h}_+ \\ \boldsymbol{h}_- \end{pmatrix}$ does not belong to its column space.

Proof. See the end of the proof of Theorem 5.3. □

5.3 Two-sided exit

The aim of this section is to characterize the exit times of X from the strip $[-b, a]$, where $a, b \geq 0$, not simultaneously 0. In other words, we determine the following matrices

$$C(a,b) = \mathbb{E}[e^{-q\tau_a^+}; \tau_a^+ < \tau_b^-, J(\tau_a^+)] \text{ and } D(a,b) = \mathbb{E}[e^{-q\tau_b^-}; \tau_b^- < \tau_a^+, J(\tau_b^-)],$$

which are of dimensions $N \times N_+$ and $N \times N_-$ respectively.

Observe that $\tau_a^+ = \tau_b^-$ can only hold if they are both infinite. Use the strong Markov property to write

$$\begin{aligned} C(a,b) &= \mathbb{E}[e^{-q\tau_a^+}; J(\tau_a^+)] - \mathbb{E}[e^{-q\tau_a^+}; \tau_b^- < \tau_a^+, J(\tau_a^+)] \\ &= \mathbb{E}[e^{-q\tau_a^+}; J(\tau_a^+)] - D(a,b)\mathbb{E}[e^{-q\tau_{a+b}^+}; J(\tau_{a+b}^+)], \end{aligned}$$

where $\tau_a < \infty$ is implicit in the first line. Thus we get

$$\begin{aligned} C(a,b) &= \Pi^+ e^{a\Lambda^+} - D(a,b)\Pi_-^+ e^{(a+b)\Lambda^+} \\ D(a,b) &= \Pi^- e^{b\Lambda^-} - C(a,b)\Pi_+^- e^{(a+b)\Lambda^-}, \end{aligned} \tag{5.12}$$

where $\Pi^\pm$ and $\Lambda^\pm$ correspond to the first passage of a process killed with rate $q \geq 0$. The above display can be rewritten as

$$[C(a,b), D(a,b)] \begin{pmatrix} \mathbb{I}^+ & \Pi_+^- e^{B\Lambda^-} \\ \Pi_-^+ e^{B\Lambda^+} & \mathbb{I}^- \end{pmatrix} = [\Pi^+ e^{a\Lambda^+}, \Pi^- e^{b\Lambda^-}]. \tag{5.13}$$

Assume Q is transient, or it is recurrent and $\kappa \neq 0$, then the second matrix on the left is invertible according to Corollary 5.7. In this case one can express $C(a,b)$ and $D(a,b)$ as follows:

$$\begin{aligned} C(a,b) &= (\Pi^+ e^{a\Lambda^+} + \Pi^- e^{b\Lambda^-}\Pi_-^+ e^{(a+b)\Lambda^+})(\mathbb{I}^+ - \Pi_+^- e^{(a+b)\Lambda^-}\Pi_-^+ e^{(a+b)\Lambda^+})^{-1} \\ D(a,b) &= (\Pi^- e^{b\Lambda^-} + \Pi^+ e^{a\Lambda^+}\Pi_+^- e^{(a+b)\Lambda^-})(\mathbb{I}^- - \Pi_-^+ e^{(a+b)\Lambda^+}\Pi_+^- e^{(a+b)\Lambda^-})^{-1}. \end{aligned} \tag{5.14}$$

If $\kappa = 0$ then the above matrix has rank $N_+ + N_- - 1$. Hence (5.13) does not determine the matrices $C(a,b)$ and $D(a,b)$ uniquely. An additional equation is required. We derive this equation using our technique based on Jordan chains of $F(\alpha)$.

Application of the technique

Pick an arbitrary eigenvalue λ of $F(\alpha) - q\mathbb{I}$ and a corresponding Jordan chain $\boldsymbol{v}^0, \ldots, \boldsymbol{v}^r$. Following the same steps as in the proof of Lemma 4.11, but using $\tau = \tau_a^+ \wedge \tau_b^-$ instead of τ_x, we obtain

$$\sum_{i=0}^{j} \frac{1}{i!}\mathbb{E}\left[X(\tau)^i e^{\lambda X(\tau)}; \tau < e_q, J(\tau)\right] \boldsymbol{v}^{j-i} - \boldsymbol{v}^j = \mathbf{0}.$$

This reasoning relies on the dominated convergence theorem and the finiteness of τ.

The above equation can be rewritten as

$$C(a,b)\sum_{i=0}^{j}\frac{1}{i!}a^i e^{\lambda a}\boldsymbol{v}_+^{j-i} + D(a,b)\sum_{i=0}^{j}\frac{1}{i!}(-b)^i e^{-\lambda b}\boldsymbol{v}_-^{j-i} = \boldsymbol{v}^j.$$

Letting (V,Γ) be a Jordan pair of $F(\alpha)-q\mathbb{I}$ corresponding to the eigenvalue λ, and using (A.1), we arrive at

$$C(a,b)V_+e^{a\Gamma} + D(a,b)V_-e^{-b\Gamma} = V. \tag{5.15}$$

This equation can be immediately extended to a spectral pair of $F(\alpha)-q\mathbb{I}$, which we again denote through (V,Γ). Finally, we rewrite it in a form similar to (5.5):

$$[C(a,b),D(a,b)]\begin{pmatrix}V_+e^{(a+b)\Gamma}\\ V_-\end{pmatrix} = Ve^{b\Gamma}. \tag{5.16}$$

Note that (5.15) holds with (V,Γ) replaced by (V^+,Γ^+), which is defined in Section 5.1. After a trivial transformation we get

$$C(a,b) = V^+e^{-a\Gamma^+}(V_+^+)^{-1} - D(a,b)V_-^+e^{-(a+b)\Gamma^+}(V_+^+)^{-1},$$

which is exactly the first equation in (5.12) according to Lemma 5.1. The second equation is obtained by considering (V^-,Γ^-).

Observe that in the case $\kappa=0$ the above procedure results in the loss of the equation associated to the null Jordan chain $(\mathbf{1},\boldsymbol{h})$. This equation is

$$C(a,b)(a\mathbf{1}_+ + \boldsymbol{h}_+) + D(a,b)(-b\mathbf{1}_- + \boldsymbol{h}_-) = \boldsymbol{h}. \tag{5.17}$$

Corollary 5.7 shows that (5.13) together with this equation determines the matrices $C(a,b)$ and $D(a,b)$ uniquely. Here we also used the fact that $\mathbf{1}$ is in the column space of the matrix appearing in Corollary 5.7. We thus arrive at the following theorem.

Theorem 5.8. *The matrices $C(a,b)$ and $D(a,b)$ are uniquely determined by Equation (5.16). Alternatively, they are uniquely determined by (5.13) unless $\kappa=0$, in which case Equation (5.17) should be added.*

5.4 Stationary distribution revisited

Time reversion is an important technique in the study of fluctuations of MAPs. For example, considering one-sided reflection we have shown that (R^*,J^*) has the same distribution as $(\overline{\hat{X}}(\infty), \hat{J}(0))$, see Theorem 2.22. This then immediately yields that $(R^*|J^* = i)$ is phase type. It turns out that time reversion allows to determine the distribution of (R^*, J^*) in the case of two-sided reflection in a rather simple way as well. The basic relation reads

$$\mathbb{P}(R^* \geq x|J^* = i) = \hat{\mathbb{P}}_i(X(\tau_{[x-B,x)}) \geq x), \tag{5.18}$$

where $\tau[u, v) = \inf\{t \geq 0 : X(t) \notin [u, v)\}$ and $\hat{\mathbb{P}}$ denotes the law of the time-reversed process. In particular, if $B = \infty$ the right hand side of (5.18) reduces to $\hat{\mathbb{P}}_i(\overline{X} \geq x)$. This type of representation was first noted in Lindley [1959] in the case of a random walk with two reflecting barriers. A short derivation of its continuous-time analogue is given in Asmussen [2003, Prop. 3.7, Ch. XIV], see also Asmussen and Pihlsgård [2007] for the case of Markov additive input.

Let us illustrate this result with two figures. Firstly, it is convenient to depict two-sided reflection by plotting the free process X and shifting the barriers accordingly. For example, when the process hits the upper barrier, it starts pushing the reflecting strip up, see Figure 5.1. Consider the time reversal of X at some time T,

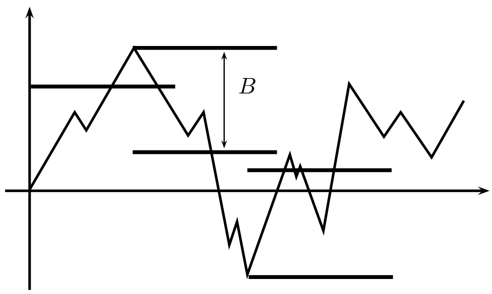

Figure 5.1 Two-sided reflection.

see Figure 5.2. Assume that the time-reversed process exits $[x - B, x)$ through x. In this case the original process cannot shift the reflecting strip high enough (just before T) to make $R(T) < x$. The converse is shown similarly.

In order to simplify the notation in the following, we consider $\hat{\mathbb{P}}(R^* \geq x|J^* = i)$, and note that reversing time twice results in the original process. It is well-known that an MMBM can not hit a level without passing it, so we obtain

$$\hat{\mathbb{P}}(R^* \geq x|J^* = i) = \mathbb{P}_i(\tau_x^+ < \tau_{B-x}^-), \text{ where } x \in (0, B]. \tag{5.19}$$

One should treat the case $x = 0$ with care. Note that this identity indeed does not hold for $x = 0$ and $i \in E_\downarrow$.

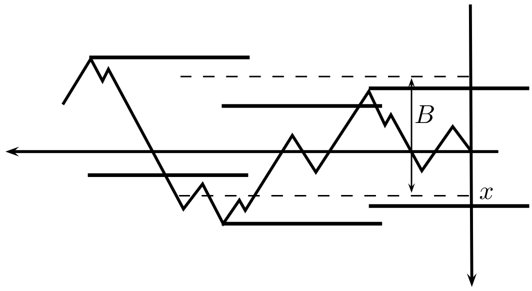

Figure 5.2 Two-sided reflection and the axes corresponding to the time-reversed path.

We have reduced our problem to the two-sided exit problem, which is discussed in depth in Section 5.3. That is, we have

$$\hat{\mathbb{P}}(R^* \geq x | J^* = i) = C_i(x, B - x)\mathbf{1}$$

for $x \in (0, B]$, which is then determined using Theorem 5.8. In the following we provide some explicit formulas under the assumption that $\kappa \neq 0$.

Theorem 5.9. *If $\kappa \neq 0$ then it holds that*

$$\hat{\mathbb{P}}(R^* \geq x | J^*)^\top = (\Pi^+ e^{x\Lambda^+} - \Pi^- e^{(B-x)\Lambda^-} \Pi_-^+ e^{B\Lambda^+}) K^+ \mathbf{1}, \qquad x \in (0, B],$$
$$\hat{\mathbb{P}}(R^* \leq x | J^*)^\top = (\Pi^- e^{(B-x)\Lambda^-} - \Pi^+ e^{x\Lambda^+} \Pi_+^- e^{B\Lambda^-}) K^- \mathbf{1}, \qquad x \in [0, B),$$

where

$$K^+ := (\mathbb{I}^+ - \Pi_+^- e^{B\Lambda^-} \Pi_-^+ e^{B\Lambda^+})^{-1}, \quad K^- := (\mathbb{I}^- - \Pi_-^+ e^{B\Lambda^+} \Pi_+^- e^{B\Lambda^-})^{-1}. \quad (5.20)$$

Proof. The first equation is a consequence of (5.14). The second equation follows by a coupling argument. Consider the two-sided reflection $\tilde{R}(t)$ of $(-X(t), J(t))$ in $[0, B]$. Assuming $X(0) = 0$ and $\tilde{X}(0) = B$ it is easy to see that

$$\mathbb{P}(R(t) \leq x | J(t)) = \mathbb{P}(\tilde{R}(t) \geq B - x | J(t)).$$

Letting $t \to \infty$ we obtain $\hat{\mathbb{P}}(R^* \leq x | J^*) = \hat{\mathbb{P}}(\tilde{R}^* \geq B - x | J^*)$. Note that Λ^+, Π^+ become Λ^-, Π^-, because the MMBM $(-X(t), J(t))$ is used to construct $(\tilde{R}, J)$. □

Note that the two equations in Theorem 5.9 lead to two different representations of the density $\hat{\mathbb{P}}(R^* \in \mathrm{d}x | J^*)$. In addition, one easily obtains the point masses at 0 and B:

$$\hat{\mathbb{P}}(R^* = 0 | J^*)^\top = (\Pi^- - \Pi^+ \Pi_+^-) e^{B\Lambda^-} K^- \mathbf{1},$$
$$\hat{\mathbb{P}}(R^* = B | J^*)^\top = (\Pi^+ - \Pi^- \Pi_-^+) e^{B\Lambda^+} K^+ \mathbf{1}.$$

Finally, $\Pi_+^+ = \mathbb{I}^+$ and $\Pi_-^- = \mathbb{I}^-$ and hence R^* has no mass at 0 (respectively B) given J^* is in E_+ (respectively E_-), which is intuitively clear.

The distribution at an exponential epoch

The above ideas can be used to get insight into the transient behavior of the reflected process. More precisely, one can identify the distribution of the reflected process at an independent exponential time, that is, $\mathbb{P}_i(R(e_q) \leq x, J(e_q) = j)$ given $X(0) \in \{0, B\}$. It is important that X starts at a boundary. If this is not the case the distribution of $R(e_q)$ may not have an explicit form. Then one has to resort to the Laplace transform method as in Section 5.2.

An inspection of the proof of Asmussen [2003, Prop. 3.7, Ch. XIV] reveals that a representation similar to (5.18) holds true for a finite time T:

$$\mathbb{P}_i(R(T) \geq x | J(T) = j) = \hat{\mathbb{P}}_j(\tau_{[x-B,x)} \leq T, X(\tau_{[x-B,x)}) \geq x | J(T) = i),$$

where we assumed that $X(0) = 0$. Note that $\pi_i \mathbb{P}_i(J(T) = j) = \pi_j \hat{\mathbb{P}}_j(J(T) = i)$ to arrive at the following equation

$$\hat{\mathbb{P}}_i(R(e_q) \geq x, J(e_q) = j) = \mathbb{P}_j(\tau_x^+ < e_q, \tau_x^+ < \tau_{B-x}^-, J(e_q) = i)\frac{\pi_j}{\pi_i},$$

which in matrix form can be written as

$$\hat{\mathbb{P}}(R(e_q) \geq x, J(e_q))^\top = \Delta_\pi \mathbb{P}^q(\tau_x^+ < \tau_{B-x}^-, J(\tau_x^+))\mathbb{P}_+(J(e_q))\Delta_\pi^{-1},$$

where $\mathbb{P}^q$ denotes the law of (X, J) killed at rate $q > 0$. Noting that $\mathbb{P}(J(e_q)) = q(q\mathbb{I} - Q)^{-1}$ we find

$$\hat{\mathbb{P}}(R(e_q) \geq x, J(e_q))^\top = q\Delta_{\boldsymbol{\pi}}(\Pi^+ e^{x\Lambda^+} - \Pi^- e^{(B-x)\Lambda^-}\Pi_-^+ e^{B\Lambda^+})K^+[(q\mathbb{I} - Q)^{-1}]_+ \Delta_{\boldsymbol{\pi}}^{-1},$$

where $x \in (0, B]$ and all the occurrences of matrices $\Lambda^\pm$ and $\Pi^\pm$ refer to the q-killed versions. Finally, one can derive a symmetric equation for the case $X(0) = B$.

5.5 On alternative approaches

Consider an MMBM reflected to stay in a strip and its stationary distribution. The problem of characterizing this distribution received some attention in the literature, where a number of different approaches can be found. Some methods are based on the theory of generators of Markov processes, and require solving second-order differential equations. The main work in this direction is Rogers [1994]. A similar method is contained in Karandikar and Kulkarni [1995], which was later extended to the case of two barriers in Ang and Barria [2000]. Alternatively, one can determine the Laplace transform of the stationary distribution. This transform was determined in Asmussen and Kella [2000] up to two vectors of unknown constants. These loss vectors are obtained in Section 5.2, completing the analysis. Finally, one can use time reversal as has been done in Section 5.4. The use of a time reversal argument is also suggested in Asmussen [2003, Sec. XIV.3]. Below we examine the first method, which is based on differential equations, and related results in more detail. We are interested in connecting those results with the statement of Theorem 5.9. Throughout it is assumed that $\kappa \neq 0$.

Differential equations

The drawback of this method lies in the fact that the differential equations are derived under the assumption that the stationary density is smooth enough. One has to check that the resulting solution is, indeed, a density. This technical problem is mentioned in Rogers [1994], though positivity of the solution is not established there. This point is not addressed in Karandikar and Kulkarni [1995], nor in Ang and Barria [2000].

Importantly, the second-order differential equations corresponding to our problem are associated to the second-order matrix polynomial $\hat{F}(\alpha)$, where $\hat{F}(\alpha)$ is the matrix exponent of the time-reversed process. This naturally leads to the theory of matrix polynomials and generalized Jordan chains, see Gohberg et al. [1982]. Let a column vector $\boldsymbol{p}(x)$ denote a vector of densities $\mathbb{P}(R^* \in \mathrm{d}x | J^* = i)$. Assuming that all the $N_+ + N_-$ zeros of $\det(\hat{F}(\alpha))$ in $\mathbb{C}$ are distinct, it is shown in Ang and Barria [2000] that

$$\boldsymbol{p}(x) = \sum_{i=0}^{N_+ + N_-} c_i e^{-\lambda_i x} \boldsymbol{v}_i, \tag{5.21}$$

where $\hat{F}(\lambda_i)\boldsymbol{v}_i = \mathbf{0}$. It is argued that the constants c_i could be determined using the boundary conditions. It is, though, not shown that the corresponding equations lead to a unique solution. Finally, we note that Equation (5.21) follows trivially from Theorem 5.9 using Representation (5.1).

Let us now consider the results presented in Rogers [1994, Section 7]. It is assumed there that the fluid evolves as an independent sum of a Markov modulated linear drift and a standard Brownian motion. At first sight, this is a rather special case of an MMBM. Note, however, that the process $(X(t), J(t))$ can be time-changed without changing the distribution of $(R^* | J^*)$ in the following way. We scale time by $c_i > 0$ while $J(t)$ is in state i, that is, we consider a new

MMBM specified by the transition rate matrix $\Delta_{\boldsymbol{c}}^{-1}Q$ and parameters $\sigma_i^2/c_i, a_i/c_i$. It is easy to see that this new MMBM leads to the same distribution of $(R^*|J^*)$. Hence, Rogers [1994], in fact, does not assume more than this: all the variance parameters are strictly positive. In other words $E_+ = E_- = E$ and hence $\Pi^\pm = \mathbb{I}$. Under this assumption the first equation of Theorem 5.9 results in the following:

$$\boldsymbol{p}(x) = -(e^{x\hat{\Lambda}^+}\hat{\Lambda}^+ + e^{(B-x)\hat{\Lambda}^-}\hat{\Lambda}^- e^{B\hat{\Lambda}^+})(\mathbb{I} - e^{B\hat{\Lambda}^-}e^{B\hat{\Lambda}^+})^{-1}\mathbf{1},$$

which is (7.13) of Rogers [1994] up to the minus sign. Here $\hat{\Lambda}^\pm = \hat{\Gamma}_\mp$, because of the different definition of time-reversal (3.3) of Rogers [1994]. The missing minus sign is a consequence of a mistake in a normalization in Rogers [1994].

Chapter 6

MMBM in a strip: inverse of a regulator

Consider an MMBM reflected to stay in a strip $[0, B]$ as defined in Section 5.2. This chapter is devoted to the study of the transient behavior of the corresponding processes $(t, X(t), J(t), R(t), L(t), U(t))$. More concretely, we characterize the joint law of these processes at inverse local times $\tau_x^L, x \geq 0$ and $\tau_x^U, x \geq 0$, where

$$\tau_x^L = \inf\{t \geq 0 : L(t) > x\} \text{ and } \tau_x^U = \inf\{t \geq 0 : U(t) > x\} \tag{6.1}$$

for any $x \geq 0$. In particular, by considering the model at τ_0^U we obtain important identities for the first passage of the process reflected at 0 over the level B. This chapter is essentially independent from Chapter 5, apart from some basic definitions given in Section 5.1.

The results of this chapter, see Theorem 6.1, allow us to answer a number of important questions. In queueing terminology some of these questions are the following. Given $X(0) \in [0, B]$ and $J(0) \in E$, when does the buffer become empty for the first time and what is the state of $J(t)$ at this time? What is the amount of lost fluid until then? Mathematically speaking, we are interested in $(\tau_0^L, J(\tau_0^L), U(\tau_0^L))$. What is the length of an arbitrary busy period? What is the amount of lost fluid during a busy period given there was a loss? Moreover, we condition on the state i of $J(t)$ right before this busy period starts and the state j at which it finishes. These quantities are described by the jumps of τ_x^L and $U(\tau_x^L)$ given there is a corresponding transition of $J(\tau_x^L)$.

This chapter is organized as follows. First, we present our main result, Theorem 6.1, and then give a proof of it. As a simple application of Theorem 6.1, we show that the loss vectors can be easily obtained from this general result; by doing so we recover Corollary 5.6. Finally, we consider a special case of a simple Brownian motion and recover the results from Williams [1992], where as an easy consequence an asymptotic variance of the overflow process is obtained. It should be mentioned that some related results can be found in Asmussen et al. [2004] and in the recent paper Breuer [2011].

The analysis of the two-sided reflection problem, compared to the one-sided one, is considerably harder. The main problem lies in the fact that there are *two* regulators, which are interrelated in an intricate way. The crucial idea is to study the set of points $x \geq 0$ such that $X(\tau_x^L) = y$ for a fixed $y \in \mathbb{R}$, see Lemma 6.3. This idea in a simpler form also appears in Rogers [1994, Section 5], where it was used to derive the point masses of $R(t)$ at 0 and B in stationarity.

6.1 Main results

We start by making the following observations:

- τ_x^L and hence also $J(\tau_x^L), X(\tau_x^L), U(\tau_x^L)$ are right-continuous in x;
- $L(\tau_x^L) = x$ by the continuity of $L(t)$;
- $R(\tau_x^L) = 0$, because τ_x^L is a point of increase of $L(t)$;
- $U(\tau_x^L)$ is piece-wise constant;
- $X(\tau_x^L) = U(\tau_x^L) - x$ is piece-wise linear.

Moreover, the strong Markov property of $(X(t), J(t))$ implies that $(X(\tau_x^L), J(\tau_x^L))$ is a MAP, and in particular $J(\tau_x^L)$ is a Markov chain. Note also that $J(\tau_x^L)$ is an irreducible Markov chain taking values in E_-. The additive component $X(\tau_x^L)$ has no negative jumps, and hence there exists a $N_- \times N_-$ matrix function $F^L(\alpha)$ for all $\alpha \in \mathbb{C}^{\mathrm{Re}\leq 0}$, such that

$$\mathbb{E}_-[e^{\alpha X(\tau_x^L)}; J(\tau_x^L)] = e^{F^L(\alpha)x}. \tag{6.2}$$

Clearly, similar statements hold true with respect to τ_x^U.

Consider the reflected system under the time change $t = \tau_x^L$ (and similarly $t = \tau_x^U$) and note, using the above observations, that it is enough to characterize the trivariate process $(\tau_x^L, X(\tau_x^L), J(\tau_x^L))$. This process is a MAP with 2-dimensional additive component, and hence is uniquely specified by the following quantity:

$$\mathbb{E}_{x_0}[e^{\alpha X(\tau_x^L) - q\tau_x^L}; J(\tau_x^L)] = \mathbb{E}_{x_0}[e^{\alpha X(\tau_0^L) - q\tau_0^L}; J(\tau_0^L)]e^{F^L(\alpha,q)x}, \tag{6.3}$$

where $\mathbb{E}_{x_0}[e^{\alpha X(\tau_0^L)-q\tau_0^L}; J(\tau_0^L)]$ describes the initial distribution, and $F^L(\alpha, q)$ is the corresponding matrix exponent. Note that $q \geq 0$ can be seen as the rate of an independent exponential killing of the original process. Formula (6.3) generalizes (6.2) by allowing for arbitrary $q \geq 0$, $X(0) = x_0 \in [0, B]$ and $J(0) \in E$. In the following we do not explicitly write the killing rate q in order to simplify the notation.

As it was shown above, the joint law of $(t, X(t), J(t), L(t), U(t))$ observed at $t = \tau_x^L, x \geq 0$ is uniquely characterized by the quantities $\mathbb{E}_{x_0}[e^{\alpha X(\tau_0^L)}; J(\tau_0^L)]$ and $F^L(\alpha)$ (q is implicit here). Hence our goal is to determine these quantities, as well as those corresponding to τ_x^U. Define $N \times N_-$ and $N \times N_+$ dimensional matrices $M^L(\alpha)$ and $M^U(\alpha)$ through

$$M^L(\alpha) = \mathbb{E}_{x_0}[e^{\alpha X(\tau_0^L)}; J(\tau_0^L)](F^L(\alpha))^{-1},$$
$$M^U(\alpha) = \mathbb{E}_{x_0}[e^{\alpha X(\tau_0^U)}; J(\tau_0^U)](F^U(\alpha))^{-1}$$

for those values of $q \geq 0$ and $\alpha \leq 0$ for which the inverses are well-defined. Letting $x_0 = 0$ and restricting the rows of $M^L(\alpha)$ to E_- we get $(F^L(\alpha))^{-1}$. Hence $\mathbb{E}_{x_0}[e^{\alpha X(\tau_0^L)}; J(\tau_0^L)]$ and $F^L(\alpha)$ are readily obtained from $M^L(\alpha)$ (a similar statement is true with respect to τ_x^U). We, therefore, aim to determine the matrices $M^L(\alpha)$ and $M^U(\alpha)$.

Let ρ^+, ρ^- and $k^L(\alpha), k^U(\alpha)$ be the Perron-Frobenius eigenvalues of Λ^+, Λ^- and $F^L(\alpha), F^U(\alpha)$ respectively. Observe that $\rho^+, \rho^- \leq 0$, see Appendix A.4. If $q > 0$, then the inequalities are strict. If $q = 0$ and $\kappa \neq 0$ then one of the inequalities is strict depending on the sign of κ. In the following we exclude the delicate case when $q = 0$ and $\kappa = 0$. We are ready to present our main result, which will be proven in Section 6.2.

Theorem 6.1. *Let $\alpha \in (\rho^-, -\rho^+)$. Then $k^L(\alpha), k^U(\alpha) < 0$ and $M^L(\alpha), M^U(\alpha)$ are uniquely specified by*

$$[M^L(\alpha), M^U(\alpha)] \begin{pmatrix} \mathbb{I}^- & -\Pi_-^+ e^{B\Lambda^+} \\ -\Pi_+^- e^{B\Lambda^-} & \mathbb{I}^+ \end{pmatrix}$$
$$= [\Pi^-(\Lambda^- - \alpha\mathbb{I}^-)^{-1} e^{x_0\Lambda^-}, \Pi^+(\Lambda^+ + \alpha\mathbb{I}^+)^{-1} e^{(B-x_0)\Lambda^+}]. \quad (6.4)$$

We make some comments concerning this theorem. Firstly, $k^L(\alpha) < 0$ and $k^U(\alpha) < 0$ ensure that the matrices $M^L(\alpha)$ and $M^U(\alpha)$ are well-defined. Secondly, a simple algebraic manipulation shows that (6.4) can be equivalently rewritten as

$$M^L(\alpha) = \left(\Pi^-(\Lambda^- - \alpha\mathbb{I}^-)^{-1}e^{x_0\Lambda^-} + \Pi^+(\Lambda^+ + \alpha\mathbb{I}^+)^{-1}e^{(B-x_0)\Lambda^+}\Pi^-_+ e^{B\Lambda^-}\right)K^-,$$
$$M^U(\alpha) = \left(\Pi^+(\Lambda^+ + \alpha\mathbb{I}^+)^{-1}e^{(B-x_0)\Lambda^+} + \Pi^-(\Lambda^- - \alpha\mathbb{I}^-)^{-1}e^{x_0\Lambda^-}\Pi^+_- e^{B\Lambda^+}\right)K^+, \tag{6.5}$$

where K^- and K^+ are given in the statement of Theorem 5.9 and are well-defined unless $q = \kappa = 0$, which was excluded from our consideration.

Remark 6.2. Theorem 6.1 can be rewritten in a very concise way using the spectral pair (V, Γ) of $F(\alpha)$, see Definition 5.2. It follows from (5.1) that

$$[M^L(\alpha), M^U(\alpha)]\begin{pmatrix} -V_- \\ V_+e^{B\Gamma}\end{pmatrix} = [V(\alpha\mathbb{I} - \Gamma)^{-1}e^{x_0\Gamma}].$$

This expression may be used in practice when one is interested in computing the matrices $M^L(\alpha)$ and $M^U(\alpha)$.

6.2 Proof of Theorem 6.1

The crucial idea of the proof of Theorem 6.1 is to consider the points $x \geq 0$ such that $X(\tau^L_x) = y$ for a fixed $y \in \mathbb{R}$. Hence we define $x^{(0)} = \inf\{x \geq 0 : X(\tau^L_x) = y\}$ and $x^{(n)} = \inf\{x > x^{(n-1)} : X(\tau^L_x) = y\}$ for $n \geq 1$. Recall that $X(\tau^L_x) = U(\tau^L_x) - x$ is piecewise linear with slope -1, so $x^{(n)}$ is strictly larger than $x^{(n-1)}$. Equivalently, we can look at the time points $t \geq 0$, such that the regulator L is increasing and X is at a fixed level y at the time t. The following lemma provides the connection between the above mentioned points and some quantities which are easily computable.

Lemma 6.3. *Let*

$$\varsigma_y = \inf\{t > \tau^+_{B+y} : X(t) < y\}.$$

It holds a.s. that

$$\tau^L_{x^{(0)}} = \begin{cases} \tau^-_{|y|} & \text{if } y \leq 0, \\ \varsigma_y & \text{if } y > 0. \end{cases}$$

Moreover, for $y = 0$ and $X(0) = 0, J(0) \in E_-$ it holds a.s. that

$$\tau^L_{x^{(1)}} = \varsigma_0.$$

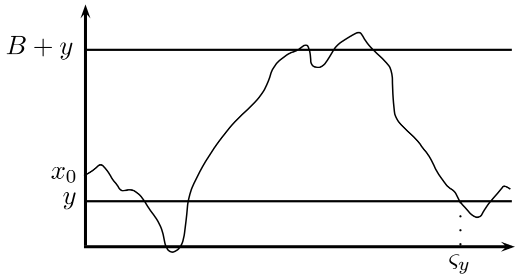

Figure 6.1 A sample path of $X(t)$.

Let us provide some explanation of this result, see also Figure 6.1. If $y \leq 0$, then the first passage time of the level y must be a point of increase of $L(t)$, as otherwise $R(t)$ becomes negative. If $y > 0$ then the first passage time of the level y may not be a point of increase of $L(t)$. It is necessary that the buffer is empty when $X(t)$ passes level y, which is only possible if an overflow has occurred before. Hence $X(t)$ should drop by at least B at the time of hitting y, which implies $\tau^L_{x(0)} = \varsigma_y$. In order to characterize $\tau^L_{x(n)}, n \geq 1$ we use the strong Markov property. Hence we only need to consider the case of $y = 0$ and $X(0) = 0, J(0) \in E_-$. In this case $\tau^L_{x(0)} = 0$ and $\tau^L_{x(1)} = \varsigma_0$ by a similar argument as above. We only present a rigorous proof of this latter result.

Proof of Lemma 6.3. We assume that $y = X(0) = 0, J(0) \in E_-$ and let $\tau := \tau^L_{x(1)}$. First, we show that $\tau < \infty$ implies $\varsigma_0 \leq \tau$. Observe that $0 = R(\tau) = X(\tau) + L(\tau) - U(\tau)$ and so $U(\tau) = L(\tau) > 0$, because $\tau > \tau_0^- = 0$. Hence there was reflection from above before τ. Let $\hat{\tau} = \sup\{t \in (0, \tau) : R(t) = B\}$ then $R(\hat{\tau}) = B, U(\tau) - U(\hat{\tau}) = 0$ and $L(\tau) - L(\hat{\tau}) \geq 0$, because L is non-decreasing. Thus $B = R(\hat{\tau}) - R(\tau) = X(\hat{\tau}) + L(\hat{\tau}) - L(\tau)$ and so $X(\hat{\tau}) \geq B$. But $X(t)$ cannot hit B without passing it with probability 1, hence $\varsigma_0 \leq \tau$ a.s.

Using the first part, note that if $\varsigma_0 = \infty$ then $\tau = \infty$. Assuming $\varsigma_0 < \infty$ one can easily see that ς_0 is a point of increase of $L(t)$. But $X(\varsigma_0) = 0$ so by the definition of τ we have $\tau \leq \varsigma_0$, which concludes the proof. □

Observe that

$$\begin{aligned} \mathbb{P}(J(\tau^-_{|y|})) &= \Pi^- e^{(x_0-y)\Lambda^-} && \text{if } y \leq 0, \\ \mathbb{P}(J(\varsigma_y)) &= \Pi^+ e^{(B+y-x_0)\Lambda^+} \Pi^-_+ e^{B\Lambda^-} && \text{if } y \geq 0, \end{aligned}$$

where $x_0 \in [0, B]$. Moreover, K^- defined in (5.20) can be written in the following way:

$$K^- = \sum_{n=0}^{\infty} \left(\Pi_-^+ e^{B\Lambda^+} \Pi_+^- e^{B\Lambda^-} \right)^n .$$

These facts and the strong Markov property lead to the following corollary.

Corollary 6.4. *It holds that*

$$\mathbb{E} \sum_{x \geq 0} 1_{\{X(\tau_x^L)=y, J(\tau_x^L)\}} = \Pi^- e^{(x_0-y)\Lambda^-} K^-, \qquad y \leq 0,$$

$$\mathbb{E} \sum_{x \geq 0} 1_{\{X(\tau_x^L)=y, J(\tau_x^L)\}} = \Pi^+ e^{(B+y-x_0)\Lambda^+} \Pi_+^- e^{B\Lambda^-} K^-, \qquad y > 0.$$

The final step of the proof is given in the following lemma, which states that we can interchange the 'integrals'.

Lemma 6.5. *For any measurable non-negative function f it holds a.s. that*

$$\int_0^\infty f(X(\tau_x^L)) 1_{\{J(\tau_x^L)=j\}} \mathrm{d}x = \int_{-\infty}^\infty f(y) \sum_{x \geq 0} 1_{\{X(\tau_x^L)=y, J(\tau_x^L)=j\}} \mathrm{d}y.$$

Proof. Recall that $X(\tau_x^L) = U(\tau_x^L) - x$ and $U(\tau_x^L)$ is piecewise constant. Suppose $U(\tau_x^L) = C$ for all $x \in [S, T)$ then it is immediate that

$$\int_{-\infty}^\infty f(y) \sum_{x \in [S,T)} 1_{\{C-x=y, J(\tau_x^L)=j\}} \mathrm{d}y$$

$$= \int_{-\infty}^\infty f(y) 1_{\{(C-y) \in [S,T), J(\tau_{C-y}^L)=j\}} \mathrm{d}y = \int_S^T f(C-x) 1_{\{J(\tau_x^L)=j\}} \mathrm{d}x.$$

Summing over all such intervals yields the statement of the lemma. ☐

Proof of Theorem 6.1. Apply Lemma 6.5 with $f(y) = e^{\alpha y}$ to Corollary 6.4 to obtain

$$\int_0^\infty \mathbb{E}[e^{\alpha X(\tau_x^L)}; J(\tau_x^L)] \mathrm{d}x$$

$$= \Pi^- \int_0^\infty e^{y(\Lambda^- - \alpha \mathbb{I}^-)} \mathrm{d}y e^{x_0 \Lambda^-} K^- + \Pi^+ \int_0^\infty e^{y(\Lambda^+ + \alpha \mathbb{I}^+)} \mathrm{d}y e^{(B-x_0)\Lambda^+} \Pi_+^- e^{B\Lambda^-} K^-.$$

Consider the MMBM $(-X(t), J(t))$ started in $B - x_0$ to find that

$$\int_0^\infty \mathbb{E}[e^{\alpha X(\tau_x^U)}; J(\tau_x^U)] \mathrm{d}x$$

$$= \Pi^+ \int_0^\infty e^{y(\Lambda^+ + \alpha \mathbb{I}^+)} \mathrm{d}y e^{(B-x_0)\Lambda^+} K^+ + \Pi^- \int_0^\infty e^{y(\Lambda^- - \alpha \mathbb{I}^-)} \mathrm{d}y e^{x_0 \Lambda^-} \Pi_-^+ e^{B\Lambda^+} K^+.$$

The integrals on the right hand sides converge if $\rho^+ + \alpha < 0$ and $\rho^- - \alpha < 0$, that is $\alpha \in (\rho^-, -\rho^+)$. Hence the left hand sides converge for such α. Use (6.3) to see that $k^L(\alpha) < 0, k^U(\alpha) < 0$ and (6.5) holds. ☐

6.3 Back to the loss vectors

In this section we reconsider the loss vectors $\boldsymbol{\ell}_-$ and $\boldsymbol{u}_+$ defined in Section 5.2. Our objective is to show that the loss vectors can be easily obtained from Theorem 6.1; by doing so we recover Corollary 5.6. Recall that $J(\tau_x^L)$ and $J(\tau_x^U)$ are irreducible recurrent Markov chains. Denote the corresponding stationary distributions through $\boldsymbol{\pi}^L$ and $\boldsymbol{\pi}^U$. Define $\lim_{t\to\infty} L(t)/t = \kappa^L$ and similarly $\lim_{t\to\infty} U(t)/t = \kappa^U$, which do not depend on the initial distribution of $(X(0), J(0))$. Using the change of variable $s = \tau_x^L$ we write

$$\kappa^L \pi_i^L = \lim_{t\to\infty} \frac{1}{t} \int_0^{L(t)} 1_{\{J(\tau_x^L)=i\}} \mathrm{d}x = \lim_{t\to\infty} \frac{1}{t} \int_0^t 1_{\{J(s)=i\}} \mathrm{d}L(s) = l_i, \tag{6.6}$$

i.e., one can interpret the vectors $\kappa^L \boldsymbol{\pi}^L$ and $\kappa^U \boldsymbol{\pi}^U$ as the mean unused capacity and the mean overflow in a unit of time in stationarity.

In the rest of this section we show that Corollary 5.6, which identifies the loss vectors when $\kappa \neq 0$, is an easy consequence of Theorem 6.1. It is noted that the loss vectors for an arbitrary MAP are considered in Asmussen and Pihlsgård [2007]. The results of this paper, however, depend on the stationary distribution, which has no explicit solution in most cases. Moreover, the authors make a restrictive assumption about the number of roots of a certain equation.

Without loss of generality we assume that $\kappa \geq 0$. Firstly, observe that

$$-M^L(0) = \int_0^\infty \mathbb{P}(\tau_x^L < e_q, J(\tau_x^L)) \mathrm{d}x = \mathbb{E} \int_0^{L(e_q)} 1_{\{J(\tau_x^L)\}} \mathrm{d}x.$$

Noting that qe_q is an exponential random variable of rate 1 and using (6.6) we get

$$l_j = -\lim_{q\downarrow 0} q \left[M^L(0)\right]_{ij} \text{ for any } i \in E, j \in E_-.$$

A similar identity holds true for $J(\tau_x^U)$, and, moreover, for $J(\tau_x^+)$. That is,

$$-\Pi^+(\Lambda^+)^{-1} = \int_0^\infty \mathbb{P}(\tau_x^+ < e_q, J(\tau_x^+)) \mathrm{d}x = \mathbb{E} \int_0^{\overline{X}(e_q)} 1_{\{J(\tau_x^+)\}} \mathrm{d}x.$$

It is well-known that $\lim_{t\to\infty} \overline{X}(t)/t = \kappa$, which then implies

$$\kappa(\pi_{\Lambda^+})_j = \lim_{t\to\infty} \frac{1}{t} \int_0^{\overline{X}(t)} 1_{\{J(\tau_x^+)=j\}} \mathrm{d}x = -\lim_{q\downarrow 0} q[\Pi^+(\Lambda^+)^{-1}]_{ij}$$

for any $i \in E, j \in E_+$. Finally, consider (6.4) with $q > 0$ and $\alpha = 0$, multiply both sides by q, and let $q \downarrow 0$ to obtain

$$(\boldsymbol{\ell}_-, \boldsymbol{u}_+) \begin{pmatrix} \mathbb{I}^- & -\Pi_-^+ e^{B\Lambda^+} \\ -\Pi_+^- e^{B\Lambda^-} & \mathbb{I}^+ \end{pmatrix} = (\mathbf{0}_-, \kappa \boldsymbol{\pi}_{\Lambda^+}),$$

which can be easily rewritten to match Corollary 5.6.

6.4 Special cases

An important special case arises when $E_- = E_+ = E$, that is there are no states when the process evolves deterministically (a Brownian component is always present). In this special case $\Pi^- = \Pi^+ = \mathbb{I}$ and hence (6.5) reduces to

$$M^L(\alpha) = \left((\Lambda^- - \alpha\mathbb{I})^{-1}e^{x_0\Lambda^-} + (\Lambda^+ + \alpha\mathbb{I})^{-1}e^{(B-x_0)\Lambda^+}e^{B\Lambda^-}\right)\left(\mathbb{I} - e^{B\Lambda^+}e^{B\Lambda^-}\right)^{-1}.$$

Letting $x_0 = 0$ and taking the inverse we obtain

$$F^L(\alpha) = \left(e^{-B\Lambda^-} - e^{B\Lambda^+}\right)\left((\Lambda^- - \alpha\mathbb{I})^{-1}e^{-B\Lambda^-} + (\Lambda^+ + \alpha\mathbb{I})^{-1}e^{B\Lambda^+}\right)^{-1}. \quad (6.7)$$

Moreover, observing that $X(\tau_0^L) = U(\tau_0^L)$ we find from (6.5) that

$$\begin{aligned}
&\mathbb{E}_{x_0=B}[e^{\alpha U(\tau_0^L) - q\tau_0^L}; J(\tau_0^L)] \quad (6.8)\\
&= \left((\Lambda^- - \alpha\mathbb{I})^{-1} + (\Lambda^+ + \alpha\mathbb{I})^{-1}\right)\left((\Lambda^- - \alpha\mathbb{I})^{-1}e^{-B\Lambda^-} + (\Lambda^+ + \alpha\mathbb{I})^{-1}e^{B\Lambda^+}\right)^{-1}\\
&= \left(\Lambda^- - \alpha\mathbb{I}\right)^{-1}\left(\Lambda^+ + \Lambda^-\right)\left(e^{-B\Lambda^-}(\Lambda^+ + \alpha\mathbb{I}) + (\Lambda^- - \alpha\mathbb{I})e^{B\Lambda^+}\right)^{-1}\left(\Lambda^- - \alpha\mathbb{I}\right),
\end{aligned}$$

where in the last step we used the fact that $(\Lambda^+ + \alpha\mathbb{I})^{-1}$ and $e^{B\Lambda^+}$ commute. A number of other useful transforms can be found using similar algebraic manipulations.

Next, we restrict ourself to the case of a single state, that is, we consider a Brownian motion with variance σ^2 and drift μ. Without real loss of generality it is assumed that $\sigma^2 = 1$. According to Corollary 4.15, $\lambda = \Lambda^\pm$ is a solution of $1/2\lambda^2 \mp \mu\lambda - q = 0$. Moreover, $\Lambda^\pm$ is negative unless $q = 0$ and $\pm\mu \geq 0$, in which case it is 0. Thus $\Lambda^+ = \mu - \gamma$ and $\Lambda^- = -\mu - \gamma$, where $\gamma = \sqrt{\mu^2 + 2q}$. Now the right side of (6.8) reduces to $-2\gamma/[e^{B(\mu+\gamma)}(\mu + \alpha - \gamma) - e^{B(\mu-\gamma)}(\mu + \alpha + \gamma)]$ and so

$$\mathbb{E}_B[e^{\alpha U(\tau_0^L) - q\tau_0^L}] = \frac{e^{-B\mu}}{\cosh(B\gamma) - \frac{\mu+\alpha}{\gamma}\sinh(B\gamma)}, \quad (6.9)$$

where $\alpha \in (-\mu-\gamma, -\mu+\gamma)$ according to Theorem 6.1. Fix $q > 0$ for a moment, so that $\gamma > 0$. Multiply both sides of the equation by the denominator in the right hand side and observe that the Laplace transform is analytic in $\text{Re}(\alpha) < -\mu + \gamma$. Thus the latter equality holds in this domain. This shows that (6.9) holds for all $q > 0, \alpha \leq 0$. By symmetry we obtain

$$\mathbb{E}_0[e^{-\alpha L(\tau_0^U) - q\tau_0^U}] = \frac{e^{B\mu}}{\cosh(B\gamma) + \frac{\mu+\alpha}{\gamma}\sinh(B\gamma)}, \qquad q > 0, \alpha \geq 0,$$

which is Equation (7) in Williams [1992]. Taking limits on both sides we show that this equation holds true also for $q = 0$, unless $\mu = 0$, in which case $\mathbb{E}e^{-\alpha L(\tau_0^U)} = 1/(1+\alpha B)$ by L'Hôpital's rule. It is noted that in Williams [1992] a very different approach is used. It uses stochastic integration and relies on a sophisticated guess of the right form of a certain function. Our approach, however, is direct and is based on simple probabilistic arguments.

Let us conclude by making some additional comments about the Brownian motion with two reflecting barriers. Firstly, the strong Markov property of $X(t)$ implies that the nondecreasing piecewise constant process $U(\tau_x^L)$ has memory-less jumps and inter-arrival times, implying that it is a Poisson process with exponential jumps. Let us confirm this and find the corresponding rates. Note that (6.7) can be rewritten as

$$F^L(\alpha, q) = \frac{2(\frac{1}{2}\alpha^2 + \mu\alpha - q)}{\gamma \coth(B\gamma) - (\mu + \alpha)}.$$

Hence if $\mu \neq 0$, then

$$\log \mathbb{E}e^{\alpha U(\tau_1^L)} = F^L(\alpha) + \alpha = \frac{-\alpha\mu(\coth(\mu B) + 1)}{\alpha - \mu(\coth(\mu B) - 1)},$$

which implies that $U(\tau_x^L)$ jumps with rate $\mu(\coth(\mu B)+1) = 2\mu/(1-e^{-2\mu B})$, and the jumps are exponential of rate $\mu(\coth(\mu B) - 1) = 2\mu/(e^{2\mu B} - 1)$. If $\mu = 0$ then $\log \mathbb{E}e^{\alpha U(\tau_1^L)} = \alpha/(1 - \alpha B)$, that is, both rates become $1/B$.

Chapter 7

The scale matrix

Let (X, J) be a spectrally negative MAP as defined in Section 2.2. This chapter is devoted to the study of the two-sided exit problem. That is, for $a, b \geq 0$ with $a+b > 0$ we aim to determine the probability that X, started at 0, exits the interval $[a, -b]$ through a. In fact, we will refine this result to include the transform of the exit time and the state of J at this time. The key role in this study is played by a matrix-valued function W, which is a generalization of the scale function of a spectrally negative Lévy process. Hence we call W a *scale matrix*. It is noted that Kyprianou and Palmowski [2008] claimed the existence of W, but did not succeed in determining its transform. The corresponding theory in the case of Lévy processes can be found in Bertoin [1996, Ch. VII] and Kyprianou [2006, Ch. 8].

Recall that the first passage times are defined through

$$\tau_x^{\pm} = \inf\{t \geq 0 : \pm X(t) > x\}$$

for all $x \geq 0$. Let us immediately formulate one of the main results.

Theorem 7.1. *Assume that $N = N_+$. Then for all $q \geq 0$ there exists a unique continuous function $W^q : [0, \infty) \to \mathbb{R}^{N \times N}$ such that $W^q(x)$ is invertible for all $x > 0$,*

$$\mathbb{E}[e^{-q\tau_a^+}; \tau_a^+ < \tau_b^-, J(\tau_a^+)] = W^q(b)W^q(a+b)^{-1} \text{ for all } a, b \geq 0 \text{ with } a+b > 0,$$

and

$$\int_0^\infty e^{-\alpha x} W^q(x) \mathrm{d}x = (F(\alpha) - q\mathbb{I})^{-1}$$

for all $\alpha > \eta(q) := \max\{\mathrm{Re}(z) : z \in \mathbb{C}, \det(F(z) - q\mathbb{I}) = 0\}$.

It is noted that the parameter $q \geq 0$ arises from the exponential killing of the process (X, J). In the following we often suppress the killing rate $q \geq 0$, where it does not cause confusion; see Remark 2.18.

Remark 7.2. The proof of Theorem 7.1 establishes some additional properties of $W(x)$. In the first place, $W^q(x)$ is continuous in $q \geq 0$ for every $x \geq 0$. Moreover, $W(x) = e^{-\Lambda x}\boldsymbol{L}(x)$, where the entries of $N \times N$ matrix $\boldsymbol{L}(x)$ are non-negative increasing functions of $x \geq 0$. These entries can be interpreted as expected local times at 0 up to the first passage time over x, which is made precise in Section 7.3.

In addition, later in this chapter we generalize Theorem 7.1 to allow $N_+ < N$, see Theorem 7.20. This more general result is, however, less clean. It is noted that one may often eliminate the states in $E_\downarrow$, see Section 7.1, to avoid the need in this extra generality.

The two-sided exit problem becomes considerably more intricate in the MAP setting compared to the case of a Lévy process. The crucial ideas used in the latter case fail to work. This chapter presents a number of novel ideas and relations, which lead to the construction of the scale matrix, and allow to establish some important properties. Let us illustrate some of the problems faced. In the case of a Lévy process $W(x)$ is taken to be proportional to $\mathbb{P}(\underline{X} \geq -x)$, which is essential to obtain the transform of W, see Bertoin [1996, Ch. VII] and Kyprianou [2006, Ch. 8]. This kind of probabilistic construction (or its variations) does not work in the MAP setting. A much more elaborate object is used instead, see (7.7). The transform is then obtained using occupation densities. Furthermore, W is a non-negative increasing function in the case of a Lévy process, which allows to apply the extended continuity theorem for Laplace transforms to treat the delicate case when $q = 0$. In the MAP setting the best we can show is that the entries of $e^{\Lambda x}W(x)$ are non-negative increasing functions. In addition, invertibility of $W(x)$ is a non-trivial issue. Finally, a MAP can evolve as a CPP in some time intervals. It is well known that CPPs exhibit uncommon behavior as far as path properties are concerned. This leads to a number of complications in the analysis of such MAPs.

This chapter is organized as follows. We start by presenting an auxiliary MAP obtained by collapsing certain states of the Markov chain J. Then we discuss some important path properties of certain Lévy processes in Section 7.2. In particular, we review so-called occupation densities, and present some properties of hitting times. Occupation densities are then generalized to the MAP case in Section 7.3. In addition, we define some fundamental objects and obtain important relations in the same section. In Section 7.4 we reconsider the two-sided exit problem and construct a candidate for the scale matrix. The proof of Theorem 7.1 is given in Section 7.5. This result is then generalized in Section 7.6. Finally, we present some examples in Section 7.7.

Before we proceed let us define the *first hitting time* of a set. For any open or closed Borel set $B \subset \mathbb{R}$ we define

$$\tau^B = \inf\{t > 0 : X(t) \in B\}.$$

The random time τ^B is a stopping time, which can be shown following the steps of Bertoin [1996, Cor. I.8]. Moreover, $X(\tau^B) \in \overline{B}$ a.s. on $\{\tau^B < \infty\}$. In the following B will be either a half-interval, e.g. $(-\infty, -b]$, or a single point $\{-b\}$. Observe that the first passage times τ_x^+ and τ_x^- coincide with $\tau^{(x,\infty)}$ and $\tau^{(-\infty,-x)}$ respectively, where $x \geq 0$.

7.1 Auxiliary process

It is sometimes convenient to modify a MAP (X, J) by reducing the set of states of the background process J. Let $\tilde{E} \subset E$ be a targeted subset of states. The idea is to construct an auxiliary MAP with the state space $\tilde{E}$, so that this process behaves exactly as (X, J) restricted to the time intervals where $J(t) \in \tilde{E}$. In other words, we collapse time intervals where $J(t) \notin \tilde{E}$ into single points, so that each evolution of $X_i, i \notin \tilde{E}$ contributes a jump of size $X_i(e_{q_i})$. If the killing rate q is non-zero, then such a jump is modified to lead to the absorbing state with probability $\mathbb{P}(e_{q_i} > e_q)$. The Laplace transform of such a jump is then given by

$$\mathbb{E}[e^{\alpha X_i(e_{q_i})}; e_{q_i} < e_q] = \mathbb{E}e^{(\psi_i(\alpha)-q)e_{q_i}} = \frac{q_i}{q_i + q - \psi_i(\alpha)}.$$

We use the subscript a to indicate the auxiliary process. It is not that straightforward to write down the matrix exponent $F_a(\alpha)$ of the auxiliary MAP in terms of $F(\alpha)$. Note, however, that

$$\int_0^\infty \mathbb{E}_i[e^{\alpha X(t)}; J(t) = j]\mathrm{d}t = \int_0^\infty \mathbb{E}_i[e^{\alpha X_a(t)}; J_a(t) = j]\mathrm{d}t$$

for any $i, j \in \tilde{E}$. Hence the signs of the Perron-Frobenius eigenvalues are related, that is, $k(\alpha) < 0$ if and only if $k_a(\alpha) < 0$, see Lemma A.9. Moreover, for such α it holds that $F_a(\alpha)^{-1}$ is equal to $F(\alpha)^{-1}$ restricted to the rows and columns in $\tilde{E}$.

There are two different choices of $\tilde{E}$ considered in this book. Firstly, $\tilde{E}$ can be a set of a single state $\{i\}$. The corresponding auxiliary process evolves as X_i with some additional jumps arriving at rate q_i. It is a Lévy process which is equal in law to the independent sum of X_i and a CPP. Note that the latter may have positive jumps. Many properties of a MAP can be understood from the behavior of this auxiliary process. We present some important properties of such a Lévy process in Section 7.2.

Another important case arises when $\tilde{E} = E_+$. That is, we collapse all the states corresponding to processes X_i with non-increasing paths. Note that this procedure only produces negative jumps, hence the auxiliary process is again a spectrally negative MAP. Furthermore, it is a 'nice' MAP in the sense that $N_a = (N_a)_+$. It is not difficult to see that $\Lambda_a = \Lambda$; recall that the killing of the original MAP with rate $q > 0$ leads to some specific killing of (X_a, J_a). Moreover, $\mathbb{P}_i(\tau_a^+ < \tau_b^-; J(\tau_a^+) = j)$, where $i, j \in E_+$, coincides with the same probability for the auxiliary process. Hence, one can represent this probability using Theorem 7.1 and the auxiliary scale matrix W_a. So in fact, the only case not covered by this theorem corresponds to $i \in E_\downarrow$.

7.2 Path properties of certain Lévy processes

In the previous section we mentioned that many path properties of a spectrally negative MAP can be understood from the behavior of a certain Lévy process. This Lévy process is obtained by collapsing all the background states of a MAP but a single one. Let us present some path properties of such a Lévy process. Recall that this process can be seen as an independent sum of a Lévy process without positive jumps and an arbitrary CPP. So throughout this section we assume that

$$X \text{ is a Lévy process with } \nu(0,\infty) < \infty. \tag{7.1}$$

Let us recall that a Lévy process characterized by (a, σ, ν) has paths of bounded variation if and only if $\sigma = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|)\nu(\mathrm{d}x) < \infty$. In this case one can talk about drift. The zero drift case is delicate, and is often treated separately. The following result is a special case of Theorem 1 in Kesten [1969].

Proposition 7.3. *Assume that X is not a process with monotone paths, nor is it a bounded variation process with zero drift. Then $\mathbb{P}(\tau^{\{x\}} < \infty) > 0$ for all $x \in \mathbb{R}$.*

Consider an open or closed Borel set B, and the probability that X hits B immediately when started in x: $\mathbb{P}_x(\tau^B = 0)$. According to Blumenthal's zero-one law this probability is necessarily 0 or 1. We say that x is *irregular* for the set B in the first case, and x is *regular* for the set B in the second case. If $B = \{0\}$ then we simply say that x is (ir)regular without mentioning B. Hence, 0 is said to be regular if X returns to the origin immediately. The next result follows directly from Theorem 8 in Bretagnolle [1971].

Proposition 7.4. *Assume that X is not a CPP.*

- *If X is of unbounded variation then 0 is regular.*
- *If X is of bounded variation then 0 is irregular.*

Occupation densities

In this part of the section we discuss occupation densities of a Lévy process X. It is assumed that X is not monotone, neither is it a processes of bounded variation with zero drift. According to Proposition 7.3, $\mathbb{P}(\tau^{\{x\}} < \infty) > 0$ for all $x \in \mathbb{R}$. This is sufficient, as stated by Bertoin [1996, Prop. II.16, Thm. V.1], for the existence of the 'occupation density' $L(x,t)$, which we define in the following.

First, suppose $X(t)$ has paths of unbounded variation. In this case 0 is regular according to Proposition 7.4, and then for every $x \in \mathbb{R}$

$$L^{\epsilon}(x,t) = \frac{1}{2\epsilon}\int_0^t 1_{\{|X(s)-x|<\epsilon\}}\mathrm{d}s$$

converges uniformly on compact intervals of time t, in $L^2(\mathbb{P})$ as $\epsilon \downarrow 0$, see Bertoin [1996, Prop. V.2]. This limit is denoted through $L(x,t)$, which consequently is continuous in t a.s. It can be shown that for each $x \in \mathbb{R}$, $L(x,\cdot)$ is an increasing $\mathcal{F}_t$-adapted process, which increases only if $X = x$. Importantly, the process $L(x,t)$ satisfies the following property.

Proposition 7.5. *For every y and every stopping time T with $X(T) = y$ on $\{T < \infty\}$, the shifted process $L(x, T+t) - L(x,T), t \geq 0$ is independent of $\mathcal{F}_T$ under $\mathbb{P}(\cdot|T < \infty)$, and has the same law as $L(x-y,t), t \geq 0$ under $\mathbb{P}$.*

Proof. It is enough to note that $L(x,t)$ is the limit of the processes which satisfy this property, see also Bertoin [1996, Prop. V.4]. □

The above property shows that $L(0,\cdot)$, in fact, coincides up to a constant factor with the *local time* of $X(t)$ at 0, see Bertoin [1996, Prop. V.4]. We often call $L(0,\cdot)$ the local time of $X(t)$ at 0.

Secondly, suppose $X(t)$ has paths of bounded variation a.s. (recall that we exclude processes with zero linear drift $d = 0$). In this case 0 is irregular, which allows us to define

$$L(x,t) = |d|^{-1}N(x,t),$$

where $N(x,t) = \#\{s \in [0,t) : X(s) = x\}$, see also Fitzsimmons and Port [1990]. In this case the strong Markov property immediately shows that Proposition 7.5 still holds. Moreover, for both cases considered above the following is true. For each $t \geq 0$, $L(\cdot, t)$ is measurable and the identity

$$\int_0^t f(X(s))\mathrm{d}s = \int_{\mathbb{R}} f(x)L(x,t)\mathrm{d}x \tag{7.2}$$

holds for all measurable functions $f \geq 0$ a.s., see Bertoin [1996, Eqn. V.(2)]. This identity is called 'occupation density formula' and will play a crucial role in the following.

We conclude this section by a corollary of Proposition 7.5.

Corollary 7.6. *It holds that*

$$\mathbb{E}L(x,\infty) = \mathbb{P}(\tau^{\{x\}} < \infty)\mathbb{E}L(0,\infty).$$

Proof. Observe that $L(x,t)$ has the distribution of $1_{\{\tau^{\{x\}}<t\}}\tilde{L}(0, t - \tau^{\{x\}})$, where $\tilde{L}(0,\cdot)$ is a copy of $L(0,\cdot)$ independent of $\tau^{\{x\}}$. Take the expectations and apply the monotone convergence theorem. □

Hitting a level

In this part we present some properties of the first hitting time $\tau^{\{x\}}$. The next proposition concerns the following question. Can the first hitting time of a level be strictly less than the first passage over this level? Note that this question is different from the question about 'creeping', where one asks if a level can be passed by hitting it.

Proposition 7.7. *Assume that X is not a CPP. Then for any $x \geq 0$ it holds that $\tau_x^+ \leq \tau^{\{x\}}$ and $\tau_x^- \leq \tau^{\{-x\}}$ a.s.*

Proof. We only prove that $\tau_x^+ \leq \tau^{\{x\}}$ for $x \geq 0$. The second claim can be shown following exactly the same steps. If 0 is regular for $(0,\infty)$ then we apply the strong Markov property at $\tau^{\{x\}}$ on the event $\{\tau^{\{x\}} < \infty\}$ to deduce the result.

In the rest we assume that 0 is irregular for $(0,\infty)$. We will rely on the fact that the local extrema of X are all distinct, except in the CPP case, see Bertoin [1996, Prop. VI.4]. So $\tau^{\{0\}} < \tau_0^+$ cannot hold a.s. We are only left to prove the claim for $x > 0$. Since 0 is irregular for $(0,\infty)$, it holds that $\overline{X}(t)$ is a step process, where the time to the next maximum and the corresponding jump are independent of the past, and all have the same law. Let us show that $\overline{X}$ cannot hit a fixed x

a.s., which would complete the proof. Suppose on the contrary that $\overline{X}$ hits x with positive probability. This implies that the jump distribution of $\overline{X}$ has an atom at some $\epsilon > 0$.

Consider an independent CPP $X_{-\epsilon}$, which has deterministic jumps of size $-\epsilon$. Note that this process has exactly one jump before $\overline{X}$ jumps with positive probability. Hence the Lévy process $X + X_{-\epsilon}$ can achieve its supremum without immediately passing it with positive probability. But this is only possible for a CPP. So X would then be a CPP too, which is not the case. □

Let us present another result, which is an immediate consequence of Bertoin [1996, Prop. I.15].

Proposition 7.8. *Let an exponential random variable e_q be independent of a Lévy process X. Then $X(e_q)$ has no atoms if X is not a CPP.*

Proof. We use the notation of Bertoin [1996]. Fix x and let $f(y) = 1_{\{y=x\}}$. Then

$$\frac{1}{q}\mathbb{P}(X(e_q) = x) = \int_0^\infty e^{-qt}\mathbb{P}(X(t) = x)\mathrm{d}t$$
$$= U^q f(0) = \int f(y)U^q(0, \mathrm{d}y) = U^q(0, \{y\}) = 0,$$

where the last equality follows from Bertoin [1996, Prop. I.15]. □

The following proposition will only be needed to extend the main result of this chapter to the case when some X_i are bounded variation processes with zero drift.

Lemma 7.9. *Let X be a Lévy process with no positive jumps. Assume X is not a process with non-increasing paths. Then for any $x' \in \mathbb{R}\backslash\{0\}$ it holds that $\tau^{\{x\}}$ is continuous at x' a.s.*

Proof. Using quasi-left-continuity, see Bertoin [1996, Proposition I.7], one can show that X does not jump downwards at a time at which it is about to hit a level $x \neq 0$ for the first time. To see this use the following sequence of stopping times

$$T_n = \inf\{t > 0 : X(t) \in (x - 1/n, x + 1/n)\}.$$

Hence it is enough to consider an arbitrary small time interval centered at $\tau^{\{x\}}$, and to show that during this time interval X hits all the points in some small neighborhood of x.

If 0 is regular for $(0, \infty)$ then the absence of positive jumps implies that X does hit the points larger than x in some neighborhood. Moreover, the claim is also true for points smaller than x, if 0 is regular for $(-\infty, 0)$ and also it is regular for $\{0\}$ (so the process has to return and to hit all the points on the way). If X has paths of unbounded variation then it satisfies the above requirements, see Proposition 7.4 and Bertoin [1996, p. 167]. So it is left to consider processes of bounded variation with positive drift. It is enough to show that X cannot hit a level by jumping onto it. This follows from Proposition 7.7, and the fact that 0 is irregular for $(-\infty, 0)$. □

7.3 Occupation densities of a MAP

Let (X, J) be a spectrally negative MAP, such that none of the X_i is a bounded variation process with zero drift. The structure of MAP allows for immediate generalization of the concept of occupation densities presented in Section 7.2. Note that X observed at times when $J \in j$ is a Lévy process, which satisfies the assumptions of Section 7.2. Let us define $L(x, j, t)$, the occupation density of (X, J) at (x, j) up to time t. If X_j is such that 0 is irregular we let

$$L(x,j,t) = \frac{1}{|d_j|}\#\{s \in [0,t) : X(s) = x, J(s) = j\};$$

otherwise $L(x, j, t)$ is the limit of

$$L^\epsilon(x,j,t) = \frac{1}{2\epsilon}\int_0^t 1_{\{|X(s)-x|<\epsilon, J(s)=j\}}\mathrm{d}s$$

as $\epsilon \downarrow 0$.

Observe that $L(x, j, t)$ is an increasing $\mathcal{F}_t$-adapted process, which increases only if $X = x, J = j$. Moreover, Proposition 7.5 reads as follows.

Proposition 7.10. *For every y, i and every stopping time T with $X(T) = y$ on $\{J(T) = i\}$, the shifted process $L(x, j, T + t) - L(x, j, T), t \geq 0$ is independent of $\mathcal{F}_T$ under $\mathbb{P}(\cdot|J(T) = i)$, and has the same law as $L(x - y, j, t), t \geq 0$ under $\mathbb{P}_i$.*

Finally, the occupation density formula becomes

$$\int_0^t f(X(s))1_{\{J(s)=j\}}\mathrm{d}s = \int_{\mathbb{R}} f(x)L(x,j,t)\mathrm{d}x.$$

Let us illustrate the use of occupation densities. Let $\boldsymbol{L}$ be a $N \times N$ matrix with (i, j)-th element equal to $\mathbb{E}_i L(0, j, \infty)$. Recall that $\mathbb{P}(J(t))$ denotes an $N \times N$ matrix with elements $\mathbb{P}_i(J(t) = j)$, whereas $\mathbb{P}_+(J(t))$ denotes its restriction to the rows corresponding to E_+, see Section 2.2. Using Proposition 7.10 we generalize Corollary 7.6 to obtain

$$\mathbb{E}L(x,j,\infty) = \mathbb{P}(J(\tau^{\{x\}}))\mathbb{E}L(0,j,\infty). \tag{7.3}$$

Moreover, observe that $\boldsymbol{L}$ has strictly positive entries. Let us present the following important result.

Theorem 7.11. *For all* $\alpha \geq 0$*, such that* $k(\alpha) < 0$*, it holds that*

$$\int_{\mathbb{R}} e^{\alpha x}\mathbb{P}(J(\tau^{\{x\}}))\mathrm{d}x\boldsymbol{L} = -F(\alpha)^{-1}.$$

Proof. Firstly, by the monotone convergence theorem we get

$$\mathbb{E}\lim_{t\to\infty}\int_0^t e^{\alpha X(s)}e_{J(s)}\mathrm{d}s = \int_0^\infty e^{F(\alpha)s}\mathrm{d}s = -F(\alpha)^{-1},$$

where we used the fact that the real parts of the eigenvalues of $F(\alpha)$ are all negative, see also Lemma A.9. Using the occupation density formula we obtain

$$\lim_{t\to\infty}\int_0^t e^{\alpha X(s)}1_{\{J(s)=j\}}\mathrm{d}s = \lim_{t\to\infty}\int_{\mathbb{R}} e^{\alpha x}L(x,j,t)\mathrm{d}x = \int_{\mathbb{R}} e^{\alpha x}L(x,j,\infty)\mathrm{d}x.$$

Identity (7.3) completes the proof. □

It is noted that the theory of occupation densities presented above holds for a killed MAP as well.

Remark 7.12. If there is an α satisfying the conditions of Theorem 7.11 then both $\boldsymbol{L}$ and $\int_{\mathbb{R}} e^{\alpha x}\mathbb{P}(J(\tau^{\{x\}})\mathrm{d}x$ must have finite entries. Furthermore, $\boldsymbol{L}$ is then invertible. This is the case when the killing rate q is strictly positive, because then $k(0) < 0$. So $q\int_0^\infty e^{-qt}\mathbb{E}_i L(0,j,t)\mathrm{d}t < \infty$, where L is the local time in the case of no killing. This further implies that $\mathbb{E}_i L(0,j,t) < \infty$ for any deterministic t. Recall that X drifts to ∞ or $-\infty$ if $\kappa \neq 0$. Using the regenerative structure of L, it is not difficult to show that $\boldsymbol{L}$ has finite entries also in the case of no killing when $\kappa \neq 0$; an argument of this type will also be used in the proof of Proposition 7.13.

In the rest of this section we assume that $N = N_+$. Let us introduce another matrix $\boldsymbol{L}(x), x \geq 0$, which will play an important role in the study of the scale matrix. Let

$$\boldsymbol{L}_{ij}(x) = \mathbb{E}_i L(0,j,\tau_x^+)$$

for $x > 0$. That is, $\boldsymbol{L}(x)$ is the expected local time at 0 up to the first passage time over x. In view of Proposition 7.10 the matrix $\boldsymbol{L}(x), x > 0$ can be expressed as

$$\boldsymbol{L}(x) = \boldsymbol{L} - \mathbb{P}(J(\tau_x^+))\mathbb{P}(J(\tau^{\{-x\}}))\boldsymbol{L}, \tag{7.4}$$

given that $\boldsymbol{L}$ has finite entries. In addition, we would like to define $\boldsymbol{L}(0)$ so that the Identity (7.4) holds true for $x = 0$ when $\boldsymbol{L}$ has finite entries. Note that $\tau_0^+ = 0$ and use the definition of the first hitting time $\tau^{\{0\}}$ to see that $\boldsymbol{L}(0)$ is given by

$$\boldsymbol{L}_{ij}(0) = \mathbb{E}_i L(0, j, 0+) = \begin{cases} 1/d_i & \text{if } i = j \text{ and } X_i \text{ is of bounded variation,} \\ 0 & \text{otherwise.} \end{cases} \tag{7.5}$$

The second equality follows from the construction of the occupation density, see Section 7.3. It is also noted that $d_i > 0$ for a Lévy process X_i of bounded variation, because of the assumption that $N = N_+$. Observe also that the entries of $\boldsymbol{L}(x)$ are non-negative and increasing in x.

Proposition 7.13. *For any $u > 0$ it holds that $\int_0^\infty e^{-ux}\boldsymbol{L}(x)\mathrm{d}x < \infty$.*

Proof. Let e_u be an exponential random variable of rate u independent of everything else. It is enough to show that $\mathbb{E}\boldsymbol{L}(e_u) < \infty$. Note that $\mathbb{E}\boldsymbol{L}(e_u)$ is the expected local time at 0 up to hitting the random level e_u. Hence the corresponding entries of $\mathbb{E}\boldsymbol{L}(e_u)$ are bounded from above by the entries of $\boldsymbol{L}$. In view of Remark 7.12 we only need to consider the case of no killing with $\kappa = 0$.

Our problem can be reduced to the same problem for a Lévy process by restricting attention to only those intervals of time where $J = j$, see Section 7.2. The reasoning below can be also used in the context of MAPs; in the case of Lévy processes, however, the notation is much simpler. In the following X is a Lévy process obtained from the original MAP, and $L(0, t)$ is the local time of X at 0 up to time t. Similarly to Remark 7.12 we can show that $\mathbb{E}L(0, t) < \infty$ for any deterministic t. Let $\tau = \inf\{s \geq t : X(s) = 0\}$ be the first hitting time of zero after time t. Let also $p = \mathbb{P}(\tau_{e_u}^+ > \tau)$ be the probability that the level e_u is not reached before τ. The regenerative structure of $L(0, \cdot)$, see Proposition 7.5, implies that $\mathbb{E}\boldsymbol{L}(e_u) \leq \mathbb{E}L(0, \tau)\sum_{i=0}^\infty p^i$. The proof is completed by noting that $\mathbb{E}L(0, \tau) = \mathbb{E}L(0, t) < \infty$ and $p < 1$, because $\kappa = 0$ implies that every level is reached a.s., see Proposition 2.15. □

7.4 Two-sided exit

This section presents some of the basic ideas in constructing the scale matrix. Firstly, we show that the event $\{\tau_a^+ < \tau_b^-\}$ is closely related to $\{\tau_a^+ < \tau^{\{-b\}}\}$. Secondly, using the strong Markov property we obtain a key representation of the latter event through the first hitting times.

Lemma 7.14. *For any $a, b \geq 0$ such that $a + b > 0$ the events $\{\tau_a^+ < \tau^{(-\infty,-b]}\}$ and $\{\tau_a^+ < \tau^{\{-b\}}\}$ coincide a.s.*

Proof. It is only required to show that $\tau_a^+ < \tau^{\{-b\}}$ implies $\tau_a^+ < \tau^{(-\infty,-b]}$. Suppose on the contrary it holds that $\tau^{(-\infty,-b]} \leq \tau_a^+ < \tau^{\{-b\}}$. Observe that $X(\tau^{(-\infty,-b]}) \leq -b$, but then X should hit level $-b$ before reaching a, because of absence of positive jumps. □

The following lemma will allow us to include and exclude boundary points of the interval $[a, -b]$ in most cases when considering the two-sided exit problem.

Lemma 7.15. *For any $x \geq 0$ it holds $\mathbb{P}_i -$ a.s. that $\tau_x^+ = \tau^{[x,\infty)}$ and $\tau_x^- = \tau^{(-\infty,-x]}$, unless X_i is a CPP.*

Proof. We only need to show that the events $\{\tau^{\{x\}} < \tau_x^+\}$ and $\{\tau^{\{-x\}} < \tau_x^-\}$ have probability 0. Consider the first event together with $J(\tau^{\{x\}} = j)$. Suppose it has a positive probability. Then $\{\tau^{\{x\}} < \tau_x^+\}$ is also true for a Lévy process obtained by observing X only at time intervals where $J = j$. This is impossible according to Proposition 7.7 unless X_j is a CPP. Assume that X_j is a CPP and j is not the starting index: $i \neq j$. The distribution of X at the time of the first jump of J is diffuse according to Proposition 7.8. Therefore, X cannot hit any fixed level while J is in state j. □

Let us now introduce the basic representation of the probability of the event $\{\tau_a^+ < \tau^{\{-b\}}\}$. Pick arbitrary $a, b \geq 0$ such that $a + b > 0$ and consider the following equations, which are an immediate consequence of the strong Markov property:

$$\mathbb{P}(J(\tau_a^+)) = \mathbb{P}(\tau_a^+ < \tau^{\{-b\}}; J(\tau_a^+)) + \mathbb{P}(\tau^{\{-b\}} < \tau_a^+; J(\tau^{\{-b\}}))\mathbb{P}(J(\tau_{a+b}^+)),$$
$$\mathbb{P}(J(\tau^{\{-b\}})) = \mathbb{P}(\tau^{\{-b\}} < \tau_a^+; J(\tau^{\{-b\}})) + \mathbb{P}(\tau_a^+ < \tau^{\{-b\}}; J(\tau_a^+))\mathbb{P}_+(J(\tau^{\{-a-b\}})).$$

Using the relation $\mathbb{P}(J(\tau_a^+)) = \Pi e^{\Lambda a}$ we get

$$\mathbb{P}(\tau_a^+ < \tau^{\{-b\}}; J(\tau_a^+))(\mathbb{P}_+(J(\tau^{\{-a-b\}}))\Pi e^{\Lambda(a+b)} - \mathbb{I}^+)$$
$$= \mathbb{P}(J(\tau^{\{-b\}}))\Pi e^{\Lambda(a+b)} - \Pi e^{\Lambda a}.$$

Right-multiply by $e^{-\Lambda(a+b)}$ and note that $\Pi_+ = \mathbb{I}^+$ to arrive at

$$\mathbb{P}(\tau_a^+ < \tau^{\{-b\}}; J(\tau_a^+))\tilde{W}_+(a+b) = \tilde{W}(b), \tag{7.6}$$

where we define $\tilde{W}(x)$ for all $x \geq 0$ through

$$\tilde{W}(x) = \Pi e^{-\Lambda x} - \mathbb{P}(J(\tau^{\{-x\}}))\Pi. \tag{7.7}$$

Note that the above calculations are still valid in the case of a killed MAP. In that case the first term of (7.6) should be read as $\mathbb{E}[e^{-q\tau_a^+}; \tau_a^+ < \tau^{\{-b\}}, J(\tau_a^+)]$.

We finish this section by making a note on how (7.7) simplifies in the case of a simple Lévy process. Recall that $\underline{X}$ denotes the all time infimum of X. Suppose that $\kappa > 0$ then $\Lambda = 0$ and hence $\tilde{W}(x) = 1 - \mathbb{P}(\tau^{\{-x\}} < \infty) = \mathbb{P}_x(\underline{X} > 0)$, which is the same as $\mathbb{P}_x(\underline{X} \geq 0)$ by Proposition 7.7. Exactly this construction is found in Kyprianou [2006, Ch. 7].

7.5 The scale matrix and its transform

In this section we present a proof of Theorem 7.1. Throughout this section we assume that $N_+ = N$, that is, none of the processes X_i is non-increasing. This excludes processes of bounded variation with zero drift, which are special in many ways. Note that (7.7) becomes $\tilde{W}(x) = e^{-\Lambda x} - \mathbb{P}(J(\tau^{\{-x\}}))$, where the killing rate $q \geq 0$ is implicit. Moreover, Lemma 7.14 and Lemma 7.15 allow to rewrite (7.6) as follows

$$\mathbb{P}(\tau_a^+ < \tau_b^-, J(\tau_a^+))\tilde{W}(a+b) = \tilde{W}(b).$$

Note that this identity is preserved if $\tilde{W}(x)$ is multiplied on the right by a constant matrix.

Assume for a moment that the killing rate is strictly positive: $q > 0$. We keep q implicit where it does not cause too much confusion. Recall also the definition of η as given in Theorem 7.1.

Lemma 7.16. *For $q > 0$ and $\alpha > \eta$ it holds that*

$$\int_0^\infty e^{-\alpha x}\tilde{W}(x)\mathrm{d}x\boldsymbol{L} = F(\alpha)^{-1}.$$

Proof. Using (7.7) we write

$$\int_0^\infty e^{-\alpha x}\tilde{W}(x)\mathrm{d}x = \int_0^\infty e^{(-\Lambda-\alpha\mathbb{I})x}\mathrm{d}x - \int_0^\infty e^{-\alpha x}\mathbb{P}(J(\tau^{\{-x\}}))\mathrm{d}x.$$

Recall that the set of eigenvalues of $-\Lambda$ coincides with the set of zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$. Hence the first integral on the right converges absolutely and is equal to $(\alpha\mathbb{I}+\Lambda)^{-1}$ for $\alpha > \eta$, see also Lemma A.9. In addition, Theorem 7.11 gives

$$-F(\alpha)^{-1} = \int_0^\infty e^{\alpha x}\mathbb{P}(J(\tau^{\{x\}}))\mathrm{d}x\boldsymbol{L} + \int_0^\infty e^{-\alpha x}\mathbb{P}(J(\tau^{\{-x\}}))\mathrm{d}x\boldsymbol{L} \tag{7.8}$$

for $\alpha \geq 0$ with $k(\alpha) < 0$. Note that $k(\alpha)$ is the Perron-Frobenius eigenvalue of $F(\alpha)$. So for $q > 0$ we have that $k(0) < 0$. Then the continuity of $k(\alpha)$ implies that there exists $\epsilon > 0$ such that $k(\alpha) < 0$ for all $\alpha \in [0, \epsilon)$. Remark 7.12 shows that $\boldsymbol{L}$ has finite entries. Furthermore, Equation (7.8) can be rewritten as

$$-\int_0^\infty e^{-\alpha x}\mathbb{P}(J(\tau^{\{-x\}}))\mathrm{d}x\boldsymbol{L} = F(\alpha)^{-1} - (\alpha\mathbb{I} + \Lambda)^{-1}\boldsymbol{L}$$

for $\alpha \in [0, \epsilon)$, because the real parts of all the eigenvalues of $(\alpha\mathbb{I} + \Lambda)$ are negative. The proof is complete as soon as we show that the latter identity can be continued to $\alpha > \eta$. To see this, multiply both sides by $F(\alpha)$ from the right and by $(\alpha\mathbb{I} + \Lambda)$ from the left. Then both sides are analytic for $\alpha \in \mathbb{C}^{\mathrm{Re}>0}$, see also Section A.7. Hence the equality holds for these α. □

Lemma 7.16 shows that we can define $W(x) = \tilde{W}(x)\boldsymbol{L}$ when $q > 0$. Using the matrix $\boldsymbol{L}(x)$ this can be further rewritten as follows

$$W(x) = \tilde{W}(x)\boldsymbol{L} = e^{-\Lambda x}[\boldsymbol{L} - \mathbb{P}(J(\tau_x^+))\mathbb{P}(J(\tau^{\{-x\}}))\boldsymbol{L}] = e^{-\Lambda x}\boldsymbol{L}(x), \tag{7.9}$$

see Section 7.3. Representation (7.9) of $W(x)$ in terms of $\boldsymbol{L}(x)$ is essential. The matrix $\boldsymbol{L}(x)$ has many nice properties, which allow to derive further properties of $W(x)$. In particular, the entries of $\boldsymbol{L}(x)$ are positive and increasing in $x \geq 0$. Moreover, $e^{-\Lambda x}\boldsymbol{L}(x)$ is continuous in $q \geq 0$. Hence this expression can be used to define $W(x)$ corresponding to $q = 0$. On the contrary, if $\kappa = 0$ then $\tilde{W}(x)$ becomes singular and the elements of $\boldsymbol{L}$ tend to ∞ as $q \downarrow 0$.

Definition 7.17. If $N = N_+$ then the scale matrix is defined through $W^q(x) = e^{-\Lambda(q)x}\boldsymbol{L}^q(x)$ for all $x, q \geq 0$.

Observe that

$$\mathbb{E}[e^{-q\tau_a^+}; \tau_a^+ < \tau_b^-, J(\tau_a^+)]W^q(a + b) = W^q(b)$$

for all $q \geq 0$ as required. Next we consider the transform of $W^q(x)$.

Lemma 7.18. *For any $q \geq 0$ and $\alpha > \eta(q)$ it holds that*

$$\int_0^\infty e^{-\alpha x}W^q(x)\mathrm{d}x = (F(\alpha) - q\mathbb{I})^{-1}.$$

Moreover, $W^q(x)$ is continuous in $x \geq 0$ and hence is uniquely identified by its transform.

Proof. Continuity of $W^q(x)$ in x follows from the continuity of $\boldsymbol{L}^q(x)$. The latter is true, because τ_x^+ is continuous at a fixed $x \geq 0$ with probability 1. In view of Lemma 7.16 it is only left to identify the transform for $q = 0$.

From the definition of $\boldsymbol{L}^q(x)$ it follows that $0 \leq \boldsymbol{L}^q(x) \leq \boldsymbol{L}^0(x)$. Moreover, according to Lemma A.12 there exists $\lambda > 0$ such that for all small enough $q \geq 0$ the elements of the matrix $e^{-\lambda x}e^{-\Lambda(q)x}, x \geq 0$ are bounded. Proposition 7.13 allows to apply the dominated convergence theorem to finish the proof for large α. Finally, observe that for all $\alpha > \eta(0)$ it holds that $\|e^{-\alpha x}e^{-\Lambda(0)x}\|_\infty$ is bounded by a constant, which can be shown similarly to the proof of Lemma A.9. This fact and Proposition 7.13 allow to extend the result to $\alpha > \eta(0)$, see Appendix A.7. □

Proof of Theorem 7.1. In view of Lemma 7.18 it only remains to show that $W(x)$ is invertible for any $x > 0$; $q \geq 0$ is implicit here. It is enough to establish that $\boldsymbol{L}(x)$ is invertible. For this observe that

$$\boldsymbol{L}(x+y) = \boldsymbol{L}(x) + \mathbb{P}(J(\tau_x^+))\mathbb{P}(\tau^{\{-x\}} < \tau_y^+, J(\tau^{\{-x\}}))\boldsymbol{L}(x+y).$$

Notice that the matrix $\mathbb{I} - \mathbb{P}(J(\tau_x^+))\mathbb{P}(\tau^{\{-x\}} < \tau_y^+, J(\tau^{\{-x\}}))$ is invertible for any $x > 0$, because it is irreducibly diagonally dominant, see Appendix A.1. So if for any $x > 0$ there is a vector $\boldsymbol{v}$, such that $\boldsymbol{L}(x)\boldsymbol{v} = \boldsymbol{0}$ then $\boldsymbol{L}(y)\boldsymbol{v} = \boldsymbol{0}$ for all $y > 0$. But then $F(\alpha)^{-1}\boldsymbol{v} = \boldsymbol{0}$ for large enough α according to Lemma 7.18, which is a contradiction. □

Let us conclude with a lemma, which identifies $W^q(0)$. The corresponding result for a Lévy process is given in Kyprianou [2006, Lem. 8.6].

Lemma 7.19. *For all $q \geq 0$ it holds that the (i,j)-th entry of $W^q(0)$ is 0 unless $i = j$ and X_i is a bounded variation process. In the latter case the corresponding entry is $1/d_i$, where d_i is the drift of X_i.*

Proof. Observe that $W^q(0) = \boldsymbol{L}^q(0)$ and use (7.5). □

7.6 The general case

Theorem 7.1 is stated under the assumption that $N_+ = N$. In this section we generalize this result by allowing $N_+ < N$. We write $\begin{pmatrix}\mathbb{I}^+\\ \mathbb{O}\end{pmatrix}$ to denote an $N \times N_+$ matrix, whose restriction to the rows in E_+ and $E_\downarrow$ is equal to $\mathbb{I}^+$ and $\mathbb{O}$ respectively.

Theorem 7.20. *For all $q \geq 0$ there exists a unique càdlàg function $W^q : [0,\infty) \to \mathbb{R}^{N\times N^+}$ such that the restriction $W_+^q(x)$ is invertible for all $x > 0$,*

$$\mathbb{E}[e^{-q\tau_a^+}; \tau_a^+ < \tau_b^-, J(\tau_a^+)] = W^q(b)W_+^q(a+b)^{-1} \text{ for all } a, b \geq 0 \text{ with } a+b>0,$$

and

$$\int_0^\infty e^{-\alpha x} W^q(x)\mathrm{d}x = (F(\alpha) - q\mathbb{I})^{-1}\begin{pmatrix}\mathbb{I}^+\\ \mathbb{O}\end{pmatrix}$$

for all $\alpha > \eta(q)$. *The function* $W_{ij}^{(q)}(x)$ *is continuous in* $x \geq 0$ *unless* X_i *is a CPP, whose distribution of jumps has atoms.*

The proof of this theorem is split into two parts presented in the following two subsections. First, we assume that none of the X_i is a process of bounded variation with zero drift, which allows to use the results of Section 7.3. In the second part we use a limiting argument to conclude the proof. An important role in this proof is played by an auxiliary MAP, which is obtained by restricting (X, J) to the time intervals when $J \in E_+$. Then the results of Section 7.5 can be used with respect to this auxiliary process.

Part I

Throughout this part we assume that none of the X_i is a process of bounded variation with $d_i = 0$. Recall the definition of $\tilde{W}(x)$ given in Section 7.4: $\tilde{W}(x) = \Pi e^{-\Lambda x} - \mathbb{P}(J(\tau^{\{-x\}}))\Pi$. The above assumption allows us to generalize Lemma 7.16 to get

$$\int_0^\infty e^{-\alpha x}\tilde{W}(x)\mathrm{d}x = F(\alpha)^{-1}\boldsymbol{L}^{-1}\Pi \tag{7.10}$$

for $q > 0$ and $\alpha > \eta$. Here we used $[\Pi(\Lambda + \alpha\mathbb{I})^{-1}, \mathbb{O}]\Pi = \Pi(\Lambda + \alpha\mathbb{I})^{-1}$, as well as Remark 7.12 to claim invertibility of $\boldsymbol{L}$.

Let us now construct an auxiliary MAP, which behaves exactly as (X, J) restricted to the time intervals where $J(t) \in E_+$, see Section 7.1. We use the subscript a to refer to the auxiliary MAP. Recall that (X_a, J_a) is a spectrally negative MAP with $N_a = (N_a)_+$, so Theorem 7.1 applies. In addition, $F_a(\alpha)^{-1}$ is equal to $F(\alpha)^{-1}$ restricted to the rows and columns in E_+. Moreover, $\Lambda = \Lambda_a$. Many other relations between (X, J) and (X_a, J_a) hold true. Note that $\mathbb{P}_+(J(\tau^{\{-x\}}))\Pi$ coincides with $\mathbb{P}_a(J(\tau^{\{-x\}}))$, and so $\tilde{W}_+(x) = \tilde{W}_a(x)$. With respect to the matrix $\boldsymbol{L}$ we have the following identity:

$$\Pi\boldsymbol{L}_a = \boldsymbol{L}\begin{pmatrix}\mathbb{I}^+\\ \mathbb{O}\end{pmatrix}, \text{ and so } \boldsymbol{L}^{-1}\Pi = \begin{pmatrix}\mathbb{I}^+\\ \mathbb{O}\end{pmatrix}\boldsymbol{L}_a^{-1}.$$

This is to be expected in view of (7.10) and the relation between $\tilde{W}(x), F(\alpha)$ and $\tilde{W}_a(x), F_a(\alpha)$.

Next we can define $W(x) = \tilde{W}(x)\boldsymbol{L}_a$ to get $W_+(x) = W_a(x)$ and

$$\int_0^\infty e^{-\alpha x} W(x)\mathrm{d}x = F(\alpha)^{-1}\begin{pmatrix}\mathbb{I}^+\\ \mathbb{O}\end{pmatrix} \tag{7.11}$$

for $q > 0$ and $\alpha > \eta$. Observe also that

$$W^q(x) = \mathbb{E}[e^{-q\tau_y^+}; \tau_y^+ < \tau_x^-, J(\tau_y^+)]W_+^q(y+x), \tag{7.12}$$

which converges as $q \downarrow 0$, because so does $W_+^q(y+x) = W_a^q(y+x)$. This limit defines $W^0(x), x \geq 0$. Identity (7.11) can be extended to $q = 0$ by taking the limit as $q \downarrow 0$. Here we use (7.12) with $y = 0$ and $x > 0$ to show that the dominated convergence theorem applies here; it does so for $\int_0^\infty e^{-\alpha x}W_a(x)\mathrm{d}x$, see the proof of Lemma 7.18.

Let us conclude the proof of Theorem 7.20 under the assumption that none of the X_i is a process of bounded variation with zero drift. Recall that $W_+(x) = W_a(x)$ for all $x \geq 0$. Hence $W_+(x)$ is invertible for $x > 0$ and continuous at $x \geq 0$, which is immediate from Theorem 7.1. Finally, the continuity of $W(x)$, $x \geq 0$ follows from the continuity of the first term on the right of (7.12), see also Lemma 7.15.

Part II

This part presents a limiting argument which finishes the proof of Theorem 7.1. It is a rather tedious and technical argument. Hence we only provide a sketch of it.

Consider the case when some X_i are bounded variation processes with $d_i = 0$. As before we can still define $W_+(x) = W_a(x)$. Then we use (7.12) with $y = 0$ to define $W(x)$ for $x > 0$. Observe that for $a \geq 0$ and $b > 0$ it holds (as is required) that

$$\mathbb{P}(\tau_a^+ < \tau_b^-, J(\tau_a^+)) = \mathbb{P}(\tau_0^+ < \tau_b^-, J(\tau_0^+))W_a(b)W_a(a+b)^{-1} = W(b)W_+(a+b)^{-1}.$$

Moreover, $W_+(x)$ is continuous. It is not difficult to see that $W(x)$ is a càdlàg function. This can be further used to define $W(0)$, and to show that the above identity holds true for $a > 0$ and $b = 0$.

In the following we sketch a limiting argument, which shows that the transform of $W(x)$ is as given in Theorem 7.20. We add some small negative drift $d < 0$ to all of the processes with $d_i = 0$, and then let $d \uparrow 0$. Firstly, this modification changes the auxiliary process X_a by perturbing its jumps at transition epochs of J_a. Using Lemma 7.9 observe that $e^{\Lambda_a x}$ and $\boldsymbol{L}_a(x)$ converge as $d \uparrow 0$ to the corresponding objects associated with $d = 0$. Hence so does $W_a(x)$.

It is not hard to see that $\mathbb{P}(\tau_0^+ < \tau^{(-\infty,-x]}, J(\tau_0^+))$ converges as $d \uparrow 0$ to the corresponding probability for $d = 0$. This probability coincides with $\mathbb{P}(\tau_0^+ < \tau_x^-, J(\tau_0^+))$ for $d \neq 0$, see Lemma 7.15. So $\mathbb{P}(\tau_0^+ < \tau_x^-, J(\tau_0^+))$ and hence $W(x)$ converge for at least those $x > 0$, for which τ_x^- and $\tau^{(-\infty,-x]}$ coincide a.s. for the original model, that is, when $d = 0$. Use Lemma 7.15 to see that these stopping times coincide $\mathbb{P}_i$-a.s. for every $x > 0$, unless X_i is a CPP, whose distribution of jumps has atoms. In the latter case the above claim holds for every $x > 0$ apart from a countable set.

In any case, it is sufficient to show that the limit and the integral can be interchanged when considering the transform of $W(x)$ as $d \uparrow 0$. This is done using the generalized dominated convergence theorem, see Kallenberg [2002, Thm. 1.21], where the bound depends on d as well. Here we rely on the fact that for $u > 0$ the transform $\int_0^\infty e^{-ux}\boldsymbol{L}_a(x)\mathrm{d}x$ converges as $d \uparrow 0$ to the transform corresponding to $d = 0$, which is shown using the extended continuity theorem for Laplace transforms, see Appendix A.7.

7.7 First examples

A scale matrix is a fundamental object appearing in various identities concerning path properties of a spectrally negative MAP. Theorem 7.1 uniquely identifies the scale matrix of a MAP through its transform. Inversion of this transform is not a trivial task. So, for example, numerical inversion may exhibit slow convergence in practice. It is thus important to have some explicit examples of scale matrices. Even in the setting of Lévy processes there is a very limited number of known examples of scale functions, see Kyprianou and Hubalek [2011] and Kuznetsov et al. [2011]. Moreover, the scale functions of the underlying Lévy processes of a MAP do not immediately yield the scale matrix.

A scale matrix can be constructed explicitly in the case of MMBM. In the following we use the notation of Chapter 5. If it is not the case that $q = 0$ and $\kappa = 0$ then the scale matrix is given by

$$\Pi^+ e^{-\Lambda^+ x} - \Pi^- e^{\Lambda^- x}\Pi_-^+ \tag{7.13}$$

up to a multiplication from the right with a constant invertible matrix. Before we address this question in more detail let us comment on the following. Recall that a spectrally negative MAP with phase-type jumps can be reduced to an MMBM without losing any information, see Section 2.7. This observation can be used to construct a scale matrix for such a MAP. One only needs to restrict the rows of the auxiliary scale matrix (the scale matrix of the MMBM) to those corresponding to the original process. Hence the form of this scale matrix very much resembles (7.13). It should also be noted that the entries of the scale matrix of an MMBM have rational transforms according to Theorem 7.1. So one can alternatively invert these transforms using partial fractions. These observations agree with the results of Asmussen et al. [2004], where a Lévy process with phase-type jumps in both directions is considered.

In the rest of this chapter we consider an MMBM. In this case many identities can be written explicitly. In particular, Section 5.3 presents a solution to the two-sided exit problem. In the following we exclude the delicate case when $q = 0$ and $\kappa = 0$. If $q > 0$ and none of the X_i is identically 0 then the construction of the scale matrix presented in the current chapter shows that $W(x) = \tilde{W}(x)\boldsymbol{L}_a$, where $\tilde{W}(x)$ is given in (7.13). In fact, these assumptions are unnecessary. That is, $W(x)$ has the same form unless $q = 0$ and $\kappa = 0$. This follows from the continuity arguments used in Section 7.6.

Alternatively, we can just evaluate the transform of $\tilde{W}(x)$. Observe that for $\alpha > \eta$ it holds that

$$\int_0^\infty e^{-\alpha x}\tilde{W}(x)\mathrm{d}x = \Pi^+(\alpha\mathbb{I} + \Lambda^+)^{-1} + \Pi^-(\Lambda^- - \alpha\mathbb{I})^{-1}\Pi^+_-.$$

In addition, Corollary 4.15 shows that

$$\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\Pi^+(\Lambda^+)^2 - \Delta_{\boldsymbol{\mu}}\Pi^+\Lambda^+ + Q\Pi^+ = \mathbb{O}.$$

This immediately yields

$$\frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}\Pi^+(\Lambda^+ - \alpha\mathbb{I})(\Lambda^+ + \alpha\mathbb{I}) - \Delta_{\boldsymbol{\mu}}\Pi^+(\Lambda^+ + \alpha\mathbb{I}) = -F(\alpha)\Pi^+,$$

which is used to express $\Pi^+(\alpha\mathbb{I} + \Lambda^+)^{-1}$. Similarly, one obtains $\Pi^-(\Lambda^- - \alpha\mathbb{I})^{-1}$. This leads to

$$\int_0^\infty e^{-\alpha x}\tilde{W}(x)\mathrm{d}x = F(\alpha)^{-1}\Xi,$$

with

$$\Xi = [\Delta_{\boldsymbol{\mu}}(\Pi^+ - \Pi^-\Pi^+_-) - \frac{1}{2}\Delta^2_{\boldsymbol{\sigma}}(\Pi^+(\Lambda^+ - \alpha\mathbb{I}) + \Pi^-(\Lambda^- + \alpha\mathbb{I})\Pi^+_-)].$$

Let us examine the matrix Ξ. The i-th row of Ξ is

- $\Xi_i = -\frac{1}{2}\sigma_i^2(\Lambda^+_i + \Lambda^-_i\Pi^+_-)$, if $\sigma_i > 0$;
- $\Xi_i = \mu_i(\boldsymbol{e}_i^\top - \Pi^-_i\Pi^+_-)$, if $\sigma_i = 0$ and $\mu_i > 0$;

- $\Xi_i = \mathbf{0}^\top$, if $\sigma_i = 0$ and $\mu_i \leq 0$.

So, indeed, Ξ does not depend on α and $\Xi_\downarrow = \mathbb{O}$, which proves that $\tilde{W}(x)$ is given by (7.13). Moreover, $\boldsymbol{L}_a = (\Xi_+)^{-1}$.

Chapter 8

Further exit problems

We have solved two fundamental exit problems for a general spectrally negative MAP. Firstly, the theory of the first passage over a positive level was presented in Chapter 4. Secondly, it served as a basis to construct a scale matrix and to solve the problem of the exit from an interval over the upper boundary in Chapter 7. Using these results in the present chapter we address further exit problems. Throughout this chapter we assume that $N_+ = N$. This simplifies notation and allows to avoid certain technicalities, which can be rather unpleasant to deal with as is demonstrated in Section 7.6. It is noted that the scale matrix defined in Theorem 7.1 plays a fundamental role in what follows.

Let us give a brief outline of this chapter. Section 8.1 addresses exit problems over a negative level, including the exit from an interval over the lower boundary. In both cases we are interested in the passage time jointly with the corresponding overshoot. In this context it is convenient to define a *second* scale matrix denoted by Z. Section 8.2 considers the first passage process killed upon arrival of an excursion from the maximum exceeding height $B > 0$. This object is known to play an important role in different problems concerning Lévy processes, see, for example, Pistorius [2004] and Avram et al. [2007], Loeffen [2008], where reflected processes and the dividend problem are considered respectively. In Section 8.3 we extend the results of Chapter 6, and characterize the two-sided reflection at inverse local times at the upper boundary. Using these results in Section 8.4 we solve the first passage problem for reflected processes.

The level crossing problem for reflected processes is important in applications. For example, it arises in queueing theory when one considers a Markov-modulated queue with a finite buffer. Another example is the dividend problem mentioned in the Introduction. The corresponding results for spectrally positive and spectrally negative Lévy processes were obtained in Avram et al. [2004] and Pistorius [2004] respectively. See also Korolyuk [1974] for an early work on CPPs, and Doney [2005] for an alternative proof. Finally, Kyprianou [2006, Sec. 8.5] provides a textbook introduction to the problem. The proofs of these results rely on Itô's excursion theory, stochastic integration and martingale calculations. It is far from straightforward to generalize these proofs to the MAP setting. In fact, the author did not succeed in doing so. We use an alternative approach based on a number of easy to understand observations in line with the ideas presented in Ivanovs [2011] for the case of a Lévy process. This simplicity allows us to solve the problem for MAPs in a very similar fashion.

8.1 First passage over a negative level

Let us start by defining the matrix function:

$$Z^q(\alpha, x) = e^{\alpha x}\left(\mathbb{I} - \int_0^x e^{-\alpha y} W^q(y)\mathrm{d}y(F(\alpha) - q\mathbb{I})\right) \text{ for } \alpha, q, x \geq 0.$$

This matrix, also called second scale matrix, appears in a number of exit identities along with the matrix $W^q(x)$. Note that $Z^q(\alpha, x)$ is continuous in x with $Z^q(\alpha, 0) = \mathbb{I}$, and is analytic in $\alpha \in \mathbb{C}^{\mathrm{Re}>0}$. In the case of a single background state we obtain $Z^q(0, x) = 1 + q\int_0^x W^q(y)\mathrm{d}y$, which is a common definition of the Z function corresponding to a spectrally negative Lévy process, see, for example, Definition 3 in Avram et al. [2004].

Let us illustrate this with the following proposition, which identifies the transform of the first passage over $-x$ and the corresponding overshoot.

Proposition 8.1. *Let $\Upsilon^q = \Delta_{\boldsymbol{\pi}}^{-1}\hat{\Lambda}(q)^\top \Delta_{\boldsymbol{\pi}}$. Then for all $x, \alpha \geq 0$ and $q > 0$, such that $\Upsilon^q(\alpha)$ is non-singular, it holds that*

$$\mathbb{E}[e^{-q\tau_x^- + \alpha(X(\tau_x^-)+x)}; J(\tau_x^-)] = Z^q(\alpha, x) - W^q(x)(\Upsilon^q + \alpha\mathbb{I})^{-1}(F(\alpha) - q\mathbb{I}).$$

Proof. Let us compute the transform of $\mathbb{E}[e^{-q\tau_x^- + \alpha X(\tau_x^-)}; J(\tau_x^-)]$ for $\alpha \geq 0, q > 0$. A similar computation can be also found in Kyprianou and Palmowski [2008]. First, observe that

$$\mathbb{E}[e^{\alpha X(e_q)}; \underline{X}(e_q) < -x, J(e_q)] = \mathbb{E}[e^{\alpha X(\tau_x^-)}; \tau_x^- < e_q, J(\tau_x^-)]\mathbb{E}[e^{\alpha X(e_q)}; J(e_q)].$$

Taking transforms on both sides yields for $\theta \geq 0$

$$\begin{aligned}&\int_0^\infty e^{-\theta x}\mathbb{E}[e^{-q\tau_x^- + \alpha X(\tau_x^-)}; J(\tau_x^-)]\mathrm{d}x\mathbb{E}[e^{\alpha X(e_q)}; J(e_q)]\\&= \mathbb{E}[(\int_0^{-\underline{X}(e_q)} e^{-\theta x}\mathrm{d}x)e^{\alpha X(e_q)}; J(e_q)]\\&= \frac{1}{\theta}(\mathbb{E}[e^{\alpha X(e_q)}; J(e_q)] - \mathbb{E}[e^{\theta\underline{X}(e_q)+\alpha X(e_q)}; J(e_q)]).\end{aligned}$$

Note that $\mathbb{E}[e^{\alpha X(e_q)}; J(e_q)] = -q(F(\alpha) - q\mathbb{I})^{-1}$ for small enough $\alpha \geq 0$. Moreover, according to Corollary 4.21, see also Remark 4.22, it holds for small enough $\alpha \geq 0$ that

$$\mathbb{E}[e^{\theta\underline{X}(e_q)+\alpha X(e_q)}; J(e_q)] = -q(F(\alpha+\theta) - q\mathbb{I})^{-1}(\mathbb{I} + \theta(\Upsilon^q + \alpha\mathbb{I})^{-1}).$$

Hence

$$\begin{aligned}&\int_0^\infty e^{-\theta x}\mathbb{E}[e^{-q\tau_x^- + \alpha X(\tau_x^-)}; J(\tau_x^-)]\mathrm{d}x\\&= \frac{1}{\theta}\mathbb{I} - (F(\alpha+\theta) - q\mathbb{I})^{-1}(\frac{1}{\theta}\mathbb{I} + (\Upsilon^q + \alpha\mathbb{I})^{-1})(F(\alpha) - q\mathbb{I}).\end{aligned}$$

But for large enough $\theta \geq 0$ the above expression coincides with

$$\int_0^\infty e^{-\theta x}e^{-\alpha x}[Z^q(\alpha, x) - W^q(x)(\Upsilon^q + \alpha\mathbb{I})^{-1}(F(\alpha) - q\mathbb{I})]\mathrm{d}x.$$

The latter computation is based on Fubini's theorem and Theorem 7.1. This proves the result for small $\alpha \geq 0$, see also Section A.7. Use analyticity to extend the result to all $\alpha \geq 0$. □

Recall that for a fixed $x \geq 0$ the matrix $W^q(x)$ is continuous in $q \geq 0$. This can be used to show that $Z^q(\alpha, x)$ is continuous in $q \geq 0$ as well. Hence the result of Proposition 8.1 can be extended to $q = 0$ by taking the limit as $q \downarrow 0$. Moreover, it is noted that $(\Upsilon^q + \alpha\mathbb{I})^{-1}(F(\alpha) - q\mathbb{I})$ reduces to $(\psi(\alpha) - q)/(\alpha - \Phi(q))$ in the Lévy case. This leads to the known identity for a Lévy process:

$$\mathbb{E}e^{-q\tau_x^-} = Z^q(0, x) - \frac{q}{\Phi(q)}W^q(x),$$

see Kyprianou [2006, Thm. 8.1].

The following corollary identifies the transform of the first passage over $-b$ and the corresponding overshoot on the event that the process has not been above a. This result complements the other exit problem, see Theorem 7.1.

Corollary 8.2. *For any $q \geq 0$ and $\alpha, a, b \geq 0$ with $a+b>0$ it holds that*

$$\mathbb{E}[e^{-q\tau_b^- +\alpha(X(\tau_b^-)+b)}; \tau_b^- < \tau_a^+, J(\tau_b^-)] = Z^q(\alpha, b) - W^q(b)W^q(a+b)^{-1}Z^q(\alpha, a+b).$$

Proof. Using the strong Markov property observe that

$$\mathbb{E}[e^{-q\tau_b^- +\alpha(X(\tau_b^-)+b)}; \tau_b^- < \tau_a^+, J(\tau_b^-)] = \mathbb{E}[e^{-q\tau_b^- +\alpha(X(\tau_b^-)+b)}; J(\tau_b^-)]$$
$$- \mathbb{E}[e^{-q\tau_a^+}; \tau_a^+ < \tau_b^-, J(\tau_a^+)]\mathbb{E}[e^{-q\tau_{a+b}^- +\alpha(X(\tau_{a+b}^-)+a+b)}; J(\tau_{a+b}^-)].$$

Use Theorem 7.1 and Proposition 8.1 to get the result for $q > 0$. Let $q \downarrow 0$ to obtain the result for $q = 0$. □

8.2 Arrival of the first excursion exceeding a certain height

The Markov chain $J(\tau_x^+), x \geq 0$, whose transition rate matrix we denote by Λ, plays a fundamental role in the study of MAPs. In this section we introduce an important class of Markov chains, which contains the above one as a boundary element. The idea is to kill $J(\tau_x^+)$ upon arrival of the first excursion of X from the maximum exceeding a certain height $B > 0$. This idea is best illustrated using the concept of reflection, see Section 2.8.

Let R be the reflection of X corresponding to $(-\infty, 0]$. That is, $R(t) = X(t) - U(t)$, where $U(t) = \overline{X}(t)$ is the upper regulator at 0. The process U is often called the local time at the maximum. Let also $\tau_B^R = \inf\{t \geq 0 : R(t) < -B\}$ be the first passage time of R over level $-B$. Then

$$\zeta^B = U(\tau_B^R)$$

is the (local) time of the arrival of the first excursion (from the maximum) with height exceeding B. The strong Markov property implies that $J(\tau_x^+)$ sent to the absorbing state ∂ at $x = \zeta^B$ is a Markov chain. We denote its transition rate matrix through Λ^B. Hence

$$\mathbb{P}(x < \zeta^B, J(\tau_x^+)) = e^{\Lambda^B x} \text{ for all } x \geq 0. \tag{8.1}$$

The following proposition identifies Λ^B in terms of the scale matrix.

Proposition 8.3. *It holds that $W'(B+) = \lim_{\epsilon \downarrow 0}(W(B+\epsilon) - W(B))/\epsilon$ exists and $\Lambda^B = -W'(B+)W(B)^{-1}$.*

Proof. Observe that for any $0 < \epsilon < \delta$ it holds that

$$\mathbb{P}(\epsilon < \zeta^B, J(\tau_\epsilon^+)) \leq \mathbb{P}(\tau_\epsilon^+ < \tau_B^-, J(\tau_\epsilon^+)) \leq \mathbb{P}(\epsilon < \zeta^{B+\delta}, J(\tau_\epsilon^+)).$$

Subtract $\mathbb{I}$, divide by ϵ and let $\epsilon \downarrow 0$ to obtain in view of (8.1) the bounds

$$\Lambda^B \leq \lim_{\epsilon \downarrow 0}(W(B)W(B+\epsilon)^{-1} - \mathbb{I})/\epsilon \leq \Lambda^{B+\delta}.$$

It is only left to show that $\Lambda^{B+\delta} \to \Lambda^B$ as $\delta \downarrow 0$, because then the result follows from the continuity of $W(x)$. But $e^{\Lambda^{B+\delta}x} = \mathbb{P}(x < \zeta^{B+\delta}, J(\tau_x^+)) \to \mathbb{P}(x < \zeta^B, J(\tau_x^+)) = e^{\Lambda^B x}$ converges for every x, which concludes the proof in view of Proposition A.11. □

Let us comment on the existence of $W'(B-)$.

Remark 8.4. Similarly, one can kill the Markov chain $J(\tau_x^+)$ upon arrival of the first excursion with height $\geq B$. The corresponding transition probability matrix is given by Λ^{B-}. Moreover, we can mimic the proof of Proposition 8.3 to get

$$\mathbb{P}(\epsilon < \zeta^{B-\delta}, J(\tau_\epsilon^+)) \leq \mathbb{P}(\tau_\epsilon^+ < \tau_{B-\epsilon}^-, J(\tau_\epsilon^+)) \leq \mathbb{P}(\epsilon < \zeta^{B-}, J(\tau_\epsilon^+)),$$

where we used Lemma 7.15. This further shows that $W'(B-)$ exists and $\Lambda^{B-} = -W'(B-)W(B)^{-1}$. Finally, $W'(B+)$ and $W'(B-)$ coincide if so do Λ^B and Λ^{B-}. For this it is sufficient to require that every X_i is of unbounded variation. Then for every i the point 0 is regular for $(-\infty, 0)$ for X_i, which implies $\Lambda^B = \Lambda^{B-}$. On the contrary, if one of the X_i is the sum of a positive drift and a CPP with jumps of size $-B$ then $W'(B-) \neq W'(B+)$.

8.3 Two-sided reflection

In this section we reconsider reflection of X and alter it by placing a lower barrier at the level $-B$. That is, we put $R(t) = X(t) + L(t) - U(t)$, where L and U are the regulators at $-B$ and 0 respectively, see Section 2.8. This model (up to translation by B) in the case of MMBM is considered in Chapter 6. The aim of this section is to generalize the results in that chapter.

The process U can be seen as the local time of R at 0. Note that U is non-decreasing and continuous. Hence its inverse τ_x^U satisfies $U(\tau_x^U) = x$ and $R(\tau_x^U) = 0$. Then the strong Markov property shows that $(L(\tau_x^U), J(\tau_x^U))$ is a MAP itself. As before we observe that, in fact, $(L(\tau_x^U), J(\tau_x^U))$ is a Markov-modulated

CPP, because it has no jumps in some interval with positive probability. We denote its matrix exponent through $F^B(\alpha)$, that is, for $\alpha \geq 0$ it holds that

$$\mathbb{E}[e^{-\alpha L(\tau_x^U)}; J(\tau_x^U)] = e^{F^B(\alpha)x}.$$

Theorem 8.5. *For all $\alpha \geq 0$ and $x_0 \in [-B, 0]$ it holds that $Z(\alpha, B)$ is invertible and*

$$F^B(\alpha) = W(B)F(\alpha)Z(\alpha, B)^{-1} - \alpha\mathbb{I},$$
$$\mathbb{E}_{x_0}[e^{-\alpha L(\tau_0^U)}; J(\tau_0^U)] = Z(\alpha, B + x_0)Z(\alpha, B)^{-1}.$$

The following lemma will play a crucial role in the proof of Theorem 8.5.

Lemma 8.6. *It holds for $q > 0$ and small enough $\alpha \geq 0$ that*

$$\int_x^\infty e^{-\alpha y}\mathbb{P}(J(\tau^{\{-y\}}))\mathrm{d}y$$
$$= -F(\alpha)^{-1}\boldsymbol{L}^{-1} + (\Lambda + \alpha\mathbb{I})^{-1}e^{-(\Lambda+\alpha\mathbb{I})x} + \int_0^x e^{-\alpha y}W(y)\mathrm{d}y\boldsymbol{L}^{-1},$$

where $\boldsymbol{L}$ is defined in Section 7.3.

Proof. The construction of $W(x)$, see Section 7.5, shows that

$$\int_0^x e^{-\alpha y}\mathbb{P}(J(\tau^{\{-y\}}))\mathrm{d}y = (\Lambda + \alpha\mathbb{I})^{-1}(\mathbb{I} - e^{-(\Lambda+\alpha\mathbb{I})x}) - \int_0^x e^{-\alpha y}W(y)\mathrm{d}y\boldsymbol{L}^{-1}$$

for small enough $\alpha \geq 0$. Letting $x \to \infty$ we obtain

$$\int_0^\infty e^{-\alpha x}\mathbb{P}(J(\tau^{\{-x\}}))\mathrm{d}x = -F(\alpha)^{-1}\boldsymbol{L}^{-1} + (\alpha\mathbb{I} + \Lambda)^{-1}.$$

Combine these equalities to complete the proof. □

Proof of Theorem 8.5. A number of arguments in this proof are taken from the proof of Theorem 6.1. Hence we only present a sketch. It is implicitly assumed that the killing rate $q > 0$ is positive. Letting N_y be a matrix with (i, j)-th component specified by

$$N_y = \#\{t \geq 0 : X(t) = y, t = \tau_x^U, x \geq 0, J(t) = j\} \text{ given } J(0) = i,$$

we find that

$$\mathbb{E}_{x_0}N_y = (1_{\{y\geq 0\}}\mathbb{P}_{x_0}(J(\tau_y^+)) + 1_{\{y<0\}}C_{B-y+x_0})\sum_{i=0}^\infty (C_B)^i,$$

where $C_x = \mathbb{P}(J(\tau^{\{-x\}}))\mathbb{P}(J(\tau_B^+))$. This readily leads to

$$\int_{\mathbb{R}} e^{\alpha y}\mathbb{E}_{x_0}N_y \mathrm{d}y$$
$$= \left(\int_0^\infty e^{\alpha y}e^{\Lambda(y-x_0)}\mathrm{d}y + \int_{-\infty}^0 e^{\alpha y}\mathbb{P}(J(\tau^{\{-B+y-x_0\}}))\mathrm{d}y e^{\Lambda B}\right)(\mathbb{I}-C_B)^{-1}.$$

The first term inside the large brackets is $-(\Lambda+\alpha\mathbb{I})^{-1}e^{-\Lambda x_0}$ for small enough $\alpha \geq 0$. The second term is $e^{\alpha(B+x_0)}\int_{B+x_0}^\infty e^{-\alpha y}\mathbb{P}(J(\tau^{\{-y\}}))\mathrm{d}y e^{\Lambda B}$. Use Lemma 8.6 to show that the expression in the brackets is equal to

$$e^{\alpha(B+x_0)}\left(\int_0^{B+x_0} e^{-\alpha y}W(y)\mathrm{d}y - F(\alpha)^{-1}\right)\boldsymbol{L}^{-1}e^{\Lambda B}.$$

Finally, the construction of $W(x)$ shows that $\mathbb{I}-C_B = W(B)\boldsymbol{L}^{-1}e^{\Lambda B}$, see Section 7.5. This immediately yields

$$\int_{\mathbb{R}} e^{\alpha y}\mathbb{E}_{x_0}N_y \mathrm{d}y = -Z(\alpha, B+x_0)F(\alpha)^{-1}W(B)^{-1}.$$

Observe that $X(\tau_x^U) = x - L(\tau_x^U)$ is piecewise constant with slope 1, so we have

$$\int_{\mathbb{R}} e^{\alpha y}\mathbb{E}_{x_0}N_y \mathrm{d}y = \int_0^\infty \mathbb{E}_{x_0}[e^{\alpha X(\tau_x^U)}; J(\tau_x^U)]\mathrm{d}x$$
$$= \mathbb{E}_{x_0}[e^{-\alpha L(\tau_0^U)}; J(\tau_0^U)]\int_0^\infty e^{\alpha x}e^{F^B(\alpha)x}\mathrm{d}x.$$

But the last integral is $-(F^B(\alpha)+\alpha\mathbb{I})^{-1}$. The result now follows for small enough $\alpha \geq 0$ and $q > 0$ by noting that $\tau_0^U = 0$ a.s. when $x_0 = 0$. Use analyticity in $\alpha > 0$ and continuity in $q \geq 0$ to complete the proof. □

8.4 Exit of the reflected process

In this section we consider the processes X and $-X$ reflected at 0, and determine their first passage times over level B. In fact, for each reflected process we are interested in the joint Laplace transform of the first passage time, the overshoot, and the corresponding value of the regulator. Furthermore, it is assumed that the initial value of the process is shifted to an arbitrary $r_0 \in [0, B]$.

It is crucial to note that these problems can be reformulated in terms of two-sided reflection discussed in Section 8.3. Considering X reflected at 0, observe that its first passage over B and the corresponding value of the regulator are exactly τ_0^U and $L(\tau_0^U)$ given $x_0 = r_0 - B$; the overshoot is clearly 0. Hence

$\mathbb{E}_{x_0}[e^{-q\tau_0^U-\alpha L(\tau_0^U)};J(\tau_0^U)]$ is the required transform, which was identified in Theorem 8.5. Consider $-X$ reflected at 0. Its first passage over level B, the overshoot, and the corresponding value of the regulator are exactly $\tau_0^L, L(\tau_0^L)$, and $U(\tau_0^L)$ given $x_0 = -r_0$. Hence we need to identify

$$\mathbb{E}_{x_0}[e^{-q\tau_0^L-\theta U(\tau_0^L)-\alpha L(\tau_0^L)};J(\tau_0^L)] \text{ for } q,\theta,\alpha \geq 0,$$

which we do in the rest of this section. As before, we do not write the killing rate $q \geq 0$ explicitly in what follows.

Before we proceed let us present a result on a Markov-modulated CPP, which we need in the analysis of a reflected MAP. We exclude the process identical to 0.

Lemma 8.7. *Let (X, J) be a Markov-modulated CPP without negative jumps, such that $\mathbb{E}[e^{-\alpha X(t)};J(t)] = e^{F(\alpha)t}$. Let also $T = \{t \geq 0 : X(t) \neq 0\}$ be the epoch of the first jump of X. Then for $q > 0$ and $\alpha \geq 0$ it holds that*

$$\mathbb{E}[e^{-qT-\alpha X(T)};J(T)] = \mathbb{I} - (F(\infty) - q\mathbb{I})^{-1}(F(\alpha) - q\mathbb{I}).$$

Proof. Using the strong Markov property and memoryless property of e_q we write

$$\mathbb{E}[e^{-\alpha X(e_q)};J(e_q)] = \mathbb{P}(e_q < T, J(e_q)) + \mathbb{E}[e^{-\alpha X(T)};e_q > T, J(T)]\mathbb{E}[e^{-\alpha X(e_q)};J(e_q)].$$

Note that $\mathbb{E}[e^{-\alpha X(e_q)};J(e_q)] = q\int_0^\infty e^{(F(\alpha)-q\mathbb{I})t}\mathrm{d}t$, which converges and is equal to $-q(F(\alpha) - q\mathbb{I})^{-1}$ for all $q > 0$ and $\alpha \geq 0$. In addition,

$$\mathbb{P}(e_q < T, J(e_q)) = \mathbb{P}(X(e_q) = 0, J(e_q)) = \lim_{\alpha\to\infty} \mathbb{E}[e^{-\alpha X(e_q)};J(e_q)],$$

which completes the proof. □

Let ζ be the epoch of the first jump of $L(\tau_x^U)$, that is,

$$\zeta = \inf\{x \geq 0 : L(\tau_x^U) > 0\}.$$

Assume for a moment that $x_0 = 0$. Then ζ is exactly ζ^B, the (local) time of the arrival of the first excursion with height exceeding B, because one-sided and two-sided reflections coincide before an arrival of such an excursion. Moreover, $L(\tau_x^U), x \geq 0$ is a Markov-modulated CPP with matrix exponent $F^B(\alpha)$, which is identified in Theorem 8.5. So we can write

$$\lim_{\alpha\to\infty} e^{F^B(\alpha)x} = \mathbb{P}_0(L(\tau_x^U) = 0, J(\tau_x^U)) = \mathbb{P}_0(x < \zeta, J(\tau_x^+)) = e^{\Lambda^B x},$$

which implies that $F^B(\infty) = \Lambda^B$, see Proposition A.11. Lemma 8.7 then shows that

$$\mathbb{E}_0[e^{-\theta\zeta-\alpha L(\tau_\zeta^U)};J(\tau_\zeta^U)] = \mathbb{I} - (\Lambda^B - \theta\mathbb{I})^{-1}(F^B(\alpha) - \theta\mathbb{I}) \qquad (8.2)$$

for all $\theta > 0$ and $\alpha \geq 0$.

Let x_0 be an arbitrary number in $[-B, 0]$. It is important to understand the meaning of τ_ζ^U. It is the first time the reflected process $R(x)$ hits the upper barrier at 0 after it has hit the lower barrier at $-B$. So $\tau_0^L < \tau_\zeta^U$ and $\zeta = U(\tau_0^L)$, but then the strong Markov property implies

$$\mathbb{E}_{x_0}[e^{-\theta\zeta-\alpha L(\tau_\zeta^U)}; J(\tau_\zeta^U)] = \mathbb{E}_{x_0}[e^{-\theta U(\tau_0^L)-\alpha L(\tau_0^L)}; J(\tau_0^L)]\mathbb{E}_{-B}[e^{-\alpha L(\tau_0^U)}; J(\tau_0^U)].$$

Alternatively, this expectation can be computed by considering the event $\{\tau_0^+ < \tau_B^-\}$ and its complement:

$$\begin{aligned}\mathbb{E}_{x_0}[e^{-\theta\zeta-\alpha L(\tau_\zeta^U)}; J(\tau_\zeta^U)] = \mathbb{P}_{x_0}(\tau_0^+ < \tau_B^-, J(\tau_0^+))\mathbb{E}_0[e^{-\theta\zeta-\alpha L(\tau_\zeta^U)}; J(\tau_\zeta^U)] \\ +\mathbb{E}_{x_0}[e^{-\alpha L(\tau_0^U)}; \tau_B^- < \tau_0^+, J(\tau_0^U)].\end{aligned}$$

The last term reduces to $\mathbb{E}_{x_0}[e^{-\alpha L(\tau_0^U)}; J(\tau_0^U)] - \mathbb{P}_{x_0}(\tau_0^+ < \tau_B^-, J(\tau_0^+))$. Use the last three displays to express $\mathbb{E}_{x_0}[e^{-\theta U(\tau_0^L)-\alpha L(\tau_0^L)}; J(\tau_0^L)]$. Finally, note that $Z(\alpha, 0) = \mathbb{I}$ and apply Theorem 7.1, Theorem 8.5, and Lemma 8.7 to get the following result.

Theorem 8.8. *It holds for $\theta, \alpha \geq 0$ and $x_0 \in [-B, 0]$ that*

$$\begin{aligned}&\mathbb{E}_{x_0}[e^{-\theta U(\tau_0^L)-\alpha L(\tau_0^L)}; J(\tau_0^L)] \\ &= Z(\alpha, B + x_0) + W(B + x_0)[W'(B) + \theta W(B)]^{-1}[W(B)F(\alpha) - (\alpha + \theta)Z(\alpha, B)].\end{aligned}$$

It is noted that the killing rate $q \geq 0$ is implicit in this theorem, which adds $-q\tau_0^L$ to the transform.

In the final part of this section we compare our result, given in Theorem 8.8, to the result of Avram et al. [2004], concerning reflection of a spectrally positive Lévy process. Let $R(t)$ be the reflection of $-X(t)$ at 0, where X is a spectrally negative Lévy process with Laplace exponent $\psi(\alpha)$. Assume that $R(0) = r_0 \in [0, B]$ and consider the first passage time of R over level B, which we denote by τ. Then according to [Avram et al., 2004, Theorem 1] one has

$$\mathbb{E}[e^{-q\tau-\alpha R(\tau)}] = e^{-\alpha r_0}\left(Z_*(B - r_0) - W_*(B - r_0)\frac{(q - \psi(\alpha))W_*(B) + \alpha Z_*(B)}{W_*'(B) + \alpha W_*(B)}\right),$$

where $W_*(x) = e^{-\alpha x}W^q(x)$ and $Z_*(x) = 1 + (q - \psi(\alpha))\int_0^x e^{-\alpha y}W^q(y)\mathrm{d}y = e^{-\alpha x}Z^q(\alpha, x)$. This formula can be rewritten in the following form

$$\mathbb{E}[e^{-q\tau-\alpha(R(\tau)-B)}] = Z^q(\alpha, B - r_0) + \frac{W^q(B - r_0)}{(W^q(B))'}[W^q(B)(\psi(\alpha) - q) - \alpha Z^q(B)].$$

Note that $R(\tau) - B$ is the overshoot of R over level B at first passage. So, indeed, this formula coincides with the statement of Theorem 8.8 with $\theta = 0$ and $x_0 = -r_0$.

Appendix

A.1 Location of eigenvalues

Most of the concepts and facts from linear algebra used in the present book can be found in Horn and Johnson [1985]. Let $A \in \mathbb{C}^{n\times n}$ be an $n \times n$ matrix with complex elements a_{ij}. Define

$$a_i = \sum_{j\neq i} |a_{ij}|, \qquad\qquad D_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq a_i\}.$$

The following is the celebrated Gershgorin's theorem. It states that the eigenvalues of a matrix lie in the union of certain disks in the complex plane, see Horn and Johnson [1985, Thm. 6.1.1].

Theorem A.1 (Gershgorin)**.** *The eigenvalues of A lie in $\cup_{i=1}^{n} D_i$. Moreover, if m of the disks D_i are isolated from the other $n - m$ disks, then there are precisely m eigenvalues of A in their union.*

Next we discuss irreducibility. Let Γ be a directed graph on n nodes induced by A, where the arc between i and j is present iff $a_{ij} \neq 0$.

Definition A.2. A matrix A is said to be *irreducible* if Γ is strongly connected, that is, for every i and j there is a directed path from i to j.

Let us present the concept of diagonal dominance.

Definition A.3. A matrix A is said to be *diagonally dominant* if $|a_{ii}| \geq a_i$ for all i. It is *strictly diagonally dominant* if the above inequality is strict for all i. Finally, A is *irreducibly diagonally dominant* if

- A is irreducible,
- A is diagonally dominant,
- for at least one i it holds that $|a_{ii}| > a_i$.

The following theorem can be found in Horn and Johnson [1985], see Thm. 6.1.10 and Thm. 6.2.27.

Theorem A.4. *Strictly diagonally dominant and irreducibly diagonally dominant matrices are invertible.*

The spectral radius of A is defined through

$$\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

The Perron-Frobenius theory is summarized by the following theorem, see Horn and Johnson [1985, Thm. 8.4.4].

Theorem A.5 (Perron-Frobenius)**.** *Let A be an irreducible matrix with non-negative elements. Then*

- *$\rho(A)$ is a simple eigenvalue of A,*
- *there is a strictly positive (in every element) eigenvector $\boldsymbol{x}$ corresponding to $\rho(A)$: $A\boldsymbol{x} = \rho(A)\boldsymbol{x}$.*

A.2 Jordan normal form

Every square matrix can be reduced to an 'almost diagonal' matrix by a similarity transform, see Horn and Johnson [1985, Thm. 3.1.11]. A *Jordan block* $\Gamma_k(\lambda)$ is a $k \times k$ matrix with λ on the diagonal, 1 on the upper diagonal, and 0 everywhere else.

Theorem A.6. *Let $A \in \mathbb{C}^{n\times n}$ be a given complex matrix. There is an invertible matrix V such that $A = V\Gamma V^{-1}$, where $\Gamma = \text{diag}(\Gamma_{k_1}(\lambda_1), \ldots, \Gamma_{k_m}(\lambda_m))$ is a block-diagonal matrix, and $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of A (not necessarily distinct). The* Jordan matrix *Γ is unique up to permutations of the diagonal Jordan blocks. If A is a real matrix with only real eigenvalues, then the similarity matrix V can be taken to be real.*

Let us discuss the structure of a Jordan matrix. Consider an arbitrary Jordan block $J_k(\lambda)$, and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ be the corresponding columns of V. Then it holds that $A\boldsymbol{v}_1 = \lambda\boldsymbol{v}_1$ and $A\boldsymbol{v}_i = \lambda\boldsymbol{v}_i + \boldsymbol{v}_{i-1}$ for $1 < i \leq k$. A sequence of vectors satisfying this property is called a *Jordan chain* corresponding to an eigenvalue λ. The matrix A is *diagonalizable* iff $m = n$, that is, all the Jordan blocks are scalars. The number of Jordan blocks corresponding to a given eigenvalue λ is the *geometric multiplicity* λ, which is the dimension of the null space of $A - \lambda\mathbb{I}$. The sum of orders k_i of the Jordan blocks corresponding to λ is the *algebraic multiplicity* of λ, which is the multiplicity of λ as a zero of the characteristic polynomial $\det(A - \lambda\mathbb{I})$.

Finally, a real symmetric matrix has a very special Jordan form, see Horn and Johnson [1985, Thm. 4.1.5].

Theorem A.7. *If A is a real symmetric matrix then Γ is a real diagonal matrix and V can be taken to be a real orthonormal matrix: $V^T V = \mathbb{I}$.*

An immediate consequence of the above theorem is that all the eigenvalues of a real symmetric matrix are real with algebraic and geometric multiplicities being the same.

A.3 Matrix norms and convergence

Consider a vector norm on $\mathbb{C}^n$. The most frequent examples are the *Euclidean norm* $\|\boldsymbol{v}\|_2 = \left(\sum |v_i|^2\right)^{1/2}$ and the *max norm* $\|\boldsymbol{v}\|_\infty = \max\{|v_i|\}$. The following result, see also Horn and Johnson [1985, Cor. 5.4.5], shows that all vector norms (on a finite-dimensional real or complex vector space) are equivalent. That is, convergence with respect to one norm implies convergence with respect to any other norm. This allows to talk about convergence without mentioning a specific norm.

Lemma A.8. *Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two vector norms, then there exist finite positive constants C_1 and C_2 such that $C_1\|\boldsymbol{v}\|_\alpha \leq \|\boldsymbol{v}\|_\beta \leq C_2\|\boldsymbol{v}\|_\alpha$ for all $\boldsymbol{v} \in \mathbb{C}^n$.*

Consider a vector norm $\|\cdot\|$ on a vector space $\mathbb{C}^{n\times n}$. In this case it is often useful to add the *submultiplicative axiom* $\|AB\| \leq \|A\|\|B\|$. A norm satisfying this axiom is called a *matrix norm*. For example, the Euclidean norm $\|A\|_2$, the max norm $n\|A\|_\infty$, and $\sup_{\boldsymbol{v}\in\mathbb{C}^n}\{\|A\boldsymbol{v}\|/\|\boldsymbol{v}\|\}$ are all matrix norms. Any matrix norm $\|\cdot\|$ serves as an upper bound for the spectral radius $\rho(A)$, that is $\rho(A) \leq \|A\|$.

Let $A(x)$ be a matrix-valued function. Assume that each element is measurable and consider the (entrywise) integral $\int A(x)\mu(\mathrm{d}x)$. We say that this integral converges absolutely if $\int |a_{ij}(x)|\mu(\mathrm{d}x) < \infty$ for all i, j. But this is the same as $\int \|A(x)\|\mu(\mathrm{d}x) < \infty$, because the max norm controls the size of the entries, and all the norms are equivalent. Note that any norm applied to $A(x)$ produces a measurable function by continuity of the norm. Suppose now that $M(\mathrm{d}x) = (\mu_{ij}(\mathrm{d}x))$ is a matrix where each element is a measure. Then $\int M(\mathrm{d}x)A(x)$ denotes a matrix with (i,j)-th element being $\sum_{k=1}^n \int a_{kj}(x)\mu_{ik}(\mathrm{d}x)$. We say that it converges absolutely if $\int |a_{kj}|(x)\mu_{ik}(\mathrm{d}x) < \infty$ for all i,j,k.

Assume $f(x) = \sum_{i=0}^\infty a_i x^i$ is a power series, which is absolutely convergent for all $|x| < r$. Then also $f(A) = \sum_{i=0}^\infty a_i A^i$ is absolutely convergent for all A with $\|A\| < r$ for some matrix norm $\|\cdot\|$, which implies $\rho(A) < r$. This defines a mapping from $\mathbb{C}^{n\times n}$ to $\mathbb{C}^{n\times n}$. Representing A through its Jordan normal form $A = V\Gamma V^{-1}$ one obtains $f(A) = Vf(\Gamma)V^{-1}$. Importantly, $f(\Gamma)$ has a simple block-diagonal form, where the blocks are given by

$$f(\Gamma_k(\lambda)) = \begin{pmatrix} f(\lambda) & f'(\lambda) & \dots & f^{(k-1)}(\lambda)/(k-1)! \\ 0 & f(\lambda) & \dots & f^{(k-2)}(\lambda)/(k-2)! \\ \dots & & & \\ 0 & 0 & \dots & f(\lambda) \end{pmatrix}. \tag{A.1}$$

Note also that f maps the eigenvalues of A *onto* the eigenvalues of $f(A)$.

An important example of the above mapping is the matrix exponential $e^A = \sum_{i=0}^\infty A^i/i!$, which is absolutely convergent for all A. Note also that $e^{A+B} = e^A e^B$ if A and B commute. So e^A is non-singular with the inverse given by e^{-A}. Observe also that λ is an eigenvalue of A of multiplicity m if and only if e^λ is an eigenvalue of e^A of multiplicity m. The integral $\int_0^\infty e^{Ay}\mathrm{d}y$ appears often in this book. The following lemma provides a condition for its absolute convergence.

Lemma A.9. *Let $\rho_r(A) = \max\{\mathrm{Re}(\lambda) : \lambda$ is an eigenvalue of $A\}$. Then $\rho_r(A) < 0$ is a necessary and sufficient condition for $\int_0^\infty e^{Ay}\mathrm{d}y$ to converge absolutely; its limit then is $-A^{-1}$.*

Proof. Let $A = V\Gamma V^{-1}$, where Γ is a Jordan matrix. Observe that the non-zero elements of $e^{\Gamma y}$ are given by $y^i e^{\lambda y}/i!$, where λ is an eigenvalue of A. Hence $\int_0^\infty e^{\Gamma y}\mathrm{d}y$ converges absolutely if and only if $\rho_r(A) < 0$. But the former is equivalent to the absolute convergence of $\int_0^\infty e^{Ay}\mathrm{d}y$. Finally, $\partial A^{-1}e^{Ay}/\partial y = e^{Ay}$, which shows that the integral is equal to $-A^{-1}$. □

Proposition A.10. *Let A have rank $n-1$, and let $\boldsymbol{u}$ be a vector such that $\boldsymbol{u}^T$ is not in the row space of A. If $(\boldsymbol{h}^i)$ is a sequence of vectors such that $A\boldsymbol{h}^i \to \boldsymbol{v}$ and $\boldsymbol{u}^T\boldsymbol{h}^i \to c$ for all i, then there exists a vector $\boldsymbol{h}$, such that $A\boldsymbol{h} = \boldsymbol{v}, \boldsymbol{u}^T\boldsymbol{h} = c$ and $\boldsymbol{h}^i \to \boldsymbol{h}$.*

Proof. Choose $n-1$ linearly independent rows of the matrix A, and replace the other one by $\boldsymbol{u}^T$ to obtain an invertible matrix A_1. So we have $A_1\boldsymbol{h}^i \to \boldsymbol{v}_1$, where $\boldsymbol{v}_1$ is the vector obtained from $\boldsymbol{v}$ by replacing a specific entry with c. Define $\boldsymbol{h} = A_1^{-1}\boldsymbol{v}_1$ and observe that $\boldsymbol{h}^i \to \boldsymbol{h}$ to complete the proof. □

A.4 Transition rate matrix

Let Q be an $n \times n$ *transition rate matrix* associated to some (continuous-time) Markov chain. Then it satisfies the following properties:

$$q_{ij} \geq 0 \text{ for all } i \neq j, \qquad\qquad q_{ii} \leq -\sum_{j\neq i} q_{ij} \text{ for all } i.$$

It is said to be *recurrent* if $Q\mathbf{1} = \mathbf{0}$ and *transient* otherwise. See the book Norris [1998] for a detailed exposition of the theory of Markov chains.

The following properties of Q follow from Appendix A.1. Firstly, Gershgorin's theorem shows that all the eigenvalues of Q belong to $\mathbb{C}^{\mathrm{Re}<0} \cup \{0\}$. It is common to assume that Q is irreducible, which we do in the following. If Q is recurrent then 0 is an eigenvalue. If Q is transient then Q is irreducibly diagonally dominant and hence invertible, which implies that all the eigenvalues of Q are in $\mathbb{C}^{\mathrm{Re}<0}$.

Proposition A.11. *Let $J, J_m, m \geq 1$ be Markov chains on n states with corresponding transition rate matrices Q, Q_m. Then J_m converges to J in the sense of finite-dimensional distributions (for each initial state) if and only if $Q_m \to Q$.*

Proof. Convergence of finite-dimensional distributions is equivalent to $e^{Q_m t} \to e^{Qt}$ for all $t \geq 0$. In the following we prove that the latter happens if and only if $Q_m \to Q$. Use the definition of the matrix exponential and the dominated convergence theorem to prove the 'if' part. Next, assume that $e^{Q_m t} \to e^{Qt}$ for all $t \geq 0$. For any $q > 0$ we have $\int_0^\infty e^{(Q_m - q\mathbb{I})t}\mathrm{d}t = -(Q_m - q\mathbb{I})^{-1}$ according to Lemma A.9. But then the dominated convergence theorem applies showing that $(Q_m - q\mathbb{I})^{-1} \to (Q - q\mathbb{I})^{-1}$, which in turn implies $Q_m \to Q$. □

Lemma A.12. *There exists a constant C_n such that the elements of the matrix $e^{-\lambda t}e^{-Qt}$ are bounded in absolute value by C_n for any $t \geq 0$, $n \times n$ transition rate matrix Q, and $\lambda > -\sum_{i=1}^n q_{ii}$.*

Proof. Let $P(t) = e^{Qt}$, so $e^{-\lambda t}e^{-Qt} = e^{-\lambda t}P(t)^{-1}$. Observe that

$$\det(P(t)) = \prod_{i=1}^n e^{\lambda_i t} = e^{\mathrm{tr}(Q)t},$$

where λ_i are the eigenvalues of Q, and $\mathrm{tr}(Q)$ is the trace of Q. Moreover, the entries of transition probability matrix $P(t)$ are bounded by 1, hence the cofactors of $P(t)$ are bounded by C_n, a constant which only depends on n. According to Cramer's rule we get a bound $e^{-(\lambda+\mathrm{tr}(Q))t}C_n$ for every element of the matrix $e^{-\lambda t}e^{-Qt}$. The claim now follows immediately. □

A.5 Analytic functions of a complex variable

The basic facts summarized in this section can be found in any standard textbook on complex analysis, see e.g. Hahn and Epstein [1996]. A complex function f defined on some neighborhood of a point $z_0 \in \mathbb{C}$ is said to be *analytic* at z_0 if there exists $r > 0$ and a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, which is absolutely convergent and equal to $f(z)$ for all $|z - z_0| < r$. Let f be defined on an open set $D \subseteq \mathbb{C}$. We say that f is *analytic on* D if it is analytic at every point of D. The function f is analytic on D if and only if it is complex differentiable on D (holomorphic), in which case it is infinitely many times differentiable on D. In addition, the coefficients a_n of the above power series are given by $a_n = f^{(n)}(z_0)/n!$.

In the following we assume that f is analytic on an open connected set $D \subseteq \mathbb{C}$. Let us present some basic properties of the zeros of f. Firstly, if the set of zeros of f has an accumulation point inside D then f is zero everywhere on D. Hence if two functions f and g are analytic on D and coincide on a subset $S \subseteq D$ which has an accumulation point then f and g coincide on the whole of D. This observation is in the basis of *analytic continuation*. The next theorem states the so-called *argument principle* in the case of no poles.

Theorem A.13 (Argument principle)**.** *Let γ be a simple closed curve in D such that f has no zeros on γ. Then*

$$\frac{1}{2\pi \mathrm{i}} \oint_{\gamma} \frac{f'(z)}{f(z)} \mathrm{d}z = N,$$

where N is the number of zeros of f in the interior of γ.

The argument principle provides a tool for studying the behavior of zeros of a convergent sequence of analytic functions, which is summarized in the following theorem.

Theorem A.14 (Hurwitz)**.** *Let f_n be a sequence of analytic functions on D that converges uniformly to f on compact subsets of D. Then f is analytic on D. Moreover, if D_0 is a disc in D such that f has no zeros on its boundary then the* *functions f and f_n have the same number of zeros (counting multiplicities) in D_0 for large enough n.*

The Hurwitz's theorem immediately yields the following result concerning convergence of eigenvalues. If a sequence (A_k) of $\mathbb{C}^{n\times n}$ matrices converges to A then the eigenvalues of A_k converge to the eigenvalues of A preserving algebraic multiplicities. To see this consider the functions $\det(A_k - z\mathbb{I})$.

A.6 Differentiation under the integral sign

Consider an integral of the form $\int f(t,x)\mu(\mathrm{d}x)$. When differentiating this integral with respect to t, it is often possible to bring the differential operator inside the integral. Legitimacy of doing so is well established in many cases, as for example in the case of the Laplace transform, see Appendix A.7. Sometimes the dominated convergence theorem provides an easy proof of such a result. Otherwise, we use the following proposition, which is based on the fundamental theorem of calculus, see also Williams [1991, A16].

Proposition A.15. *Let μ be a sigma-finite measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Let $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ be a function measurable in the second argument. If $f'(t,x)$, the derivative in t, is continuous on $[a,b]$ and $\int|f'(t,x)|\mu(\mathrm{d}x) < C$ for some $C>0$ and all $t\in[a,b]$, then*

$$\frac{\partial}{\partial t}\int f(t,x)\mu(\mathrm{d}x) = \int f'(t,x)\mu(\mathrm{d}x)$$

for $t\in(a,b)$.

We remark that a similar result holds true for $f:\mathbb{C}\times\mathbb{R}\to\mathbb{C}$ and *complex* derivative. Let us demonstrate the applicability of Proposition A.15. Consider the Laplace exponent $\psi(\alpha)$ of a Lévy process without positive jumps as given in (2.1). Let us show that $\psi(\alpha)$ is analytic in $\mathbb{C}^{\mathrm{Re}>0}$ by proving the following identity for $\alpha\in\mathbb{C}^{\mathrm{Re}>0}$:

$$\frac{\partial}{\partial\alpha}\int_{-\infty}^{0}(e^{\alpha x}-1-\alpha x 1_{\{x>-1\}})\nu(\mathrm{d}x) = \int_{-\infty}^{0}(xe^{\alpha x}-x1_{\{x>-1\}})\nu(\mathrm{d}x).$$

It is only required to show that $\int_{-\infty}^{0}|xe^{\alpha x}-x1_{\{x>-1\}}|\nu(\mathrm{d}x) < C$ for all α in some disc in $\mathbb{C}^{\mathrm{Re}>0}$. But this follows from the requirement that $\int_{-\infty}^{0}(1\wedge x^2)\nu(\mathrm{d}x)<\infty$, see also Lemma A.19.

A.7 The Laplace transform

Let μ be a sigma-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. First, assume that μ is concentrated on $[0, \infty)$. Consider the integral

$$f(s) = \int_0^\infty e^{-sx} \mu(\mathrm{d}x)$$

for $s \in \mathbb{R}$, where the interval of integration is closed. Observe that if $f(r) < \infty$ for some $r \in \mathbb{R}$ then $f(s) < \infty$ for all $s \geq r$. It is assumed in the following that such r exists. Note that if μ is a finite measure then $f(s) < \infty$ for all $s \geq 0$. The function $f(s)$ defined for all $s \geq r$ is called *Laplace transform* of the measure μ. If μ has a density m then $f(s) = \int_0^\infty e^{-sx} m(x)\mathrm{d}x$ is also called the ordinary Laplace transform of m. As usual we stretch the language and speak about 'the Laplace transform of a random variable X', meaning $\mathbb{E}e^{-sX}$. We often deal with negative random variables, e.g. jumps of a MAP, in which case we mean $\mathbb{E}e^{sX}$.

It is known that a measure is uniquely determined by its Laplace transform. Moreover, convergence of measures is closely related to the convergence of the corresponding Laplace transforms, see also Feller [1966, Thm. 2a, p. 410].

Theorem A.16 (Extended continuity theorem)**.** *Let μ_n be a measure with Laplace transform f_n. If $f_n(s) \to f(s)$ for all $s > r$, then f is the Laplace transform of a measure μ, and $\mu_n \to \mu$. Conversely, if $\mu_n \to \mu$ and the sequence $f_n(r)$ is bounded, then $f_n(s) \to f(s)$ for all $s > r$.*

Let us discuss the analytic character of the Laplace transform. Observe that if $f(r) < \infty$ then $\int_0^\infty e^{-sx}\mu(\mathrm{d}x)$ is absolutely convergent for all complex s with $\mathrm{Re}(s) \geq r$. In fact, if X is a non-negative random variable with law μ, then picking $s = -\theta\mathrm{i}$ we obtain the characteristic function of X, that is $f(-\theta\mathrm{i}) = \mathbb{E}e^{\mathrm{i}\theta X}$. Define $r = \inf\{s \in \mathbb{R} : f(s) < \infty\}$, which is assumed to belong to $[-\infty, \infty)$. This number r is called *abscissa of convergence*. It is easy to see using Proposition A.15 that $f(s)$ is analytic on the half-plane specified by $\mathrm{Re}(s) > r$, and $f'(s) = -\int_0^\infty xe^{-sx}\mu(\mathrm{d}x)$ for s in this half-plane. The following proposition, see Theorem 5b in Widder [1941, Ch. II], allows to apply analytic continuation to Laplace transforms.

Proposition A.17. *If r is the abscissa of convergence of $\int_0^\infty e^{-sx}\mu(\mathrm{d}x)$ then $f(s)$ cannot be analytic at r. That is, r is a singularity of $f(s)$.*

Let us demonstrate analytic continuation at work. Suppose we can establish that $\int_0^\infty e^{-sx}\mu(\mathrm{d}x) = g(s)$ for all $s > r_0$. Assume there exists $r < r_0$ such that $g(s)$ is analytic on $\mathrm{Re}(s) > r$. According to Proposition A.17 the number r is not smaller then the corresponding abscissa of convergence. So the transform is analytic on $\mathrm{Re}(s) > r$. This further implies that the above equality holds true for all s with $\mathrm{Re}(s) > r$, see Appendix A.5.

It was assumed above that the measure μ is concentrated on $[0, \infty)$. If this is not the case then one speaks about *bilateral Laplace transform* defined by

$$f(s) = \int_{-\infty}^{\infty} e^{-sx} \mu(\mathrm{d}x).$$

The region of convergence (its interior) of this integral is given by a strip $\mathrm{Re}(s) \in (r_1, r_2)$. This and many other facts can be established using unilateral transforms. Note that the above strip may be an empty set. We exclude this case and assume that $r_1 < r_2$. Then the transform is analytic in this strip. We also note that a measure is uniquely determined by its (bilateral) Laplace transform given that the strip of convergence is not empty.

Finally, let us consider the transform $f(s) = \int_0^\infty e^{-sx} m(x)\mathrm{d}x$, where $m(x)$ is not necessarily non-negative. We assume that $m(x)$ is càdlàg, and consequently Borel measurable with countably many discontinuities. We are interested in the domain where the integral converges absolutely, that is, $\int_0^\infty |e^{-sx} m(x)|\mathrm{d}x < \infty$. As before this domain is a half plane $\mathrm{Re}(s) > r$. It is tacitly assumed that such $r < \infty$ exists. The transform $f(s)$ is analytic on this half plane. Moreover, $m(x)$ is uniquely determined by its Laplace transform. These facts can be found in Widder [1941, Ch. II]. Finally, we remark that one has to be careful with analytic continuation of such transforms. It is not the case in general that the abscissa of absolute convergence is a singularity of $f(s)$, that is, the analog of Proposition A.17 does not hold in general.

A.8 Various relations

Lemma A.18. *Fix $\epsilon > 0$ and $i \in \mathbb{N}$. Then there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$ and all $\alpha = a + b\mathrm{i}, a, b \in \mathbb{R}$ it holds that*

$$|x^i e^{\alpha x}| \leq e^{(a-\epsilon)x} + Ce^{ax} + e^{(a+\epsilon)x}.$$

Proof. Pick $C_0 > 0$ large enough so that $x^i \leq e^{\epsilon x}$ for all $x > C_0$ then

$$\begin{aligned} |x^i e^{\alpha x}| &= (-x)^i e^{ax} 1_{\{x < -C_0\}} + |x|^i e^{ax} 1_{\{-C_0 \leq x \leq C_0\}} + x^i e^{ax} 1_{\{x > C_0\}} \\ &\leq e^{(a-\epsilon)x} 1_{\{x < -C_0\}} + C_0^i e^{ax} 1_{\{-C_0 \leq x \leq C_0\}} + e^{(a+\epsilon)x} 1_{\{x > C_0\}} \end{aligned}$$

for any $x \in \mathbb{R}$. The result follows with $C = C_0^i$. □

Lemma A.19. *There exists a constant $C > 0$ such that for all $z \in \mathbb{C}^{\mathrm{Re}\le 0}$ it holds that*

$$|e^z - 1 - z| \le C|z|^2, \qquad |e^z - 1| \le C|z|.$$

Proof. We only show the first bound. If $|z| \ge 1$ then $|e^z - 1 + z| \le 2 + |z| \le 3|z| \le 3|z|^2$. If, however, $|z| < 1$ then using a power series expansion we have $|e^z - 1 - z| = |z^2/2! + z^3/3! + \dots| \le |z|^2(1/2! + 1/3! + \dots) \le 3|z|^2$. So we can pick $C \ge 3$. □

Lemma A.20. *Let $x, y, z_0, z_1, \dots$ be real numbers then*

$$\sum_{k=0}^{r} \frac{(-y)^{r-k}}{(r-k)!} \sum_{i=0}^{k} \frac{1}{i!}(x+y)^i z_{k-i} = \sum_{j=0}^{r} \frac{1}{j!} x^j z_{r-j}.$$

Proof. Consider the terms on the left side involving $x^j z_l$. These terms sum up to

$$\sum_{k=l+j}^{r} \frac{(-y)^{r-k}}{(r-k)!} \frac{1}{(k-l)!} \binom{k-l}{j} x^j y^{k-l-j} z_l,$$

because the second sum contains only one such term, which corresponds to $i = k-l$. This expression simplifies to

$$\frac{1}{j!} x^j y^{r-l-j} z_l \sum_{k=l+j}^{r} \frac{(-1)^{r-k}}{(r-k)!(k-l-j)!}.$$

Moreover, the latter sum is $\sum_{i=0}^{s} \frac{(-1)^{s-i}}{i!(s-i)!} = (1-1)^s/s!$, where $s = r - l - j$. Hence it is non-zero only if $r = l + j$, in which case it is 1. Therefore, the above summand is $\frac{1}{j!} x^j z_{r-j}$. □

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List of symbols

càdlàg	right-continuous with left limits (French "continue à droite, limitée à gauche")
$\wedge, \vee$	minimum, maximum: $a \wedge b = \min(a, b), a \vee b = \max(a, b)$
$\mathbb{C}^{\mathrm{Re}>0}$	$\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$; $\mathbb{C}^{\mathrm{Re}\geq 0}, \mathbb{C}^{\mathrm{Re}<0}, \mathbb{C}^{\mathrm{Re}\leq 0}$ are defined similarly
$f(t-), f(t+)$	left and right limits of f at t : $\lim_{s\uparrow t} f(s), \lim_{s\downarrow t} f(s)$;
$o(f(h))$	denotes a real function $g(h)$ such that $\lim_{h\downarrow 0} g(h)/f(h) = 0$

Vectors and matrices

$\mathbf{1}, \mathbf{0}$	vectors of 1s and 0s respectively
$\boldsymbol{e}_i$	coordinate vector with 1 in the i-th position and 0 everywhere else
$M^\top$	transpose of a matrix M
$\mathbb{I}, \mathbb{O}$	identity matrix and matrix of 0s of appropriate dimensions
$\mathrm{diag}(D_1, \ldots, D_n)$	(block-)diagonal matrix with D_i on the diagonal
$\boldsymbol{\Delta}_{\boldsymbol{v}}$	diagonal matrix $\mathrm{diag}(v_1, \ldots, v_n)$
$[M_1, \ldots, M_2]$	matrix obtained by merging the columns of matrices M_i
$\Gamma_k(\lambda)$	Jordan block of size $k \times k$ with eigenvalue λ, see Appendix A.2
$A \circ B$	entrywise (Hadamard) matrix multiplication

Probability

a.s.	almost surely
1_A	indicator of an event A
e_q	exponential random variable of rate $q \geq 0$; $e_0 = \infty$

Lévy processes

ψ	Laplace exponent
$a, \sigma, \nu(\mathrm{d}x)$	Lévy triple appearing in Lévy-Khintchine formula
Φ	right inverse of ψ
d	drift of a Lévy process; defined when $\int_{-1}^{0} \lvert x\rvert \nu(\mathrm{d}x) < \infty$
∂	absorbing 'cemetery' state

MAPs

(X, J)	spectrally negative MAP; J is a Markov chain
$Q, \boldsymbol{\pi}$	transition rate matrix and the stationary distribution of J
X_i, ψ_i	underlying Lévy process and its Laplace exponent
U_{ij}, G_{ij}	jump of X at transition of J from i to j, and its transform
q	killing rate
(∂_X, ∂_J)	absorbing 'cemetery' state
E	state space of J
$E_+, E_\downarrow$	partition of E with $E_\downarrow = \{i \in E : X_i \text{ is non-increasing}\}$
$N, N_+, N_\downarrow$	cardinalities of $E, E_+, E_\downarrow$
M_+	restriction of a matrix M with N rows to the rows indexed by E_+, see Definition 2.6
$\mathbb{I}^+$	$N_+ \times N_+$ identity matrix
$\mathbb{P}_i, \mathbb{E}_i$	law of (X, J) given $\{X(0) = 0, J(0) = i\}$, and the corresponding expectation
$\mathbb{E}[Z; J]$	matrix composed of $\mathbb{E}_i[Z; J = j]$
$\mathbb{P}(A, J)$	matrix composed of $\mathbb{P}_i(A, J = j)$
$\mathbb{E}_{x_0}[Z; J]$	the same as $\mathbb{E}[Z; J]$ with $X(0) = x_0$
$F(\alpha)$	matrix exponent of a MAP: $\mathbb{E}[e^{\alpha X(t)}; J(t)] = e^{F(\alpha)t}$
$k(\alpha)$	Perron-Frobenius eigenvalue of $F(\alpha)$
κ	asymptotic drift; defined when Q is recurrent (no killing)
$(\hat{X}, \hat{J}), \hat{\mathbb{P}}$	time-reversed MAP and its law

First passage

$\tau_x, x \geq 0$	first passage time: $\inf\{t \geq 0 : X(t) > x\}$
$\overline{X}(t), \underline{X}(t)$	supremum and infimum processes: $\overline{X}(t) = \sup_{0 \leq s \leq t}\{X(s)\}$, $\underline{X}(t) = \inf_{0 \leq s \leq t}\{X(s)\}$
$\overline{X}, \underline{X}$	$\overline{X}(\infty)$ and $\underline{X}(\infty)$

$\Lambda(q)$	matrix exponent of the first passage process, see Section 2.6
$\Pi(q)$	matrix of initial distributions of the first passage process
$\boldsymbol{\pi}_\Lambda$	the stationary distribution of $\Lambda(0)$; defined when $\kappa \geq 0$
$\rho(q)$	Perron-Frobenius eigenvalue of $\Lambda(q)$
(V, Γ)	right spectral pair of $F(\alpha)$, see Definition 4.10

MMBM

$E_-, E_\uparrow$	partition of E with $E_\uparrow = \{i \in E : X_i(t) = a_i t, a_i \geq 0\}$
$M^\pm$	equivalent to an object M for $(\pm X, J)$; e.g. $\Lambda^+ = \Lambda$
$M_-, M_\uparrow$	restriction of the rows of M to E_- and $E_\uparrow$, see Section 5.1
(V, Γ)	spectral pair of $F(\alpha)$, see Definition 5.2
$K^\pm$	matrices defined in (5.20)

Reflection

$L(t), U(t)$	regulators at the lower and the upper barriers
$\boldsymbol{\ell}, \boldsymbol{u}$	loss vectors corresponding to $L(t)$ and $U(t)$, see Section 5.2 and Section 6.3
$R(t)$	reflection of $X(t) : R(t) = X(t) + L(t) - U(t)$
(R^*, J^*)	refers to the stationary distribution of $(R(t), J(t))$
τ_x^L, τ_x^U	first passage times of L and U over level $x \geq 0$, see (6.1)

Scale matrices

$\tau_x^\pm$	first passage times: $\inf\{t \geq 0 : \pm X(t) > x\}$
$W^q(x)$	scale matrix
$Z^q(\alpha, x)$	second scale matrix, see Section 8.1
τ^B	first hitting time of a set $B : \quad \inf\{t > 0 : X(t) \in B\}$
$L(x, j, t)$	occupation density of (X, J) at (x, j) up to time t; also called local time
$\boldsymbol{L}$	matrix composed of $\mathbb{E}_i L(0, j, \infty)$; i.e. expected local times at 0
$\boldsymbol{L}(x)$	matrix of expected local times at 0 up to first passage time over x, see Section 7.3
$\eta(q)$	$\max\{\mathrm{Re}(z) : z \in \mathbb{C}, \det(F(z) - q\mathbb{I}) = 0\}$

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Summary

Lévy processes, that is, processes with stationary and independent increments, have become a classical model in applied probability. Their use is widespread, ranging from biology problems to storage models, insurance risk and financial mathematics. Many real-world problems, however, exhibit non-stationary behavior in longer time intervals. One may think about seasonality of prices, recurring patterns of activity, burst arrivals, occurrence of events in phases and so on. This motivates the interest in regime-switching models, where the process under consideration is modulated by an exogenous background process. Markov Additive Processes (MAPs) form a natural generalization of Lévy processes to regime-switching models.

The focus of this thesis is on the path properties of MAPs. Both MAPs and their reflections at constant boundaries are considered. We address the basic exit problems, such as first passage over a level and first exit from an interval. Most of the results appear in the form of Laplace transforms.

In many applications it is natural to assume that a process of interest jumps only in one direction, i.e., it is spectrally one-sided. As shown in the recent literature on path properties of Lévy processes, under this assumption it is often possible to derive substantially more explicit and transparent results. In this monograph we restrict ourselves to spectrally one-sided MAPs and generalize some of these results. Importantly, so-called phase-type distributions fit naturally in the framework of MAPs. Arbitrary phase-type jumps can be added to the model keeping the analysis tractable. This is achieved by enlarging the number of states of the background process and replacing phase-type jumps by linear stretches.

This book can be split into three parts. In the first part, Chapter 3 and Chapter 4, we study the first passage process. We show that the theory of analytic

matrix functions is a natural tool in the analysis of MAPs. In addition, we consider the supremum and the infimum processes, which are closely related to the reflection at zero. The latter process can be used, for example, to model the workload evolution in a regime-switching queue. We identify the stationary distribution of the reflected process, which leads to a generalization of the celebrated Pollaczek-Khintchine formula.

The second part of this book, Chapter 5 and Chapter 6, concerns MAPs without jumps. Such processes have piecewise Brownian paths with drift and variance parameters determined by the Markovian background process. Hence they are called Markov-Modulated Brownian Motions (MMBMs). This special case results in further substantial simplifications. Consider an MMBM reflected to stay in a strip $[0, B]$; one may think about this as a queue with a finite buffer. We determine the corresponding stationary distribution and the so-called loss vectors. Moreover, we characterize this model at inverse local times at both barriers, which readily leads to the solution of the first passage problem for an MMBM reflected at zero.

In the last part, Chapter 7 and Chapter 8, we extend some results on MMBMs to spectrally one-sided MAPs. In particular, we solve the two-sided exit problem for such a MAP, and the first passage problem for a MAP reflected at zero. These results are based on so-called scale matrices. The corresponding theory is given in the beginning of this part, which includes the construction of a scale matrix and identification of its transform. We show that a scale matrix is closely related to expected local times at zero, which enables to prove some essential properties of the former.

Finally, a newcomer to the theory of MAPs and their path properties may benefit from reading Chapter 2 on basic theory and the Appendix, where we discuss fundamental concepts, tools and definitions.

Samenvatting

Lévy-processen vormen een klassiek model uit de toegepaste kansrekening, en worden gekenmerkt door stationaire en onafhankelijke incrementen. Ze kennen vele toepassingen, bijvoorbeeld bij het modelleren van biologische processen, voorraadsystemen, bij risicomanagement van verzekeringen en binnen de financiële wiskunde in het algemeen. Veel processen uit de praktijk vertonen echter niet-stationair gedrag, waardoor Lévy-processen hier niet van toepassing zijn. Denk hierbij bijvoorbeeld aan seizoensafhankelijke prijzen, terugkerende activiteitspatronen, gecorreleerde aankomsten en gefaseerde gebeurtenissen. Dit soort verschijnselen vormen de aanleiding tot het bestuderen van zogenaamde regime-switching-modellen, waarin een bepaald proces wordt gemoduleerd door een zeker achtergrondproces.

In dit proefschrift bekijken we Markov Additive Processes (MAPs), die beschouwd kunnen worden als een natuurlijke uitbreiding van Lévy-processen naar regime-switching-modellen. De nadruk ligt hierbij op padeigenschappen, waarbij aandacht wordt besteed aan zowel MAPs als MAPs gereflecteerd op één of twee vaste randen. We richten ons op de gebruikelijke vraagstukken op het gebied van bereikingstijden, zoals de tijd die het duurt totdat een bepaald niveau bereikt wordt, en de tijd die het duurt totdat een interval verlaten wordt. De meeste resultaten zijn gegeven in de vorm van Laplace-getransformeerden.

Bij veel toepassingen maakt het onderliggende proces sprongen in slechts één richting; deze processen worden spectraal-eenzijdig genoemd. Voor Lévy-processen is recentelijk aangetoond dat deze beperking leidt tot expliciete uitdrukkingen. Wij richten ons dan ook richten op spectraal-eenzijdige MAPs, met als doel resultaten die reeds afgeleid zijn voor Lévy-processen te generaliseren naar MAPs. We kunnen daarnaast de spectraal-eenzijdige MAPs uitbreiden met zogenaamde

fase-type verdeelde sprongen. Deze uitbreiding kan worden gedaan door het aantal toestanden van het achtergrondsproces uit te breiden, en de fase-type sprongen te vervangen door lineare segmenten.

Dit proefschrift is onderverdeeld in drie stukken. Het eerste onderdeel bestaat uit hoofdstukken 3 en 4, en betreft het zgn. 'first-passage proces'. We tonen aan dat MAPs op natuurlijke wijze kunnen worden bestudeerd aan de hand van analytische matrix-functies. Daarnaast bekijken we de supremum- en infimum-processen, welke beide gerelateerd zijn aan de reflectie van het proces op de ondergrens 0. Dit proces kan onder meer worden gebruikt om het gedrag van het werklastproces van een wachtrij met verschillende regimes te bestuderen. We bepalen de stationaire verdeling van het gereflecteerde proces, en generaliseren aldus de bekende Pollaczek-Khintchine formule.

Het tweede onderdeel bestaat uit hoofdstuk 5 en hoofdstuk 6. Hierin wordt gekeken naar MAPs zonder sprongen, wat neerkomt op stuksgewijze Brownse paden met een drift en variantie die afhangen van het achtergrondproces. Deze processen worden ook wel Markov-Modulated Brownian Motions (MMBMs) genoemd, en geven aanleiding tot een significante vereenvoudiging van de resultaten op het gebied van spectraal-positieve MAPs. We bekijken een MMBM die wordt geflecteerd aan de randen van het interval $[0, B]$; dit model kan gezien worden als een wachtrij met eindige buffer. We bepalen de stationaire verdeling van dit proces, en leiden de zogenaamde verliesvectoren af. Daarnaast geven we een beschrijving van dit proces op 'inverse local times' op beide grenzen, wat direct leidt tot de oplossing van het 'first passage problem' voor een MMBM gereflecteerd op 0.

Het laatste deel van dit proefschrift wordt gevormd door hoofdstukken 7 en 8, waarin enkele resultaten betreffende MMBMs worden uitgebreid naar spectraal-positieve MAPs. Zo lossen we zowel het 'two-sided exit problem' op voor zulke MAPs, als het 'first passage problem' voor een MAP gereflecteerd op 0. De resultaten zijn gebaseerd op zogenaamde 'scale-matrices'; de theorie hierachter wordt aan het begin van dit onderdeel uiteen gezet. Dit betreft de constructie van een scale-matrix, maar ook het identificeren van diens getransformeerde. We tonen aan dat een scale-matrix sterk verband houdt met verwachte local times in 0, waaruit enkele essentiële eigenschappen van scale-matrices kunnen worden afgeleid.

Hoofdstuk 2 tenslotte biedt lezers die onbekend zijn met MAPs en hun padeigenschappen een inleiding tot dit onderwerp. In de bijlage bespreken we enkele fundamentele concepten, wiskundige technieken en definities.

About the author

Jevgenijs Ivanovs was born in Daugavpils, Latvia in July 1983. He attended Daugavpils Secondary School No. 4. Throughout school years he was exposed to advanced mathematical training and took part in various mathematical contests.

In 2001 he moved to Riga and entered University of Latvia. He obtained BSc degree (with distinction) in computer science and discrete mathematics in 2005.

During the last years of university Jevgenijs combined studies with work. He worked as a software developer, first in a small local company Baltic Instruments Ltd., and later in Accenture. The summer of 2004 he spent in USA working as a lifeguard in a swimming pool. This international experience led him, in 2005, to the Netherlands, where he entered applied computing science program at Utrecht University and received Huygens scholarship for international master students. During his master studies Jevgenijs was engaged in research concerning Bayesian networks under supervision of A. Feelders. His thesis entitled 'Recognition of Human Poses Using Pictorial Structures' and supervised by D. Gavrila was based on an external project involving the Dutch Railways. He received MSc degree (cum laude) in 2007.

In 2007 Jevgenijs started his PhD work at KdV, University of Amsterdam and EURANDOM, Eindhoven University of Technology in the area of applied probability under advice of M. Mandjes and O. Boxma. His research resulted in a number of publications and conference talks. He collaborated with B. D'Auria, O. Kella, and Z. Palmowski. Jevgenijs was involved in teaching, supervised a bachelor project, organized a reading seminar and served as a referee for scientific publications.