

# One-Step Piecewise Polynomial Galerkin Methods for Initial Value Problems\*

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**Abstract.** A new approach to the numerical solution of systems of first-order ordinary differential equations is given by finding local Galerkin approximations on each subinterval of a given mesh of size  $h$ . One step at a time, a piecewise polynomial, of degree  $n$  and class  $C^0$ , is constructed, which yields an approximation of order  $O(h^{2n})$  at the mesh points and  $O(h^{n+1})$  between mesh points. In addition, the  $j$ th derivatives of the approximation on each subinterval have errors of order  $O(h^{n-j+1})$ ,  $1 \leq j \leq n$ . The methods are related to collocation schemes and to implicit Runge-Kutta schemes based on Gauss-Legendre quadrature, from which it follows that the Galerkin methods are  $A$ -stable.

**1. Introduction.** In this paper, we show how Galerkin's method can be employed to devise one-step methods for systems of nonlinear first-order ordinary differential equations. The basic idea is to find local  $n$ th degree polynomial Galerkin approximations on each subinterval of a given mesh and to match them together continuously, but not smoothly.

For each  $n \geq 1$ , a method is defined (Section 2) which uses an  $n$ -point Gauss-Legendre quadrature formula to evaluate certain inner products in the Galerkin equations. For sufficiently small step size  $h$ , a unique numerical solution exists and may be found by successive substitution (Section 3). After showing that these Galerkin methods are also collocation methods (Section 4) and implicit Runge-Kutta methods (Section 5), we show that the mesh point errors are of the order  $O(h^{2n})$ , and the global errors are of the order  $O(h^{n+1})$  for the approximate solution and  $O(h^{n-j+1})$ ,  $1 \leq j \leq n$ , for its  $j$ th derivatives (Section 6). A proof of the  $A$ -stability of the methods is given in Section 7, and numerical results are presented in Section 8.

Discrete one-step methods based on quadrature, other than the classical Runge-Kutta methods, have been studied by several authors, including the explicit schemes in [12, p. 101], [13], [14], [22] and the implicit schemes in [1], [2], [3], [6, Chapters 4, 9], [10], [12, p. 159]. Also, discrete block implicit methods are given in [21], [24], [25]. The methods of this paper, however, yield continuous piecewise polynomial approximations with the inherent benefit of derivative approximations. Earlier uses of piecewise polynomials may be found in [4], [5], [11], [15], [16], [17], [26].

Finally, we remark that recent "semidiscrete" Galerkin methods [7], [9], [18], [19], [23] reduce initial-boundary value problems to systems of ordinary differential equations. When combined with such methods, our techniques open the possibility of "fully discrete" Galerkin methods for these problems.

Received June 14, 1971.

AMS 1969 subject classifications. Primary 6561.

*Key words and phrases.* Galerkin method, initial value problems, ordinary differential equations, piecewise polynomials, Gauss-Legendre quadrature, collocation methods, implicit Runge-Kutta methods,  $A$ -stable.

\* This work was supported by the United States Atomic Energy Commission.

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2. Piecewise Polynomial Galerkin Methods. We consider the numerical solution of only a single nonlinear ordinary differential equation

$$(1) \quad u'(t) = f(t, u(t)), \quad t_0 \leq t,$$

$$(2) \quad u(t_0) = u_0$$

on a finite interval  $[t_0, t_N]$ , although the results carry over to systems of such equations. We assume that  $f(t, x) \in C^{2n}$  in  $[t_0, t_N] \times (-\infty, \infty)$ , so that the exact solution  $u(t) \in C^{2n+1}[t_0, t_N]$ ,  $n \geq 1$ , and we also assume that  $f$  has a Lipschitz constant  $L$  in this same region.

Let  $\pi: t_i = t_0 + ih, 0 \leq i \leq N$ , be a uniform mesh for the sake of simplicity. (It will be seen that our arguments do not depend crucially on this assumption since our method is a one-step method and step size changes are easy.) Then, we may approximate  $u(t)$  on each subinterval by an  $n$ th degree polynomial

$$(3) \quad y(t) = \sum_{j=1}^{n+1} b_j^{(i)} \varphi_{i+j}(t), \quad t_i \leq t \leq t_{i+1}, \quad 0 \leq i \leq N-1,$$

where  $\varphi_{i+j}(t)$  are basis functions which are  $n$ th degree polynomials on each  $[t_i, t_{i+1}]$ . For example,  $\{\varphi_k\}_{k=1}^{N+1}$  might be the  $n$ th degree  $B$ -spline basis functions of Schoenberg [20] or some other piecewise polynomial basis. Since the  $b_j$  may change from one subinterval to the next,  $y(t)$  need not be as smooth as the  $\varphi_k(t)$ .

We require that  $y(t)$  be continuous on  $[t_0, t_N]$  and that it provide a local Galerkin approximation on each subinterval  $[t_i, t_{i+1}]$ ,  $0 \leq i \leq N-1$ . Accordingly, on each subinterval, we write the following  $n+1$  equations (one linear,  $n$  nonlinear) for the  $b_j^{(i)}$ ,  $1 \leq j \leq n+1$ ,

$$(4) \quad \begin{aligned} y_{i+} &= y_{i-}, & i &\geq 1, \\ &= u_0, & i &= 0, \end{aligned}$$

$$(5) \quad (y' - f(t, y), \varphi_{i+k})_i = 0, \quad 2 \leq k \leq n+1, \quad 0 \leq i \leq N-1,$$

using the notation

$$(v, w)_i = \int_{t_i}^{t_{i+1}} v(t)w(t) dt.$$

To obtain a computational form of (4)–(5), we assume that the  $(\varphi'_{i+j}, \varphi_{i+k})_i$  in (5) are computed exactly, i.e., analytically or by an exact quadrature formula, while the inner products  $(f, \varphi_{i+k})_i$  are replaced by the  $n$ -point Gauss-Legendre quadrature formula having the form

$$(6) \quad \int_{t_i}^{t_{i+1}} v(t) dt = h \sum_{k=1}^n w_k v(\sigma_{i,k}) + O(h^{2n+1}),$$

$$(7) \quad \sigma_{i,k} = t_i + \theta_k h, \quad 1 \leq k \leq n,$$

where  $w_k > 0$  and  $\theta_k$  are the weights and abscissae for  $[0, 1]$ . The result is that (4)–(5) are replaced by the following set of  $N$  systems of  $n+1$  nonlinear equations to be solved in succession

$$(8) \quad \mathbf{A}b^{(i)} = \mathbf{c}^{(i)}(\mathbf{b}^{(i)}), \quad 0 \leq i \leq N-1,$$

where

$$\begin{aligned}
 (9) \quad \mathbf{b}^{(i)} &= \{b_1^{(i)}, b_2^{(i)}, \dots, b_{n+1}^{(i)}\}^T, \\
 (10) \quad A_{k,i} &= \varphi_{i+j}(t_i), \quad k = 1, \\
 &= (\varphi_{i+k}, \varphi'_{i+j})_i, \quad 2 \leq k \leq n + 1, 1 \leq j \leq n + 1, \\
 c_k^{(i)}(\mathbf{b}^{(i)}) &= y_i, \quad k = 1, \\
 (11) \quad &= h \sum_{m=1}^n w_m f\left(\sigma_{i,m}, \sum_{j=1}^{n+1} b_j^{(i)} \varphi_{i+j}(\sigma_{i,m})\right) \varphi_{i+k}(\sigma_{i,m}), \quad 2 \leq k \leq n + 1.
 \end{aligned}$$

We consider only the cases where  $\mathbf{A}$  is nonsingular. Certainly,  $\mathbf{A}$  will be nonsingular when  $\{\varphi_{i+k}\}_{k=2}^{n+1}$  span  $\mathcal{P}_{n-1}$ , the class of  $(n - 1)$ st degree polynomials. For then,  $\mathbf{A}\mathbf{b}^{(i)} = \mathbf{0}$  implies  $y(t_i) = 0$  and  $(y', \varphi_{i+k})_i = 0, 2 \leq k \leq n + 1$ , which, in turn, imply  $y' \equiv 0, y \equiv 0$  and  $\mathbf{b}^{(i)} = \mathbf{0}$ . However, this condition is not necessary, since  $\mathbf{A}$  is nonsingular in the case of the cubic ( $n = 3$ )  $B$ -spline basis functions used for the computations given in Section 8, but  $\{\varphi_{i+k}\}_{k=2}^4$  do not span  $\mathcal{P}_2$ . Since we may multiply (8) by  $\mathbf{A}^{-1}$ , our numerical method depends on the solution of

$$(12) \quad \mathbf{b}^{(i)} = \mathbf{A}^{-1} \mathbf{c}^{(i)}(\mathbf{b}^{(i)}), \quad 0 \leq i \leq N - 1.$$

**3. Existence and Uniqueness of the Numerical Solution.** Having let  $L$  denote the Lipschitz constant for  $f$  in  $[t_0, t_N] \times R$ , where  $R \equiv (-\infty, \infty)$ , we use the  $l_\infty$ -norm to show that the right side of (12) is a contraction mapping on  $R^{n+1}$  when  $h$  is sufficiently small. Since

$$\|\mathbf{A}^{-1} \mathbf{c}^{(i)}(\mathbf{b}) - \mathbf{A}^{-1} \mathbf{c}^{(i)}(\mathbf{b}^*)\|_\infty \leq \|\mathbf{A}^{-1}\|_\infty \|\mathbf{c}^{(i)}(\mathbf{b}) - \mathbf{c}^{(i)}(\mathbf{b}^*)\|_\infty$$

and

$$\|\mathbf{c}^{(i)}(\mathbf{b}) - \mathbf{c}^{(i)}(\mathbf{b}^*)\|_\infty \leq hQ_1 L \|\mathbf{b} - \mathbf{b}^*\|_\infty,$$

where

$$(13) \quad Q_1 = \max_{2 \leq k \leq n+1} \sum_{m=1}^n w_m |\varphi_{i+k}(\sigma_{i,m})| \sum_{j=1}^{n+1} |\varphi_{i+j}(\sigma_{i,m})|,$$

it is clear that

$$\|\mathbf{A}^{-1} \mathbf{c}^{(i)}(\mathbf{b}) - \mathbf{A}^{-1} \mathbf{c}^{(i)}(\mathbf{b}^*)\|_\infty \leq hQ_2 \|\mathbf{b} - \mathbf{b}^*\|_\infty,$$

where

$$(14) \quad Q_2 = Q_1 L \|\mathbf{A}^{-1}\|_\infty.$$

Thus, we have a contraction mapping, and (12) has a unique solution which may be found by successive substitution when

$$(15) \quad h < Q_2^{-1}.$$

**4. The Galerkin Method as a Collocation Method.** We show here that the approximate solution  $y(t)$  satisfies (1) at the quadrature points in each subinterval. Using (11), we may write (12) as

$$(16) \quad b_j^{(i)} = A_{j,1}^{-1}y_i + \sum_{m=1}^n \gamma_{j,m}f(\sigma_{i,m}, y(\sigma_{i,m})), \quad 1 \leq j \leq n + 1,$$

where

$$\gamma_{j,m} = hw_m \sum_{k=2}^{n+1} A_{j,k}^{-1}\varphi_{i+k}(\sigma_{i,m}).$$

Then, from substituting (16) into (3), we have at the quadrature points

$$(17) \quad y'(\sigma_{i,k}) = \alpha_k y_i + \sum_{m=1}^n \beta_{m,k}f(\sigma_{i,m}, y(\sigma_{i,m})), \quad 1 \leq k \leq n,$$

where

$$\alpha_k = \sum_{j=1}^{n+1} A_{j,1}^{-1}\varphi'_{i+j}(\sigma_{i,k})$$

and

$$\beta_{m,k} = \sum_{j=1}^{n+1} \gamma_{j,m}\varphi'_{i+j}(\sigma_{i,k}).$$

In the following, we make use of the fact that whenever  $f$  is independent of  $u$  and  $f \in \mathcal{P}_{n-1}$ , the exact solution  $u \in \mathcal{P}_n$  and  $y \equiv u$ . This follows because the quadrature (6) is exact for  $v \in \mathcal{P}_{2n-1}$ , in this case  $f\varphi_{i+k} \in \mathcal{P}_{2n-1}$ , and the exact computation of  $(f, \varphi_{i+k})_i$  means (8) is equivalent to (4)–(5). Since  $u$  satisfies (4)–(5) and  $y$  satisfies (8), they satisfy equivalent equations in this case and, by uniqueness,  $u \equiv y$ .

Let  $q(t) \in \mathcal{P}_n$  be defined by  $q(t_i) = 1, q'(\sigma_{i,k}) = 0, 1 \leq k \leq n$ , and let  $f = q'$  so that  $u' = f, u(t_i) = 1$  leads to  $u = q = y$  on  $[t_i, t_{i+1}]$ . Substituting  $y = q$  and  $f = q'$  into (17) yields

$$(18) \quad \alpha_k = 0, \quad 1 \leq k \leq n.$$

Now for each  $r, 1 \leq r \leq n$ , let  $q_r(t) \in \mathcal{P}_n$  be defined by  $q_r(t_i) = 0, q'_r(\sigma_{i,k}) = \delta_{r,k}, 1 \leq k \leq n$ , and let  $f = q'_r$  and  $u(t_i) = 0$  so that  $u = q_r = y$ . This time, substituting  $y = q_r$  and  $f = q'_r$  into (17) shows that

$$(19) \quad \beta_{r,k} = \delta_{r,k}, \quad 1 \leq r, k \leq n.$$

Consequently, (17) becomes the collocation equation

$$(20) \quad y'(\sigma_{i,k}) = f(\sigma_{i,k}, y(\sigma_{i,k})), \quad 1 \leq k \leq n,$$

showing that one-step collocation to (1) at the quadrature points by means of a continuous piecewise  $n$ th degree polynomial is equivalent to the Galerkin method.

Notice that the proof of this collocation property depends on the use of exactly  $n$  points in a quadrature formula (6) which is exact for  $v \in \mathcal{P}_{2n-1}$ . The proof would break down if (6) had more than  $n$  points or different weights and abscissae.

**5. The Galerkin Method as an Implicit Runge-Kutta Method.** Wright [26] has shown that any one-step collocation method is equivalent to some implicit Runge-Kutta method. Having already shown that the Galerkin method is equivalent to a

certain one-step collocation method, we now derive the *particular* implicit Runge-Kutta method to which they are both equivalent. Of course, the Galerkin and collocation methods yield continuous approximations, so “equivalent” here means “matches the discrete values” of the Runge-Kutta approximation.

From (3) and (16), we have

$$(21) \quad y_{i+1} = \bar{\alpha}y_i + \sum_{m=1}^n \bar{\beta}_m f(\sigma_{i,m}, y(\sigma_{i,m})),$$

where

$$\bar{\alpha} = \sum_{j=1}^{n+1} A_{j,1}^{-1} \varphi_{i+j}(t_{i+1})$$

and

$$\bar{\beta}_m = \sum_{j=1}^{n+1} \gamma_{j,m} \varphi_{i+j}(t_{i+1}).$$

If we let  $f = 0$ ,  $u(t_i) = 1$  so that  $u = 1 = y$ , then substituting  $y = 1$  and  $f = 0$  into (21) produces

$$(22) \quad \bar{\alpha} = 1.$$

Next, for each  $r$ ,  $1 \leq r \leq n$ , let  $q_r(t) \in \mathcal{P}_n$  be defined as in Section 4. Now, substitution of  $y = q_r$  and  $f = q'_r$  into (21) leads to

$$q_r(t_{i+1}) = \bar{\beta}_r.$$

Since the  $n$ -point Gauss-Legendre formula (6) is exact for elements of  $\mathcal{P}_{n-1}$ , we also have

$$q_r(t_{i+1}) = \int_{t_i}^{t_{i+1}} q'_r(t) dt = h \sum_{k=1}^n w_k q'_r(\sigma_{i,k}) = hw_r,$$

from which it follows that

$$(23) \quad \bar{\beta}_r = hw_r, \quad 1 \leq r \leq n.$$

Together, (21)–(23) imply

$$(24) \quad y_{i+1} = y_i + h \sum_{m=1}^n w_m f(\sigma_{i,m}, y(\sigma_{i,m})),$$

and this is simply the implicit Runge-Kutta method based on the  $n$ -point Gauss-Legendre formula (6). Again, the proof of (24) depends on the fact that (6) is a Gauss-Legendre formula with exactly  $n$  points.

Thus, each of Butcher’s implicit Runge-Kutta methods based on  $n$ -point Gauss-Legendre quadrature [2] has a corresponding “equivalent” Galerkin method using  $n$ th degree piecewise polynomials.

**6. Error Bounds.** In the following, a technique similar to that used by Shampine and Watts [21], [25] is employed to obtain asymptotic error bounds for the *discrete values* given by an implicit Runge-Kutta method. We view the Galerkin method as

a discrete one-step method and use Henrici's theory [12, Chapter 2] of such methods. Continuous error bounds are then obtained from the discrete ones.

First, we need to define an increment function. Since, from (20),  $y'(t)$  interpolates  $f(t, y(t))$  at  $\sigma_{i,k}$ ,  $1 \leq k \leq n$ , the Lagrangian representation for  $y'(t)$  is

$$(25) \quad y'(t) = \sum_{k=1}^n l_k(t) f(\sigma_{i,k}, y(\sigma_{i,k})), \quad t_i \leq t \leq t_{i+1},$$

where

$$l_k(t) = \prod_{j=1: j \neq k}^n \frac{(t - \sigma_{i,j})}{(\sigma_{i,k} - \sigma_{i,j})}, \quad 1 \leq k \leq n.$$

Integrating (25) leads to

$$(26) \quad y(t) = y_i + \sum_{k=1}^n f(\sigma_{i,k}, y(\sigma_{i,k})) \int_{t_i}^t l_k(s) ds, \quad t_i \leq t \leq t_{i+1}.$$

Using (26), we now may write the Runge-Kutta form of the Galerkin method (24), in terms of an increment function  $\Phi$ ,

$$(27) \quad y_{i+1} = y_i + h\Phi(t_i, y_i; h), \quad 0 \leq i \leq N-1,$$

where  $\Phi$  satisfies

$$(28) \quad \Phi(t_i, y_i; h) = \sum_{m=1}^n w_m g_m(t_i, y_i; h)$$

and

$$(29) \quad g_m(t_i, y_i; h) = f(\sigma_{i,m}, y(\sigma_{i,m})) = f\left(t_i + \theta_m h, y_i + \sum_{k=1}^n g_k(t_i, y_i; h) \int_{t_i}^{t_i + \theta_m h} l_k(s) ds\right), \quad 1 \leq m \leq n.$$

In order for Henrici's theory to apply, we must show that  $\Phi$  is Lipschitz continuous with respect to  $y$  in  $\Omega \equiv [t_0, t_N] \times R \times [0, h_0]$ . If, for any  $i$ ,  $0 \leq i \leq N-1$ , and any  $y_i^* \in R$ ,  $y^*(t)$  is the Galerkin approximate solution to  $u' = f(t, u)$ ,  $u(t_i) = y_i^*$ ,  $t_i \leq t \leq t_{i+1}$ , then (26) holds for  $y^*$

$$(26') \quad y^*(t) = y_i^* + \sum_{k=1}^n f(\sigma_{i,k}, y^*(\sigma_{i,k})) \int_{t_i}^t l_k(s) ds, \quad t_i \leq t \leq t_{i+1}.$$

Letting  $B_0$  be a constant such that

$$(30) \quad \sum_{k=1}^n \max_{t_i \leq t \leq t_{i+1}} \left| \int_{t_i}^t l_k(s) ds \right| \leq h B_0, \quad 0 \leq i \leq N-1,$$

and subtracting (26) and (26') leads to

$$(31) \quad \max_{t_i \leq t \leq t_{i+1}} |y(t) - y^*(t)| \leq \frac{1}{1 - h_0 B_0 L} |y_i - y_i^*|, \quad 0 \leq i \leq N-1,$$

where  $0 \leq h \leq h_0 < (B_0 L)^{-1}$ . The Lipschitz condition then follows from (28), (29) and (31) since, for  $0 \leq h \leq h_0$  and  $0 \leq i \leq N-1$ ,

$$\begin{aligned}
 |\Phi(t_i, y_i; h) - \Phi(t_i, y_i^*; h)| &\leq \sum_{m=1}^n w_m |g_m(t_i, y_i; h) - g_m(t_i, y_i^*; h)| \\
 (32) \qquad \qquad \qquad &\leq L \sum_{m=1}^n w_m |y(\sigma_{i,m}) - y^*(\sigma_{i,m})| \\
 &\leq \frac{L}{1 - h_0 B_0 L} |y_i - y_i^*|,
 \end{aligned}$$

where  $\sum_{m=1}^n w_m = 1$ .

Now, we may prove

**THEOREM 1.** *Assume that  $f(t, x) \in C^{2n}$  in  $[t_0, t_N] \times R$  so that  $u(t) \in C^{2n+1}[t_0, t_N]$ , and denote by  $L$  the Lipschitz constant for  $f$  in this region. Let the Galerkin method be defined as in Section 2 for some piecewise polynomial basis functions of degree  $n \geq 1$  and the  $n$ -point Gauss-Legendre quadrature formula (6). If  $Q_2$  and  $B_0$  are defined by (14) and (30), respectively, and  $0 < h \leq h_0$  where  $0 < h_0 < \min(Q_2^{-1}, (B_0 L)^{-1})$ , then there exists a constant  $M$  such that*

$$(33) \qquad \qquad \qquad |u_i - y_i| \leq M h^{2n}, \quad 0 \leq i \leq N.$$

*Proof.* The local truncation error  $\tau_i$  is defined from (24) by

$$u_{i+1} = u_i + h \sum_{m=1}^n w_m f(\sigma_{i,m}, u(\sigma_{i,m})) + \tau_i.$$

Thus,

$$\tau_i = \int_{t_i}^{t_{i+1}} f(t, u(t)) dt - h \sum_{m=1}^n w_m f(\sigma_{i,m}, u(\sigma_{i,m}))$$

and, from (6),  $|\tau_i| \leq K h^{2n+1}$ , where  $K$  is a constant depending on the maximum of  $u^{(2n+1)}(t)$  on  $[t_0, t_N]$ . The bound (33) follows immediately from Henrici's Theorem 2.2 [12]. Q.E.D.

The discrete error bounds (33) agree with those for Butcher's methods [2].

We obtain continuous error bounds in

**THEOREM 2.** *Let the hypotheses of Theorem 1 hold. Then there exist constants  $E_j, 0 \leq j \leq n$ , such that*

$$(34) \qquad \qquad \qquad \max_{t_0 \leq t \leq t_N} |u(t) - y(t)| \leq E_0 h^{n+1},$$

and

$$(35) \qquad \max_{t_i \leq t \leq t_{i+1}} |u^{(j)}(t) - y^{(j)}(t)| \leq E_j h^{n-j+1}, \quad 1 \leq j \leq n, 0 \leq i \leq N - 1.$$

*Proof.* We write  $u(t)$  in the same form as  $y(t)$  in (26) by using the  $n$ -point Lagrangian quadrature found there

$$\begin{aligned}
 (36) \qquad u(t) &= u_i + \int_{t_i}^t f(s, u(s)) ds \\
 &= u_i + \sum_{k=1}^n f(\sigma_{i,k}, u(\sigma_{i,k})) \int_{t_i}^t l_k(s) ds + R_n(t), \quad t_i \leq t \leq t_{i+1},
 \end{aligned}$$

where  $R_n(t) = O(h^{n+1})$ . Subtracting from (26), we find that

$$\max_{t_i \leq t \leq t_{i+1}} |u(t) - y(t)| \leq \frac{1}{1 - h_0 B_0 L} |u_i - y_i| + O(h^{n+1}), \quad 0 \leq i \leq N - 1;$$

and (34) follows from (33). If we differentiate (26) and (36)  $j$  times using  $R_n^{(j)}(t) = O(h^{n-j+1})$  and subtract, we can show that

$$\max_{t_i \leq t \leq t_{i+1}} |u^{(j)}(t) - y^{(j)}(t)| \leq L B_j \max_{1 \leq k \leq n} |u(\sigma_{i,k}) - y(\sigma_{i,k})| + O(h^{n-j+1}),$$

for  $1 \leq j \leq n, 0 \leq i \leq N - 1$ , where

$$\sum_{k=1}^n \max_{t_i \leq t \leq t_{i+1}} |l_k^{(j-1)}(t)| \leq B_j.$$

Then (35) follows from (34). Q.E.D.

**7. A-Stability of the Galerkin Methods.** Dahlquist [8] defines  $A$ -stability as follows.

*Definition.* A  $k$ -step method is called  $A$ -stable, if all its solutions tend to zero, as  $i \rightarrow \infty$ , when the method is applied with fixed positive  $h$  to any differential equation of the form  $u' = \lambda u$ , where  $\lambda$  is a complex constant with negative real part.

Butcher's implicit Runge-Kutta methods based on Gauss-Legendre quadrature [2] have been shown by Ehle [10] to be  $A$ -stable. Ehle observed that the  $n$ -stage method, applied to  $u' = \lambda u$ , yields  $y_{i+1} = P_{n,n}(\lambda h)y_i$ , where  $P_{n,n}(\lambda h)$  is the  $n$ th diagonal Padé rational approximation to  $\exp(\lambda h)$ .  $A$ -stability follows from the fact that  $|P_{n,n}(\lambda h)| < 1$  for  $\text{Re}(\lambda h) < 0$ . Our Galerkin methods, which from (24) give discrete values  $y_i$  identical to those of Butcher's methods [2], are therefore  $A$ -stable.

We should remark that Axelsson [1] has used similar properties of subdiagonal and diagonal Padé rational approximations to prove  $A$ -stability for implicit Runge-Kutta methods based on Radau and Lobatto quadratures. It is natural then to ask whether a Galerkin method which uses these quadratures rather than Gauss-Legendre would yield corresponding "equivalent" methods. The answer is no. If (6) were an  $n$ -point Radau formula with  $\sigma_{i,n} = t_{i+1}$ , it would be exact only for  $v \in \mathcal{P}_{2n-2}$ . The quadratures for  $(f, \varphi_{i+k})_i$  would not be exact for  $f \in \mathcal{P}_{n-1}$ ,  $y$  would not be exact for  $u \in \mathcal{P}_n$ , (24) would not hold, and the order of the Galerkin method would be  $O(h^{n-1})$ , whereas Axelsson [1] and Butcher [3] have shown that an  $n$ -stage implicit Runge-Kutta method based on Radau quadrature has the order  $O(h^{2n-1})$ . Similar results are true of Lobatto quadrature.

**8. Numerical Examples.** In this section, we give numerical results of an  $A$ -stable piecewise cubic ( $n = 3$ ) Galerkin scheme of order 6. We have employed Schoenberg's [20] cubic  $B$ -spline basis functions where  $\varphi_{i+j}$  has its support on  $[t_{i+j-4}, t_{i+j}]$ . The calculations were performed on a CDC 6600, which has about 14 decimal digits, using a successive substitution iteration at each step to solve (12).

First, we consider problems for single equations.

*Problem 1.*  $u' = -2tu^2, u(0) = 1, u(t) = 1/(1 + t^2), 0 \leq t \leq 1$ .

*Problem 2.*  $u' = 1/(1 + \tan^2 u), u(0) = 0, u(t) = \arctan t, 0 \leq t \leq 1$ .



*Problem 3.*  $u' = u - (2t/u)$ ,  $u(0) = 1$ ,  $u(t) = (2t + 1)^{1/2}$ ,  $0 \leq t \leq 1$ .

*Problem 4.*  $u' = u$ ,  $u(0) = 1$ ,  $u(t) = e^t$ ,  $0 \leq t \leq 10$ .

Several uniform meshes are used for each problem. Tables 1–4 are designed to

TABLE 1  
*Error Norms for Problem 1*

$h$	$\ e(t; h)\ '$	$\ e'(t; h)\ '$	$\ e''(t; h)\ '$	$\ e'''(t; h)\ '$
1	$3.55(10)^{-4}$	$7.94(10)^{-2}$	$8.20(10)^{-1}$	$3.49(10)^0$
$2^{-1}$	$8.54(10)^{-6}$ (5.38)	$1.30(10)^{-2}$ (2.61)	$3.36(10)^{-1}$ (1.29)	$4.10(10)^0$ (−0.23)
$2^{-2}$	$1.18(10)^{-7}$ (6.18)	$2.65(10)^{-3}$ (2.30)	$1.29(10)^{-1}$ (1.38)	$2.71(10)^0$ (0.60)
$2^{-3}$	$1.79(10)^{-9}$ (6.04)	$3.75(10)^{-4}$ (2.82)	$3.61(10)^{-2}$ (1.84)	$1.46(10)^0$ (0.89)
$2^{-4}$	$2.81(10)^{-11}$ (6.00)	$4.83(10)^{-5}$ (2.96)	$9.29(10)^{-3}$ (1.96)	$7.45(10)^{-1}$ (0.97)
$2^{-5}$	$3.45(10)^{-13}$ (6.35)	$6.09(10)^{-6}$ (2.99)	$2.34(10)^{-3}$ (1.99)	$3.74(10)^{-1}$ (0.99)
$2^{-6}$	$1.85(10)^{-13}$ (0.90)	$7.62(10)^{-7}$ (3.00)	$5.86(10)^{-4}$ (2.00)	$1.87(10)^{-1}$ (1.00)

TABLE 2  
*Error Norms for Problem 2*

$h$	$\ e(t; h)\ '$	$\ e'(t; h)\ '$	$\ e''(t; h)\ '$	$\ e'''(t; h)\ '$
1	$2.48(10)^{-5}$	$3.04(10)^{-2}$	$3.62(10)^{-1}$	$1.62(10)^0$
$2^{-1}$	$1.28(10)^{-7}$ (7.60)	$4.76(10)^{-3}$ (2.67)	$1.11(10)^{-1}$ (1.70)	$6.61(10)^{-1}$ (1.29)
$2^{-2}$	$5.79(10)^{-9}$ (4.46)	$5.88(10)^{-4}$ (3.02)	$2.84(10)^{-2}$ (1.97)	$5.70(10)^{-1}$ (0.21)
$2^{-3}$	$8.62(10)^{-11}$ (6.07)	$7.64(10)^{-5}$ (2.94)	$7.30(10)^{-3}$ (1.96)	$2.85(10)^{-1}$ (1.00)
$2^{-4}$	$1.34(10)^{-12}$ (6.01)	$9.48(10)^{-6}$ (3.01)	$1.82(10)^{-3}$ (2.01)	$1.46(10)^{-1}$ (0.97)
$2^{-5}$	$9.24(10)^{-14}$ (3.86)	$1.19(10)^{-6}$ (2.99)	$4.56(10)^{-4}$ (1.99)	$7.29(10)^{-2}$ (1.00)
$2^{-6}$	$1.49(10)^{-13}$ (−0.69)	$1.49(10)^{-7}$ (3.00)	$1.14(10)^{-4}$ (2.00)	$3.64(10)^{-2}$ (1.00)

TABLE 3  
*Error Norms for Problem 3*

$h$	$\ e(t; h)\ '$	$\ e'(t; h)\ '$	$\ e''(t; h)\ '$	$\ e'''(t; h)\ '$
1	$7.08(10)^{-4}$	$2.57(10)^{-2}$	$3.24(10)^{-1}$	$2.43(10)^0$
$2^{-1}$	$2.22(10)^{-5}$ (4.99)	$6.03(10)^{-3}$ (2.09)	$1.50(10)^{-1}$ (1.11)	$1.87(10)^0$ (0.37)
$2^{-2}$	$4.67(10)^{-7}$ (5.57)	$1.14(10)^{-3}$ (2.40)	$5.58(10)^{-2}$ (1.42)	$1.26(10)^0$ (0.57)
$2^{-3}$	$8.05(10)^{-9}$ (5.86)	$1.83(10)^{-4}$ (2.64)	$1.77(10)^{-2}$ (1.65)	$7.55(10)^{-1}$ (0.74)
$2^{-4}$	$1.30(10)^{-10}$ (5.96)	$2.63(10)^{-5}$ (2.80)	$5.07(10)^{-3}$ (1.81)	$4.19(10)^{-1}$ (0.85)
$2^{-5}$	$2.73(10)^{-12}$ (5.57)	$3.53(10)^{-6}$ (2.89)	$1.36(10)^{-3}$ (1.90)	$2.21(10)^{-1}$ (0.92)
$2^{-6}$	$1.48(10)^{-12}$ (0.88)	$4.59(10)^{-7}$ (2.95)	$3.53(10)^{-4}$ (1.95)	$1.14(10)^{-1}$ (0.96)

TABLE 4  
Error Norms for Problem 4

$h$	$\ e(t; h)\ '$	$\ e'(t; h)\ '$	$\ e''(t; h)\ '$	$\ e'''(t; h)\ '$
1	$2.27(10)^0$	$1.15(10)^2$	$1.48(10)^3$	$5.59(10)^3$
$2^{-1}$	$3.45(10)^{-2}$ (6.04)	$1.80(10)^1$ (2.67)	$4.49(10)^2$ (1.72)	$3.90(10)^3$ (0.52)
$2^{-2}$	$5.35(10)^{-4}$ (6.01)	$2.54(10)^0$ (2.83)	$1.24(10)^2$ (1.86)	$2.31(10)^3$ (0.75)
$2^{-3}$	$8.31(10)^{-6}$ (6.01)	$3.37(10)^{-1}$ (2.92)	$3.27(10)^1$ (1.93)	$1.26(10)^3$ (0.88)
$2^{-4}$	$9.32(10)^{-8}$ (6.48)	$4.34(10)^{-2}$ (2.96)	$8.38(10)^0$ (1.96)	$6.59(10)^2$ (0.94)
$2^{-5}$	$6.23(10)^{-8}$ (0.58)	$5.52(10)^{-3}$ (2.98)	$2.12(10)^0$ (1.98)	$3.37(10)^2$ (0.97)
$2^{-6}$	$1.27(10)^{-7}$ (-1.03)	$6.95(10)^{-4}$ (2.99)	$5.34(10)^{-1}$ (1.99)	$1.70(10)^2$ (0.98)

illustrate the  $O(h^6)$  mesh point accuracy of Theorem 1 as well as the  $O(h^3)$ ,  $O(h^2)$  and  $O(h)$  accuracies of the first three derivatives predicted by Theorem 2. The tables give the discrete error norms for  $y(t; h)$  and its first three derivatives

$$(37) \quad \|e^{(j)}(t; h)\|' = \max_{0 \leq i \leq N} |e^{(j)}(t_{i \pm}; h)|, \quad 0 \leq j \leq 3,$$

where  $e = u - y$  and also in parentheses the computed orders of accuracy, based on successive mesh sizes  $h_1, h_2$ ,

$$(38) \quad \omega_j = \frac{\log[\|e^{(j)}(t; h_1)\|'/\|e^{(j)}(t; h_2)\|']}{\log(h_1/h_2)},$$

i.e.,  $\|e^{(j)}(t; h)\|' \approx O(h^{\omega_j})$ ,  $0 \leq j \leq 3$ .

Next, we present in Table 5 absolute errors  $e(t; h)$  and relative errors  $e(t; h)/u(t)$

TABLE 5  
Absolute and Relative Errors for Problem 5

$t$	$e(t; 1)$	$e(t; 1)/u(t)$	$e(t; 0.5)$	$e(t; 0.5)/u(t)$
1	$3.79(10)^{-6}$	$1.03(10)^{-5}$	$5.76(10)^{-8}$	$1.57(10)^{-7}$
10	$4.68(10)^{-9}$	$1.03(10)^{-4}$	$7.11(10)^{-11}$	$1.57(10)^{-6}$
20	$4.25(10)^{-13}$	$2.06(10)^{-4}$	$6.45(10)^{-15}$	$3.13(10)^{-6}$
40	$1.75(10)^{-21}$	$4.12(10)^{-4}$	$2.67(10)^{-23}$	$6.26(10)^{-6}$
60	$5.42(10)^{-30}$	$6.19(10)^{-4}$	$8.22(10)^{-32}$	$9.39(10)^{-6}$
80	$1.49(10)^{-38}$	$8.25(10)^{-4}$	$2.26(10)^{-40}$	$1.25(10)^{-5}$
100	$3.83(10)^{-47}$	$1.03(10)^{-3}$	$5.83(10)^{-49}$	$1.57(10)^{-5}$

at selected points  $t$ , for  $h = 1$  and  $0.5$  in

**Problem 5.**  $u' = -u$ ,  $u(0) = 1$ ,  $u(t) = e^{-t}$ ,  $0 \leq t \leq 100$ , in order to illustrate the stability of the method.

Finally, we give in Tables 6 and 7 the results of the application of our method to

TABLE 6  
Error Norms for  $e_1(t; h)$  of Problem 6

$h$	$\ e_1(t; h)\ '$	$\ e_1'(t; h)\ '$	$\ e_1''(t; h)\ '$	$\ e_1'''(t; h)\ '$
1	$1.97(10)^{-3}$	$1.82(10)^{-2}$	$1.77(10)^{-1}$	$7.44(10)^{-1}$
$2^{-1}$	$2.91(10)^{-5}$ (6.08)	$2.53(10)^{-3}$ (2.85)	$5.43(10)^{-2}$ (1.71)	$4.95(10)^{-1}$ (0.59)
$2^{-2}$	$4.50(10)^{-7}$ (6.02)	$3.33(10)^{-4}$ (2.92)	$1.52(10)^{-2}$ (1.84)	$2.89(10)^{-1}$ (0.77)
$2^{-3}$	$7.01(10)^{-9}$ (6.00)	$4.29(10)^{-5}$ (2.96)	$4.01(10)^{-3}$ (1.92)	$1.57(10)^{-1}$ (0.88)
$2^{-4}$	$1.08(10)^{-10}$ (6.02)	$5.45(10)^{-6}$ (2.98)	$1.03(10)^{-3}$ (1.96)	$8.16(10)^{-2}$ (0.94)
$2^{-5}$	$2.19(10)^{-12}$ (5.62)	$6.86(10)^{-7}$ (2.99)	$2.62(10)^{-4}$ (1.98)	$4.16(10)^{-2}$ (0.97)
$2^{-6}$	$2.69(10)^{-12}$ (-1.61)	$8.61(10)^{-8}$ (2.99)	$6.59(10)^{-5}$ (1.99)	$2.10(10)^{-2}$ (0.99)

TABLE 7  
Error Norms for  $e_2(t; h)$  of Problem 6

$h$	$\ e_2(t; h)\ '$	$\ e_2'(t; h)\ '$	$\ e_2''(t; h)\ '$	$\ e_2'''(t; h)\ '$
1	$2.58(10)^{-4}$	$7.26(10)^{-3}$	$7.69(10)^{-2}$	$3.96(10)^{-1}$
$2^{-1}$	$3.82(10)^{-6}$ (6.08)	$9.40(10)^{-4}$ (2.95)	$2.17(10)^{-2}$ (1.82)	$2.21(10)^{-1}$ (0.84)
$2^{-2}$	$5.91(10)^{-8}$ (6.02)	$1.23(10)^{-4}$ (2.93)	$5.82(10)^{-3}$ (1.90)	$1.18(10)^{-1}$ (0.91)
$2^{-3}$	$9.20(10)^{-10}$ (6.00)	$1.58(10)^{-5}$ (2.96)	$1.51(10)^{-3}$ (1.95)	$6.06(10)^{-2}$ (0.96)
$2^{-4}$	$1.42(10)^{-11}$ (6.02)	$2.00(10)^{-6}$ (2.98)	$3.84(10)^{-4}$ (1.97)	$3.08(10)^{-2}$ (0.98)
$2^{-5}$	$2.13(10)^{-13}$ (6.06)	$2.52(10)^{-7}$ (2.99)	$9.68(10)^{-5}$ (1.99)	$1.55(10)^{-2}$ (0.99)
$2^{-6}$	$7.27(10)^{-13}$ (-1.77)	$3.17(10)^{-8}$ (2.99)	$2.43(10)^{-5}$ (1.99)	$7.78(10)^{-3}$ (0.99)

the system of equations in

Problem 6.  $u_1' = u_1^2 u_2$ ,  $u_2' = -1/u_1$ ,  $u_1(0) = 1$ ,  $u_2(0) = 1$ ,  $u_1 = e^t$ ,  $u_2 = e^{-t}$ ,  $0 \leq t \leq 1$ .

**Acknowledgements.** The author would like to express his gratitude to Mrs. Sharon L. Daniel for her outstanding work in implementing the methods of this paper and obtaining the above numerical results, to Dr. H. A. Watts for invaluable discussions on implicit methods, and to Dr. R. J. Thompson for helpful suggestions concerning the manuscript.

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