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By

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One-Way Essential Complements*

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Abstract

While competition between firms producing substitutes is well understood, less is known about rivalry between complementors. We study the interaction between firms in markets with one-way essential complements. One good is essential to the use of the other but not vice versa, as arises with an operating system and applications. Our interest is in the division of surplus between the two goods and the related incentive for firms to create complements to an essential good.

Formally, we study a two-good model where consumers value A alone, but can only enjoy B if they also purchase A. When one firm sells A and another sells B, the firm that sells B earns a majority of the value it creates. However, if the A firm were to buy the B firm, it would optimally charge zero for B, provided marginal costs are zero and the average value of B is small relative to A. Hence, absent strong antitrust or intellectual property protections, the A firm can leverage its monopoly into B costlessly by producing a competing version of B and giving it away. For example, Microsoft provided Internet Explorer as a free substitute for Netscape; in our model, this maximizes Microsoft's joint monopoly profits. Furthermore, Microsoft has no incentive to raise prices, even if all browser competition exits. This may seem surprising since it runs counter to the traditional gains from price discrimination and versioning. We also show that a essential monopolist has no incentive to degrade rival complementary products, which suggests that a monopoly internet service provider will offer net neutrality.

There are other means for the essential A monopolist to capture surplus from B. We consider the incentive to add a surcharge (or subsidy) to the price of B, or to act as a Stackelberg leader. We find a small gain from pricing first, but much greater profits from adding a surcharge to the price of B. The potential for A to capture B's surplus highlights the challenges facing a firm whose product depends on an essential good.

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1 Introduction

We generally think of competition as being between two substitute products. But competition also arises between two complementary products. In this paper, we study the competition between two complements where one is essential and the other is not.

An operating system and a microprocessor are both essential as neither works without the other. Here we look at the case where one good (A) is essential and the other good (B) is optional. A consumer can enjoy A without B, but not B without A. An example that fits this rule is Windows (A) and a media player (B). The media player requires Windows, while Windows has utility absent a media player. A second example is a cable modem (A) and cable telephony (B). A consumer can enjoy a cable modem without using cable telephony but cannot use cable telephony without a cable modem.

We consider the case where the sellers of A and B each have market power. We expect, all else equal, the seller of A will do better than B. Anyone who wants to enjoy B must also buy A, but not vice versa. Thus A is in a stronger position. How much does this asymmetry hurt B?

It turns out that in the Nash pricing game, firm B is able to earn more than half the increased industry profits it creates. This ability for firm B to capture surplus determines its incentive to innovate or to enter a market where its product is dependent on an incumbent firm in a complementary market.

The problem for B is that there are many ways in which firm A can capture the surplus created by B. Indeed, the best outcome for B is our baseline case, the Nash equilibrium, where the two firms price simultaneously.

Firm A can be more aggressive by entering the B market and competing directly with B. Outside the case of complements, this is rarely a profitable strategy. Firm A has to pay the costs of entering the B market, but once in the market will find itself in Bertrand competition with the incumbent. Assuming no product differentiation, price will be driven to cost and neither firm will earn any profits. The situation is different here as firm A benefits from bringing down the price of B. If customers know that they can get B for free, this gives them a greater incentive to pay a high price for A.

A second option for A is to acquire B and then set the joint profit-maximizing price. We know that this is efficient as A can eliminate the problem of double marginalization

(Cournot, 1838). The surprising result is not that joint profits rise, but how this is achieved. When the value of B is small relative to A and marginal costs are zero, we find that A would choose to price B at zero. As we explain below, this result runs contrary to the common intuition from price-discrimination and versioning. The joint monopolist would give B away and earn all of its profits in A.

This presents a problem to a firm selling B—its rival in A can earn the joint monopoly profits by driving the price of B down to zero. If firm A can enter the B market and drive the price to zero, this will be as profitable as buying B; hence firm A will compare the costs of buying B to entering the B market and will only be willing to buy B to the extent this is cheaper than entering the market.[1]

A third option for firm A is to be a price leader. When firm A sets its price before B, it raises its price over the Nash result in order to push down the price of B. While A's profits increase, the gain is relatively small.

A fourth option for A is to tack on a surcharge or a subsidy to B. In the case of a subsidy, firm A could offer a coupon for B with the purchase of A. In the case of a surcharge, this could be accomplished by creating two versions of A, one compatible with B and one discounted but incompatible version. We find that the monopolist would always seek to impose a surcharge for the purchase of an A product compatible with B and that it is never optimal to offer a subsidy.

A related problem is the incentive for an incumbent in A to influence the quality of B. Instead of making an incompatible version, the A firm can either enhance or degrade the quality of the complementary product. This issue is particularly relevant to internet-based businesses, such as the internet telephone services provided by Skype and Vonage. These services allow users to replace their land-line phone service, but they depend on the user having high-speed internet access, usually from their local cable company or telco. This has led to a policy debate over equal access. There is a concern that the broadband provider will have an incentive to degrade the quality of Skype and Vonage in order to increase the attractiveness of their own competing internet phone service. Our model shows that the A firm does not have an incentive to degrade the quality of B, even if A were to enter the B market itself. This suggests that profit-maximizing internet service providers have no

[1] While it makes the situation difficult for B, this is not a problem for consumers who generally come out ahead when the complements are sold by a joint monopolist.

incentive to disrupt net neutrality.

We believe that the one-way essential complements lead to interesting and relevant market interactions. Essential complements are prevalent and unstudied. Most software runs on an operating system without which it is useless. Thus there is often a set of dependant B goods to an essential A good. Chains of such relationships can also arise; the operating system is essential to the browser, which is essential to the search engine and the media player. We explore how the complementary firms compete and how surplus is divided in these situations.

2 Related Literature

Farrell and Katz (2000) study the incentive for innovation in a world where A and B are each essential to the other—as is the case with hardware and software. In their model, the producer of A has a monopoly. Firm A seeks to reduce the price of B so as to capture more of the surplus for itself.

In the case where all the B firms simultaneously set their price before A, firm A would like to enter the B market and charge marginal cost. If firm A can set its prices first, it may choose to enter and offer to sell B at a price below marginal cost. The goal is not to subsidize the purchase of B, but rather to force its more efficient B rivals to lower their price.

Farrell and Katz also consider the incentive for firms to engage in quality-enhancing (or cost-reducing) R&D. They show that the B firms have an efficient incentive to engage in R&D. This follows because the B firms capture all of the incremental surplus with their first-mover advantage. In contrast, the A monopolist has an excess incentive to pursue R&D. The reason is that the monopolist gains from improving its version of B, even if it doesn't sell B, because that leads rivals to lower their price of good B.

The result that B firms capture all of the incremental surplus relies on their ability to set prices first and on the assumption that consumer valuations for the A-B bundle are homogeneous. With simultaneous pricing and heterogeneous valuations, profits are split equally between the A and B sellers.[2] While this result is to be expected when costs are

[2]The profits of a B firm might be limited by competition with other similarly positioned B-good firms. Profits between A and B are equal subject to the constraint that these profits are less than the cost (or quality) advantage of B over its rivals. In particular, profits are equal when there is a monopolist in A and

equal, the equal division of profits continues to hold even when production costs are unequal across A and B; see Casadesus-Masanell, Nalebuff, and Yoffie (2006). Thus a firm only gains half of the value it creates through innovation.

In our model, good B is not essential to A and profits are no longer equally split equally. We show that when the value of A is large relative to B, the nonessential complement in B is able to capture more than half the value it creates. On the margin, this is a better situation for B than when its product is essential.[3]

It is still the case that the monopolist in A gains when the price of B is lower. This is a general result for complementary goods. But there are several further differences between one-way and two-way essential complements. When consumers must buy both A and B to enjoy either, they care only about the joint price: a subsidy on B is identical to a discount on A. It doesn't make sense to ask how a joint monopolist would price A and B separately. With one-way complements, the individual prices are relevant as some consumers purchase A without B. The linkage between the two markets suggests the possibility of price discrimination. We find that in one-way markets, price discrimination is advantageous when firm A does not own B, but not when A owns B.

Our price discrimination results are connected to the literature on versioned goods. The A-B bundle can be thought of as a high-quality version of the basic A good. It is well understood that creating different quality version of a good may allow a monopolist to engage in second-degree price-discrimantion. Deneckere and McAfee (1996) provide general conditions under which a monopolist would always sell the premium good (the A-B bundle) for strictly more than the basic good (A alone). Thus it is surprising that we reach an opposite conclusion, at least for the case where marginal costs are zero and the value of B is small relative to A. In that case, we show that the monopolist charges the same price for the premium and basic versions, so that all customers buy the premium version. The different conclusions follow from different assumptions: specifically, in our model preference are multi-dimensional. The discussion following model three provides a graph that translates our framework into a versioning model and illustrates the intuition for why the results differ.

The case of one-way essential complements is also similar to aspects of the literature on tying (Bowman 1957, Stigler 1964, Adams and Yellen 1976, Bork 1978, Whinston 1990).

a monopolist in B.

[3] Overall, firm B does worse in that it is unable to capture any of the surplus created by A.

Under tying, a monopolist in A forces consumers to buy A along with B. Here, the situation is reversed; customers can only enjoy B along with A. This "tied" relationship is not imposed on the consumer by A, but is the result of consumer preferences. While the essential monopolist does not force the joint sale, it is still able to exploit the asymmetry in consumer preferences to capture surplus created in the B market.

3 The Model and Cases

In all of our models there are two goods, A and B, where the consumption of A is essential to the enjoyment of B, but not vice-versa.[4]

3.1 Base Cases

We start our analysis with the most basic case: homogeneous values. We further assume that both goods are produced with zero costs.

Proposition 1 *When all customers value A at 1 and B at λ, any pair of non-negative prices (p_a, p_b) is a Nash equilibrium if and only if $p_a + p_b = 1 + \lambda$ and $p_b \leq \lambda$.*

The proof of this proposition and all subsequent results are contained in the Appendix.

Observe that the division of the pie is indeterminate. As a result, if the incumbent firm A is able to move first, it would have an advantage. It could set the price of A to be $1 + \lambda$ (or $1 + \lambda - \varepsilon$) and thereby capture all of the surplus from both A and B.

Next we add heterogeneity to the valuations of A. We assume that the valuations of good A are distributed uniformly on $[0, 1]$. As before, the value of B is the same for all consumers and equal to λ. We are most interested in the case where A is more valuable than B. Hence we further assume that $\lambda \leq 1/2$.

Proposition 2 *With A uniform on $[0, 1]$ and B at $\lambda \leq 1/2$, there is a unique Nash equilibrium. Firm A charges $1/2$ and firm B charges λ.*

[4] In our model, there is only one B good. In practice, one could think of analyzing the pricing of one B good in isolation. Part of the value of the A good comes from the opportunity to purchase all of the other B goods, some of which may be free.

Observe that B gets all of the value it creates. If that situation is representative, then there is no concern that a potential entrant into the complementary market would have insufficient incentives to innovate and enter.

However, it is an artificial assumption that good B is homogeneous in valuation. The results from the two base cases suggest that adding heterogeneity may change the equilibrium. Since heterogeneous valuations are the general case, we turn now to this setting.

3.2 Main Case: Uniform Consumer Valuations

Assume consumer valuations of A are distributed uniformly on $[0, 1]$, and valuations of B are distributed uniformly on $[0, \lambda]$. Further assume that consumer valuations for A and B are independent.

3.2.1 Model One: Two firms A and B, each a monopoly, and Nash Pricing

The formula for profits depend on whether $p_a + p_b \lessgtr \lambda$. The reason is that the geometry of market areas depends on whether the line defining the set of consumers who are indifferent about buying both A and B is truncated by the highest possible value for B or not.

As can be seen from the figures below, the demand for good A is the upper left rectangle combined with the shaded trapezoid to the right. Demand for good B is limited to the shaded trapezoid.[5]

[5] Note that area of the box is λ. Thus the population density is normalized to $1/\lambda$ so that the total population stays constant at 1.

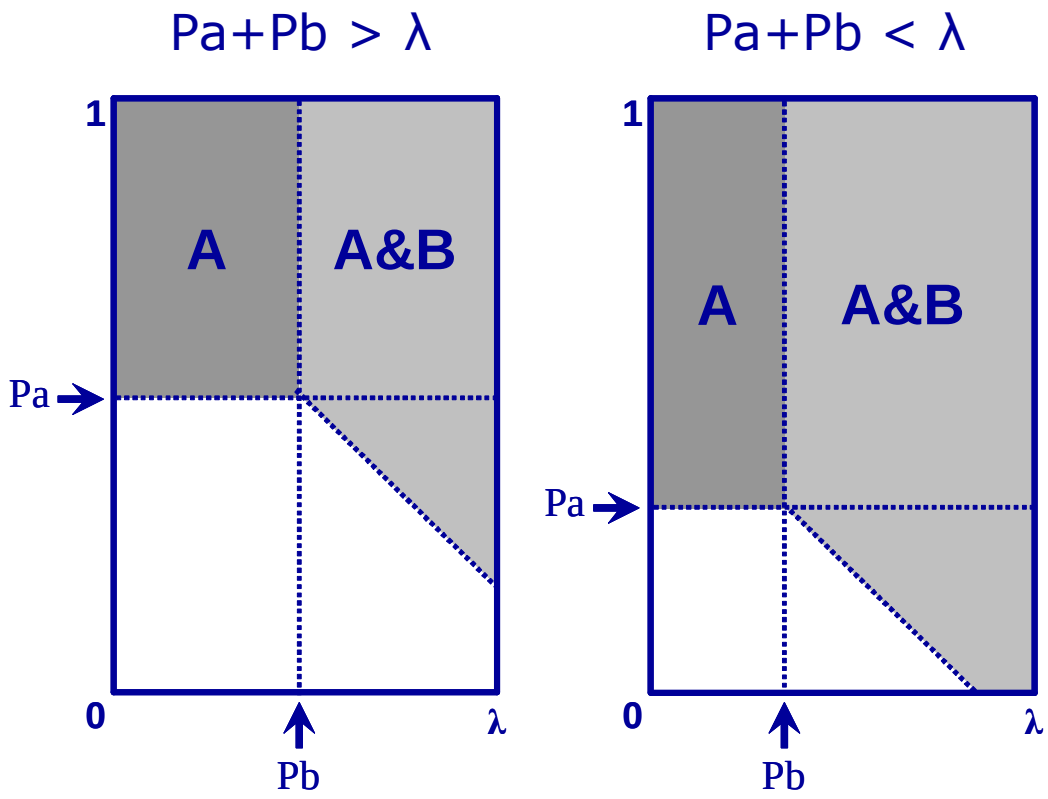


Figure 1

With a uniform density, we have a closed-form expression for profits. Firm A's profits are

$$\left\{ \begin{array}{cc} \frac{p_a}{\lambda}(\lambda(1-p_a)+\frac{1}{2}(\lambda-p_b)^2) & \text{when} \quad p_a+p_b \geq \lambda \\ \frac{p_a}{\lambda}(\lambda-(p_a*p_b)-\frac{1}{2}{p_a}^2) & \text{when} \quad p_a+p_b \leq \lambda \end{array} \right\}. \tag{1}$$

Firm B's profits are

$$\left\{ \begin{array}{cc} \frac{p_b}{\lambda}((\lambda-p_b)(1-p_a)+\frac{1}{2}(\lambda-p_b)^2) & \text{when} \quad p_a+p_b \geq \lambda \\ \frac{p_b}{\lambda}(\lambda-p_b-\frac{1}{2}{p_a}^2) & \text{when} \quad p_a+p_b \leq \lambda \end{array} \right\}. \tag{2}$$

In the case of primary interest λ is small, so we expect $p_a + p_b \geq \lambda$. The solution of the general model reveals that $\lambda = 1$ is the crossover point where $p_a + p_b$ shifts from being bigger to being smaller than λ.

Lemma 1 *When $\lambda = 1$ there is a closed-form solution: $p_a = 2 - \sqrt{2}$ and $p_b = \sqrt{2} - 1$*

Note that at $\lambda = 1$, $p_a + p_b = 1 = \lambda$. In general, the joint solution to the first-order conditions has no simple analytic form. In such cases, our approach is to graph the equilibrium results. The underlying equations for p_a and p_b are provided in the appendix.

Figure 2 below shows that as λ increases, profits rise for both firms. Firm A is able to capture some of the value created by B. In contrast, if the two goods were perfect complements (with two-way essentiality) then profits would be split evenly between the two firms. Here, firm B gets more than half of the surplus which it creates, but none of the surplus associated with A. As B become increasingly valuable, eventually firm B earns more profits than A, even though A is essential for B. Finally, observe that A captures relatively more of the incremental surplus created by B when λ is small ($\lambda < 1$) than when λ is large.

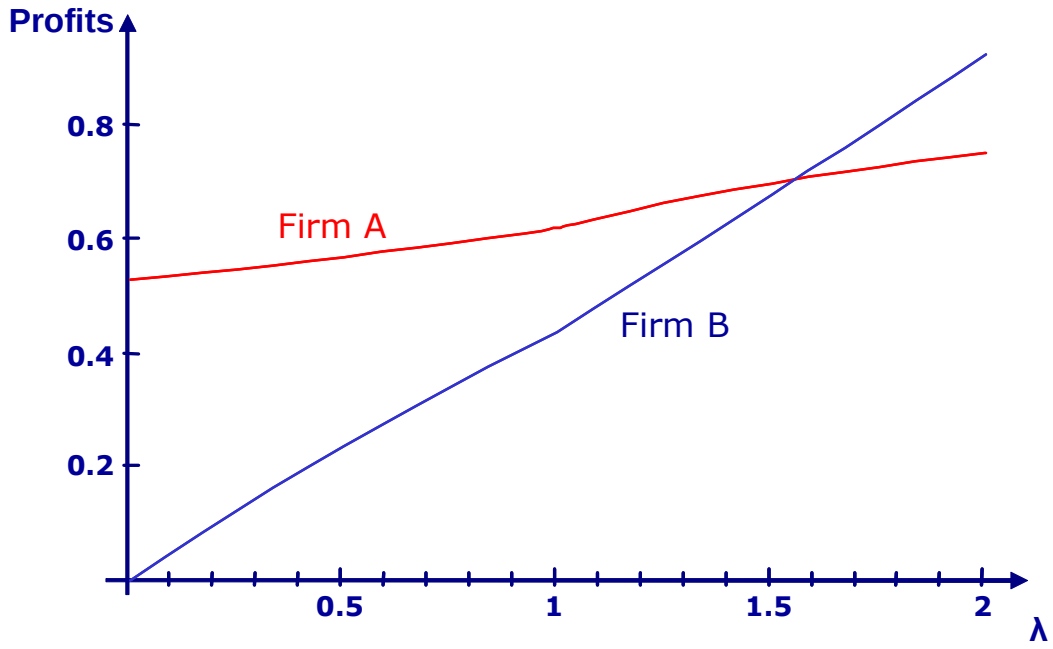


Figure 2: Profits in the Nash Pricing Game

Given that good A is essential to consumers wishing to enjoy B, there are several ways that firm A can increase the amount of B surplus it captures. First, we investigate the strategy of introducing competition into the B market, driving the price of good B down to cost. Recall that when Microsoft introduced Explorer, it introduced competition into the browser market dominated by Netscape. Later, we will compare this to alternative strategies such as purchasing or merging with the B monopolist, charging a different price for a version of A which is compatible with B, or entering the B market and purposely degrading (perhaps through barriers to compatibility) the quality of rival B products.

3.2.2 Model Two: Firm A is a monopolist over A and good B is supplied competitively

In the competition to capture surplus, firm A has an incentive to lower the price of B. This might happen exogenously if there is competition in the B market. It could also happen if A were to enter the B market and drive the price of B down to cost, in this case zero.

Proposition 3 *Equilibrium prices are:*

$$p_a = \left\{ \begin{array}{ll} \frac{1}{4}\lambda + \frac{1}{2} & when \quad \lambda \leq \frac{2}{3} \\ \sqrt{\frac{2}{3}\lambda} & when \quad \lambda \geq \frac{2}{3} \end{array} \right\}, \tag{3}$$

$$p_b = 0. \tag{4}$$

The resulting profits are:

$$\begin{array}{ll} \frac{1}{16}(2+\lambda)^2 & \text{when} \quad \lambda \leq \frac{2}{3} \\ (\frac{2}{3})^{\frac{3}{2}}\lambda^{\frac{1}{2}} & \text{when} \quad \lambda \geq \frac{2}{3} \end{array} . \tag{5}$$

It is interesting to note that the price of A is linear in λ up to the point where $\lambda = \frac{2}{3}$ (and $p_a = \frac{2}{3}$). At $\lambda = \frac{2}{3}$, the monopolist's profits are $\frac{4}{9}$. Note that the maximum possible surplus in this case is $\frac{5}{6}$ $(= (1 + \frac{2}{3})/2)$ and so the monopolist is able to capture $\frac{8}{15}$ or slightly more than half of the total.

3.2.3 Model Three: Firm A is a monopolist over both products.

One strategy for the A firm to capture more of the surplus generated by the B good is to buy or merge with the B firm, becoming a monopolist over both markets. This new joint monopolist will be able to increase industry profits compared to the Nash equilibrium pricing game. Total profits rise as the joint monopolist can solve the horizontal equivalent of the double-marginalization problem (Sonnenschein 1968, Nalebuff 2000).

While combined profits are higher, there remains the question of how much the A monopolist is willing to pay to acquire the B firm. The answer to that question depends on its profits as a joint monopolist compared to its other options. These other options include entering the B market, offering subsidies or surcharges for the B product, degrading the quality of the B good, or moving first to set price.

We have just presented the results for the case where A enters to make the B market competitive. A benefits when the price of B is as low as possible—at least when A doesn't

own B. To the extent that a joint monopolist would do better than the outcome with a competitive B market, this gives the monopolist an incentive to purchase B rather than compete.

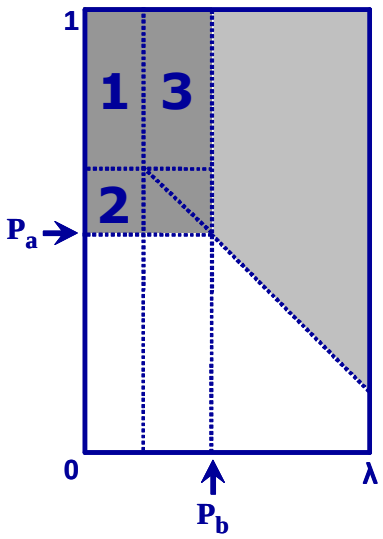
To compare options, we need to understand how a joint monopolist would maximize profits and how much it would earn. Along the way, we gain insight on how the joint monopolist chooses to extract the surplus through A and B.

In the case of complements sold by a joint monopolist, we often see one product being given away. For example, Adobe gives away a program to read pdf files and charges for the program to write them.[6] We find that under a range of conditions, the complement to the essential good is given away. For λ up to $2/3$, the monopolist takes *all* the surplus via A.

To demonstrate this result, we first establish that the optimal $p_b = 0$ when the optimal p_a is small, specifically for $p_a \leq \frac{2}{3}$. In that case, the previous solution for $p_a(\lambda)$ when $p_b = 0$ holds, so that $p_a = \frac{1}{4}\lambda + \frac{1}{2}$. It then follows that $p_a \leq 2/3$ holds for $\lambda \leq \frac{2}{3}$.

Lemma 2 *If the optimal $p_a \leq 2/3$, then the optimal $p_b = 0$.*

The outline of the proof for this lemma is found in Figure 3.



Consider lowering P_b by some small Δ and raising P_a by Δ (so as to hold $P_a + P_b$ constant).

There are three effects:

1) Firm gains Δ on customers just buying A.

$$\textbf{Gain} = \Delta * (1 - P_a) * P_b$$

2) Firm loses some sales of A.

$$\textbf{Loss} = P_a * \Delta * P_b$$

3) Firm gains some sales of B

$$\textbf{Gain} = P_b * \Delta * (1 - P_a)$$

Total effect = $\Delta * P_b * [(1 - P_a) - P_a + (1 - P_a)] =$

$$\Delta * P_b * (2 - 3\,P_a)$$

So long as $P_a = 2/3$, we would want to lower P_b and raise P_a until P_b=0.

Figure 3

[6] This may be done for several reasons, not covered in our model. The motivations include the ability to charge consumers versus firms. Also note that the case of Adobe is a two-way essentiality in that without an encoder, there is no value to a reader.

Using this lemma, we can solve for optimal prices when consumer valuations are distributed uniformly. This is our paper's first main result. (Theorem 7 shows that this result extends to general value distributions.)

Theorem 1 *The optimal monopoly prices are given by:*

$$p_a = \left\{ \begin{array}{ccc} \frac{1}{4}\lambda + \frac{1}{2} & when & \lambda \leq \frac{2}{3} \\ \frac{2}{3} & when & \lambda \geq \frac{2}{3} \end{array} \right\}, \tag{6}$$

$$p_b = \left\{ \begin{array}{ccc} 0 & when & \lambda \leq \frac{2}{3} \\ \frac{1}{2}\lambda - \frac{1}{3} & when & \lambda \geq \frac{2}{3} \end{array} \right\}. \tag{7}$$

These prices (graphed in Figure 4) lead to the monopoly profits below (graphed in Figure 5).

$$\text{Profits} = \begin{array}{cc} \frac{1}{16}(2+\lambda)^2 & \text{when} \quad \lambda \leq \frac{2}{3} \\ \frac{1}{12}(4+3\lambda) - \frac{1}{27\lambda} & \text{when} \quad \lambda \geq \frac{2}{3} \end{array}. \tag{8}$$

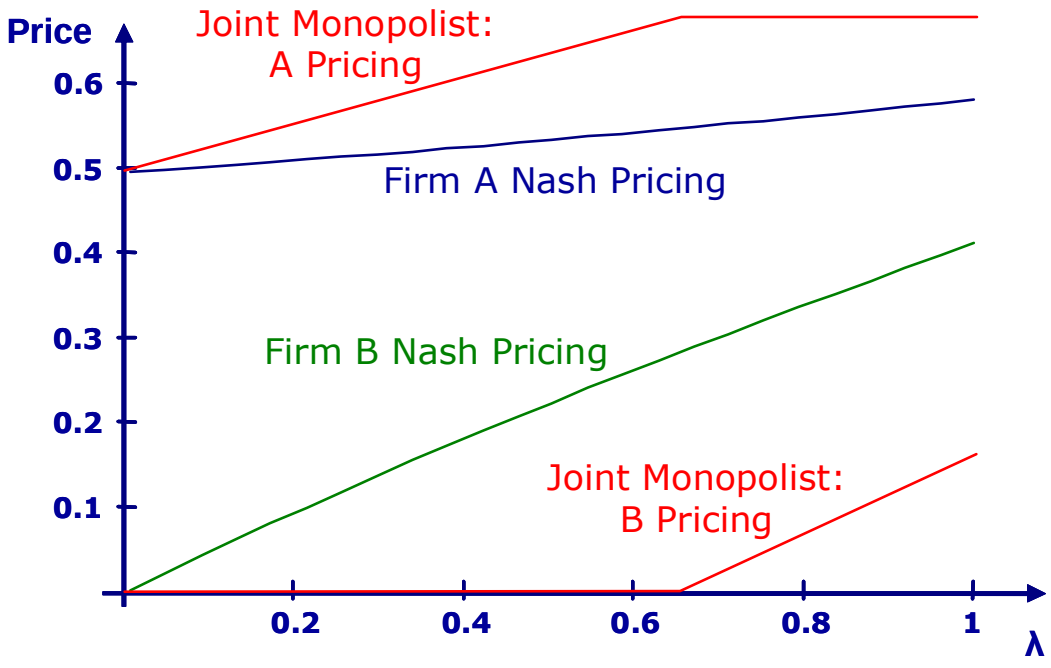


Figure 4: Nash Pricing and Joint A & B Monopolist Pricing.

Observe that the monopolist makes its full joint monopoly profits by entering the B market when $\lambda \leq 2/3$. There is no gain from buying firm B compared to competing with firm B and driving price down to cost. Thus the most the A monopolist would be willing to pay to

purchase B is its cost of entry into the B market (assuming that entry into the B market is possible).

It is not the case that the B good is given away for all values of B—the give away requires that $\lambda \leq 2/3$. When the average value of B becomes large, the complement is no longer given away.

Figure 5 graphs profits as a share of total surplus. We compare the combined A & B profits when the firms price independently (Nash) to profits when a monopolist coordinates pricing. We also illustrate the monopolist's profits when it pushes P_b to 0. Note that when λ is bigger than 2/3rds, the monopolist does nearly as well entering the B market (and pushing p_b down to 0) compared to buying B and setting p_b optimally. This remains true until λ becomes larger than 1, so that the value of B is more than A.

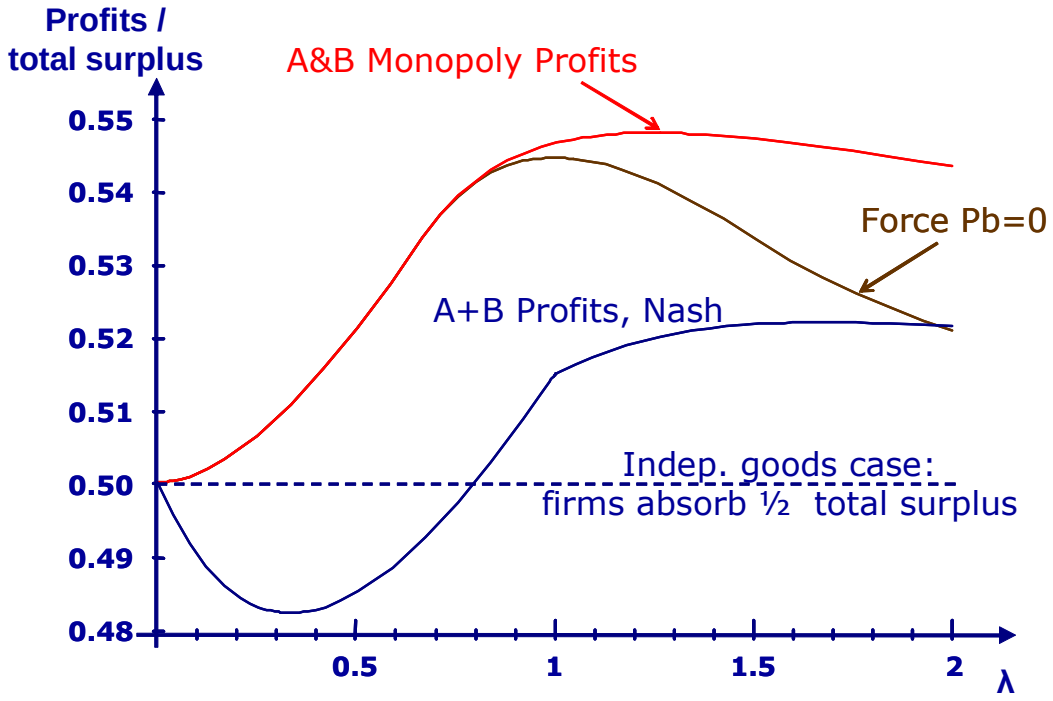


Figure 5: Profits as a Share of Total Social Surplus.

3.2.4 Intuition for why $P_b = 0$

The result that the unified firm would not charge for B until its value becomes large might seem contrary to a basic price-discrimination intuition. The combination of goods A & B can be thought of as a premium version of A. Our result says that the firm will price the premium version the same as the basic version, thereby pooling customers.

One reason there is no opportunity to price discriminate is that the marginal value the customer places on B is independent of how he values A. The firm can't use the price of B to charge more to those customers who have a high value for A. This still leaves open the question as to why the joint monopolist does not set a positive p_b to directly extract surplus generated by B.

Indeed, if it could just the raise the price of B to those customers with the largest values of B this would be a good idea. The problem is that it has to raise the price of B uniformly to all customers. Because customers need to buy A in order to enjoy B, for most customers raising the price of B is identical to raising the price of A. If the price of B is raised from 0 to Δ, then for all customers whose value of B exceeds Δ, they will still buy both A and B or they will buy nothing. For these customers, there is no difference between raising the price of A by Δ or raising the price of B up to Δ.

Hence the desirability of raising the price of B centers around the effect it has on customers with a *low* value for B. For these customers, raising the price of B has a cost without a corresponding gain. Some of these customers switch to only buying A, while others stop buying A & B altogether, imposing a loss. By comparison, raising the price of A leads to increased profits on the inframarginal customers at the usual cost of some decreased sales. It turns out that the lost A sales due to the increase in p_b are exactly half of the lost sales when p_a rises. Thus, if the inframarginal gains on the low-value B, high-value A customer group are at least half the average gains, it will be better to raise p_a rather than p_b. In our model, when p_a=2/3 the inframarginal gains are $1-p_a = 1/3$, which is just half of the average inframarginal gain across all B customers. This helps explain why p_b remains at 0 until p_a reaches 2/3.

3.2.5 Relationship with versioning results

It is straightforward to translate our essential complements model to the case of versioning. We relabel the A and B bundle as the premium good and A alone as the standard good. Since all of the customers value the premium good more than the standard good, customers all lie above the 45-degree line.

Figure 6 illustrates the effect on profits from raising the price of p_b from 0 to Δ while lowering p_a by the same amount, thus holding the price of $A+B$ constant. The monopolist loses $\Delta^2(1-p_a)/\lambda$ on the customers that switch to buying the lower-priced A. It gains

$\Delta^2/2 * p_a/\lambda$ from the new customers attracted to the lower price on A. This is only profitable if $p_a > 2/3$.

Note that if all of the customers were located along the 45-degree line (or any other line), then the gain from new customers would be proportional to $\Delta * p_a$, while the loss would be proportional to $\Delta(1-p_a)$. Thus price discrimination would be profitable whenever $p_a > 1/2$ – which arises for all $\lambda > 0$. It is because preferences are two dimensional that the new customer group is only half as large as the case with one-dimensional preferences. The reason is that half of the marginal customers for A were already buying the A-B bundle. As the price of B rises, these customers transition to buying A alone and thus are not new customers to the monopolist.

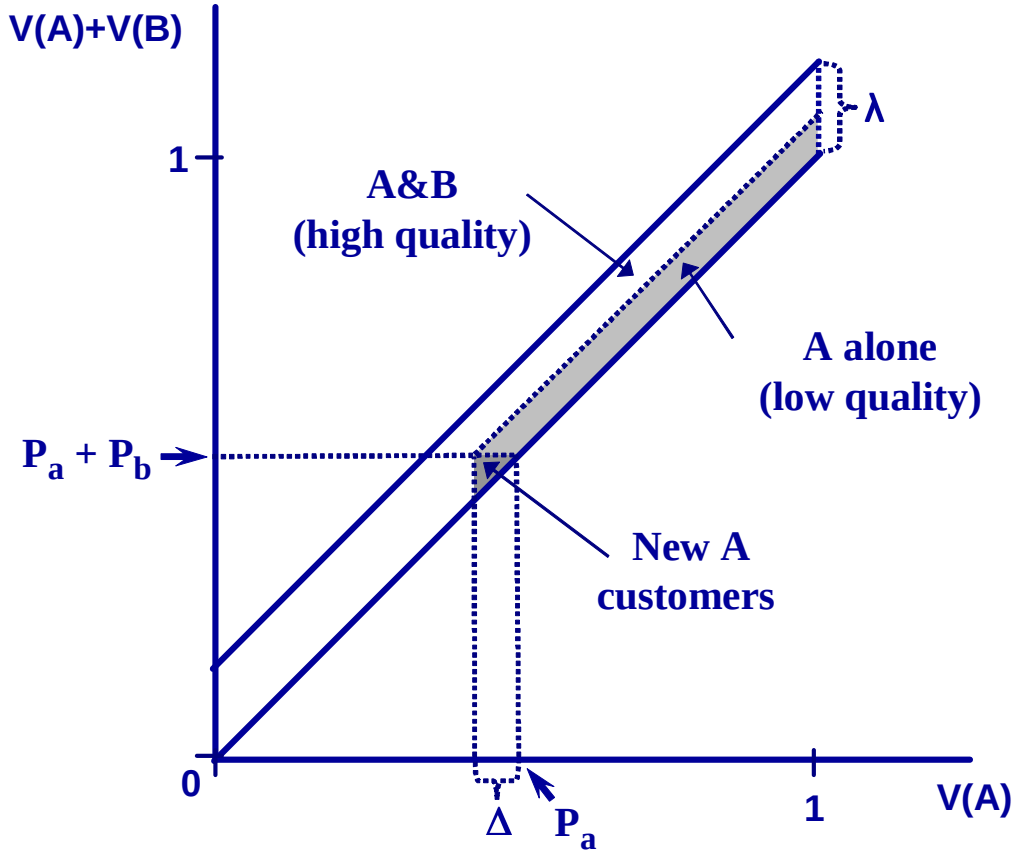


Figure 6

3.2.6 Model Four: Firm B has a positive cost of production

We have assumed that marginal costs are zero. We think this is a reasonable assumption for software and many other information good industries where one-way complementarities arise. Of course, in other circumstances, production costs matter; thus we now turn to

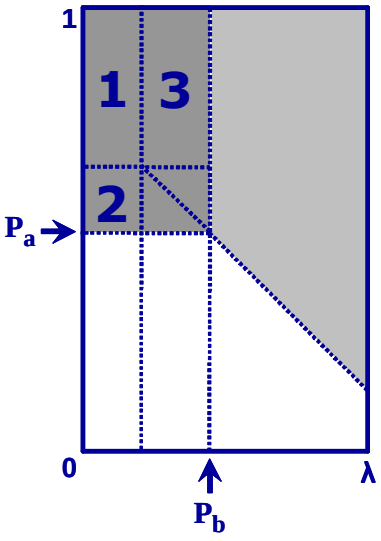
consider how a joint monopolist would price B when costs are a factor.

We find that it is no longer the case that B is optimally priced at cost over a wide range of λs. Instead, we find that B's price equals cost at $\lambda = c$ and then grows very slowly: $\frac{\partial p_b}{\partial \lambda} = 0$ at $\lambda = c$. The positive cost model also allows us to explore the question of whether an A monopolist would want to subsidize or surcharge the B good (when A doesn't own B). We turn to this question after looking at the joint monopoly solution with positive costs.

Suppose that the costs of production are some positive amount, $c > 0$. In this case, an independent B firm would clearly never sell B below c; the question remains whether a firm with a monopoly over both goods would ever chose to subsidize sales of B by selling it for a price below c. The answer is no.

Theorem 2 *A monopolist over both A and B always sets $p_b \geq c$ and $p_a \geq 1/2$.*

The intuition for this result follows from Figure 7.



To add positive costs, note we only need to change profits from added sales of B.

Our three effects are:

1) Firm gains Δ on customers just buying A:

$$\textbf{Gain} = \Delta * (1 - P_a) * P_b$$

2) Firm loses some sales of A:

$$\textbf{Loss} = P_a * \Delta * P_b$$

3) Firm gains some sales of B:

$$\textbf{Gain} = (P_b - c) * \Delta * (1 - P_a)$$

FOC implies $P_b = c * (1 - P_a) / (2 - 3\,P_a)$

$P_a = 1/2$ implies $P_b = c$, and increasing afterwards.
Hence, the joint monopolist never subsidizes B.

Figure 7

Our next question is whether the joint monopolist's optimal strategy of pricing B at 0 for small λ extends to this case. Would the firm price B at c for some range of $\lambda > c$? To the first-order it does: $p_b = c$ when $\lambda = c$ and increases slowly while $\lambda - c$ is small.

Theorem 3 *As λ increases from c, p_b increases slowly: $\frac{\partial p_b}{\partial \lambda} = 0$ at $\lambda = c$.*

Turning to the price of A, in contrast to the zero cost case the price of A initially rises, but slowly.

Corollary 1 *For positive c, as λ increases from c, p_a increases slowly: $\frac{\partial p_a}{\partial \lambda} = 0$ at $\lambda = c > 0$. In contrast, at $c = 0$, $\frac{\partial p_a}{\partial \lambda} = 1/4$.*

This result is not the direct analog of Theorem 1. With positive costs it is no longer the case that there is an interval where the price of B remains at cost. The reason for the difference is that there is now a finite group of customers who value A but to whom it is not efficient to supply B. If we had assumed that the distribution of B values was from c to λ (rather than 0 to λ), then our previous result would have carried through directly. Until λ equals $2/3 + c$, the price of B would remain at c. However, we believe it is more reasonable to assume that customer valuations range from 0 to λ. Thus as λ increases to just beyond c, the set of customers that are interested in B is small, and this provides the monopolist with an incentive to charge above cost.

3.2.7 Model Five: Firms A sets two prices, for versions compatible and incompatible with B.

Another strategy that firm A can use to capture surplus is to directly influence the price of B. It can do this in either direction. The monopolist can offer a discount to its consumers who choose to buy B or the monopolist can impose a surcharge on its consumers who choose to buy B. To offer a discount, the A monopolist could provide a coupon with each purchase of A that entitles the buyer to get some amount off good B. A motivation for doing this is that if the firm were a joint monopolist, A would be giving B away (at least for small values of λ). From A's perspective, the price of B is too high and the discount coupon will lower the price of B and thereby make the purchase of A more attractive.

Alternatively, the monopolist could impose a surcharge on B. In order to add a surcharge to the price of B, the A monopolist can create two versions of its good, one which is compatible with B and one which is not. By charging more for the compatible version, this has the effect of adding a surcharge to the price of B.

The results below show that A would never want to subsidize B. Rather, firm A always finds it optimal to add a surcharge to the price of B.

To consider the question of surcharges versus subsidies, we introduce the notation that Firm A charges p_a for the incompatible version of A and p_{ac} for the compatible versions of A. When $p_{ac} > p_a$, the only consumers that buy the more expensive compatible version

are those who also plan to buy B. In contrast, when $p_{ac} < p_a$, we interpret this as the case of a discount coupon. Here we "force" the customers buying the compatible version to also buy B. Thus in both cases, the compatible version is always bought alongside good B and the incompatible version is always bought alone.

Theorem 4 *The A firm always charges strictly more for the compatible version.*

The general intuition from the bundling literature (MacAfee et. al. 1989) is that a monopolist will generally wish to sell a bundle at discount relative to the individual prices. But this intuition for a bundle discount relies on the firm owning both goods A and B. Here the monopolist only owns A.

The intuition for why the monopolist wants to impose a surcharge on B is based on price discrimination. Imagine that the monopolist could charge a different price for A depending on the customer's valuation for B. Then the A monopolist would want to charge a higher price for customers with high B valuations. The reason is that as the value of B increases, there are more customers buying the bundle and thus more inframarginal customers. Thus the value of raising the price to these customers is larger, while the cost of raising the price (the incremental lost customer) remains the same.

In Figure 8, we graph the prices that firms A and B would charge for B and the compatible and incompatible versions of A, when consumer valuations are distributed uniformly.

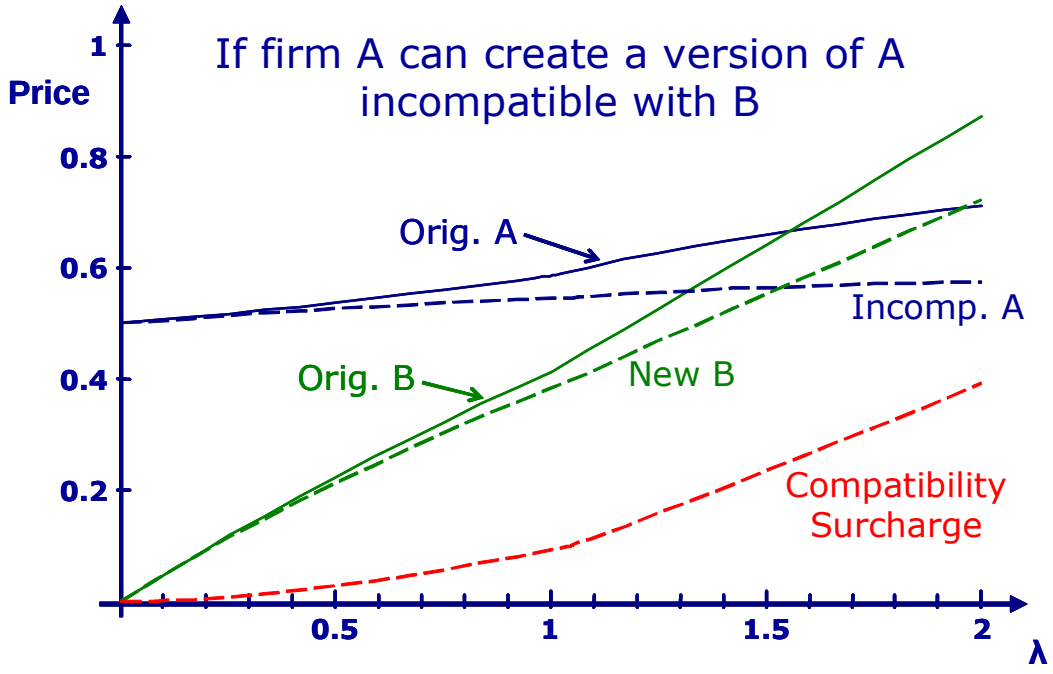


Figure 8

3.2.8 Model Six: Firms A and B play a Stackelberg price game with A moving first

Another option for the monopolist to capture surplus is to move first and preemptively set the price of A. The goal is to induce firm B to lower its price. This is done by raising the price of A. While moving first does increase profits of A, the effect is relatively small. We graph these prices below in Figure 9; the equations are in the appendix.

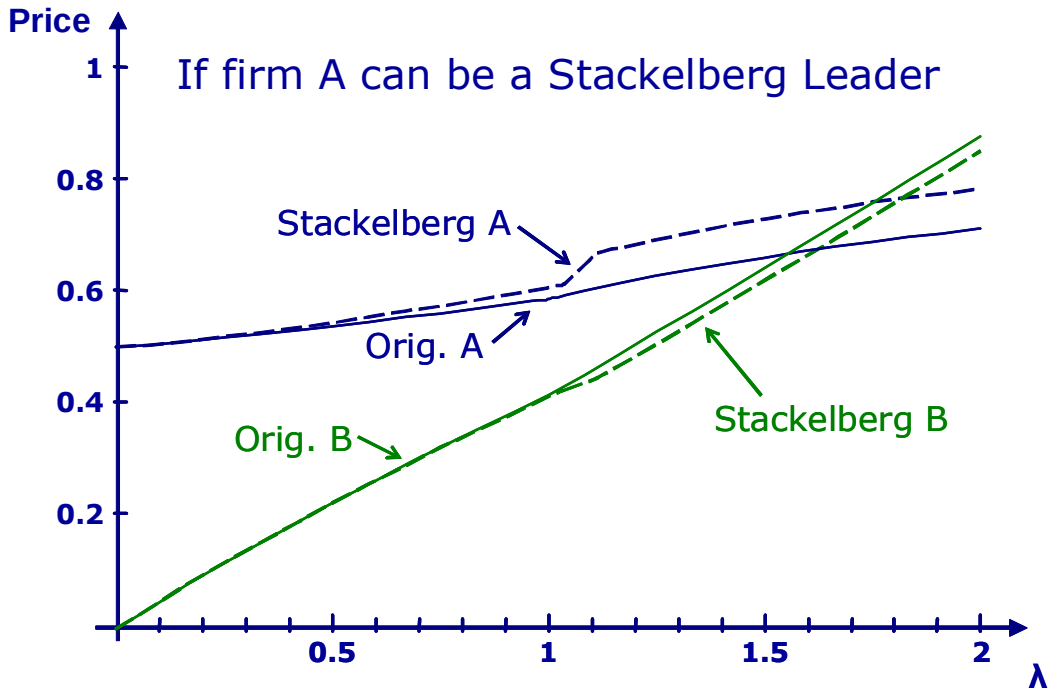


Figure 9

From the graph it is clear that firm A is unable to lower the price of B very much by being a price leader. Indeed when $\lambda < 1$, the Stackelberg prices are nearly indistinguishable from the simultaneous-move Nash prices.

3.2.9 Model Seven: Firms A can degrade the value of competing B products by θ

If moving first does not allow firm A to significantly alter prices, another strategy is to introduce a B good while discouraging the use of B goods manufactured by other parties. This could be accomplished by making it difficult for other firms to build compatible add-ons. An example of this is a high-speed internet provider degrading the packet quality of other-party supplied internet telephone service, while providing priority to A's internet

telephony service.

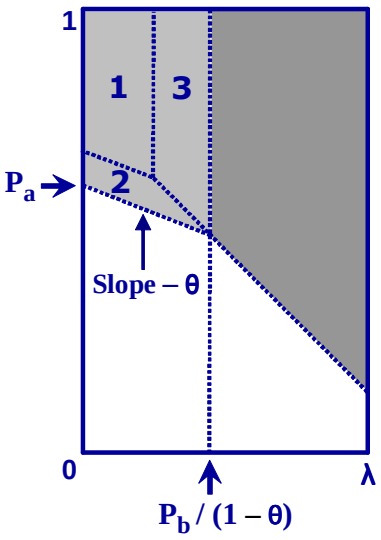
Here we assume that there are two versions of good B. The degraded $\underline{B}$ has value discounted by $\theta < 1$ and is priced at $p_{\underline{b}}$ and the premium B product is priced at p_b. Our main question is whether the A firm would ever *purposely* degrade the quality of rival B products. If θ were a choice variable for firm A, would it have an incentive to lower θ? We first examine a situation where firm A does not produce a B product and show that if B was priced above cost, firm A would want to enter with a premium (undegraded) B and drive the price of $\underline{B}$ down to 0. Formally,

Theorem 5 *If $\underline{B}$ is non-competitively supplied so that $p_{\underline{b}} > 0$, then for λ low enough (as long as $p_a \leq 2/3$), firm A will profit from driving the price $p_{\underline{b}}$ to 0.*

We next show that once $p_{\underline{b}} = 0$, for λ low enough, the profit-maximizing price of the undegraded B is 0. This implies that firm A has no incentive to purposely lower θ in the first place, since it will never be optimal to take any of its profits through the price of B. Formally:

Theorem 6 *If $\underline{B}$ is competitively supplied so that $p_{\underline{b}} = 0$, then for λ low enough (as long as $p_a \leq 2/3$), firm A will charge 0 for B.*

The intuition for this theorem is provided in Figure 10.



Consider lowering P_b by some small Δ and raising P_a by Δ so as to hold $P_a + P_b$ constant.

1) Firm gains Δ on customers just buying A:

$$\Delta * [1 - P_a + (\theta P_b) / 2(1 - \theta)] * P_b / (1 - \theta)$$

2) Firm loses some sales of A:

$$P_a * \Delta * P_b / (1 - \theta)$$

3) Firm gains some sales of B:

$$P_b * \Delta / (1 - \theta) * [1 - P_a + (\theta P_b) / (1 - \theta)]$$

This raises profits if:

$$\Delta * P_b / (1 - \theta) * [2 - 3P_a + 3(\theta P_b) / 2(1 - \theta)] > 0$$

Hence, the optimal $P_b = 0$ when $P_a = 2/3$.

Figure 10

3.3 Extensions to General Distributions

We now show that Theorem 1 extends to general value distributions when consumer valuations for A and B are independent. Consider a setting where a consumer's value v_a for good A is described by the CDF $F(\cdot)$ and v_b by $G(\cdot)$ where both F and G are distributed on $[0,1]$. Assume that consumer valuations for A and B are distributed independently, where both densities are smooth with full support on $[0,1]$. Now consider a set of distributions for v_b indexed by λ such that:

$$\forall \lambda \in (0,1] : G_\lambda(x) = G(\frac{1}{\lambda}x). \tag{9}$$

Denote the optimal prices charged by a monopolist facing valuations $F(\cdot)$ and $G_\lambda(\cdot)$ by $p_a(\lambda)$ and $p_b(\lambda)$. Then we have the following generalization of our main result:

Theorem 7 *If one firm is a monopoly over both goods A and B, $\forall\, G(\cdot), \exists\, U > 0$ such that for all $\lambda \in (0, U], p_b(\lambda) = 0$.*

As the range of possible values for B becomes small (relative to the range for A), we find that the joint monopolist will quite generally set the price of B at zero and earn all of its profits through A. Although general densities are allowed, the result requires that both densities are smooth with full support.

Without this smoothness assumption, it is possible to construct a simple counterexample. Consider the case where A is uniform on $[0,1]$ and B has a 2-point distribution, half the consumers value B at zero and half value B at λ.[7] In that case, the optimal pricing is $p_a = 1/2$ and $p_b = \lambda/2$. The consumers who don't value B are sold A at the regular monopoly price and the half that value B at λ pay $\lambda/2$ for the add on. This leads to profits of $(1 + \lambda + \lambda^2/2)/4$. In contrast, when $p_b = 0$, the highest profits arise with $p_a = 1/2 + \lambda/4$ and these profits are slightly lower at $(1 + \lambda + \lambda^2/4)/4$. This two-point distribution is very much a knife-edge case; even for U-shaped distributions arbitrarily close to the two-point distribution, it is still the case that the optimal $p_b = 0$ as the range of B values becomes small.

Our earlier results on subsidies and surcharges also extend to the general distribution case. Suppose firm A is a monopolist over the production of good A and firm B is a monopolist over good B. Further, suppose that firm A can charge two prices, p_a for a

[7] We thank Dmitri Kuksov for suggesting this example.

baseline A, and $p_{ac} = p_a + \delta$ for a version of A that is compatible with B (that is, charge a different price for consumers who also wish to buy good B). Consumer valuations are again distributed with smooth CDFs F and G with full support on $[0, 1]$ and $[0, \lambda]$ respectively. We then have:

Theorem 8 *If F satisfies the hazard rate condition that $\frac{f(x)}{1-F(x)}$ is strictly increasing in x, then the A firm will always charge more for its compatible version, setting $p_{ac} > p_a$.*

4 Conclusion

The interaction between an essential good and its complement is naturally lopsided. And yet in the Nash pricing game, the complementary product is able to capture over half of the surplus it creates.

The greatest challenge for the complementor firm arises if A is able to enter the B market. While the A firm is not able to earn any profits in the B market, it doesn't have to. Firm A can raise the price of A in response to the reduced price of B. The surprising result is that giving away B leads to joint profit maximization under a range of conditions. This is a problem for a B firm hoping to be bought out by A. There is no incentive for A to purchase B in order to set its price optimally. In these cases, the most A is willing to pay is its cost of entry.

If A is not able to enter the B market, there is only a small gain to setting the price of A before B. There is a larger gain from creating two versions of A, one compatible and one incompatible with B, and charging more for the compatible version.

The multiple avenues for A to capture B's surplus highlights the challenges facing a firm whose product depends on an essential good.

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5 Appendix

Proposition 1 *When all customers value A at* 1 *and B at* λ*, any pair of non-negative prices* (p_a, p_b) *is a Nash equilibrium if and only if* $p_a + p_b = 1 + \lambda$ *and* $p_b \leq \lambda$*.*

Proof. If $p_a + p_b = 1 + \lambda$ and $p_b \leq \lambda$ then neither firm can unilaterally increase profits. Lowering price does not increase demand; raising price leads to zero demand and zero profits. If $p_a + p_b < 1 + \lambda$ then firm A can raise profits by increasing p_a, as its demand will remain unchanged. If $p_a + p_b > 1 + \lambda$, then either $p_a > 1$ or $p_b > \lambda$. If $p_a > 1$, then firm A will increase its profits up from zero by setting $p_a = 1$. If $p_a \leq 1$ and $p_b > \lambda$, then firm B will increase its profits up from zero by setting $p_b = \lambda$. Finally, given that $p_a + p_b = 1 + \lambda$ in any Nash Equilibrium, if $p_b > \lambda$ then consumers will only be buying A. Thus firm B could increase its profits up from zero by setting $p_b = \lambda$. ■

Proposition 2 *With A uniform on* $[0, 1]$ *and B at* $\lambda \leq 1/2$*, there is a unique Nash equilibrium. Firm A charges* $1/2$ *and firm B charges* λ*.*

Proof. First we show it is an equilibrium, then show it is unique.

If firm B charges λ, then A faces a demand curve $1 - p_a$ and A's optimal response is to charge $1/2$. If A charges $1/2$, then firm B faces a demand curve of $1+\lambda-1/2-p_b = 1/2+\lambda-p_b$ for $p_b \leq \lambda$ and zero demand if $p_b > \lambda$. (If firm B charges more than λ, no customer will buy B as an add on.)

The optimal price for B is $p_b = Min[(1/4 + \lambda/2), \lambda]$. The min is λ provided $\lambda \leq 1/2$. This demonstrates that $(1/2, \lambda)$ is a Nash Equilibrium for $\lambda \leq 1/2$.

To see that this equilibrium is unique, observe that were A to charge some other prices than $1/2$, firm B would then face a demand curve of $1 + \lambda - p_a - p_b$ which leads to an optimal price of $p_b = min[(1 + \lambda - p_a)/2, \lambda]$.

If $p_a < 1/2$, B will settle at λ which leads to $p_a = 1/2$. Thus the only possible alternative equilibria arise when $p_a \geq 1/2$ and $p_b = (1 + \lambda - p_a)/2$.

In that case, the residual demand for A is $1 + \lambda - p_b - p_a$ Thus the profit-maximizing price for A is

$$p_a = (1 + \lambda - p_b)/2 = (1 + \lambda - (1 + \lambda - p_a)/2)/2. \tag{10}$$

Simplifying leads to $p_a = (1 + \lambda)/3 \leq 1/2$ as $\lambda \leq 1/2$. Thus p_a can never be greater than $1/2$ which leads p_b to λ, demonstrating the uniqueness of the Nash equilibrium. ■

Lemma 1 *When $\lambda = 1$ there is a closed-form solution: $p_a = 2 - \sqrt{2}$ and $p_b = \sqrt{2} - 1$*

Proof. Firm A's first-order conditions are:

$$\left\{ \begin{array}{ccc} p_b^2 + \lambda(1 - 4p_a + \lambda) - 2p_b = 0 & \text{when} & p_a + p_b \geq \lambda \\ p_a(3p_a + 4p_b) - 2\lambda = 0 & \text{when} & p_a + p_b \leq \lambda \end{array} \right\}, \tag{11}$$

and firm B's first-order conditions are

$$\left\{ \begin{array}{ccc} 3p_b^2 + 4p_b(p_a - 1 - \lambda) + \lambda(2 - 2p_a + \lambda) = 0 & \text{when} & p_a + p_b \geq \lambda \\ p_a^2 + 4p_b - 2\lambda = 0 & \text{when} & p_a + p_b \leq \lambda \end{array} \right\}. \tag{12}$$

For $\lambda = 1$, firm A's first order conditions simplify to

$$\begin{array}{ccc} p_b^2 + (2 - 4p_a) - 2p_b = 0 & \text{when} & p_a + p_b \geq 1 \\ p_a(3p_a + 4p_b) - 2 = 0 & \text{when} & p_a + p_b \leq 1 \end{array}, \tag{13}$$

and firm B's first-order conditions simplify to:

$$\begin{array}{ccc} 3p_b^2 + 4p_b(p_a - 2) + (3 - 2p_a) = 0 & \text{when} & p_a + p_b \geq 1 \\ p_a^2 + 4p_b = 2 & \text{when} & p_a + p_b \leq 1 \end{array}. \tag{14}$$

Looking first at the case where $p_a + p_b \geq 1$, we have three solutions for p_a and p_b. The solutions are $(p_a = 9/16, p_b = 3/2)$, $(p_a = 2 - \sqrt{2}, p_b = \sqrt{2} - 1)$, and $(p_a = 2 + \sqrt{2}, p_b = -1 - \sqrt{2})$. The first solution violates $p_b \leq \lambda$, and the third condition has a negative value of p_b. The middle solution exactly satisfies $p_a + p_b = 1$. This shows there is a unique solution for $p_a + p_b \geq 1$.

Looking next at the case where $p_a + p_b \leq 1$, again we have three solutions for p_a and p_b. The solutions are $(p_a = -1, p_b = 1/4)$, $(p_a = 2 - \sqrt{2}, p_b = \sqrt{2} - 1)$, and $(p_a = 2 + \sqrt{2}, p_b = -1 - \sqrt{2})$. The first solution violates $p_a \geq 0$, and the third condition has a negative value of p_b. The middle solution exactly satisfies $p_a + p_b = 1$. This shows there is a unique solution for $p_a + p_b \leq 1$. These two solutions coincide. ■

Figure Notes 1 (Support for Figure 2)

Simplifying the first-order conditions from equations 11 and 12, we find that for $p_a + p_b \geq \lambda$, p_a and p_b solve equations:

$$p_a = \frac{1}{4\lambda}(p_b^2 + 2\lambda - 2p_b\lambda + \lambda^2) \tag{15}$$

$$p_b = \frac{1}{3}(2 - 2p_a + 2\lambda - \sqrt{4 + 4p_a^2 + \lambda(2 + \lambda) - 2p_a(4 + \lambda)}) \tag{16}$$

and that for $p_a + p_b \leq \lambda$, p_a and p_b solve the equations:

$$p_a = \frac{1}{3}(-2p_b + \sqrt{4p_b^2 + 6\lambda}), \tag{17}$$

$$p_b = \frac{1}{4}(-p_a^2 + 2\lambda). \tag{18}$$

We know from lemma 1 that $p_a + p_b = \lambda$ at $\lambda = 1$. Graphing these solutions, we find that $p_a + p_b \geq \lambda$ if and only if $\lambda \leq 1$. Figure 2 graphs the profits corresponding to these solutions.

Proposition 3 *Equilibrium prices are:*

$$p_a = \left\{ \begin{array}{ll} \frac{1}{4}\lambda + \frac{1}{2} & when \quad \lambda \leq \frac{2}{3} \\ \sqrt{\frac{2}{3}\lambda} & when \quad \lambda \geq \frac{2}{3} \end{array} \right\}, \tag{19}$$

$$p_b = 0. \tag{20}$$

Proof. To solve this case, we look at firm A's first-order conditions where $p_b = 0$. Prices are the solution to firm A's first-order conditions below:

$$\begin{array}{ll} \lambda(2 - 4p_a + \lambda) = \lambda & \text{when} \quad \lambda \leq 1 \\ p_a(3p_a) = 2\lambda & \text{when} \quad \lambda \geq 1 \end{array} . \tag{21}$$

■

Lemma 2 *If the optimal $p_a \leq 2/3$, then the optimal $p_b = 0$.*

Proof. Assume $p_b > 0$. Consider lowering p_b by Δ and raising p_a by Δ, where Δ is small. This has three first-order effects. First we charge more to those who continue to buy only A. The gain is:

$$\Delta * p_b * (1 - p_a). \tag{22}$$

Second, we lose some customers who used to buy A. The loss is

$$-p_a * p_b * \Delta. \tag{23}$$

Third, we sell B to more customers. The gain is

$$p_b * \Delta * (1 - p_a). \tag{24}$$

Combining these effects, our price change has a first-order effect on profits of:

$$\Delta * p_b * (2(1 - p_a) - p_a), \tag{25}$$

which is positive when $2 - 3p_a > 0$.

This implies that so long as $p_a < 2/3$, the monopolist would increase its profits by lowering p_b and raising p_a by the same amount. But if for any positive p_b the firm would want to lower p_b, then the optimal p_b must be 0. By continuity $p_b = 0$ when $p_a = 2/3$. ■

Theorem 1 *The optimal monopoly prices are given by:*

$$p_a = \left\{ \begin{array}{ccc} \frac{1}{4}\lambda + \frac{1}{2} & when & \lambda \leq \frac{2}{3} \\ \frac{2}{3} & when & \lambda \geq \frac{2}{3} \end{array} \right\}, \tag{26}$$

$$p_b = \left\{ \begin{array}{ccc} 0 & when & \lambda \leq \frac{2}{3} \\ \frac{1}{2}\lambda - \frac{1}{3} & when & \lambda \geq \frac{2}{3} \end{array} \right\}. \tag{27}$$

Proof. Lemma two shows that for all $p_a \leq 2/3$, the optimal $p_b = 0$. As $\lambda \to 0$, the optimal $p_a \to 1/2$. (This follows because the B market becomes irrelevant and the monopoly prices to maximize profits on A alone.) Since $p_a(0) = 1/2$, for some range of λ around 0, the optimal $p_a < 2/3$, so by our lemma the optimal p_b will be 0 in that range. To demarche that range, observe that so long as $p_b = 0$, the first-order conditions that give the optimal p_a are:

$$\begin{array}{rcl} 1 - 2p_a + \frac{\lambda}{2} = 0 & \text{for} & p_a > \lambda \\ \lambda - \frac{3}{2}p_a^2 = 0 & \text{for} & p_a < \lambda \end{array}. \tag{28}$$

The solution to the second condition is $p_a = \sqrt{\frac{2}{3}\lambda}$, which violates $p_a < \lambda$ when $0 < \lambda < 2/3$. The solution to the first condition is $p_a = \frac{1}{4}\lambda + \frac{1}{2}$, which satisfies $p_a < \lambda$ when $\lambda < 2/3$, and $p_a = \frac{2}{3}$ at $\lambda = 2/3$. Also note that $\frac{1}{4}\lambda + \frac{1}{2} < 2/3$ for $\lambda < 2/3$, justifying the assumption that $p_b = 0$. Therefore $p_a = \frac{1}{4}\lambda + \frac{1}{2}$ and $p_b = 0$ is the unique solution while $\lambda \leq \frac{2}{3}$.

Turning to the case of $\lambda > \frac{2}{3}$, it is possible that the optimal $p_b > 0$. If so, then we are at an interior solution for p_a and p_b and both first-order conditions are satisfied. As in the Nash case, the geometry of monopoly profits changes depending on whether $p_a + p_b \lessgtr \lambda$. The profits the joint monopoly will earn are:

$$\frac{1}{\lambda}(p_a(\lambda - (p_a * p_b) - \frac{1}{2}p_a^2) + p_b(\lambda - p_b - \frac{1}{2}p_a^2)), \text{ and} \tag{29}$$

$$\frac{1}{\lambda}(p_a(\lambda - (1 - p_a) + \frac{1}{2}(\lambda - p_b)^2) + p_b((\lambda - p_b)(1 - p_a) + \frac{1}{2}(\lambda - p_b)^2)) \tag{30}$$

for $p_a + p_b < \lambda$ and $p_a + p_b > \lambda$, respectively. Differentiating these with respect to p_a and

p_b leads to two first-order conditions per case. The first-order conditions for p_a are:

$$\begin{array}{rll} 3p_a(p_a+2p_b)=2\lambda & \text{when} & p_a+p_b<\lambda \\ 3p_b^2+\lambda(2-4p_a+\lambda)=4p_b\lambda & \text{when} & p_a+p_b>\lambda \end{array}. \tag{31}$$

The first-order conditions for good B are:

$$\begin{array}{rll} 3p_a^2+4p_b=2\lambda & \text{when} & p_a+p_b<\lambda \\ p_b(6p_a+3p_b-4)+\lambda(2+\lambda)=4(p_a+p_b)\lambda & \text{when} & p_a+p_b>\lambda \end{array}. \tag{32}$$

When $p_a+p_b\geq\lambda$, the first-order conditions are only satisfied if:

$$p_a=\frac{2}{3} \text{ and } p_b=\frac{1}{3}(2\lambda+\sqrt{2\lambda+\lambda^2}), \tag{33}$$

$$p_a=\frac{2}{3} \text{ and } p_b=\frac{1}{3}(2\lambda-\sqrt{2\lambda+\lambda^2}), \tag{34}$$

or if

$$p_a=\frac{1}{4}\lambda+\frac{1}{2} \text{ and } p_b=0. \tag{35}$$

Only the first solution does not violate $p_a+p_b\geq\lambda$ when $\lambda>2/3$.

When $p_a+p_b\leq\lambda$, the first-order conditions are only satisfied if:

$$p_a=\sqrt{\frac{2}{3}\lambda} \text{ and } p_b=0, \tag{36}$$

or if:

$$p_a=\frac{2}{3} \text{ and } p_b=\frac{1}{2}\lambda-\frac{1}{3}. \tag{37}$$

Both solutions satisfy $p_a+p_b\leq\lambda$ when $\lambda>2/3$. Therefore we are left with three candidate solutions. Calling these solutions 1, 2 and 3 respectively, we have corresponding profits Π_1,Π_2, and Π_3 of:

$$\Pi_1 = \frac{1}{27}\lambda(6-2\sqrt{\lambda(2+\lambda)})+\lambda(3+\lambda-\sqrt{\lambda(2+\lambda)} \tag{38}$$

$$\Pi_2 = (\frac{2}{3}\lambda)^{3/2} \tag{39}$$

$$\Pi_3 = -\frac{1}{27}+\frac{1}{12}\lambda(4+3\lambda). \tag{40}$$

To establish which solution maximizes profits, we compare their profits directly. Comparing solutions 1 and 2,

$$\Pi_1<\Pi_2 \Leftrightarrow \lambda(6(1-\sqrt{6\lambda})+\lambda^2-(2+\lambda)\sqrt{\lambda(2+\lambda)}+3\lambda)<0, \tag{41}$$

which is true for all $\lambda > 2/3$. ($\Pi_1 < \Pi_2$ for all λ greater than approximately 0.13.) Therefore only the second and third solutions are possible optima. Looking at Π_2 and Π_3,

$$\Pi_2 < \Pi_3 \Leftrightarrow 4 < 3\lambda(12 - 8\sqrt{6\lambda} + 9\lambda) \Leftrightarrow \lambda > 2/3. \tag{42}$$

This confirms that our third solution is unique for all $\lambda > 2/3$, and proves our theorem. ■

Theorem 2 *A monopolist over both A and B always sets $p_b \geq c$ and $p_a \geq 1/2$.*

Proof. As in the case without costs, consider lowering p_b by Δ and raising p_a by Δ, where Δ is small. This has three first-order effects.

First we charge more to those who continue to buy only A. The gain is

$$\Delta * p_b * (1 - p_a). \tag{43}$$

Second, we lose some customers who used to buy A. The loss is

$$-p_a * p_b * \Delta. \tag{44}$$

Third, we sell B to more customers. The gain is

$$(p_b - c) * \Delta * (1 - p_a). \tag{45}$$

Combining these effects, our price change has a first-order effect on profits of

$$\Delta * \{(p_b(2 - 3p_a) - c(1 - p_a)\}. \tag{46}$$

Since Δ can be either positive or negative, we know that

$$p_b = \frac{c(1 - p_a)}{2 - 3p_a}. \tag{47}$$

Note that at $p_a = 1/2$ we have $p_b = c$, and $p_b > c$ if and only if $p_a > 1/2$.[8] Therefore, either $p_a \geq 1/2$ and $p_b \geq c$ or $p_a < 1/2$ and $p_b < c$. To establish our theorem we only need show that the monopolist would never choose the second combination, namely $p_a < 1/2$ and $p_b < c$.

But the monopolist would never want to charge a price below 1/2 for A when it is also subsidizing B. It would do better by raising the price of A up to 1/2. This raises profits on A. The gain from existing customers is more than 1/2 (as A's market area exceeds 1/2

[8] Observe that p_b is monotonic in p_a.

all the way to the price of 1/2) and the loss from reduced demand is less than p_a, which is always less than 1/2. Thus the net effect on A is positive. A side effect of increasing the price of A is that sales of B will fall, but as the monopolist was losing money on each B sale, that, too, increases profits. ■

Theorem 3 *As λ increases from c, p_b increases slowly: $\frac{\partial p_b}{\partial \lambda} = 0$ at $\lambda = c$.*

Proof. We prove this by inspection of the first-order conditions. The first-order condition for p_a leads to:

$$p_a = \frac{1}{2} + \frac{(\lambda - p_b)^2}{4\lambda} - \frac{(p_b - c)}{2\lambda}(\lambda - p_b), \text{ and} \tag{48}$$

$$\frac{dp_a}{d\lambda} = \frac{\lambda^2 + 2cp_b - 3p_b^2}{4\lambda^2} + (\frac{3p_b - c}{2\lambda} - 1)\frac{dp_b}{d\lambda}. \tag{49}$$

From equation 47, we know that

$$\frac{dp_b}{d\lambda} = \frac{c}{(2 - 3p_a)^2}\frac{dp_a}{d\lambda}. \tag{50}$$

Hence

$$\frac{dp_a}{d\lambda}(1 - \frac{3p_b - c - 2\lambda}{2\lambda}\frac{c}{(2 - 3p_a)^2}) = \frac{\lambda^2 + 2cp_b - 3p_b^2}{4\lambda^2}. \tag{51}$$

From our previous theorem we know that $p_b \to c$ as $\lambda \to c$. Therefore as $\lambda \to c$,

$$\frac{dp_a}{d\lambda}(1 - \frac{c - \lambda}{\lambda}\frac{c}{(2 - 3p_a)^2}) = \frac{\lambda^2 - c^2}{4\lambda^2}. \tag{52}$$

Now observe that if $c > 0$, then as $\lambda \to c$, $\frac{dp_a}{d\lambda} = 0$, while for $c = 0$, $\frac{dp_a}{d\lambda} = 1/4$. Hence, from equation 50, it follows that $\frac{\partial p_b}{\partial \lambda} = 0$ for both $c > 0$ and $c = 0$. ■

Corollary 1 *For positive c, as λ increases from c, p_a increases slowly: $\frac{\partial p_a}{\partial \lambda} = 0$ at $\lambda = c > 0$. At $c = 0$, $\frac{\partial p_a}{\partial \lambda} = 1/4$.*

Proof. The corollary follows from the proof of theorem 3. ■

Theorem 4 *The A firm always charges strictly more for the compatible version.*

Proof. Suppose we are in the coupon case so that we give a discount to people who buy both goods, i.e. $p_{ac} < p_a$. Consider lowering p_a until it equals p_{ac}. This has two effects. First, we charge less to those who used to buy the incompatible A. This costs:

$$(p_a - p_{ac}) * (1 - p_a) * (p_b + p_{ac} - p_a). \tag{53}$$

Second, we gain some customers who now buy the incompatible A. This increases profits by:

$$p_{ac} * (p_a - p_{ac}) * (p_b + \frac{p_{ac} - p_a}{2}). \tag{54}$$

Some customers switch from buying the compatible version to the incompatible version of A, but this is only a loss to firm B. Note that $(p_b + \frac{p_{ac}-p_a}{2}) > (p_b + p_{ac} - p_a)$, so this increases A's profits if $p_{ac} > (1-p_a)$. But this must be true since $p_{ac} > 1/2$. Thus discounts are never optimal.

Now, suppose $p_a = p_{ac}$. Consider lowering p_a by Δ. This has two first-order effects. First, firm A charges less to those who used to buy only A. This costs:

$$\Delta * p_b * (1 - p_{ac}). \tag{55}$$

Second, firm A gains customers who now buy A. This increases profits by:

$$p_{ac} * p_b * \Delta. \tag{56}$$

Firm A loses Δ on customers who switch from buying the compatible version to buying the incompatible version, but this is a second-order effect.

Combining these effects, the price change has a first-order effect on profits of:

$$\Delta * p_b * (p_{ac} - (1 - p_{ac})), \tag{57}$$

which is positive as long as $p_{ac} > 1/2$.

This shows that firm A would never subsidize a version that is compatible with B, and in fact always wants to apply a surcharge. Prices solve:

$$p_a = \left\{ \begin{array}{ll} \frac{1}{4\lambda}(p_{ac} + p_b - \lambda)(3p_{ac} + p_b - \lambda) + \frac{1}{2} & \text{when} \quad \lambda \leq \frac{126}{121} \\ \frac{1}{3}(-3p_{ac} - 2p_b + \sqrt{(3p_{ac} + 2p_b)^2 + 6\lambda}) & \text{when} \quad \lambda \geq \frac{126}{121} \end{array} \right\}, \tag{58}$$

$p_{ac} =$

$$\begin{array}{ll} \frac{1}{3}(2 - 3p_a - 2p_b + 2\lambda - \sqrt{4 + 3p_a(3p_a - 4) + p_b^2 - 2p_b(1+\lambda) + \lambda(2+\lambda)}) & \text{when} \quad \lambda \leq \frac{126}{121}, \\ \text{and} \quad \frac{1}{4}(-3p_a^2 - 2p_b + 2\lambda) & \text{when} \quad \lambda \geq \frac{126}{121}, \end{array} \tag{59}$$

and $p_b =$

$$\begin{array}{ll} \frac{1}{3}(2 - 2p_a - 2p_{ac} + 2\lambda - \sqrt{4 + 4p_a^2 - 2p_{ac} + 2p_a(p_{ac} - 4 - \lambda) + (p_{ac} - \lambda)^2 + 2\lambda}) & \text{when} \quad \lambda \leq \frac{126}{121}, \\ \text{and} \quad \frac{1}{4}(-p_a^2 - 2p_{ac} + 2\lambda) & \text{when} \quad \lambda \geq \frac{126}{121}. \end{array} \tag{60}$$

■

Figure Notes 2 (Support for Figure 8)

In some cases the closed-form solutions for p_a and p_b are more than several pages. These cases are omitted and available from the authors upon request. Optimal prices solve:

$$p_a = \left\{ \begin{array}{ccc} \text{omitted} & \text{when} & \lambda \leq x \\ \text{omitted} & \text{when} & x \leq \lambda \leq \frac{10}{9} \\ p_a^3 + \lambda = \frac{3}{2}p_a^2 + p_a\lambda & \text{when} & \lambda \geq \frac{10}{9} \end{array} \right\}, \tag{61}$$

and

$$p_b = \left\{ \begin{array}{ccc} \frac{1}{3}(2 - 2p_a + 2\lambda - \sqrt{4 - 8p_a + 4p_a^2 + 2\lambda - 2p_a\lambda + \lambda^2}) & \text{when} & \lambda \leq \frac{10}{9} \\ \text{omitted} & \text{when} & x \leq \lambda \leq \frac{10}{9} \\ \frac{1}{4}(-p_a^2 + 2\lambda) & \text{when} & \lambda \geq \frac{10}{9} \end{array} \right\}. \tag{62}$$

where x solves $2x^3 - 19x^2 + 60x - 44 = 0$ ($x \approx 1.036$).

Theorem 5 *If $\underline{B}$ is non-competitively supplied so that $p_{\underline{b}} > 0$, then for λ low enough (as long as $p_a \leq 2/3$), firm A will profit from driving the price $p_{\underline{b}}$ to 0.*

Proof. Note that consumers who value good B at value $\frac{p_b - p_{\underline{b}}}{1-\theta}$ are indifferent between buying $\underline{B}$ and B. Therefore we can restrict attention to the case where $\frac{p_b - p_{\underline{b}}}{1-\theta} > \frac{p_{\underline{b}}}{\theta}$, since if this does not hold, no consumers will buy $\underline{B}$.

Consider lowering $p_{\underline{b}}$ all the way down to 0, lowering p_b by the same amount, and raising p_a by the same amount. This has two effects. First, we lose sales of A from the price increase:

$$-p_a * \frac{1}{2} * \frac{p_{\underline{b}}}{\theta} * p_{\underline{b}}. \tag{63}$$

Second, we gain on those who were buying A and not buying B:

$$p_{\underline{b}} * \frac{p_b - p_{\underline{b}}}{1-\theta} * (1 - p_a + \theta\frac{p_b - p_{\underline{b}}}{1-\theta}). \tag{64}$$

Since $\frac{p_b - p_{\underline{b}}}{1-\theta} > \frac{p_{\underline{b}}}{\theta}$, if we rewrite the second force substituting for $\frac{p_b - p_{\underline{b}}}{1-\theta}$, we make the overall gain smaller. Therefore profits are less than:

$$p_{\underline{b}} * \frac{p_{\underline{b}}}{\theta} * (1 - p_a + \theta\frac{p_{\underline{b}}}{\theta}) - p_a * \frac{1}{2} * \frac{p_{\underline{b}}}{\theta} * p_{\underline{b}}. \tag{65}$$

Simplifying, this is equal to:

$$\frac{p_{\underline{b}}^2}{\theta}(1 + p_{\underline{b}} - \frac{3}{2}p_a). \tag{66}$$

Examining this expression shows that for small p_a, the firm earns positive profits from lowering $p_{\underline{b}}$. In particular, as long as $p_a < \frac{2}{3}$ the firm strictly profits from forcing competition into the market for $\underline{B}$, driving $p_{\underline{b}}$ to 0. Since for small λ, p_a is close to $\frac{1}{2}$, the firm would choose to do so for small λ. ■

Theorem 6 *If $\underline{B}$ is competitively supplied so that $p_{\underline{b}} = 0$, then for λ low enough (as long as $p_a \leq 2/3$), firm A will charge 0 for B.*

Proof. Assume $p_{\underline{b}} = 0$ and $p_b > 0$.

Consider lowering p_b by Δ and raising p_a by Δ. This has three effects. First the firm charge more to those who continue to buy only A:

$$\Delta * \frac{p_b}{1-\theta} * (1 - p_a + \frac{1}{2} p_b \frac{\theta}{1-\theta}). \tag{67}$$

Second, it loses some customers who used to buy A:

$$-p_a * \Delta * \frac{p_b}{1-\theta}. \tag{68}$$

Third, it sells B to more customers:

$$p_b * \frac{\Delta}{1-\theta} * (1 - p_a + p_b \frac{\theta}{1-\theta}). \tag{69}$$

Combining these effects, the effect on profits is:

$$\Delta * \frac{p_b}{1-\theta} * (2(1-p_a) - p_a + \frac{3}{2} p_b \frac{\theta}{1-\theta}), \tag{70}$$

which is positive as long as $2 - 3p_a + \frac{3}{2} p_b \frac{\theta}{1-\theta} > 0$.

In particular, if $p_a < \frac{2}{3}$ then lowering p_b by Δ and raising p_a by Δ raises profits. Since for small λ, p_a is close to $\frac{1}{2}$; hence the firm would set $p_b = 0$. ■

Theorem 7 *If one firm is a monopoly over both goods A and B, $\forall\, G(\cdot), \exists\, U > 0$ such that for all $\lambda \in (0, U], p_b(\lambda) = 0$.*
In words, for any smooth distributions of values for A and B, if the distribution of values for B is compressed enough relative to A, then the monopolist would charge 0 for B.

Proof. Define $p_a(0)$ and $p_b(0)$ as the $\lim_{\lambda \to 0} p_a(\lambda)$ and the $\lim_{\lambda \to 0} p_b(\lambda)$ as λ goes to 0 from above. First, note that $p_b(0) = 0$ follows from that fact that for all λ, $p_b(\lambda) \leq \lambda$. Next,

$0 < p_a(0) < 1$ follows trivially from the fact that $f(\cdot)$ has full support, since profits are 0 at both $p_a = 0$ and $p_a = 1$.

Given these limit conditions, we consider the effect on profits of the following deviation: starting at the optimal prices $p_a(\lambda)$ and $p_b(\lambda)$, lower the price of B to 0, and raise the price of A by $p_b(\lambda)$ so as to keep the sum of the two prices constant. We show that for small enough λ this deviation will increase profits for any $0 < p_b \leq \lambda$. But since $p_b(\lambda)$ was supposed to be profit maximizing, it must be that for λ small enough, $p_b(\lambda) = 0$.

To prove our theorem by contradiction, assume that there exists a smooth $G(\cdot)$ such that $\forall \lambda > 0, p_b(\lambda) > 0$. First, we establish that our proposed deviation is possible. Since $p_a(\lambda)$ is converging to $p_a(0)$ and $p_b(\lambda)$ is converging to 0, we must be able to choose a $U_1 > 0$ such that $\forall \lambda \in (0, U_1]$:

$$0 < p_b(\lambda) \leq \lambda < (1 - p_a(\lambda)), \text{ so that } p_b(\lambda) + p_a(\lambda) \leq 1. \tag{71}$$

We want to do this so that for all $\lambda \in (0, U_1]$ we can consider the deviation of lowering the price of B to 0 and raising the price of A by $p_b(\lambda)$ to offset it. Writing the effect on profits, there are a number of customers who used to purchase (only) good A that now stop purchasing A. They represent lost profits of:

$$Losses = p_a(\lambda) \int_0^{p_b(\lambda)} (F(p_a(\lambda) + p_b(\lambda) - x) - F(p_a(\lambda))) * g(x)dx. \tag{72}$$

Offsetting this loss, there are a number of customers who used to buy only A and now buy both A and B, at the new higher price of A. This represents a gain in profits of:

$$Gains = p_b(\lambda) \int_0^{p_b(\lambda)} (1 - F(p_a(\lambda) + p_b(\lambda) - x)) * g(x)dx. \tag{73}$$

We can bound these expression with simpler expressions for the bounding rectangles either larger or smaller than the exact integrals. Looking first at our gains, we see that:

$$Gains \geq (1 - F(p_a(\lambda) + p_b(\lambda))) * G(p_b(\lambda)) * p_b(\lambda). \tag{74}$$

Adding the corresponding rectangular lower bound, we can bound our gains above and below:

$$1 - F(p_a(\lambda) + p_b(\lambda)) \leq \frac{Gains}{G(p_b(\lambda)) * p_b(\lambda)} \leq 1 - F(p_a(\lambda)). \tag{75}$$

Note that we are looking at the ratio of gains to $G(p_b(\lambda)) * p_b(\lambda)$, which by assumption is strictly positive for all $\lambda > 0$. Intuitively, as λ goes to 0 the change in profits from our

proposed deviation will go to zero; however the ratio of this change to $G(p_b(\lambda)) * p_b(\lambda)$ will converge to a strictly positive sum. This will establish that for small enough λ, our proposed deviation will strictly increase profits.

Now as $\lambda \to 0$, the two bounds in equation 75 converge to the same limit, so we have that:

$$\lim_{\lambda \to 0} \frac{Gains}{G(p_b(\lambda)) * p_b(\lambda)} = 1 - F(p_a(0)). \tag{76}$$

Turning to losses, we will show that subtracting an upper bound of losses from our gains results in no change in profits as λ goes to 0. We then show that for small enough λ, our real losses are strictly less then their upper bound, which establishes our result.

Applying the same rectangular upper bound to our losses as we did to gains, we see that:

$$\frac{Losses}{G(p_b(\lambda)) * p_b(\lambda)} \leq p_a(\lambda) \frac{F(p_a(\lambda) + p_b(\lambda)) - F(p_a(\lambda))}{p_b(\lambda)}. \tag{77}$$

Since we've assumed that f and g have full support, as $\lambda \to 0$ we have:

$$\lim_{\lambda \to 0} \frac{Losses}{G(p_b(\lambda)) * p_b(\lambda)} \leq p_a(0) * f(p_a(0)). \tag{78}$$

Note that when $\lambda = 0$, the first-order condition for the optimality of p_a is:

$$1 - F(p_a(0)) - p_a(0) * f(p_a(0)) = 0. \tag{79}$$

Combining equations 76, 78, and 79, we see that as $\lambda \to 0$, our proposed deviation is weakly a good idea. That is, taking gains, subtracting our upper bound on losses and applying the first-order condition for p_a gives us:

$$\lim_{\lambda \to 0} \frac{Gains - Losses}{G(p_b(\lambda)) * p_b(\lambda)} \geq 0. \tag{80}$$

Recall that to establish our result we need this inequality to be strict; we now show that our losses converge to a number strictly less than their upper bound.

Since we have assumed that g is smooth with full support, we can rewrite our losses from equation 72 using a first-order approximation to the integral. This gives us:

$$\lim_{\lambda \to 0} Losses = \lim_{\lambda \to 0} p_a(\lambda) * f(p_a(\lambda))[p_b(\lambda) * G(p_b(\lambda)) - \int_0^{p_b(\lambda)} x * g(x) dx]. \tag{81}$$

Putting these losses over $G(p_b(\lambda)) * p_b(\lambda)$ and simplifying gives us:

$$\lim_{\lambda \to 0} \frac{Losses}{G(p_b(\lambda)) * p_b(\lambda)} = \lim_{\lambda \to 0} p_a(\lambda) * f(p_a(\lambda)) - \frac{p_a(\lambda) * f(p_a(\lambda)) * \int_0^{p_b(\lambda)} x * g(x) dx}{p_b(\lambda) * G(p_b(\lambda))}. \tag{82}$$

The first term of our losses cancels with our gains by equation 79. We can also use equation 79 to substitute for $p_a(\lambda) * f(p_a(\lambda))$, giving us:

$$\lim_{\lambda \to 0} \frac{Gains - Losses}{G(p_b(\lambda)) * p_b(\lambda)} = \lim_{\lambda \to 0} \frac{[1 - F(p_a(\lambda))] * \int_0^{p_b(\lambda)} x * g(x)dx}{p_b(\lambda) * G(p_b(\lambda))}. \tag{83}$$

If this limit is strictly greater that 0, then there must be a $U_2 > 0$ such that $Gains - Losses$ is positive for all $\forall \lambda \in (0, U_2]$, and taking $U = \min(U_1, U_2)$ would finish our proof by contradiction. But $1 - F(p_a(0)) > 0$; therefore we are done if:

$$\lim_{\lambda \to 0} \frac{\int_0^{p_b(\lambda)} x * g(x)dx}{p_b(\lambda) * G(p_b(\lambda))} > 0. \tag{84}$$

Since both the numerator and denominator of this ratio go to 0, by L'Hospital's rule we have that:

$$\lim_{\lambda \to 0} \frac{\int_0^{p_b(\lambda)} x * g(x)dx}{p_b(\lambda) * G(p_b(\lambda))} = \lim_{p_b(\lambda) \to 0} \frac{p_b(\lambda) * g(p_b(\lambda))}{G(p_b(\lambda)) + p_b(\lambda) * g(p_b(\lambda))}. \tag{85}$$

If $g(0) > 0$ then in the limit $G(p_b(\lambda)) = p_b(\lambda) * g(p_b(\lambda))$, and

$$\lim_{p_b(\lambda) \to 0} \frac{p_b(\lambda) * g(p_b(\lambda))}{G(p_b(\lambda)) + p_b(\lambda) * g(p_b(\lambda))} = \frac{1}{2}, \tag{86}$$

which would complete our proof. If $g(0) = 0$, applying L'Hospital's rule again gives us:

$$\lim_{\lambda \to 0} \frac{p_b(\lambda) * g(p_b(\lambda))}{G(p_b(\lambda)) + p_b(\lambda) * g(p_b(\lambda))} = \lim_{\lambda \to 0} \frac{g'(p_b(\lambda)) * p_b(\lambda) + g(p_b(\lambda)) * p_b(\lambda)}{2g(p_b(\lambda)) + p_b(\lambda) * g'(p_b(\lambda))}, \tag{87}$$

so we would have our result if $g'(p_b(\lambda)) > 0$.

Repeating this process, we see that our limit will be strictly positive as long as at least one higher order derivative of G at 0 is positive. But this follows from the fact that g has full support. This proves our theorem. ■

Theorem 8 *If F satisfies the hazard rate condition that $\frac{f(x)}{1-F(x)}$ is strictly increasing in x, then the A firm will always charge more for its compatible version, setting $p_{ac} > p_a$.*

Proof. We show that the optimal surcharge is strictly positive by contradiction. Assume that the optimal prices satisfy $p_{ac} \leq p_a$. Consider lowering p_a by Δ.

This has three effects. First, the A firm gains new A customers who are now willing to buy the incompatible A; this gain is:

$$p_a * \Delta * f(p_a) * G(p_b - p_a + p_{ac}). \tag{88}$$

Second, the firm loses money by lowering prices for existing customers who were buying the incompatible A:

$$-\Delta * (1 - F(p_a)) * G(p_b - p_a + p_{ac}). \tag{89}$$

Third, some A customers who used to buy the compatible A switch to buying the incompatible version of A. This effect is:

$$(p_a - p_{ac}) * (1 - F(p_a)) * \Delta * g(p_b - p_a + p_{ac}). \tag{90}$$

When $p_{ac} < p_a$, this effect is a gain to the firm, and when $p_a = p_{ac}$, then this has no first-order effect (and a second-order loss).

It is sufficient for lowering p_a to increase profits that the gains in equation 88 outweigh the losses in equation 89, since equation 90 is to the first-order weakly positive. Therefore, ignoring equation 90, lowering p_a by a small amount will have a strictly positive effect if:

$$p_a * f(p_a) - (1 - F(p_a)) > 0. \tag{91}$$

This would contradict the optimality of p_a; therefore, it follows that:

$$p_{ac} \leq p_a \Rightarrow p_a * f(p_a) - (1 - F(p_a)) \leq 0. \tag{92}$$

Now consider raising both p_a and p_{ac} by Δ.

This has four effects. First, by raising p_a the firm gains on those customers who continue buying the incompatible A:

$$\Delta * (1 - F(p_a)) * G(p_b - p_a + p_{ac}). \tag{93}$$

Second, the firm loses some incompatible A customers:

$$-p_a * \Delta * f(p_a) * G(p_b - p_a + p_{ac}). \tag{94}$$

Third, by raising p_{ac}, the firm gains on those customers who continue buying the compatible A. This gain is:

$$\Delta * \int_{p_b - p_a + p_{ac}}^{\lambda} (1 - F(p_b + p_{ac} - x)) * g(x)dx. \tag{95}$$

Fourth, the firm loses some compatible A customers:

$$-p_{ac} * \Delta * \int_{p_b - p_a + p_{ac}}^{\lambda} f(p_b + p_{ac} - x) * g(x)dx. \tag{96}$$

By equation 92, the gain from equation 93 is weakly greater than the loss from equation 94. This leaves us to consider the combined effect of equations 95 and 96:

$$\Delta * \int_{p_b - p_a + p_{ac}}^{\lambda} [1 - p_{ac} * \frac{f(p_b + p_{ac} - x)}{1 - F(p_b + p_{ac} - x)}] * (1 - F(p_b + p_{ac} - x)) * g(x)dx. \quad (97)$$

Our hazard rate condition guarantees that this is strictly positive (as it is weakly positive at $x = p_b - p_a + p_{ac}$). Therefore, raising both prices results in a strict increase in profits to firm A, contradicting the optimality of p_a and p_{ac}. Therefore $p_{ac} > p_a$. The monopolist who can produce both a compatible and incompatible version of A would always charge strictly more for the compatible version. ■