Technical Appendix to: Firm-Specific Capital, Nominal Rigidities and the Business Cycle

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Contents

These are the technical notes the paper whose title appears above.

1. Firms

1.1. General Setup

The intermediate good producer's technology is:

$$y_t(i) = \epsilon_t K_t(i) f\left(\frac{z_t h_t(i)}{K_t(i)}\right) - \phi z_t^*,$$

where ϵ_t has mean unity and

$$\frac{z_t}{z_{t-1}} = \mu_{zt},$$

and

$$z_t^* = \Upsilon_t^{\frac{\alpha}{1-\alpha}} z_t,$$

Let

$$\frac{\Upsilon_t}{\Upsilon_{t-1}} = \mu_{\Upsilon t},\ \ \mu_{z^*t} = \frac{z_t^*}{z_{t-1}^*}.$$

The time series representations of μ_{zt} and $\mu_{\Upsilon t}$ are provided below. Note that

$$\mu_{z^*t} = (\mu_{\Upsilon t})^{\frac{\alpha}{1-\alpha}} \mu_{zt},$$

so that

$$\hat{\mu}_{z^*t} = \frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon t} + \hat{\mu}_{zt}.$$

A hat over a variable, say γ_t, means $\hat{\gamma}_t = d\gamma_t/\gamma$, where γ is the value of the variable in nonstochastic steady state.

Also, K_t denotes the services of capital:

$$K_t = u_t \bar{K}_t.$$

The law of motion for capital has the following form:

$$\bar{K}_{t+1}(i) = (1-\delta)\bar{K}_t(i) + F(I_t(i), I_{t-1}(i)).$$

In addition, investment adjustment costs are given by:

$$F(I_t(i), I_{t-1}(i)) = (1 - S\left(\frac{I_t(i)}{I_{t-1}(i)}\right))I_t(i).$$

The function, S, is restricted to satisfy the following properties: $S(\mu_\Upsilon \mu_{z^*}) = S'(\mu_\Upsilon \mu_{z^*}) = 0$, and $\varkappa \equiv S''(\mu_\Upsilon \mu_{z^*}) > 0$. For checking purposes, the following S function was used:

$$\begin{aligned} S\left(\frac{I_t(i)}{I_{t-1}(i)}\right) &= S\left(\frac{i_t(i)\mu_{z^*t}\mu_{\Upsilon t}}{i_{t-1}(i)}\right) \\ S(x) &= (\mu_{z^*}\mu_\Upsilon)^2 [S''] \left(\frac{x^2}{2(\mu_{z^*}\mu_\Upsilon)^2} - \frac{x}{\mu_{z^*}\mu_\Upsilon} + \frac{1}{2}\right) \end{aligned}$$

The present discounted value of profits of the intermediate good firm are:

$$E_t \sum_{j=0}^{\infty} \beta^j \Lambda_{t+j} \{ P_{t+j}(i) y_{t+j}(i) - P_{t+j} R_{t+j}(\nu) w_{t+j}(i) h_t(i) - P_{t+j} \Upsilon_{t+j}^{-1} I_{t+j}(i) - P_{t+j} \left[a(u_{t+j}) \Upsilon_{t+j}^{-1} \right] \bar{K}_{t+j} \},$$

where ν denotes the fraction of the wage bill that must be financed in advance, and Λ_{t+j} is the Lagrange multiplier on currency in the Lagrangian representation of the household problem. If R_t is the gross nominal rate of interest, then

$$R_t(\nu) = \nu R_t + 1 - \nu.$$

Linearizing this,

$$\hat{R}_t(\nu) = \frac{\nu R}{\nu R + 1 - \nu} \hat{R}_t.$$

Here, Λ_t is the shadow value of a dollar to the household, the owner of the intermediate good firm and τ denotes a subsidy to the intermediate good firm.

Final goods are produced according to the following production function:

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{1}{\lambda_{f,t}}} dj \right]^{\lambda_{f,t}}, \ 1 \leq \lambda_f < \infty$$

and

$$P_t = \left[\int_0^1 P_t(i)^{\frac{1}{1-\lambda_{f,t}}} di \right]^{1-\lambda_{f,t}}.$$

The the intermediate good firm must satisfy the demand curve:

$$\left(\frac{P_t}{P_t(i)} \right)^{\theta} Y_t = y_t(i), \ \theta = \frac{\lambda_f}{\lambda_f - 1}$$

To see where the aggregate condition involving prices comes from, take each side of the above to the power $1/\lambda_f$ and integrate:

$$Y_t^{\frac{1}{\lambda_f}} \int_0^1 \left(\frac{P_t}{P_t(i)} \right)^{\frac{1}{\lambda_f - 1}} di = \int_0^1 y_t(i)^{\frac{1}{\lambda_f}} di.$$

Now, raise each side to the power λ_f :

$$Y_t \left[\int_0^1 \left(\frac{P_t}{P_t(i)} \right)^{\frac{1}{\lambda_f - 1}} di \right]^{\lambda_f} = \left[\int_0^1 y_t(i)^{\frac{1}{\lambda_f}} di \right]^{\lambda_f} = Y_t.$$

Then,

$$\begin{aligned} P_t^{\frac{\lambda_f}{\lambda_f - 1}} \left[\int_0^1 \left(\frac{1}{P_t(i)} \right)^{\frac{1}{\lambda_f - 1}} di \right]^{\lambda_f} &= 1, \\ \left[\int_0^1 \left(\frac{1}{P_t(i)} \right)^{\frac{1}{\lambda_f - 1}} di \right]^{\lambda_f} &= P_t^{\frac{-\lambda_f}{\lambda_f - 1}} \\ \left[\int_0^1 (P_t(i))^{\frac{1}{1-\lambda_f}} di \right]^{1-\lambda_f} &= P_t \end{aligned}$$

In working with the firm's problem, it is useful to substitute out for hours worked in terms of the amount of output produced, the capital stock and the technology shocks:

$$\begin{aligned} \frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)} &= f\left(\frac{z_t h_t(i)}{K_t(i)}\right) \\ h_t(i) &= \frac{K_t(i)}{z_t} f^{-1}\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)}\right) \end{aligned}$$

We will be differentiating f^{-1} so it will be useful to have an expression for this. Thus, let $y = f(x)$, so that $dy/dx = f'(x)$. Now, $x = f^{-1}(y)$, so that $dx/dy = (f^{-1}(y))' = 1/(f'(x))$.

Writing the intermediate good firm's objective in Lagrangian form, and letting $\lambda_{t+j} = \Lambda_{t+j} P_{t+j}$,

$$E_t \sum_{j=0}^{\infty} \beta^j \lambda_{t+j} \{ p_{t+j}(i) y_{t+j}(i) - R_{t+j}(\nu) w_{t+j} h_{t+j}(i) - \Upsilon_{t+j}^{-1} I_{t+j}(i) - \left[a(u_{t+j}) \Upsilon_{t+j}^{-1} \right] \bar{K}_{t+j}$$
$$+\mu_{t+j} \left[(1-\delta)\bar{K}_{t+j}(i) + (1 - S\left(\frac{I_{t+j}(i)}{I_{t+j-1}(i)}\right)) I_{t+j}(i) - \bar{K}_{t+j+1}(i) \right] \}$$

Substitute out for hours worked:

$$\begin{aligned} &E_t \sum_{j=0}^{\infty} \beta^j \lambda_{t+j} \{ p_{t+j}(i) y_{t+j}(i) - R_{t+j}(\nu) w_{t+j} \frac{K_{t+j}(i)}{z_{t+j}} f^{-1}\left(\frac{y_{t+j}(i)+\phi z_{t+j}^*}{\epsilon_{t+j} K_{t+j}(i)}\right) \\ &-\Upsilon_{t+j}^{-1} I_{t+j}(i) - \left[a(u_{t+j}(i)) \Upsilon_{t+j}^{-1} \right] \bar{K}_{t+j}(i) \\ &+\mu_{t+j} \left[(1-\delta)\bar{K}_{t+j}(i) + (1 - S\left(\frac{I_{t+j}(i)}{I_{t+j-1}(i)}\right)) I_{t+j}(i) - \bar{K}_{t+j+1}(i) \right] \}. \end{aligned}$$

Next, substitute out for output using the demand function and for the physical stock of capital:

$$\begin{aligned} &E_t \sum_{j=0}^{\infty} \beta^j \lambda_{t+j} \{ p_{t+j}(i)^{1-\theta} Y_{t+j} - R_{t+j}(\nu) w_{t+j} \frac{u_{t+j}(i)\bar{K}_{t+j}(i)}{z_{t+j}} f^{-1}\left(\frac{p_{t+j}(i)^{-\theta} Y_{t+j} + \phi z_{t+j}^*}{\epsilon_{t+j} u_{t+j}(i) \bar{K}_{t+j}(i)}\right) \\ &-\Upsilon_{t+j}^{-1} I_{t+j}(i) - \left[a(u_{t+j}(i)) \Upsilon_{t+j}^{-1} \right] \bar{K}_{t+j}(i) \\ &+\mu_{t+j} \left[(1-\delta)\bar{K}_{t+j}(i) + (1 - S\left(\frac{I_{t+j}(i)}{I_{t+j-1}(i)}\right)) I_{t+j}(i) - \bar{K}_{t+j+1}(i) \right] \}. \end{aligned}$$

The functional form for a used when performing checks is:

$$\begin{aligned} a(u) &= au^2 + bu + c \\ a &= 0.5\tilde{\rho}\sigma_a \\ b &= \tilde{\rho}(1-\sigma_a) \\ c &= \tilde{\rho}((\sigma_a/2) - 1) \end{aligned}$$

We adopt the following scaling of variables:

$$\begin{aligned}
C_t &= c_t z_t^* \\
I_t &= i_t \Upsilon_t z_t^* \\
Y_t &= y_t z_t^* \\
\bar{K}_{t+1} &= \bar{k}_{t+1} z_t^* \Upsilon_t \\
w_t &= z_t^* \tilde{w}_t, \\
q_t &= \frac{Q_t}{z_t^* P_t}
\end{aligned}$$

1.2. Capital Utilization Decision (First-failed-Try)

Consider the first order condition with respect to $u_{t+j}(i)$:

$$\begin{aligned}
&\{-R_{t+j}(\nu) w_{t+j}(i) \frac{\bar{K}_{t+j}(i)}{z_{t+j}} f^{-1}\left(\frac{p_{t+j}(i)^{-\theta} Y_{t+j} + \phi z_{t+j}^*}{\epsilon_{t+j} u_{t+j}(i) \bar{K}_{t+j}(i)}\right) \\
&+R_{t+j}(\nu) w_{t+j} \frac{u_{t+j}(i) \bar{K}_{t+j}(i)}{z_{t+j}} \frac{1}{f'\left(f^{-1}\left(\frac{p_{t+j}(i)^{-\theta} Y_{t+j} + \phi z_{t+j}^*}{\epsilon_{t+j} u_{t+j}(i) \bar{K}_{t+j}(i)}\right)\right)} \frac{p_{t+j}(i)^{-\theta} Y_{t+j} + \phi z_{t+j}^*}{\epsilon_{t+j} u_{t+j}(i) \bar{K}_{t+j}(i)^2} \\
&-a'(u_{t+j}(i)) \Upsilon_{t+j}^{-1} \bar{K}_{t+j}(i)\} \\
=&\ 0
\end{aligned}$$

Let's specialize a little to see if it simplifies....

$$f = x^{1-\alpha}, \text{ so } f' = (1-\alpha) x^{-\alpha} \text{ and } f^{-1}(y) = y^{1/(1-\alpha)}.$$

Then,

$$\begin{aligned}
&\{-R_{t+j}(\nu) w_{t+j}(i) \frac{u_{t+j}(i)^{-\frac{1}{1-\alpha}} \bar{K}_{t+j}(i)}{z_{t+j}} \left(\frac{p_{t+j}(i)^{-\theta} Y_{t+j} + \phi z_{t+j}^*}{\epsilon_{t+j} \bar{K}_{t+j}(i)}\right)^{\frac{1}{1-\alpha}} \\
&+R_{t+j}(\nu) w_{t+j} \frac{u_{t+j}(i)^{-\frac{\alpha}{1-\alpha}}}{(1-\alpha) z_{t+j}} \left(\frac{p_{t+j}(i)^{-\theta} Y_{t+j} + \phi z_{t+j}^*}{\epsilon_{t+j} \bar{K}_{t+j}(i)}\right)^{\frac{1}{1-\alpha}} \\
&-a'(u_{t+j}(i)) \Upsilon_{t+j}^{-1} \bar{K}_{t+j}(i)\} \\
=&\ 0
\end{aligned}$$

or,

$$\begin{aligned}
&\left(\frac{p_t(i)^{-\theta} Y_t + \phi z_t^*}{\epsilon_t \bar{K}_t(i)}\right)^{\frac{1}{1-\alpha}} \frac{R_t(\nu) w_t}{z_t} \left[\frac{u_t(i)}{(1-\alpha)} - \bar{K}_t(i)\right] \\
&= u_t(i)^{\frac{1}{1-\alpha}} a'(u_t(i)) \Upsilon_t^{-1} \bar{K}_t(i).
\end{aligned}$$

or,

$$\begin{aligned} & \left(\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t\bar{K}_t(i)}\right)^{\frac{1}{1-\alpha}}\frac{R_t(\nu)w_t}{z_t}\left[\frac{u_t(i)}{(1-\alpha)}-\bar{K}_t(i)\right] \\ = \;& \delta_0\left(u_t(i)\right)^{\left(\frac{1}{1-\alpha}+\delta_1\right)}\Upsilon_t^{-1}\bar{K}_t(i). \end{aligned}$$

Note that the object to the left of the equality is $-\infty$ for $u_t(i)=0$ and converges to

$$\left(\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t\bar{K}_t(i)}\right)^{\frac{1}{1-\alpha}}\frac{R_t(\nu)w_t}{z_t}\frac{1}{(1-\alpha)}$$

as $u_t(i)\rightarrow\infty$. Tough to get anything closed form out of this!

1.2.1. A much simpler setup.

Start with perfect competition....

$$\begin{aligned} p\left(uk\right)^{\alpha}h^{1-\alpha}-a(u)k-wh,\; a(u) &= \frac{\delta_0}{1+\delta_1}u^{1+\delta_1} \\ a'(u) &= \delta_0 u^{\delta_1},\; a''(u)=\delta_0\delta_1 u^{\delta_1-1}>0. \end{aligned}$$

fonc:

$$\alpha u^{\alpha-1}pk^{\alpha}h^{1-\alpha}=\delta_0 u^{\delta_1}k,$$

so,

$$\begin{aligned} \alpha pk^{\alpha-1}h^{1-\alpha} &= \delta_0 u^{\delta_1+1-\alpha} \\ u &= \left(\frac{\alpha pk^{\alpha-1}h^{1-\alpha}}{\delta_0}\right)^{\frac{1}{\delta_1+1-\alpha}}. \end{aligned}$$

Then, the 'reduced form' problem, after substituting out for optimized capital utilization, is:

$$p\left(\frac{\alpha pk^{\alpha-1}h^{1-\alpha}}{\delta_0}\right)^{\frac{\alpha}{\delta_1+1-\alpha}}k^{\alpha}h^{1-\alpha}-\frac{\delta_0}{1+\delta_1}\left(\frac{\alpha pk^{\alpha-1}h^{1-\alpha}}{\delta_0}\right)^{\frac{1+\delta_1}{\delta_1+1-\alpha}}k-wh$$

or,

$$\begin{aligned} & p\left(\frac{\alpha p}{\delta_0}\right)^{\frac{\alpha}{\delta_1+1-\alpha}}(k)^{\alpha\left[1+\frac{\alpha-1}{\delta_1+1-\alpha}\right]}h^{(1-\alpha)\left[1+\frac{\alpha}{\delta_1+1-\alpha}\right]} \\ & -\frac{\delta_0}{1+\delta_1}\left(\frac{\alpha p}{\delta_0}\right)^{\frac{1+\delta_1}{\delta_1+1-\alpha}}(k)^{1+(\alpha-1)\frac{1+\delta_1}{\delta_1+1-\alpha}}(h)^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}}-wh \end{aligned}$$

or,

$$\begin{aligned} & p\left(\frac{\alpha p}{\delta_0}\right)^{\frac{\alpha}{\delta_1+1-\alpha}}(k)^{\alpha\frac{\delta_1}{\delta_1+1-\alpha}}h^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}} \\ & -\frac{\delta_0}{1+\delta_1}\left(\frac{\alpha p}{\delta_0}\right)^{\frac{1+\delta_1}{\delta_1+1-\alpha}}(k)^{1+(\alpha-1)\frac{1+\delta_1}{\delta_1+1-\alpha}}(h)^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}}-wh \end{aligned}$$

But,

$$\begin{aligned} & 1+(\alpha-1)\frac{1+\delta_1}{\delta_1+1-\alpha} \\ = & \frac{\delta_1+1-\alpha+(\alpha-1)(1+\delta_1)}{\delta_1+1-\alpha} \\ = & \frac{-\alpha+\alpha(1+\delta_1)}{\delta_1+1-\alpha} \\ = & \frac{\alpha\delta_1}{\delta_1+1-\alpha} \end{aligned}$$

so,

$$\begin{aligned} & p\left(\frac{\alpha p}{\delta_0}\right)^{\frac{\alpha}{\delta_1+1-\alpha}}(k)^{\alpha\frac{\delta_1}{\delta_1+1-\alpha}}h^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}} \\ & -\frac{\delta_0}{1+\delta_1}\left(\frac{\alpha p}{\delta_0}\right)^{\frac{1+\delta_1}{\delta_1+1-\alpha}}(k)^{1+(\alpha-1)\frac{1+\delta_1}{\delta_1+1-\alpha}}(h)^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}}-wh \\ = & p\left(\frac{\alpha p}{\delta_0}\right)^{\frac{\alpha}{\delta_1+1-\alpha}}(k)^{\alpha\frac{\delta_1}{\delta_1+1-\alpha}}h^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}} \\ & -\frac{\delta_0}{1+\delta_1}\left(\frac{\alpha p}{\delta_0}\right)^{\frac{1+\delta_1}{\delta_1+1-\alpha}}(k)^{\alpha\frac{\delta_1}{\delta_1+1-\alpha}}(h)^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}}-wh \\ = & \left[p\left(\frac{\alpha p}{\delta_0}\right)^{\frac{\alpha}{\delta_1+1-\alpha}}-\frac{\delta_0}{1+\delta_1}\left(\frac{\alpha p}{\delta_0}\right)^{\frac{1+\delta_1}{\delta_1+1-\alpha}}\right](k)^{\alpha\frac{\delta_1}{\delta_1+1-\alpha}}h^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}}-wh \\ = & \left[p^{\frac{\delta_1+1}{\delta_1+1-\alpha}}\left(\frac{\alpha}{\delta_0}\right)^{\frac{\alpha}{\delta_1+1-\alpha}}-\frac{\delta_0}{1+\delta_1}\left(\frac{\alpha p}{\delta_0}\right)^{\frac{1+\delta_1}{\delta_1+1-\alpha}}\right](k)^{\alpha\frac{\delta_1}{\delta_1+1-\alpha}}h^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}}-wh \\ = & \left[\left(\frac{\alpha}{\delta_0}\right)^{\frac{\alpha}{\delta_1+1-\alpha}}-\frac{\delta_0}{1+\delta_1}\left(\frac{\alpha}{\delta_0}\right)^{\frac{1+\delta_1}{\delta_1+1-\alpha}}\right]p^{\frac{\delta_1+1}{\delta_1+1-\alpha}}(k)^{\alpha\frac{\delta_1}{\delta_1+1-\alpha}}h^{(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha}}-wh \end{aligned}$$

What are the degree of returns to scale?

$$\begin{aligned} & \alpha\frac{1+\delta_1}{\delta_1+1-\alpha}+(1-\alpha)\frac{1+\delta_1}{\delta_1+1-\alpha} \\ = & \frac{1+\delta_1}{\delta_1+1-\alpha}. \end{aligned}$$

Looks like increasing returns! Note too, that there is less curvature on hours worked. For example, if $\delta_1 = 0$, then the production function is linear in hours worked.

1.3. Capital First Order Condition

$$E_t \sum_{j=0}^{\infty} \beta^j \lambda_{t+j} \{ p_{t+j}(i)^{1-\theta} Y_{t+j} - R_{t+j}(\nu) w_{t+j} \frac{u_{t+j}(i)\bar{K}_{t+j}(i)}{z_{t+j}} f^{-1}\left(\frac{p_{t+j}(i)^{-\theta} Y_{t+j} + \phi z^*_{t+j}}{\epsilon_{t+j} u_{t+j}(i) \bar{K}_{t+j}(i)}\right)$$
$$-\Upsilon_{t+j}^{-1} I_{t+j}(i) - \left[a(u_{t+j}(i))\Upsilon_{t+j}^{-1}\right] \bar{K}_{t+j}(i)$$
$$+\mu_{t+j}(i)\left[(1-\delta)\bar{K}_{t+j}(i) + (1 - S\left(\frac{I_{t+j}(i)}{I_{t+j-1}(i)}\right))I_{t+j}(i) - \bar{K}_{t+j+1}(i)\right]\}.$$

It is useful to write out the firm's objective in detail:

$$\lambda_t \{ p_t(i)^{1-\theta} Y_t - R_t(\nu) w_t \frac{u_t(i)\bar{K}_t(i)}{z_t} f^{-1}\left(\frac{p_t(i)^{-\theta} Y_t + \phi z^*_t}{\epsilon_t u_t(i)\bar{K}_t(i)}\right) - \Upsilon_t^{-1} I_t(i) - \left[a(u_t(i))\Upsilon_t^{-1}\right]\bar{K}_t(i)$$
$$+\mu_t(i)\left[(1-\delta)\bar{K}_t(i) + (1 - S\left(\frac{I_t(i)}{I_{t-1}(i)}\right))I_t(i) - \bar{K}_{t+1}(i)\right]\}$$
$$+\beta\lambda_{t+1}\{p_{t+1}(i)^{1-\theta} Y_{t+1} - R_{t+1}(\nu) w_{t+1} \frac{u_{t+1}(i)\bar{K}_{t+1}(i)}{z_{t+1}} f^{-1}\left(\frac{p_{t+1}(i)^{-\theta} Y_{t+1} + \phi z^*_{t+1}}{\epsilon_{t+1} u_{t+1}(i)\bar{K}_{t+1}(i)}\right)$$
$$-\Upsilon_{t+1}^{-1} I_{t+1}(i) - \left[a(u_{t+1}(i))\Upsilon_{t+1}^{-1}\right]\bar{K}_{t+1}(i)$$
$$+\mu_{t+1}(i)\left[(1-\delta)\bar{K}_{t+1}(i) + (1 - S\left(\frac{I_{t+1}(i)}{I_t(i)}\right))I_{t+1}(i) - \bar{K}_{t+2}(i)\right]\}$$
$$+....$$

Differentiating this with respect to $\bar{K}_{t+1}(i)$:

$$-\lambda_t \mu_t(i) + \beta\lambda_{t+1}\{-R_{t+1}(\nu) w_{t+1} \frac{u_{t+1}(i)}{z_{t+1}} f^{-1}\left(\frac{p_{t+1}(i)^{-\theta} Y_{t+1} + \phi z^*_{t+1}}{\epsilon_{t+1} u_{t+1}(i)\bar{K}_{t+1}(i)}\right)$$
$$+R_{t+1}(\nu) w_{t+1} \frac{u_{t+1}(i)\bar{K}_{t+1}(i)}{z_{t+1}} f^{-1\prime}\left(\frac{p_{t+1}(i)^{-\theta} Y_{t+1} + \phi z^*_{t+1}}{\epsilon_{t+1} u_{t+1}(i)\bar{K}_{t+1}(i)}\right) \frac{p_{t+1}(i)^{-\theta} Y_{t+1} + \phi z^*_{t+1}}{\epsilon_{t+1} u_{t+1}(i)\bar{K}_{t+1}(i)^2}$$
$$-a(u_{t+1}(i))\Upsilon_{t+1}^{-1} + \mu_{t+1}(i)(1-\delta)\}$$

Write

$$\rho_{t+1}(i) = -R_{t+1}(\nu) w_{t+1} \frac{1}{z_{t+1}} f^{-1}\left(\frac{p_{t+1}(i)^{-\theta} Y_{t+1} + \phi z^*_{t+1}}{\epsilon_{t+1} u_{t+1}(i)\bar{K}_{t+1}(i)}\right)$$
$$+R_{t+1}(\nu) w_{t+1} \frac{\bar{K}_{t+1}(i)}{z_{t+1}} f^{-1\prime}\left(\frac{p_{t+1}(i)^{-\theta} Y_{t+1} + \phi z^*_{t+1}}{\epsilon_{t+1} u_{t+1}(i)\bar{K}_{t+1}(i)}\right) \frac{p_{t+1}(i)^{-\theta} Y_{t+1} + \phi z^*_{t+1}}{\epsilon_{t+1} u_{t+1}(i)\bar{K}_{t+1}(i)^2}$$

We can think of $\rho(i)$ as the 'shadow rental rate of capital services'. This can be seen by noting that if $\rho_t(i)$ were a rental rate treated exogenously by the firm, then the firm would choose to rent $K_t(i) = u_t(i)\bar{K}_t(i)$. To see this, let

$$MP_{K,t} = \frac{dy_t(i)}{dK_t} = \frac{dy_t(i)}{u_t(i) d\bar{K}_t} = \frac{MP_{\bar{K},t}}{u_t(i)},$$

so that MP_K is the marginal product of a unit of capital services, and $MP_{\bar{K}}$ is the marginal product of a unit of physical capital. Also, MP_L is the marginal product of labor. Cost minimization by a firm which hires factors in competitive markets implies:

$$\frac{R_t(\nu)w_t(i)}{MP_{L,t}} = \frac{\rho_t(i)}{MP_{K,t}} = \frac{u_t(i)\rho_t(i)}{MP_{\bar{K},t}}.$$

In our setup,

$$\begin{aligned} MP_{L,t} &= \epsilon_t f'\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right) z_t \\ MP_{\bar{K},t} &= \epsilon_t u_t(i) f\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right) - \epsilon_t u_t(i)\bar{K}_t(i) f'\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right)\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)^2} \end{aligned}$$

Then,

$$\begin{aligned} \rho_{t+1}(i) &= -R_{t+1}(\nu)w_{t+1}\frac{1}{z_{t+1}}\left(\frac{p_{t+1}(i)^{-\theta}Y_{t+1}+\phi z_{t+1}^*}{\epsilon_{t+1}u_{t+1}(i)\bar{K}_{t+1}(i)}\right)^{\frac{1}{1-\alpha}} \\ &\quad +R_{t+1}(\nu)w_{t+1}\frac{1}{z_{t+1}}\frac{1}{1-\alpha}\left(\frac{p_{t+1}(i)^{-\theta}Y_{t+1}+\phi z_{t+1}^*}{\epsilon_{t+1}u_{t+1}(i)\bar{K}_{t+1}(i)}\right)^{\frac{\alpha}{1-\alpha}}\frac{p_{t+1}(i)^{-\theta}Y_{t+1}+\phi z_{t+1}^*}{\epsilon_{t+1}u_{t+1}(i)\bar{K}_{t+1}(i)} \end{aligned}$$

$$\begin{aligned} &\frac{MP_{\bar{K},t}}{MP_{L,t}}R_t(\nu)w_t(i) \\ &= \frac{\epsilon_t u_t(i) f\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right) - \epsilon_t u_t(i)\bar{K}_t(i) f'\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right)\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)^2}}{\epsilon_t f'\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right) z_t} R_t(\nu)w_t(i) \\ &= \left[\frac{u_t(i) f\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right)}{f'\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right) z_t} - \frac{h_t(i)}{\bar{K}_t(i)}\right] R_t(\nu)w_t(i) \\ &= \left[\frac{u_t(i) f\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right)}{f'\left(\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right) z_t} - \frac{u_t(i)}{z_t}\frac{z_t h_t(i)}{u_t(i)\bar{K}_t(i)}\right] R_t(\nu)w_t(i) \\ &= \left[\frac{1}{z_t}u_t(i) f^{-1\prime}\left(\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)}\right)\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)} - \frac{u_t(i)}{z_t} f^{-1}\left(\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)}\right)\right] R_t(\nu)w_t(i) \\ &= u_t(i)\rho_t(i), \end{aligned}$$

where $\rho_t(i)$ is as defined as above.

So, we can write the first order condition for $\bar{K}_{t+1}(i)$ as follows:

$$\lambda_t = \beta\lambda_{t+1}\frac{u_t(i)\rho_{t+1}(i) - a(u_{t+1}(i))\Upsilon_{t+1}^{-1} + \mu_{t+1}(i)(1-\delta)}{\mu_t(i)},$$

with the understanding that $\rho_{t+1}(i)$ is as defined above. Note that this is the same as the first order condition for capital obtained in CEE, where it is the household that is accumulating the capital, and identifying $\rho_{t+1}(i)$ with the market rental rate of capital. Also, $\mu_t(i)$ corresponds to the 'price of capital'.

1.4. Investment First Order Condition

$$\lambda_t\{p_t(i)^{1-\theta}Y_t - R_t(\nu)w_t\frac{u_t(i)\bar{K}_t(i)}{z_t}f^{-1}\left(\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)}\right) - \Upsilon_t^{-1}I_t(i) - \left[a(u_t(i))\Upsilon_t^{-1}\right]\bar{K}_t(i)$$
$$+\mu_t(i)\left[(1-\delta)\bar{K}_t(i) + (1-S\left(\frac{I_t(i)}{I_{t-1}(i)}\right))I_t(i) - \bar{K}_{t+1}(i)\right]\}$$
$$+\beta\lambda_{t+1}\{p_{t+1}(i)^{1-\theta}Y_{t+1} - R_{t+1}(\nu)w_{t+1}\frac{u_{t+1}(i)\bar{K}_{t+1}(i)}{z_{t+1}}f^{-1}\left(\frac{p_{t+1}(i)^{-\theta}Y_{t+1}+\phi z_{t+1}^*}{\epsilon_{t+1}u_{t+1}(i)\bar{K}_{t+1}(i)}\right)$$
$$-\Upsilon_{t+1}^{-1}I_{t+1}(i) - \left[a(u_{t+1}(i))\Upsilon_{t+1}^{-1}\right]\bar{K}_{t+1}(i)$$
$$+\mu_{t+1}(i)\left[(1-\delta)\bar{K}_{t+1}(i) + (1-S\left(\frac{I_{t+1}(i)}{I_t(i)}\right))I_{t+1}(i) - \bar{K}_{t+2}(i)\right]\}$$
$$+....$$

Differentiating the firm's objective with respect to $I_t(i)$:

$$\lambda_t\{-\Upsilon_t^{-1} + \mu_t(i)\left[1 - S\left(\frac{I_t(i)}{I_{t-1}(i)}\right) - S'\left(\frac{I_t(i)}{I_{t-1}(i)}\right)\frac{I_t(i)}{I_{t-1}(i)}\right]\}$$
$$+\beta\lambda_{t+1}\mu_{t+1}(i)S'\left(\frac{I_{t+1}(i)}{I_t(i)}\right)\left(\frac{I_{t+1}(i)}{I_t(i)}\right)^2$$

1.5. Capital Utilization First Order Condition (Second Try)

Differentiating with respect to $u_t(i)$:

$$\begin{aligned}
& -R_t(\nu)w_t\frac{\bar{K}_t(i)}{z_t}f^{-1}\left(\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)}\right) \\
& +R_t(\nu)w_t\frac{\bar{K}_t(i)}{z_t}f^{-1\prime}\left(\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)}\right)\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)} \\
& -a'(u_t(i))\Upsilon_t^{-1}\bar{K}_t(i) \\
= & \;\; 0
\end{aligned}$$

Divide by $\bar{K}_t(i)$:

$$\begin{aligned}
& -R_t(\nu)w_t\frac{1}{z_t}f^{-1}\left(\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)}\right) \\
& +R_t(\nu)w_t\frac{1}{z_t}f^{-1\prime}\left(\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)}\right)\frac{p_t(i)^{-\theta}Y_t+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)} \\
& -a'(u_t(i))\Upsilon_t^{-1} \\
= & \;\; 0,
\end{aligned}$$

or

$$\rho_t(i) = a'(u_t(i))\Upsilon_t^{-1}.$$

Interestingly, if there were a competitive rental market for capital with the rental rate of capital services being $\rho_t(i)$, then this would be the firms' efficiency condition for choosing $u_t(i)$.

1.6. Scaling and Linearizing the Firm's First Order Conditions

1.6.1. Some Useful Aggregation Results

Define the aggregate stock of physical capital:

$$\bar{K}_t = \int_0^1 \bar{k}_t(i)di,$$

so that

$$d\bar{K}_t = \int_0^1 d\bar{k}_t(i)di,$$

or,

$$\bar{K}\widehat{\bar{K}}_t = \int_0^1 \bar{k}(i)\widehat{\bar{k}}_t(i)di.$$

But, in steady state production across firms, and hence their useage of capital, is equal. As a result, $K = k(i)$ for all i, and

$$\widehat{\bar{K}}_t = \int_0^1 \widehat{\bar{k}}_t(i)di.$$

Also,

$$\begin{aligned} dY_t &= \frac{\theta}{\theta-1}\left[\int_0^1 y_t(i)^{\frac{\theta-1}{\theta}}di\right]^{\frac{\theta}{\theta-1}-1}\frac{\theta-1}{\theta}\left[\int_0^1 y_t(i)^{\frac{\theta-1}{\theta}-1}dy_t(i)\right]di \\ &= Y_t^{\frac{1}{\theta}}\left[\int_0^1 y_t(i)^{\frac{\theta-1}{\theta}}\hat{y}_t(i)\right]di. \end{aligned}$$

But, in steady state $y_t(i) = Y$ for all i, so that

$$\begin{aligned} dY_t &= Y^{\frac{1}{\theta}}\left[\int_0^1 Y^{\frac{\theta-1}{\theta}}\hat{y}_t(i)\right]di \\ &= Y^{\frac{1}{\theta}}Y^{\frac{\theta-1}{\theta}}\left[\int_0^1 \hat{y}_t(i)\right]di, \end{aligned}$$

so that,

$$\hat{Y}_t = \int_0^1 \hat{y}_t(i)di. \tag{1.1}$$

1.6.2. The Utilization Rate of Capital

The first order condition for capital is:

$$\Upsilon_t \rho_t(i) = \tilde{\rho}_t(i) = a'(u_t(i)),$$

so that

$$\tilde{\rho}\widehat{\tilde{\rho}}_t(i) = a'' \hat{u}_t(i),$$

or,

$$\widehat{\tilde{\rho}}_t(i) = \frac{a''}{\tilde{\rho}} \hat{u}_t(i) = \frac{a''}{a'} \hat{u}_t(i) = \sigma_a \hat{u}_t(i),$$

say, where

$$\sigma_a = \frac{a''}{a'}.$$

Also, note that, in steady state:

$$\tilde{\rho} = a'. \tag{1.2}$$

1.6.3. The Investment First Order Condition

Now consider the first order condition for investment:

$$\lambda_t \Upsilon_t^{-1} = \lambda_t \mu_t(i) \left[1 - S\left(\frac{I_t(i)}{I_{t-1}(i)}\right) - S'\left(\frac{I_t(i)}{I_{t-1}(i)}\right) \frac{I_t(i)}{I_{t-1}(i)}\right]$$
$$+\beta \lambda_{t+1} \mu_{t+1}(i) S'\left(\frac{I_{t+1}(i)}{I_t(i)}\right) \left(\frac{I_{t+1}(i)}{I_t(i)}\right)^2$$

First, we scale this. Multiplying by z_t^* and making use of $I_t(i) = i_t(i)\Upsilon_t z_t^*$,

$$z_t^* \lambda_t = z_t^* \lambda_t \Upsilon_t \mu_t(i) \left[1 - S\left(\frac{i_t(i)\Upsilon_t z_t^*}{i_{t-1}(i)\Upsilon_{t-1} z_{t-1}^*}\right) - S'\left(\frac{i_t(i)\Upsilon_t z_t^*}{i_{t-1}(i)\Upsilon_{t-1} z_{t-1}^*}\right) \frac{i_t(i)\Upsilon_t z_t^*}{i_{t-1}(i)\Upsilon_{t-1} z_{t-1}^*}\right]$$
$$+\beta \frac{z_t^*}{z_{t+1}^*} z_{t+1}^* \lambda_{t+1} \frac{\Upsilon_t}{\Upsilon_{t+1}} \Upsilon_{t+1} \mu_{t+1}(i) S'\left(\frac{i_{t+1}(i)\Upsilon_{t+1} z_{t+1}^*}{i_t(i)\Upsilon_t z_t^*}\right) \left(\frac{i_{t+1}(i)\Upsilon_{t+1} z_{t+1}^*}{i_t(i)\Upsilon_t z_t^*}\right)^2$$

or, using the notation introduced above:

$$\lambda_{z^*,t} = \lambda_{z^*,t} \tilde{\mu}_t(i) \left[1 - S\left(\frac{i_t(i)}{i_{t-1}(i)} \mu_{\Upsilon,t} \mu_{z^*,t}\right) - S'\left(\frac{i_t(i)}{i_{t-1}(i)} \mu_{\Upsilon,t} \mu_{z^*,t}\right) \frac{i_t(i)}{i_{t-1}(i)} \mu_{\Upsilon,t} \mu_{z^*,t}\right]$$
$$+\beta \frac{1}{\mu_{z^*,t+1}} \lambda_{z^*,t+1} \frac{1}{\mu_{\Upsilon,t+1}} \tilde{\mu}_{t+1}(i) S'\left(\frac{i_{t+1}(i)}{i_t(i)} \mu_{\Upsilon,t+1} \mu_{z^*,t+1}\right) \left(\frac{i_{t+1}(i)}{i_t(i)} \mu_{\Upsilon,t+1} \mu_{z^*,t+1}\right)^2$$

Evaluating this in steady state and taking into account that $S = S' = 0$ in steady state, we find

$$\tilde{\mu} = 1.$$

Log-linearizing this expression:

$$\lambda_{z^*}\hat{\lambda}_{z^*,t} = \lambda_{z^*}\left\{\hat{\lambda}_{z^*,t} + \widehat{\tilde{\mu}}_t(i) + \left[1 - S\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right) - S'\widehat{\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right)\frac{i_t(i)}{i_{t-1}(i)}}\mu_{\Upsilon,t}\mu_{z^*,t}\right]\right\}$$
$$+\beta\lambda_{z^*}\left[S''\right]\mu_{\Upsilon}\mu_{z^*}d\left(\frac{i_{t+1}(i)}{i_t(i)}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}\right)$$

but,

$$\begin{aligned} & d\left(\frac{i_{t+1}(i)}{i_t(i)}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}\right) \\ = \; & \mu_{\Upsilon}\mu_{z^*}\widehat{\left(\frac{i_{t+1}(i)}{i_t(i)}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}\right)} \\ = \; & \mu_{\Upsilon}\mu_{z^*}\left(\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}\right) \end{aligned}$$

Then,

$$\lambda_{z^*}\hat{\lambda}_{z^*,t} = \lambda_{z^*}\left\{\hat{\lambda}_{z^*,t} + \widehat{\tilde{\mu}}_t(i) + \left[1 - S\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right) - S'\widehat{\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right)\frac{i_t(i)}{i_{t-1}(i)}}\mu_{\Upsilon,t}\mu_{z^*,t}\right]\right\}$$
$$+\beta\lambda_{z^*}\left[S''\right]\left(\mu_{\Upsilon}\mu_{z^*}\right)^2\left[\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}\right]$$

Now, taking into account that $S = S' = 0$ when evaluated in steady state,

$$\begin{aligned} & \widehat{\left[1 - S\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right) - S'\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right)\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right]} \\ = \; & \frac{d\left[1 - S\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right) - S'\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right)\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right]}{1} \\ = \; & -S''\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right)\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}d\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right) \\ = \; & -\left[S''\right]\left(\mu_{\Upsilon}\mu_{z^*}\right)^2\widehat{\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right)} \\ = \; & -\left[S''\right]\left(\mu_{\Upsilon}\mu_{z^*}\right)^2\left[\hat{\imath}_t(i) - \hat{\imath}_{t-1}(i) + \hat{\mu}_{\Upsilon,t} + \hat{\mu}_{z^*,t}\right] \end{aligned}$$

Then,

$$\hat{\lambda}_{z^*,t} = \hat{\lambda}_{z^*,t} + \widehat{\tilde{\mu}}_t(i) - \left[S''\right]\left(\mu_{\Upsilon}\mu_{z^*}\right)^2\left[\hat{\imath}_t(i) - \hat{\imath}_{t-1}(i) + \hat{\mu}_{\Upsilon,t} + \hat{\mu}_{z^*,t}\right]$$
$$+\beta\left[S''\right]\left(\mu_{\Upsilon}\mu_{z^*}\right)^2\left[\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}\right]$$

and,

$$(****) \qquad \widehat{\tilde{\mu}}_t(i) = \left[S''\right]\left(\mu_{\Upsilon}\mu_{z^*}\right)^2\left[\hat{\imath}_t(i) - \hat{\imath}_{t-1}(i) + \hat{\mu}_{\Upsilon,t} + \hat{\mu}_{z^*,t}\right] \qquad (1.3)$$
$$-\beta\left[S''\right]\left(\mu_{\Upsilon}\mu_{z^*}\right)^2\left[\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}\right]$$

1.6.4. The Capital First Order Condition

Multiply the capital first order condition by z_t^* :

$$z_t^* \lambda_t = \beta \frac{z_t^*}{z_{t+1}^*} z_{t+1}^* \lambda_{t+1} \frac{u_{t+1}(i)\Upsilon_{t+1}\rho_{t+1}(i) - a(u_{t+1}(i)) + \Upsilon_{t+1}\mu_{t+1}(i)(1-\delta)}{\frac{\Upsilon_{t+1}}{\Upsilon_t}\Upsilon_t \mu_t(i)}.$$

Denote

$$\tilde{\mu}_{t+1}(i) = \Upsilon_{t+1}\mu_{t+1}(i),\ \mu_{z^*,t+1} = \frac{z_{t+1}^*}{z_t^*},\ \mu_{\Upsilon,t+1} = \frac{\Upsilon_{t+1}}{\Upsilon_t},\ \lambda_{z^*,t} = z_t^*\lambda_t,\ \tilde{\rho}_{t+1}(i) = \Upsilon_{t+1}\rho_{t+1}(i).$$

Then,

$$\lambda_{z^*,t} = \beta \frac{1}{\mu_{z^*,t+1}} \lambda_{z^*,t+1} \frac{u_{t+1}(i)\tilde{\rho}_{t+1}(i) - a(u_{t+1}(i)) + \tilde{\mu}_{t+1}(i)(1-\delta)}{\mu_{\Upsilon,t+1}\tilde{\mu}_t(i)},$$

or, in steady state,

$$\frac{\mu_\Upsilon \mu_{z^*}}{\beta} = \tilde{\rho} + 1 - \delta.$$

Then,

$$\begin{aligned}\hat{\lambda}_{z^*,t} &= \hat{\lambda}_{z^*,t+1} - \hat{\mu}_{z^*,t+1} - \hat{\mu}_{\Upsilon,t+1} - \widehat{\tilde{\mu}}_t(i) \\ &\quad + \left[u_{t+1}(i)\tilde{\rho}_{t+1}(i) - a(\widehat{u_{t+1}(i)}) + \tilde{\mu}_{t+1}(i)(1-\delta)\right]\end{aligned}$$

Now,

$$u_{t+1}(i)\tilde{\rho}_{t+1}(i) - a(\widehat{u_{t+1}(i)}) + \tilde{\mu}_{t+1}(i)(1-\delta) = d\frac{u_{t+1}(i)\tilde{\rho}_{t+1}(i) - a(u_{t+1}(i)) + \tilde{\mu}_{t+1}(i)(1-\delta)}{\tilde{\rho}+1-\delta},$$

where we have taken into account that in steady state, $u_t(i) = 1$, and $a(u_t(i)) = 0$. Then,

$$\begin{aligned}& u_{t+1}(i)\tilde{\rho}_{t+1}(i) - a(\widehat{u_{t+1}(i)}) + \tilde{\mu}_{t+1}(i)(1-\delta) \\ = & \frac{\tilde{\rho}\left[\hat{u}_{t+1}(i) + \widehat{\tilde{\rho}}_{t+1}(i)\right] - da(u_{t+1}(i)) + (1-\delta)\widehat{\tilde{\mu}}_{t+1}(i)}{\tilde{\rho}+1-\delta}\end{aligned}$$

But,

$$da(u_{t+1}(i)) = a'\hat{u}_{t+1}(i) = \tilde{\rho}\hat{u}_{t+1}(i),$$

where a' denotes the derivative of a, evaluated in steady state. Then,

$$\begin{aligned}& u_{t+1}(i)\tilde{\rho}_{t+1}(i) - a(\widehat{u_{t+1}(i)}) + \tilde{\mu}_{t+1}(i)(1-\delta) \\ = & \frac{\tilde{\rho}\left[\hat{u}_{t+1}(i) + \widehat{\tilde{\rho}}_{t+1}(i)\right] - \tilde{\rho}\hat{u}_{t+1}(i) + (1-\delta)\widehat{\tilde{\mu}}_{t+1}(i)}{\tilde{\rho}+1-\delta} \\ = & \frac{\tilde{\rho}\widehat{\tilde{\rho}}_{t+1}(i) + (1-\delta)\widehat{\tilde{\mu}}_{t+1}(i)}{\tilde{\rho}+1-\delta}\end{aligned}$$

Then,

$$(***)\ \hat{\lambda}_{z^*,t} = \hat{\lambda}_{z^*,t+1} - \hat{\mu}_{z^*,t+1} - \hat{\mu}_{\Upsilon,t+1} - \widehat{\tilde{\mu}}_t(i) + \frac{\tilde{\rho}\widehat{\tilde{\rho}}_{t+1}(i) + (1-\delta)\widehat{\tilde{\mu}}_{t+1}(i)}{\tilde{\rho}+1-\delta} \qquad (1.4)$$

1.6.5. The Shadow Rental Rate of Capital

Now let's go after ρ :

$$\rho_t(i) = R_t(\nu)\left(\frac{w_t}{z_t}\right) f^{-1}\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)}\right)\left[\frac{\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)}}{f'\left(f^{-1}\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)}\right)\right) f^{-1}\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)}\right)} - 1\right]$$

Let's simplify things:

$$\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)} = f\left(\frac{z_t h_t(i)}{K_t(i)}\right) = \left(\frac{z_t h_t(i)}{K_t(i)}\right)^{1-\alpha},$$

so that

$$\frac{f\left(\frac{z_{t+1}h_{t+1}(i)}{K_{t+1}(i)}\right)}{f'\left(\frac{z_{t+1}h_{t+1}(i)}{K_{t+1}(i)}\right)\frac{z_{t+1}h_{t+1}(i)}{K_{t+1}(i)}} - 1 = \frac{\alpha}{1-\alpha}$$

and

$$\frac{z_t h_t(i)}{K_t(i)} = f^{-1}\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)}\right) = \left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)}\right)^{\frac{1}{1-\alpha}}.$$

Substituting:

$$\rho_t(i) = \frac{\alpha}{1-\alpha} R_t(\nu)\left(\frac{w_t}{z_t}\right)\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)}\right)^{\frac{1}{1-\alpha}}$$

Recall

$$z_t^* = \Upsilon_t^{\frac{\alpha}{1-\alpha}} z_t,\ \ \bar{K}_{t+1} = \bar{k}_{t+1} z_t^* \Upsilon_t,\ \ z_t^* \tilde{w}_t = w_t$$

so that

$$\begin{aligned}
\rho_t(i) &= \frac{\alpha}{1-\alpha} R_t(\nu)\frac{z_t^*\tilde{w}_t}{z_t^*}\Upsilon_t^{\frac{\alpha}{1-\alpha}}\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t u_t(i)\bar{K}_t(i)}\right)^{\frac{1}{1-\alpha}} \\
&= \frac{\alpha}{1-\alpha} R_t(\nu)\tilde{w}_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t u_t(i)\bar{k}_t(i) z_t^*\left(z_{t-1}^*/z_t^*\right)\left(\Upsilon_{t-1}/\Upsilon_t\right)}\Upsilon_t^{-1}\right)^{\frac{1}{1-\alpha}} \\
&= \frac{\alpha}{1-\alpha} R_t(\nu)\tilde{w}_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\Upsilon_t^{-\frac{1}{1-\alpha}}\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t u_t(i)\bar{k}_t(i) z_t^*}\mu_{z^*,t}\mu_{\Upsilon,t}\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

Then,

$$\begin{aligned}
\tilde{\rho}_t(i) &= \Upsilon_t\rho_t(i) = \frac{\alpha}{1-\alpha} R_t(\nu)\tilde{w}_t\Upsilon_t\Upsilon_t^{\frac{\alpha}{1-\alpha}}\Upsilon_t^{-\frac{1}{1-\alpha}}\left(\frac{\tilde{y}_t(i)+\phi}{\epsilon_t u_t(i)\bar{k}_t(i)}\mu_{z^*,t}\mu_{\Upsilon,t}\right)^{\frac{1}{1-\alpha}} \\
&= \frac{\alpha}{1-\alpha} R_t(\nu)\tilde{w}_t\left(\frac{\tilde{y}_t(i)+\phi}{\epsilon_t u_t(i)\bar{k}_t(i)}\mu_{z^*,t}\mu_{\Upsilon,t}\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

where

$$y_t(i) = z_t^*\tilde{y}_t(i).$$

Log-linearizing:

$$\widehat{\rho}_t(i) = \hat{R}_t(\nu) + \widehat{\bar{w}}_t + \frac{1}{1-\alpha}\left((\widehat{\tilde{y}_t(i) + \phi}) - \hat{\epsilon}_t - \hat{u}_t(i) - \widehat{\bar{k}}_t(i) + \hat{\mu}_{z^*,t} + \hat{\mu}_{\Upsilon,t}\right)$$

Now,

$$(\widehat{\tilde{y}_t(i) + \phi}) = \frac{\tilde{y}\widehat{\tilde{y}}_t(i)}{\tilde{y} + \phi},$$

so,

$$\begin{aligned}
\widehat{\rho}_t(i) &= \hat{R}_t(\nu) + \widehat{\bar{w}}_t + \frac{1}{1-\alpha}\left(\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t(i) - \hat{\epsilon}_t - \hat{u}_t(i) - \widehat{\bar{k}}_t(i) + \hat{\mu}_{z^*,t} + \hat{\mu}_{\Upsilon,t}\right) \\
&= \hat{R}_t(\nu) + \widehat{\bar{w}}_t + \frac{1}{1-\alpha}\left(\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t(i) - \hat{\epsilon}_t - \widehat{\bar{k}}_t(i) + \hat{\mu}_{z^*,t} + \hat{\mu}_{\Upsilon,t}\right) - \frac{1}{1-\alpha}\hat{u}_t(i) \\
&= \hat{R}_t(\nu) + \widehat{\bar{w}}_t + \frac{1}{1-\alpha}\left(\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t(i) - \hat{\epsilon}_t - \widehat{\bar{k}}_t(i) + \hat{\mu}_{z^*,t} + \hat{\mu}_{\Upsilon,t}\right) - \frac{1}{1-\alpha}\frac{1}{\sigma_a}\widehat{\rho}_t(i),
\end{aligned}$$

after substituting from the utilization condition. Then,

$$(***)\ \widehat{\rho}_t(i) = \frac{\hat{R}_t(\nu) + \widehat{\bar{w}}_t + \frac{1}{1-\alpha}\left(\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t(i) - \hat{\epsilon}_t - \widehat{\bar{k}}_t(i) + \hat{\mu}_{z^*,t} + \hat{\mu}_{\Upsilon,t}\right)}{1 + \frac{1}{1-\alpha}\frac{1}{\sigma_a}} \tag{1.5}$$

1.6.6. The Capital Evolution Equation

Turn now to the capital accumulation rule:

$$\bar{K}_{t+1}(i) = (1-\delta)\bar{K}_t(i) + (1 - S\left(\frac{I_t(i)}{I_{t-1}(i)}\right))I_t(i).$$

Write this in terms of scaled variables:

$$\bar{k}_{t+1}(i)z_t^*\Upsilon_t = (1-\delta)\bar{k}_t(i)z_{t-1}^*\Upsilon_{t-1} + (1 - S\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right))i_t(i)\Upsilon_t z_t^*$$

Divide by $z_t^*\Upsilon_t$

$$\bar{k}_{t+1}(i) = \frac{(1-\delta)}{\mu_{\Upsilon,t}\mu_{z^*,t}}\bar{k}_t(i) + (1 - S\left(\frac{i_t(i)}{i_{t-1}(i)}\mu_{\Upsilon,t}\mu_{z^*,t}\right))i_t(i).$$

In steady state:

$$\left[1 - \frac{(1-\delta)}{\mu_\Upsilon\mu_{z^*}}\right] = \frac{i}{\bar{k}}$$

Log-linearizing:

$$\begin{aligned}
(***)\ \widehat{\bar{k}}_{t+1}(i) &= \frac{(1-\delta)}{\mu_\Upsilon\mu_{z^*}}\left[\widehat{\bar{k}}_t(i) - \hat{\mu}_{\Upsilon,t} - \hat{\mu}_{z^*,t}\right] + \frac{i}{\bar{k}}\hat{\imath}_t(i) \\
&= \frac{(1-\delta)}{\mu_\Upsilon\mu_{z^*}}\left[\widehat{\bar{k}}_t(i) - \hat{\mu}_{\Upsilon,t} - \hat{\mu}_{z^*,t}\right] + \left[1 - \frac{(1-\delta)}{\mu_\Upsilon\mu_{z^*}}\right]\hat{\imath}_t(i)
\end{aligned}$$

or,

$$\hat{\imath}_t(i) = \frac{\widehat{\bar{k}}_{t+1}(i) - \frac{(1-\delta)}{\mu_\Upsilon \mu_{z^*}}\left[\widehat{\bar{k}}_t(i) - \hat{\mu}_{\Upsilon,t} - \hat{\mu}_{z^*,t}\right]}{1 - \frac{(1-\delta)}{\mu_\Upsilon \mu_{z^*}}} \qquad (1.6)$$

$$= \frac{\mu_\Upsilon \mu_{z^*}\widehat{\bar{k}}_{t+1}(i) - (1-\delta)\left[\widehat{\bar{k}}_t(i) - \hat{\mu}_{\Upsilon,t} - \hat{\mu}_{z^*,t}\right]}{\mu_\Upsilon \mu_{z^*} - (1-\delta)}$$

1.7. Marginal Cost

The marginal product of labor is:

$$MP_{L,t} = (1-\alpha)\epsilon_t z_t \left(\frac{z_t h_t(i)}{K_t(i)}\right)^{-\alpha}$$

But,

$$h_t(i) = \frac{K_t(i)}{z_t}\left(\frac{y_t(i) + \phi z_t^*}{\epsilon_t K_t(i)}\right)^{\frac{1}{1-\alpha}}$$

so that,

$$MP_{L,t} = (1-\alpha)\epsilon_t z_t \left(\frac{z_t h_t(i)}{K_t(i)}\right)^{-\alpha}$$

$$= (1-\alpha)\epsilon_t z_t \left(\frac{y_t(i) + \phi z_t^*}{\epsilon_t K_t(i)}\right)^{\frac{-\alpha}{1-\alpha}}$$

Marginal cost is:

$$s_t(i) = \frac{R_t(\nu) w_t}{MP_{L,t}}$$

$$= \frac{R_t(\nu)\tilde{w}_t z_t^*}{(1-\alpha)\epsilon_t z_t}\left(\frac{y_t(i) + \phi z_t^*}{\epsilon_t K_t(i)}\right)^{\frac{\alpha}{1-\alpha}}$$

$$= \frac{R_t(\nu)\tilde{w}_t z_t^*}{(1-\alpha)\epsilon_t z_t}\left(\frac{\tilde{y}_t(i) + \phi}{\epsilon_t u_t(i)\bar{k}_t(i) z_{t-1}^* \Upsilon_{t-1}} z_t^*\right)^{\frac{\alpha}{1-\alpha}}$$

$$= \frac{R_t(\nu)\tilde{w}_t \Upsilon_t^{\frac{\alpha}{1-\alpha}} z_t}{(1-\alpha)\epsilon_t z_t}\left(\Upsilon_{t-1}\right)^{\frac{-\alpha}{1-\alpha}}\left(\frac{\tilde{y}_t(i) + \phi}{\epsilon_t u_t(i)\bar{k}_t(i)}\mu_{z^*,t}\right)^{\frac{\alpha}{1-\alpha}}$$

$$= \frac{R_t(\nu)\tilde{w}_t}{(1-\alpha)\epsilon_t}\left(\frac{\tilde{y}_t(i) + \phi}{\epsilon_t u_t(i)\bar{k}_t(i)}\mu_{z^*,t}\mu_{\Upsilon,t}\right)^{\frac{\alpha}{1-\alpha}}$$

Linearizing this:

$$\hat{s}_t(i) = \hat{R}_t(\nu) + \widehat{\tilde{w}}_t - \hat{\epsilon}_t + \frac{\alpha}{1-\alpha}\left[\widehat{\tilde{y}_t(i) + \phi} - \hat{\epsilon}_t - \hat{u}_t(i) - \widehat{\bar{k}}_t(i) + \hat{\mu}_{z^*,t} + \hat{\mu}_{\Upsilon,t}\right]$$

$$= \hat{R}_t(\nu) + \widehat{\tilde{w}}_t - \hat{\epsilon}_t + \frac{\alpha}{1-\alpha}\left[\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t(i) - \hat{\epsilon}_t - \hat{u}_t(i) - \widehat{\bar{k}}_t(i) + \hat{\mu}_{z^*,t} + \hat{\mu}_{\Upsilon,t}\right]$$

$$= \hat{R}_t(\nu) + \widehat{\tilde{w}}_t - \hat{\epsilon}_t + \frac{\alpha}{1-\alpha}\left[\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t(i) - \hat{\epsilon}_t - \frac{1}{\sigma_a}\widehat{\rho}_t(i) - \widehat{\bar{k}}_t(i) + \hat{\mu}_{z^*,t} + \hat{\mu}_{\Upsilon,t}\right]$$

It is of useful to express marginal cost in deviation from the economy-wide average:

$$\hat{s}_t^+(i) = \frac{\alpha}{1-\alpha}\left[\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t^+(i) - \frac{1}{\sigma_a}\widehat{\tilde{\rho}}_t^+(i) - \widehat{\bar{k}}_t^+(i)\right]$$

But,

$$\widehat{\tilde{\rho}}_t(i) = \frac{\hat{R}_t(\nu) + \widehat{\bar{w}}_t + \frac{1}{1-\alpha}\left(\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t(i) - \hat{\epsilon}_t - \widehat{\bar{k}}_t(i) + \hat{\mu}_{z^*,t} + \hat{\mu}_{\Upsilon,t}\right)}{1 + \frac{1}{1-\alpha}\frac{1}{\sigma_a}}$$

so,

$$\widehat{\tilde{\rho}}_t^+(i) = \frac{\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t^+(i) - \widehat{\bar{k}}_t^+(i)}{1-\alpha+\frac{1}{\sigma_a}},$$

Substituting this into the expression for marginal cost:

$$\begin{aligned}
\hat{s}_t^+(i) &= \frac{\alpha}{1-\alpha}\left[\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t^+(i) - \frac{\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t^+(i) - \widehat{\bar{k}}_t^+(i)}{\sigma_a(1-\alpha)+1} - \widehat{\bar{k}}_t^+(i)\right] \\
&= \frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left[\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_t^+(i) - \widehat{\bar{k}}_t^+(i)\right]
\end{aligned}$$

When the fixed cost is positive, then we replace it by $\phi = (\lambda_f - 1)\tilde{y}$, or,

$$\hat{s}_t^+(i) = \frac{\alpha\sigma_a}{\sigma_a(1-\alpha)+1}\left[\frac{1}{\lambda_f}\widehat{\tilde{y}}_t^+(i) - \widehat{\bar{k}}_t^+(i)\right]$$

This equation conveys some of the economics in the model. When $\sigma_a = \infty$, then the ration in front of the bracket is unity. This is the case when there is no variability in the utilization of capital. As σ_a comes down and there is variability, then the ratio falls below unity. This ratio controls the slope of the i^{th} firm's marginal cost with respect to its own production. So, with more variable capital utilization, that slope flattens out. Indeed, when utilization becomes infinitely elastic, the slope goes to zero. That is, when $\sigma_a = 0$ the ratio in front of the bracket is zero. In this case, capital specificity should have no impact on the coefficient on marginal cost. That is, ζ should be unity when $\sigma_a = 0$. Of course, driving σ_a to zero will affect the responsiveness of s_t to a shock. It would be interesting to study an object like:

$$\gamma\frac{d\hat{s}_t}{dshock_t}.$$

Here we can see that changes in model specification will have different effects on these two pieces. Driving σ_a to zero will drive γ up and the other term down.

2. Households

Maximize utility:

$$\sum_{t=0}^{\infty} \beta^t \{u(C_t - bC_{t-1}, h_t(j)) + \Lambda_t [R_t (M_t - Q_t + (x_t - 1) M_t^a) + A_{j,t} + W_{j,t} h_{j,t} \\ + Q_t + D_t - (1 + \eta (V_t)) P_t C_t - M_{t+1}]\},$$

where

$$\begin{aligned} u(C_t - bC_{t-1}, h_t(j)) &= \log (C_t - bC_{t-1}) - \zeta_t z(h_{j,t}) \\ z(h) &= \frac{h^{1+\sigma_L}}{1+\sigma_L} \psi_L \end{aligned}$$

2.1. Money Demand

The first order condition for Q_t is:

$$R_t = 1 + \eta' \left(\frac{P_t C_t}{Q_t} \right) \left(\frac{P_t C_t}{Q_t} \right)^2,$$

since R_t, P_t, C_t, Q_t are known after the monetary policy shock. Also,

$$\eta'(V), \ \eta''(V) > 0,$$

where V denotes steady state velocity. Note that in steady state,

$$R = 1 + \eta' V^2,$$

where absence of an argument means the function is evaluated in steady state. Linearizing:

$$\begin{aligned} R_t - 1 - \eta'(V_t)(V_t)^2 &= 0, \\ R\hat{R}_t - \eta''(V_t)(V_t)^2 V_t \hat{V}_t - 2\eta'(V_t)(V_t) V_t \hat{V}_t &= 0 \\ R\hat{R}_t - \left[2 + \frac{\eta'' V}{\eta'} \right] \eta' V^2 \hat{V}_t &= 0. \end{aligned}$$

Using the steady state formula for R,

$$\hat{R}_t - [2 + \sigma_\eta] \frac{R-1}{R} \hat{V}_t = 0,$$

where

$$\sigma_\eta = \frac{\eta'' V}{\eta'}$$

Since $\hat{V}_t = \hat{c}_t - \hat{q}_t$ (see below),

$$\frac{R}{R-1} \frac{1}{2+\sigma_\eta} \hat{R}_t - \hat{c}_t + \hat{q}_t = 0,$$

or,

$$\hat{q}_t = \hat{c}_t - \frac{R}{R-1}\frac{1}{2+\sigma_\eta}\hat{R}_t.$$

Another way to write a variable with a hat is, $\hat{q}_t = \log(q_t/q)$, so that the money demand equation is:

$$\log(q_t/q) = \log(c_t/c) - \frac{R}{R-1}\frac{1}{2+\sigma_\eta}\log\left(\frac{R_t}{R}\right),$$

so,

$$\frac{d\log q_t}{d\log R_t} = -\frac{R}{R-1}\frac{1}{2+\sigma_\eta}$$

What is called the 'log-log representation' of money demand is expressed in terms of the log of the net interest rate. Using the fact, $d\log(R_t) = dR_t/R_t = dr_t/R_t$, where $R_t = 1 + r_t$. Then,

$$d\log(R_t) = \frac{dr_t}{R_t} = r_t\frac{d\log(r_t)}{R_t} = (R_t - 1)\frac{d\log(r_t)}{R_t}.$$

Then,

$$\frac{d\log q_t}{d\log R_t} = \frac{R}{R-1}\frac{d\log q_t}{d\log r_t},$$

or,

$$\begin{aligned}
\frac{d\log q_t}{d\log r_t} &= \frac{R-1}{R}\frac{d\log q_t}{d\log R_t} \\
&= -\frac{R-1}{R}\frac{R}{R-1}\frac{1}{2+\sigma_\eta} \\
&= -\frac{1}{2+\sigma_\eta}.
\end{aligned}$$

The 'semi-elasticity representation' of money demand based on:

$$\frac{d\log q_t}{dR_t} = -\frac{1}{R-1}\frac{1}{2+\sigma_\eta}.$$

The interest semi-elasticity of money demand is measured as:

$$\epsilon = -\frac{100 \times d\log(q)}{400 \times dR_t},$$

so that in the model,

$$\epsilon = \frac{1}{R-1}\frac{1}{2+\sigma_\eta}\frac{1}{4}.$$

The mean interest rate over the period 1974 to 2003 (measured by the one-year treasury bill rate) is 6.99 percent. This translates into $R = 1 + 6.99/400 = 1.017$. In this case, the upper bound on ϵ (achieved with $\sigma_\eta = 0$) is 7.15. This is reasonably high, and is almost the value of 8 estimated by Lucas.

It is interesting to adopt a functional form for the transactions technology. Stefanie and Martin adopt:

$$\begin{aligned}\eta &= AV_t + \frac{B}{V_t} - 2\sqrt{AB} \\ \sigma_\eta &= \frac{\eta'' V}{\eta'} = \frac{2BV^{-2}}{A - BV^{-2}} = \frac{2B}{AV^2 - B}\end{aligned}$$

This functional form has the property, $\eta' = A - BV^{-2} = 0$ implies

$$V = \left(\frac{B}{A}\right)^{1/2}.$$

In this case, $\eta = 0$. Thus, when the nominal rate of interest is zero, velocity is set to the point where there are no transactions costs in consumption. That is, the cost of consumption is just PC.

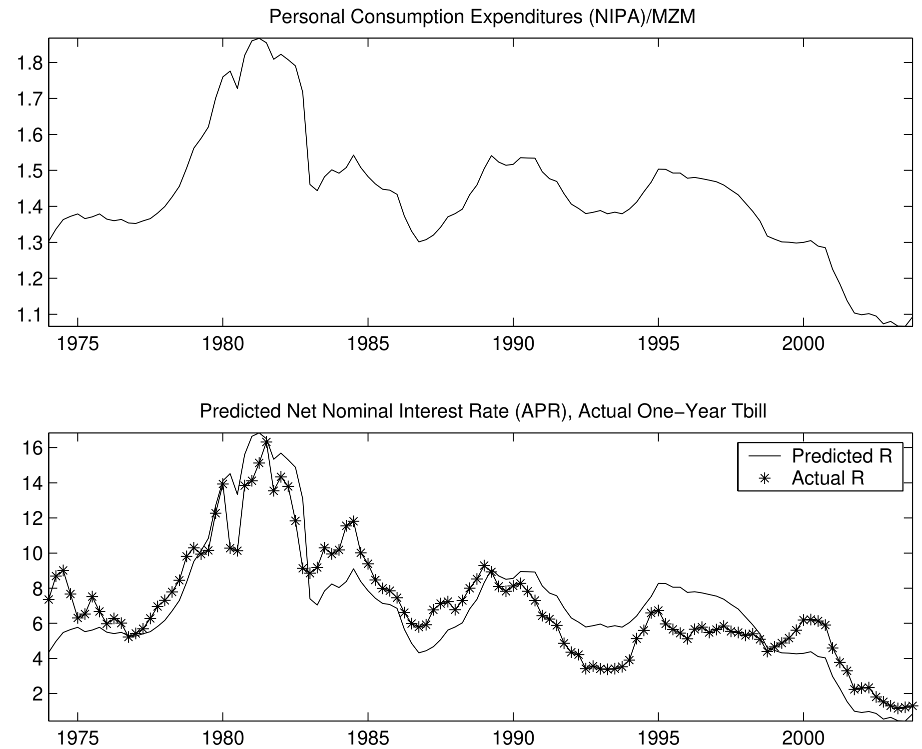
The rate of interest corresponding to a given velocity is:

$$\begin{aligned}R &= 1 + \eta'(V) \times V^2 \\ &= 1 + \left[A - BV^{-2}\right] V^2 = 1 - B + AV^2,\end{aligned}$$

or,

$$V^2 = \frac{B-1}{A} + \frac{1}{A}R.$$

I ran a regression of V^2 (where V is NIPA personal consumption expenditures (services plus nondurables, PCESV+PCND) in dollars, divided by the St. Louis Fed's MZM measure of money) on R (R was measured as the gross quarterly return on one-year T-bills). I recovered A and B from the constant and slope terms in this regression ($A = 0.0174$ and $B = 0.0187$). Using velocity, I computed the interest rate implied by this equation and, after converting it to net, annual percentage terms, compared it to the actual interest rate. The results are presented in the following graph. Velocity is displayed in the top panel. The predicted and actual interest rates are reported in the bottom panel.



The mean rate of interest in the sample is 7 percent per year. The mean level of velocity is 1.43. This is very nearly the value of V implied by the money demand equation at the mean interest rate, which is 1.44. The value of σ_η at this last level of velocity and values of A and B is 2.14. The interest rate semi-elasticity is 3.45.

In the computations, we used a different functional form:

$$\eta(V) = AV + \frac{B}{V} + C,$$

where

$$\begin{aligned} A &= \eta' \times (1 + \sigma_\eta/2), \\ B &= V^2 \eta' \sigma_\eta/2 \\ C &= \eta - AV - B/V. \end{aligned}$$

where $\eta' = \eta'(V)$, and V is the steady state value of V_t, and

$$\sigma_\eta = \frac{\eta'' V}{\eta'}.$$

2.2. First Order Condition for C_t

The first order condition for C_t is:

$$E_t\left\{\frac{1}{C_t - bC_{t-1}} - \beta b\frac{1}{C_{t+1} - bC_t} - \lambda_t\left[(1+\eta(V_t)) + \eta'(V_t)V_t\right]\right\} = 0,$$

where

$$\lambda_t = \Lambda_t P_t.$$

Multiplying by z_t^* and letting,

$$\lambda_{z^*t} \equiv z_t^*\lambda_t = z_t^*\Lambda_t P_t,$$

we obtain:

$$E_t\left\{\frac{1}{\frac{C_t}{z_t^*} - b\frac{z_{t-1}^*}{z_t^*}\frac{C_{t-1}}{z_{t-1}^*}} - \frac{\beta b}{\frac{z_{t+1}^*}{z_t^*}\frac{C_{t+1}}{z_{t+1}^*} - b\frac{C_t}{z_t^*}} - \lambda_{z^*t}\left[(1+\eta(V_t)) + \eta'(V_t)V_t\right]\right\} = 0,$$

or,

$$E_t\left\{\frac{1}{c_t - b\mu_{z_t^*}^{-1}c_{t-1}} - \frac{\beta b}{\mu_{z_{t+1}^*}c_{t+1} - bc_t} - \lambda_{z^*t}\left[(1+\eta(V_t)) + \eta'(V_t)V_t\right]\right\} = 0. \qquad (2.1)$$

Linearizing the first term in braces:

$$d\frac{1}{c_t - b\mu_{z_t^*}^{-1}c_{t-1}} = \left(\frac{1}{c\left(1 - b\mu_{z^*}^{-1}\right)}\right)^2\left[c\hat{c}_t - \frac{bc}{\mu_{z^*}}\hat{c}_{t-1} + \frac{bc}{\mu_{z^*}}\hat{\mu}_{z_t^*}\right]$$

The second terms is:

$$d\frac{\beta b}{\mu_{z_{t+1}^*}c_{t+1} - bc_t} = \beta b\left(\frac{1}{\mu_{z_{t+1}^*}c_{t+1} - bc_t}\right)^2\left[\mu_{z^*}c\left(\hat{\mu}_{z_{t+1}^*} + \hat{c}_{t+1}\right) - bc\hat{c}_t\right]$$

The last term is:

$$\begin{aligned} & d\lambda_{z^*t}\left[(1+\eta(V_t)) + \eta'(V_t)V_t\right] \\ = \; & \lambda_{z^*}\left[(1+\eta(V)) + \eta'(V)V\right]\hat{\lambda}_{z^*t} \\ & +\lambda_{z^*}\left[2 + \frac{\eta''(V)V}{\eta'(V)}\right]\eta'(V)V\hat{V}_t \end{aligned}$$

Finally,

$$\begin{aligned} V_t &= \frac{z_t^*}{z_t^*}\frac{P_tC_t}{Q_t} \\ &= \frac{c_t}{q_t}, \end{aligned}$$

so that

$$\hat{V}_t = \hat{c}_t - \hat{q}_t$$

$$
\begin{aligned}
& E_t\{-\left(\frac{1}{c\left(1-b\mu_{z^*}^{-1}\right)}\right)^2\left[c\hat{c}_t-\frac{bc}{\mu_{z^*}}\hat{c}_{t-1}+\frac{bc}{\mu_{z^*}}\hat{\mu}_{z_t^*}\right] \\
& +\beta b\left(\frac{1}{c\left(\mu_{z^*}-b\right)}\right)^2\left[\mu_{z^*}c\left(\hat{\mu}_{z_{t+1}^*}+\hat{c}_{t+1}\right)-bc\hat{c}_t\right] \\
& -\lambda_{z^*}\left[\left(1+\eta\left(V\right)\right)+\eta'\left(V\right)V\right]\hat{\lambda}_{z^*t}-\lambda_{z^*}\left[2+\frac{\eta''\left(V\right)V}{\eta'\left(V\right)}\right]\eta'\left(V\right)V\times\left(\hat{c}_t-\hat{q}_t\right)\} \\
= \; & 0.
\end{aligned}
$$

2.3. M_{t+1} First Order Condition

The first order condition for M_{t+1} is:

$$E_t\left[-\Lambda_t+\beta\Lambda_{t+1}R_{t+1}\right]=0.$$

Multiply by $z_t^*P_t$:

$$E_t\left[-\lambda_{z^*t}+\beta\frac{z_t^*P_t}{z_{t+1}^*P_{t+1}}\lambda_{z^*t+1}R_{t+1}\right]=0,$$

or,

$$E_t\left[-\lambda_{z^*t}+\beta\frac{1}{\pi_{t+1}\mu_{z^*,t+1}}\lambda_{z^*t+1}R_{t+1}\right]=0.$$

Linearly expand this:

$$E_t\left[-\lambda_{z^*}\hat{\lambda}_{z^*t}+\beta d\frac{\lambda_{z^*t+1}R_{t+1}}{\pi_{t+1}\mu_{z^*,t+1}}\right]=0$$

or,

$$E_t\left[-\lambda_{z^*}\hat{\lambda}_{z^*t}+\beta\frac{\lambda_{z^*}R}{\pi\mu_{z^*}}\frac{\widehat{\lambda_{z^*t+1}R_{t+1}}}{\pi_{t+1}\mu_{z^*,t+1}}\right]=0$$

or, dividing by λ_{z^*} and taking into account $\beta R/(\pi\mu_{z^*})=1$

$$E\left[-\hat{\lambda}_{z^*t}+\hat{\lambda}_{z^*t+1}+\hat{R}_{t+1}-\hat{\pi}_{t+1}-\hat{\mu}_{z^*,t+1}|\Omega_t\right]=0.$$

2.4. The Wage Equation

The wage rate set by the household that gets to reoptimize today is $\tilde{W}_t$. The household takes into account that if it does not get to reoptimize next period, it's wage rate then is

$$W_{t+1}=\pi_t\left(\mu_{z^*}\right)^{1-\vartheta}\left(\mu_{z^*,t+1}\right)^{\vartheta}\tilde{W}_t,$$

where μ_{z^*} is the steady state growth rate of z_t^*. Note the partial indexation to the realized growth rate of z_t^*. The only economically interesting specification is $\vartheta=0$. We allow $\vartheta=1$ in order to be in a position to compare the reduced form expression - for checking purposes - with the reduced form derived earlier when $\vartheta=0$.

In period $t+l$ the wage is:

$$\begin{aligned}
W_{t+1} &= \pi_t \left(\mu_{z^*}\right)^{1-\vartheta} \left(\mu_{z^*,t+1}\right)^{\vartheta} \tilde{W}_t \\
W_{t+2} &= \pi_{t+1}\pi_t \left(\mu_{z^*}^2\right)^{1-\vartheta} \left(\mu_{z^*,t+2}\mu_{z^*,t+1}\right)^{\vartheta} \tilde{W}_t \\
&\ldots \\
W_{t+l} &= \pi_{t+l-1}\cdots\pi_{t+1}\pi_t \left(\mu_{z^*}^l\right)^{1-\vartheta} \left(\mu_{z^*,t+l}\cdots\mu_{z^*,t+1}\right)^{\vartheta} \tilde{W}_t.
\end{aligned}$$

The demand curve that the individual household faces is:

$$h_{t+j} = \left(\frac{\tilde{W}_{t+j}}{W_{t+j}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+j} = \left(\frac{\pi_{t+j-1}\cdots\pi_{t+1}\pi_t \left(\mu_{z^*}^j\right)^{1-\vartheta} \left(\mu_{z^*,t+j}\cdots\mu_{z^*,t+1}\right)^{\vartheta} \tilde{W}_t}{\tilde{w}_{t+j} z_{t+j}^* P_{t+j}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+j}.$$

Note:

$$\begin{aligned}
P_{t+j} &= \pi_{t+j} P_{t+j-1} \\
&= \ldots = \pi_{t+j}\pi_{t+j-1}\cdots\pi_{t+1} P_t \\
z_{t+j}^* &= \mu_{z^*,t+j}\mu_{z^*,t+j-1}\cdots\mu_{z^*,t+1} z_t^*.
\end{aligned}$$

Then, the demand curve in terms of stationary variables is:

$$\begin{aligned}
h_{t+j} &= \left(\frac{\pi_{t+j-1}\cdots\pi_{t+1}\pi_t \left(\mu_{z^*}^j\right)^{1-\vartheta} \left(\mu_{z^*,t+j}\cdots\mu_{z^*,t+1}\right)^{\vartheta} \tilde{W}_t}{\tilde{w}_{t+j}\mu_{z^*,t+j}\mu_{z^*,t+j-1}\cdots\mu_{z^*,t+1} z_t^* \pi_{t+j}\pi_{t+j-1}\cdots\pi_{t+1} P_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+j} \\
&= \left(\frac{\tilde{W}_t}{\tilde{w}_{t+j} z_t^* P_t} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+j} \qquad (2.2) \\
&= \left(\frac{w_t^+ \tilde{w}_t}{\tilde{w}_{t+j}} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+j}
\end{aligned}$$

where $\tilde{W}_t$ denotes the nominal wage set by households that reoptimize in period t, W_t denotes the nominal wage rate associated with aggregate, homogeneous labor, H_t, and $w_t^+ = \tilde{W}_t / W_t$. Be careful not to confuse $\tilde{W}_t$, the wage chosen by optimizing households, and $\tilde{w}_t$, the aggregate wage, scaled by $z_t^* P_t$. Also,

$$\begin{aligned}
X_{t,j} &= \frac{\pi_{t+j-1}\cdots\pi_{t+1}\pi_t \left(\mu_{z^*}^j\right)^{1-\vartheta} \left(\mu_{z^*,t+j}\cdots\mu_{z^*,t+1}\right)^{\vartheta}}{\pi_{t+j}\pi_{t+j-1}\cdots\pi_{t+1}\mu_{z^*,t+j}\mu_{z^*,t+j-1}\cdots\mu_{z^*,t+1}},\ j>0 \\
&= 1,\ j=0.
\end{aligned}$$

Note that

$$\begin{aligned}
\hat{X}_{t,j} &= -\left(\Delta\hat{\pi}_{t+j} + \Delta\hat{\pi}_{t+j-1} + \cdots + \Delta\hat{\pi}_{t+1}\right) \qquad (2.3) \\
&\quad -(1-\vartheta)\left(\hat{\mu}_{z^*,t+j} + \hat{\mu}_{z^*,t+j-1} + \cdots + \hat{\mu}_{z^*,t+1}\right)
\end{aligned}$$

The homogeneous labor is related to household labor by:

$$H=\left[\int_0^1 (h_j)^{\frac{1}{\lambda_w}}dj\right]^{\lambda_w},\ 1\leq\lambda_w<\infty.$$

The j^{th} household that reoptimizes its wage, $\tilde{W}_t$, does so to optimize (neglecting irrelevant terms in the household objective):

$$E_t\sum_{l=0}^{\infty}(\beta\xi_w)^{l-t}\left\{-z(h_{j,t+l})+\Lambda_{t+l}W_{j,t+l}h_{j,t+l}\right\},$$

where we have taken into account that we only need worry about future histories in which the household cannot reoptimize. In the previous expression,

$$z(h)=\frac{h^{1+\sigma_L}}{1+\sigma_L}\psi_L.$$

It is useful to have the curvature of this function:

$$\frac{z''h}{z'}=\sigma_L.$$

The presence of ξ_w by the discount factor in the discounted sum reflects that in choosing its wage, the household can disregard future histories in which it reoptimizes its wage.

We now derive the first order condition for $\tilde{W}_t$. For this, we need to rewrite the household's objective in terms of this variable. Substituting out for $h_{j,t+1}$ using (2.2), and making use of the definition, $\lambda_{z^*t}\equiv\Lambda_t\left(z_t^*P_t\right)$,

$$E_t\sum_{l=0}^{\infty}(\beta\xi_w)^{l-t}\{-z(\left(\frac{\tilde{W}_t}{\tilde{w}_{t+l}z_t^*P_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}H_{t+l})+\lambda_{z^*t+l}\frac{\tilde{W}_{t+l}}{z_{t+l}^*P_{t+l}}\left(\frac{\tilde{W}_t}{\tilde{w}_{t+l}z_t^*P_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}H_{t+l}\}.$$

Here, $\tilde{W}_{t+l}$ is the wage rate in period $t+l$, of a household that optimized in period t and could not reoptimize again up to, and including, in period $t+l$. Using the fact, $\tilde{W}_{t+l}/\left(z_{t+l}^*P_{t+l}\right)=\left[\tilde{W}_t/\left(z_t^*P_t\right)\right]X_{t,l}$ and rearranging,

$$E_t\sum_{l=0}^{\infty}(\beta\xi_w)^{l}\{-z(\left(\frac{\tilde{W}_t}{\tilde{w}_{t+l}z_t^*P_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}H_{t+l})+\lambda_{z^*t+l}\left(\frac{\tilde{W}_t}{z_t^*P_t}\right)^{1+\frac{\lambda_w}{1-\lambda_w}}X_{t,l}\left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}H_{t+l}\}.$$

We now have the objective in the form that we need. Differentiate with respect to $\tilde{W}_t$:

$$\begin{aligned}
&E_t\sum_{l=0}^{\infty}(\beta\xi_w)^{l}\{-z'(\left(\frac{\tilde{W}_t}{\tilde{w}_{t+l}z_t^*P_t}X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}}H_{t+l})\frac{\lambda_w}{1-\lambda_w}\left(\frac{\tilde{W}_t}{\tilde{w}_{t+l}z_t^*P_t}X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}-1}H_{t+l}\frac{1}{\tilde{w}_{t+l}z_t^*P_t}X_{t,j}\\
&+\lambda_{z^*t+l}\left(\frac{1}{1-\lambda_w}\right)\left(\frac{\tilde{W}_t}{z_t^*P_t}\right)^{\frac{\lambda_w}{1-\lambda_w}}\frac{1}{z_t^*P_t}X_{t,l}\left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}H_{t+l}\}\\
=\ &0
\end{aligned}$$

The next step is to write this first order condition in terms of stationary variables only. Multiply by $\tilde{W}_t^{-\frac{\lambda_w}{1-\lambda_w}+1}(1-\lambda_w)/\lambda_w$:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \{-z'(\left(\frac{\tilde{W}_t}{\tilde{w}_{t+j} z_t^* P_t} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}) \left(\frac{1}{\tilde{w}_{t+j} z_t^* P_t} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}-1} H_{t+l} \frac{1}{\tilde{w}_{t+j} z_t^* P_t} X_{t,j}$$
$$+\frac{1}{\lambda_w} \tilde{W}_t \lambda_{z^*t+l} \left(\frac{1}{z_t^* P_t}\right)^{\frac{\lambda_w}{1-\lambda_w}+1} X_{t,l} \left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}\}$$
$$= \quad 0$$

Multiply by $P_t^{\frac{\lambda_w}{1-\lambda_w}}$:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \{-z'(\left(\frac{\tilde{W}_t}{\tilde{w}_{t+j} z_t^* P_t} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}) \left(\frac{1}{\tilde{w}_{t+j} z_t^*} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}$$
$$+\frac{1}{\lambda_w} \frac{\tilde{W}_t}{P_t} \lambda_{z^*t+l} \left(\frac{1}{z_t^*}\right)^{\frac{1}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}\} \quad = \quad 0.$$

Now get this in terms of stationary variables using

$$w_t^+ \equiv \frac{\tilde{W}_t}{W_t},$$

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \{-z'(\left(\frac{w_t^+ W_t}{\tilde{w}_{t+j} z_t^* P_t} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}) \left(\frac{1}{\tilde{w}_{t+j} z_t^*} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}$$
$$+\frac{1}{\lambda_w} \frac{w_t^+ W_t}{P_t} \lambda_{z^*t+l} \left(\frac{1}{z_t^*}\right)^{\frac{1}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}\} = 0.$$

and, taking into account,

$$\tilde{w}_t \equiv \frac{W_t}{z_t^* P_t},$$

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \{-z'(\left(\frac{w_t^+ \tilde{w}_t}{\tilde{w}_{t+j}} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}) \left(\frac{1}{\tilde{w}_{t+j} z_t^*} X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}$$
$$+\frac{1}{\lambda_w} w_t^+ \tilde{w}_t \lambda_{z^*t+l} \left(\frac{1}{z_t^*}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}\} \quad = \quad 0.$$

Multiply by $z_t^{*\frac{\lambda_w}{1-\lambda_w}}$ on both sides, and take into account that the technology shocks are known at the time the price decision is taken:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \{-z'(\left(\frac{w_t^+ \tilde{w}_t}{\tilde{w}_{t+j}} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}) \left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l} + \frac{1}{\lambda_w} w_t^+ \tilde{w}_t \lambda_{z^*t+l} X_{t,l} \left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}\} = 0.$$

Factor:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \, H_{t+l} \left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} \{\frac{1}{\lambda_w} w_t^+ \tilde{w}_t X_{t,l} \lambda_{z^*t+l} - z'(\left(\frac{w_t^+ \tilde{w}_t}{\tilde{w}_{t+j}} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l})\} = 0.$$

writing this out carefully:

$$\begin{aligned}
& H_t \left(\frac{1}{\tilde{w}_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} \{\frac{1}{\lambda_w} w_t^+ \tilde{w}_t \lambda_{z^*t} - z_t'\} \\
& + (\beta\xi_w) \, H_{t+1} \left(\frac{X_{t,1}}{\tilde{w}_{t+1}}\right)^{\frac{\lambda_w}{1-\lambda_w}} \{\frac{1}{\lambda_w} w_t^+ \tilde{w}_t X_{t,1} \lambda_{z^*t+1} - z_{t+1}'\} \\
& + ... \\
& + (\beta\xi_w)^l \, H_{t+l} \left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} \{\frac{1}{\lambda_w} w_t^+ \tilde{w}_t X_{t,l} \lambda_{z^*t+l} - z_{t+l}'\} \\
& + ...
\end{aligned} \tag{2.4}$$

where

$$z_{t+l}' \equiv z'(\left(\frac{w_t^+ \tilde{w}_t}{\tilde{w}_{t+l}} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l})$$

This is the household's scaled first order condition for the wage rate. We now log-linearize this expression. Note,

$$\begin{aligned}
dz_{t+l}' &= z''(\left(\frac{w_t^+ \tilde{w}_t}{\tilde{w}_{t+l}} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}) \\
& \quad \times d\left[\left(\frac{w_t^+ \tilde{w}_t}{\tilde{w}_{t+l}} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} H_{t+l}\right] \\
&= z'' H \left[\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t - \widehat{\tilde{w}}_{t+l} + \hat{X}_{t,l}\right) + \hat{H}_{t+l}\right].
\end{aligned}$$

Here, we have made use of the fact, $dx_t = x\hat{x}_t$. For now, we do not substitute out for $\hat{X}_{t,l}$.

Consider the first term in braces in (2.4):

$$\begin{aligned}
& d\{\frac{1}{\lambda_w} w_t^+ \tilde{w}_t X_{t,0} \lambda_{z^*t} - z_t'\} \\
=& \frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\left[\hat{w}_t^+ + \widehat{\tilde{w}}_t + \hat{\lambda}_{z^*t} + \hat{X}_{t,0}\right] - z'' H \left[\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t - \widehat{\tilde{w}}_t + \hat{X}_{t,0}\right) + \hat{H}_t\right] \\
=& \frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\left\{\left[\hat{w}_t^+ + \widehat{\tilde{w}}_t + \hat{\lambda}_{z^*t} + \hat{X}_{t,0}\right] - \frac{z'' H}{\frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}}\left[\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t - \widehat{\tilde{w}}_t + \hat{X}_{t,0}\right) + \hat{H}_t\right]\right\}.
\end{aligned}$$

Here, don't worry about the fact that $X_{t,0} \equiv 1$, so that $\hat{X}_{t,0} = 0$. Note that in steady state,

$\frac{1}{\lambda_w}\tilde{w}\lambda_{z^*} = z'$, so that this can be written,

$$
\begin{aligned}
& d\{\frac{1}{\lambda_w} w_t^+ \tilde{w}_t \lambda_{z^* t} - z_t'\} \\
= & \frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\left\{\left[\hat{w}_t^+ + \widehat{\tilde{w}}_t + \hat{\lambda}_{z^* t} + \hat{X}_{t,0}\right] - \sigma_L\left[\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t - \widehat{\tilde{w}}_t + \hat{X}_{t,0}\right) + \hat{H}_t\right]\right\} \\
= & \frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\left\{\left(1 - \sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t + \hat{X}_{t,0}\right) + \hat{\lambda}_{z^* t} + \sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_t - \sigma_L\hat{H}_t\right\}
\end{aligned}
$$

Now, consider the second term in braces in (2.4),

$$
\begin{aligned}
& d\{\frac{1}{\lambda_w} w_t^+ \tilde{w}_t X_{t,1}\lambda_{z^* t+1} - z_{t+1}'\} \\
= & \frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\left[\hat{w}_t^+ + \widehat{\tilde{w}}_t + \hat{X}_{t,1} + \hat{\lambda}_{z^* t+1}\right] - z''H\left[\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t - \widehat{\tilde{w}}_{t+1} + \hat{X}_{t,1}\right) + \hat{H}_{t+1}\right] \\
= & \frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\left\{\left[\hat{w}_t^+ + \widehat{\tilde{w}}_t + \hat{X}_{t,1} + \hat{\lambda}_{z^* t+1}\right] - \sigma_L\left[\frac{\lambda_w}{1-\lambda_w}\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t - \widehat{\tilde{w}}_{t+1} + \hat{X}_{t,1}\right) + \hat{H}_{t+1}\right]\right\} \\
= & \frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\left\{\left(1 - \sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\left[\hat{w}_t^+ + \widehat{\tilde{w}}_t + \hat{X}_{t,1}\right] + \hat{\lambda}_{z^* t+1} + \sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_{t+1} - \sigma_L\hat{H}_{t+1}\right\}
\end{aligned}
$$

Finally, consider the l^{th} term in braces in (2.4):

$$
\begin{aligned}
& d\left\{\frac{1}{\lambda_w} w_t^+ \tilde{w}_t X_{t,l}\lambda_{z^* t+l} - z_{t+l}'\right\} \\
= & \frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\left\{\left(1 - \sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\left[\hat{w}_t^+ + \widehat{\tilde{w}}_t + \hat{X}_{t,l}\right] + \hat{\lambda}_{z^* t+l} + \sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_{t+l} - \sigma_L\hat{H}_{t+l}\right\}.
\end{aligned}
$$

Use these results to develop the log-linear expansion of the scaled first order condition. In doing so, we take into account that we need only expand the terms in braces, and not the terms outside of the braces. The coefficients on these expansions are zero because the terms in braces are zero in steady state. Thus,

$$\begin{aligned}
& H_t\left(\frac{1}{\tilde{w}_t}\right)^{\frac{\lambda_w}{1-\lambda_w}}\{\frac{1}{\lambda_w}w_t^+\tilde{w}_t\lambda_{z^*t}-z_t'\} \\
& +(\beta\xi_w)H_{t+1}\left(\frac{X_{t,1}}{\tilde{w}_{t+1}}\right)^{\frac{\lambda_w}{1-\lambda_w}}\{\frac{1}{\lambda_w}w_t^+\tilde{w}_tX_{t,1}\lambda_{z^*t+1}-z_{t+1}'\} \\
& +... \\
& +(\beta\xi_w)^lH_{t+l}\left(\frac{X_{t,l}}{\tilde{w}_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}\{\frac{1}{\lambda_w}w_t^+\tilde{w}_tX_{t,l}\lambda_{z^*t+l}-z_{t+l}'\} \\
& +...=0\ \Rightarrow \\
& H\left(\frac{1}{\tilde{w}}\right)^{\frac{\lambda_w}{1-\lambda_w}}\frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\left\{\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\left(\hat{w}_t^++\widehat{\tilde{w}}_t+\hat{X}_{t,0}\right)+\hat{\lambda}_{z^*t}+\sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_t-\sigma_L\hat{H}_t\right\} \\
& +(\beta\xi_w)H\left(\frac{1}{\tilde{w}}\right)^{\frac{\lambda_w}{1-\lambda_w}}\frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\{\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\left[\hat{w}_t^++\widehat{\tilde{w}}_t+\hat{X}_{t,1}\right] \\
& +\hat{\lambda}_{z^*t+1}+\sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_{t+1}-\sigma_L\hat{H}_{t+1}\} \\
& +... \\
& +(\beta\xi_w)^lH\left(\frac{1}{\tilde{w}}\right)^{\frac{\lambda_w}{1-\lambda_w}}\frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}\{\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\left[\hat{w}_t^++\widehat{\tilde{w}}_t+\hat{X}_{t,l}\right] \\
& +\hat{\lambda}_{z^*t+l}+\sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_{t+l}-\sigma_L\hat{H}_{t+l}\} \\
= & \ 0
\end{aligned}$$

We can divide through by $H\left(\frac{1}{\tilde{w}}\right)^{\frac{\lambda_w}{1-\lambda_w}}\frac{1}{\lambda_w}\tilde{w}$, to obtain

$$\begin{aligned}
& \left\{\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\left(\hat{w}_t^++\widehat{\tilde{w}}_t+\hat{X}_{t,0}\right)+\hat{\lambda}_{z^*t}+\sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_t-\sigma_L\hat{H}_t\right\} \\
& +(\beta\xi_w)\left\{\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\left[\hat{w}_t^++\widehat{\tilde{w}}_t+\hat{X}_{t,1}\right]+\hat{\lambda}_{z^*t+1}+\sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_{t+1}-\sigma_L\hat{H}_{t+1}\right\} \\
& +... \\
& +(\beta\xi_w)^l\left\{\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\left[\hat{w}_t^++\widehat{\tilde{w}}_t+\hat{X}_{t,l}\right]+\hat{\lambda}_{z^*t+l}+\sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_{t+l}-\sigma_L\hat{H}_{t+l}\right\} \\
= & \ 0
\end{aligned}$$

or

$$\begin{aligned}
& \left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\left(\hat{w}_t^++\widehat{\tilde{w}}_t\right)+\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\sum_{l=1}^{\infty}(\beta\xi_w)^l\hat{X}_{t,l} \\
& +\sum_{l=0}^{\infty}(\beta\xi_w)^l\hat{\lambda}_{z^*t+l}+\sigma_L\frac{\lambda_w}{1-\lambda_w}\sum_{l=0}^{\infty}(\beta\xi_w)^l\widehat{\tilde{w}}_{t+l}-\sigma_L\sum_{l=0}^{\infty}(\beta\xi_w)^l\hat{H}_{t+l} \\
= & \ 0.
\end{aligned}$$

We need to work out the sum involving $\hat{X}_{t,l}$. Using (2.3),

$$\hat{X}_{t,j} = -\left(\Delta\hat{\pi}_{t+j} + \Delta\hat{\pi}_{t+j-1} + \cdots + \Delta\hat{\pi}_{t+1}\right) - (1-\vartheta)\left(\hat{\mu}_{z^*,t+j} + \hat{\mu}_{z^*,t+j-1} + \cdots + \hat{\mu}_{z^*,t+1}\right) \tag{2.5}$$

$$\begin{aligned}
& \hat{X}_{t,0} + (\beta\xi_w)\hat{X}_{t,1} + ... + (\beta\xi_w)^l \hat{X}_{t,l} + ... \\
& + (\beta\xi_w)\left[-\Delta\hat{\pi}_{t+1} - (1-\vartheta)\hat{\mu}_{z^*,t+1}\right] \\
& + (\beta\xi_w)^2\left[-\Delta\hat{\pi}_{t+1} - \Delta\hat{\pi}_{t+2} - (1-\vartheta)\hat{\mu}_{z^*,t+1} - (1-\vartheta)\hat{\mu}_{z^*,t+2}\right] \\
& + ... \\
& + (\beta\xi_w)^l [-\Delta\hat{\pi}_{t+1} - \Delta\hat{\pi}_{t+2} - ... - \Delta\hat{\pi}_{t+l} \\
& - (1-\vartheta)\hat{\mu}_{z^*,t+1} - (1-\vartheta)\hat{\mu}_{z^*,t+2} - ... - (1-\vartheta)\hat{\mu}_{z^*,t+l}] \\
& + ... \\
= & -\frac{\beta\xi_w}{1-\beta\xi_w}\Delta\hat{\pi}_{t+1} - \frac{(\beta\xi_w)^2}{1-\beta\xi_w}\Delta\hat{\pi}_{t+2} - ... - \frac{(\beta\xi_w)^l}{1-\beta\xi_w}\Delta\hat{\pi}_{t+l} - ... \\
& -\frac{\beta\xi_w}{1-\beta\xi_w}(1-\vartheta)\hat{\mu}_{z^*,t+1} - \frac{(\beta\xi_w)^2}{1-\beta\xi_w}(1-\vartheta)\hat{\mu}_{z^*,t+2} - ... - \frac{(\beta\xi_w)^l}{1-\beta\xi_w}(1-\vartheta)\hat{\mu}_{z^*,t+l} - ... \\
= & -\frac{1}{1-\beta\xi_w}\sum_{l=1}^{\infty}(\beta\xi_w)^l \Delta\hat{\pi}_{t+l} - (1-\vartheta)\frac{1}{1-\beta\xi_w}\sum_{l=1}^{\infty}(\beta\xi_w)^l \hat{\mu}_{z^*,t+l}.
\end{aligned}$$

Substituting this into the linearized first order condition:

$$\begin{aligned}
& \left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t\right) \\
& -\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\left[\sum_{l=1}^{\infty}(\beta\xi_w)^l \Delta\hat{\pi}_{t+l} + (1-\vartheta)\sum_{l=1}^{\infty}(\beta\xi_w)^l \hat{\mu}_{z^*,t+l}\right] \\
& + \sum_{l=0}^{\infty}(\beta\xi_w)^l \hat{\lambda}_{z^*t+l} + \sigma_L\frac{\lambda_w}{1-\lambda_w}\sum_{l=0}^{\infty}(\beta\xi_w)^l \widehat{\tilde{w}}_{t+l} - \sigma_L\sum_{l=0}^{\infty}(\beta\xi_w)^l \hat{H}_{t+l} \\
= & \; 0.
\end{aligned}$$

It is convenient to write this out in lag-operator form:

$$\begin{aligned}
& \left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t\right) \\
& -\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\frac{\beta\xi_w}{1-\beta\xi_w L^{-1}}\left[\Delta\hat{\pi}_{t+1} + (1-\vartheta)\hat{\mu}_{z^*,t+1}\right] \\
& + \frac{1}{1-\beta\xi_w L^{-1}}\hat{\lambda}_{z^*t} + \sigma_L\frac{\lambda_w}{1-\lambda_w}\frac{1}{1-\beta\xi_w L^{-1}}\widehat{\tilde{w}}_t - \sigma_L\frac{1}{1-\beta\xi_w L^{-1}}\hat{H}_t \\
= & \; 0.
\end{aligned} \tag{2.6}$$

We are now done with the linearized first order condition for the wage rate. We now turn to linearizing the relationship between the aggregate wage and the individual households' wage.

The aggregate wage equation is:

$$W_t = \left[(1-\xi_w) \left(\tilde{W}_t \right)^{\frac{1}{1-\lambda_w}} + \xi_w \left(\pi_{t-1} \left(\mu_{z^*} \right)^{1-\vartheta} \left(\mu_{z^*,t} \right)^{\vartheta} W_{t-1} \right)^{\frac{1}{1-\lambda_w}} \right]^{1-\lambda_w}$$

Dividing this by $z_t^* P_t$, we obtain:

$$\tilde{w}_t = \left[(1-\xi_w) \left(w_t^+ \tilde{w}_t \right)^{\frac{1}{1-\lambda_w}} + \xi_w \left(\frac{\pi_{t-1} \left(\mu_{z^*} \right)^{1-\vartheta} \left(\mu_{z^*,t} \right)^{\vartheta} W_{t-1}}{z_t^* P_t} \right)^{\frac{1}{1-\lambda_w}} \right]^{1-\lambda_w}$$

or,

$$\tilde{w}_t = \left[(1-\xi_w) \left(w_t^+ \tilde{w}_t \right)^{\frac{1}{1-\lambda_w}} + \xi_w \left(\frac{\pi_{t-1} \left(\mu_{z^*} \right)^{1-\vartheta} \left(\mu_{z^*,t} \right)^{\vartheta} W_{t-1}}{\left[z_t^* P_t / \left(z_{t-1}^* P_{t-1} \right) \right] z_{t-1}^* P_{t-1}} \right)^{\frac{1}{1-\lambda_w}} \right]^{1-\lambda_w}$$

or

$$\tilde{w}_t = \left[(1-\xi_w) \left(w_t^+ \tilde{w}_t \right)^{\frac{1}{1-\lambda_w}} + \xi_w \left(\frac{\pi_{t-1} \left(\mu_{z^*} \right)^{1-\vartheta} \left(\mu_{z^*,t} \right)^{\vartheta} \tilde{w}_{t-1}}{\mu_{z^*,t} \pi_t} \right)^{\frac{1}{1-\lambda_w}} \right]^{1-\lambda_w}.$$

This expression is consistent with our previous finding that the steady state value of w_t^+ must be unity. We now linearize this expression. Transform it:

$$\left(\tilde{w}_t \right)^{\frac{1}{1-\lambda_w}} = (1-\xi_w) \left(w_t^+ \tilde{w}_t \right)^{\frac{1}{1-\lambda_w}} + \xi_w \left(\frac{\pi_{t-1} \left(\mu_{z^*} \right)^{1-\vartheta} \left(\mu_{z^*,t} \right)^{\vartheta} \tilde{w}_{t-1}}{\mu_{z^*,t} \pi_t} \right)^{\frac{1}{1-\lambda_w}}$$

Now, totally differentiate:

$$\begin{aligned} \frac{1}{1-\lambda_w} \left(\tilde{w} \right)^{\frac{1}{1-\lambda_w}} \widehat{\tilde{w}}_t &= (1-\xi_w) \frac{1}{1-\lambda_w} \left(\tilde{w} \right)^{\frac{1}{1-\lambda_w}} \left(\hat{w}_t^+ + \widehat{\tilde{w}}_t \right) \\ &+ \xi_w \frac{1}{1-\lambda_w} \left(\tilde{w} \right)^{\frac{1}{1-\lambda_w}} \left(\hat{\pi}_{t-1} + \widehat{\tilde{w}}_{t-1} - (1-\vartheta) \hat{\mu}_{z^*,t} - \hat{\pi}_t \right) \end{aligned}$$

or,

$$\widehat{\tilde{w}}_t = (1-\xi_w) \left(\hat{w}_t^+ + \widehat{\tilde{w}}_t \right) + \xi_w \left(\hat{\pi}_{t-1} + \widehat{\tilde{w}}_{t-1} - (1-\vartheta) \hat{\mu}_{z^*,t} - \hat{\pi}_t \right),$$

or

$$\left(\hat{w}_t^+ + \widehat{\tilde{w}}_t \right) = \frac{1}{1-\xi_w} \widehat{\tilde{w}}_t - \frac{\xi_w}{1-\xi_w} \left(\hat{\pi}_{t-1} + \widehat{\tilde{w}}_{t-1} - (1-\vartheta) \hat{\mu}_{z^*,t} - \hat{\pi}_t \right).$$

Substitute this into (2.6):

$$
\begin{aligned}
&\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\left(\frac{1}{1-\xi_w}\widehat{\tilde{w}}_t-\frac{\xi_w}{1-\xi_w}\left(\hat{\pi}_{t-1}+\widehat{\tilde{w}}_{t-1}-(1-\vartheta)\,\hat{\mu}_{z^*,t}-\hat{\pi}_t\right)\right)\\
&-\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\frac{\beta\xi_w}{1-\beta\xi_w L^{-1}}\left[\Delta\hat{\pi}_{t+1}+(1-\vartheta)\,\hat{\mu}_{z^*,t+1}\right]\\
&+\frac{1}{1-\beta\xi_w L^{-1}}\hat{\lambda}_{z^*t}+\sigma_L\frac{\lambda_w}{1-\lambda_w}\frac{1}{1-\beta\xi_w L^{-1}}\widehat{\tilde{w}}_t-\sigma_L\frac{1}{1-\beta\xi_w L^{-1}}\hat{H}_t\\
=&\;\;0.
\end{aligned}
$$

Now, multiply by $1-\beta\xi_w L^{-1}$,

$$
\begin{aligned}
&\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma}[\frac{1}{\xi_w}\left(\widehat{\tilde{w}}_t-\beta\xi_w\widehat{\tilde{w}}_{t+1}\right)\\
&-\left(\left(\hat{\pi}_{t-1}-\beta\xi_w\hat{\pi}_t\right)+\widehat{\tilde{w}}_{t-1}-\beta\xi_w\widehat{\tilde{w}}_t-(1-\vartheta)\left(\hat{\mu}_{z^*,t}-\beta\xi_w\hat{\mu}_{z^*,t+1}\right)-\left(\hat{\pi}_t-\beta\xi_w\hat{\pi}_{t+1}\right)\right)]\\
&-\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\beta\xi_w\left[\hat{\pi}_{t+1}-\hat{\pi}_t+(1-\vartheta)\,\hat{\mu}_{z^*,t+1}\right]\\
&+\hat{\lambda}_{z^*t}+\sigma_L\frac{\lambda_w}{1-\lambda_w}\widehat{\tilde{w}}_t-\sigma_L\hat{H}_t\\
=&\;\;0,
\end{aligned}
$$

where

$$
\gamma=\frac{(1-\xi_w)(1-\beta\xi_w)}{\xi_w}.
$$

Writing it out explicitly,

$$
\tilde{\eta}_0\widehat{\tilde{w}}_{t-1}+\tilde{\eta}_1\widehat{\tilde{w}}_t+\tilde{\eta}_2\widehat{\tilde{w}}_{t+1}+\tilde{\eta}_3^{-}\hat{\pi}_{t-1}+\tilde{\eta}_3\hat{\pi}_t+\tilde{\eta}_4\hat{\pi}_{t+1}+\tilde{\eta}_5\hat{H}_t+\tilde{\eta}_6\hat{\lambda}_{z^*t}+\tilde{\eta}_7\hat{\mu}_{z^*,t}+\tilde{\eta}_8\hat{\mu}_{z^*,t+1}=0,
$$

where

$$
\begin{aligned}
\tilde{\eta}_0 &= -\left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma} \\
\tilde{\eta}_1 &= \left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma}\left(\frac{1}{\xi_w}+\beta\xi_w\right)+\sigma_L\frac{\lambda_w}{1-\lambda_w} \\
\tilde{\eta}_2 &= -\left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma}\beta \\
\tilde{\eta}_3^- &= -\left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma} \\
\tilde{\eta}_3 &= \left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma}\left(\beta\xi_w+1\right)+\frac{1}{1-\beta\xi_w}\left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\beta\xi_w \\
\tilde{\eta}_4 &= -\left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma}\beta\xi_w-\frac{1}{1-\beta\xi_w}\left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\beta\xi_w \\
\tilde{\eta}_5 &= -\sigma_L \\
\tilde{\eta}_6 &= 1 \\
\tilde{\eta}_7 &= \left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma}\left(1-\vartheta\right) \\
\tilde{\eta}_8 &= \left[-\left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma}\beta\xi_w-\left(1-\sigma_L \frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\beta\xi_w\right]\left(1-\vartheta\right)
\end{aligned}
$$

It is convenient to multiply the $\tilde{\eta}$'s by $(1-\lambda_w)$, and use:

$$
b_w \equiv \frac{\sigma_L\lambda_w-(1-\lambda_w)}{\left(1-\beta\xi_w\right)\left(1-\xi_w\right)}
$$

Note:

$$
\begin{aligned}
& \left(1-\lambda_w\right)\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right) \\
&= \left(\left(1-\lambda_w\right)-\sigma_L\lambda_w\right)\frac{\left(1-\xi_w\right)\left(1-\beta\xi_w\right)}{\left(1-\xi_w\right)\left(1-\beta\xi_w\right)} \\
&= -b_w\left(1-\xi_w\right)\left(1-\beta\xi_w\right),
\end{aligned}
$$

and,

$$
\begin{aligned}
& \left(1-\lambda_w\right)\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma} \\
&= -b_w\left(1-\xi_w\right)\left(1-\beta\xi_w\right)\frac{\xi_w}{\left(1-\xi_w\right)\left(1-\beta\xi_w\right)} \\
&= -b_w\xi_w.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\tilde{\eta}_0(1-\lambda_w) &= \left(\sigma_L\lambda_w - (1-\lambda_w)\right)\frac{1}{\gamma} \\
&= b_w\xi_w \\
\tilde{\eta}_1(1-\lambda_w) &= \left(1-\lambda_w-\sigma_L\lambda_w\right)\frac{1}{\gamma}\left(\frac{1}{\xi_w}+\beta\xi_w\right)+\sigma_L\lambda_w \\
&= -\left(\sigma_L\lambda_w-(1-\lambda_w)\right)\frac{1}{\gamma}\left(\frac{1}{\xi_w}+\beta\xi_w\right)+\sigma_L\lambda_w \\
&= -\left(\sigma_L\lambda_w-(1-\lambda_w)\right)\frac{1}{\gamma\xi_w}\left(1+\beta\xi_w^2\right)+\sigma_L\lambda_w \\
&= -b_w\left(1+\beta\xi_w^2\right)+\sigma_L\lambda_w \\
\tilde{\eta}_2(1-\lambda_w) &= -\left((1-\lambda_w)-\sigma_L\lambda_w\right)\frac{1}{\gamma}\beta \\
&= b_w\xi_w\beta \\
\tilde{\eta}_3^-(1-\lambda_w) &= -\left((1-\lambda_w)-\sigma_L\lambda_w\right)\frac{1}{\gamma} \\
&= b_w\xi_w \\
\tilde{\eta}_3(1-\lambda_w) &= \left((1-\lambda_w)-\sigma_L\lambda_w\right)\frac{1}{\gamma}\left(\beta\xi_w+1\right)+\frac{1}{1-\beta\xi_w}\left((1-\lambda_w)-\sigma_L\lambda_w\right)\beta\xi_w \\
&= -b_w\xi_w\left(\beta\xi_w+1\right)+\frac{\left((1-\lambda_w)-\sigma_L\lambda_w\right)}{\left(1-\xi_w\right)\left(1-\beta\xi_w\right)}\frac{1}{1-\beta\xi_w}\left(1-\xi_w\right)\left(1-\beta\xi_w\right)\beta\xi_w \\
&= -b_w\xi_w\left(\beta\xi_w+1\right)-b_w\left(1-\xi_w\right)\left(1-\beta\xi_w\right)\frac{1}{1-\beta\xi_w}\beta\xi_w \\
&= -b_w\xi_w\left(\beta\xi_w+1\right)-b_w\left(1-\xi_w\right)\beta\xi_w \\
&= -b_w\xi_w\left[\left(\beta\xi_w+1\right)+\left(1-\xi_w\right)\beta\right] \\
&= -b_w\xi_w \\
\tilde{\eta}_4\left(1-\lambda_w\right) &= -\left((1-\lambda_w)-\sigma_L\lambda_w\right)\frac{1}{\gamma}\beta\xi_w-\frac{1}{1-\beta\xi_w}\left((1-\lambda_w)-\sigma_L\lambda_w\right)\beta\xi_w \\
&= b_w\beta\xi_w^2-\frac{1}{1-\beta\xi_w}\frac{\left((1-\lambda_w)-\sigma_L\lambda_w\right)}{\left(1-\beta\xi_w\right)\left(1-\xi_w\right)}\left(1-\beta\xi_w\right)\left(1-\xi_w\right)\beta\xi_w \\
&= b_w\beta\xi_w^2+b_w\left(1-\xi_w\right)\beta\xi_w \\
&= b_w\beta\xi_w \\
\tilde{\eta}_5(1-\lambda_w) &= -\sigma_L(1-\lambda_w)
\end{aligned}
$$

Also,

$$\begin{aligned}
\tilde{\eta}_6(1-\lambda_w) &= (1-\lambda_w) \\
\tilde{\eta}_7(1-\lambda_w) &= (1-\lambda_w)\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma}(1-\vartheta) \\
&= -b_w\xi_w(1-\vartheta) \\
\tilde{\eta}_8(1-\lambda_w) &= (1-\lambda_w)\left[-\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{\gamma}\beta\xi_w-\left(1-\sigma_L\frac{\lambda_w}{1-\lambda_w}\right)\frac{1}{1-\beta\xi_w}\beta\xi_w\right](1-\vartheta) \\
&= \left[b_w\xi_w^2\beta+b_w(1-\xi_w)\beta\xi_w\right](1-\vartheta) \\
&= b_w\beta\xi_w(1-\vartheta)
\end{aligned}$$

Write

$$\eta_i=\tilde{\eta}_i(1-\lambda_w),\ \ i=0,...,8.$$

Then, the wage equation is:

$$\eta_0\hat{\tilde{w}}_{t-1}+\eta_1\hat{\tilde{w}}_t+\eta_2\hat{\tilde{w}}_{t+1}+\eta_{\bar{3}}\hat{\pi}_{t-1}+\eta_3\hat{\pi}_t+\eta_4\hat{\pi}_{t+1}+\eta_5\hat{H}_t+\eta_6\hat{\lambda}_{z^*t}+\eta_7\hat{\mu}_{z^*,t}+\eta_8\hat{\mu}_{z^*,t+1}=0,$$

where

$$\eta=\begin{pmatrix} b_w\xi_w \\ -b_w[1+\beta\xi_w^2]+\sigma_L\lambda_w \\ \beta\xi_w b_w \\ b_w\xi_w(1-\varphi_w) \\ -\xi_w b_w\left[1+(1-\varphi_w)\beta\right] \\ b_w\beta\xi_w \\ -\sigma_L(1-\lambda_w) \\ 1-\lambda_w \\ -b_w\xi_w(1-\vartheta) \\ b_w\beta\xi_w(1-\vartheta) \end{pmatrix}=\begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_{\bar{3}} \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \\ \eta_8 \end{pmatrix}.$$

Finally, taking into account

$$\hat{\mu}_{z^*,t}=\frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon,t}+\hat{\mu}_{z,t}$$

so that

$$\begin{aligned}
\eta_0\hat{\tilde{w}}_{t-1}+\eta_1\hat{\tilde{w}}_t+\eta_2\hat{\tilde{w}}_{t+1}+\eta_{\bar{3}}\hat{\pi}_{t-1}+\eta_3\hat{\pi}_t+\eta_4\hat{\pi}_{t+1}+\eta_5\hat{H}_t+\eta_6\hat{\lambda}_{z^*t}& \\
+\eta_7\frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon,t}+\eta_7\hat{\mu}_{z,t}+\eta_8\frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon,t+1}+\eta_8\hat{\mu}_{z,t+1} &= 0.
\end{aligned}$$

3. Market Clearing and Monetary Policy

Goods market clearing, in terms of scaled variables (careful, this aggregate relationship actually only exists in a steady state....the linearized version also exists in a neighborhood of steady state):

$$P_{t+j}\Upsilon_{t+j}^{-1}I_{t+j}(i) - P_{t+j}\left[a(u_{t+j})\Upsilon_{t+j}^{-1}\right]\bar{K}_{t+j}$$

$$z_t^* = \Upsilon_t^{\frac{\alpha}{1-\alpha}} z_t,$$

$$\left(1+\eta\left(\frac{c_t}{q_t}\right)\right)c_t z_t^* + \Upsilon_t^{-1} i_t \Upsilon_t z_t^* = \epsilon_t\left(u_t\bar{k}_t\Upsilon_{t-1}z_{t-1}^*\right)^{\alpha}(z_t h_t)^{1-\alpha} - a(u_t)\Upsilon_t^{-1}\bar{K}_t - \phi z_t^*$$

$$\left(1+\eta\left(\frac{c_t}{q_t}\right)\right)c_t + \Upsilon_t^{-1} i_t \Upsilon_t = \frac{\epsilon_t\left(u_t\bar{k}_t\Upsilon_{t-1}z_{t-1}^*\right)^{\alpha}(z_t h_t)^{1-\alpha}}{z_t^*} - a(u_t)\Upsilon_t^{-1}\frac{\bar{K}_t}{z_t^*} - \phi$$

$$\left(1+\eta\left(\frac{c_t}{q_t}\right)\right)c_t + \Upsilon_t^{-1} i_t \Upsilon_t = \epsilon_t\left(\frac{u_t\bar{k}_t\Upsilon_{t-1}z_{t-1}^*}{z_t^*}\right)^{\alpha}\left(\frac{z_t h_t}{z_t^*}\right)^{1-\alpha} - a(u_t)\Upsilon_t^{-1}\frac{\bar{K}_t}{z_t^*} - \phi$$

$$\left(1+\eta\left(\frac{c_t}{q_t}\right)\right)c_t + \Upsilon_t^{-1} i_t \Upsilon_t = \epsilon_t\left(\frac{u_t\bar{k}_t\Upsilon_{t-1}z_{t-1}^*}{z_{t-1}^*\left(z_t^*/z_{t-1}^*\right)}\right)^{\alpha}\left(\frac{z_t h_t}{\Upsilon_t^{\frac{\alpha}{1-\alpha}}z_t}\right)^{1-\alpha} - a(u_t)\frac{\bar{K}_t}{\Upsilon_t z_t^*} - \phi$$

$$\left(1+\eta\left(\frac{c_t}{q_t}\right)\right)c_t + \Upsilon_t^{-1} i_t \Upsilon_t = \epsilon_t\left(\frac{u_t\bar{k}_t\Upsilon_{t-1}z_{t-1}^*}{z_{t-1}^*\left(z_t^*/z_{t-1}^*\right)}\right)^{\alpha}\left(\frac{z_t h_t}{\Upsilon_t^{\frac{\alpha}{1-\alpha}}z_t}\right)^{1-\alpha} - a(u_t)\frac{\bar{K}_t}{\Upsilon_{t-1}z_{t-1}^*\left(\Upsilon_t/\Upsilon_{t-1}\right)\left(z_t^*/z_{t-1}^*\right)} - \phi$$

$$\left(1+\eta\left(\frac{c_t}{q_t}\right)\right)c_t + i_t = \epsilon_t\left(\frac{u_t\bar{k}_t}{\mu_{z^*t}\mu_{\Upsilon t}}\right)^{\alpha}h_t^{1-\alpha} - a(u_t)\frac{\bar{k}_t}{\mu_{z^*t}\mu_{\Upsilon t}} - \phi.$$

This is the scaled resource constraint. Log-linearize this:

$$\begin{aligned}
&\eta'\left(\frac{c_t}{q_t}\right)c_t\left(\frac{dc_t}{q_t} - \frac{c_t}{q_t^2}dq_t\right) + \left(1+\eta\left(\frac{c_t}{q_t}\right)\right)dc_t + di_t \\
= \;& \epsilon_t\left(\frac{u_t\bar{k}_t}{\mu_{z^*t}\mu_{\Upsilon t}}\right)^{\alpha}h_t^{1-\alpha}\left[\hat{\epsilon}_t + \alpha\left(\hat{u}_t + \hat{\bar{k}}_t - \hat{\mu}_{z^*t} - \hat{\mu}_{\Upsilon t}\right) + (1-\alpha)\hat{h}_t\right] \\
&-a'(u_t)\frac{\bar{k}_t}{\mu_{z^*t}\mu_{\Upsilon t}}du_t,
\end{aligned}$$

or,

$$\begin{aligned}
&\eta'\frac{c^2}{q}\left(\hat{c}_t - \hat{q}_t\right) + (1+\eta)\,c\hat{c}_t + i\hat{\imath}_t \\
= \;& \left(\frac{\bar{k}}{\mu_{z^*}\mu_{\Upsilon}}\right)^{\alpha}h^{1-\alpha}\left[\hat{\epsilon}_t + \alpha\left(\hat{u}_t + \hat{\bar{k}}_t - \hat{\mu}_{z^*t} - \hat{\mu}_{\Upsilon t}\right) + (1-\alpha)\hat{h}_t\right] \\
&-a'\frac{\bar{k}}{\mu_{z^*}\mu_{\Upsilon}}\hat{u}_t,
\end{aligned}$$

or,

$$\begin{aligned}
&\eta'\frac{c^2}{q}\left(\hat{c}_t - \hat{q}_t\right) + (1+\eta)\,c\hat{c}_t + i\hat{\imath}_t \\
= \;& (\tilde{y}+\phi)\left[\hat{\epsilon}_t + \alpha\left(\hat{u}_t + \hat{\bar{k}}_t - \hat{\mu}_{z^*t} - \hat{\mu}_{\Upsilon t}\right) + (1-\alpha)\hat{h}_t\right] \\
&-\tilde{\rho}\frac{\bar{k}}{\mu_{z^*}\mu_{\Upsilon}}\hat{u}_t = \tilde{y}\hat{\tilde{y}}_t.
\end{aligned}$$

Money market clearing requires:

$$\nu P_t w_t h_t = M_t - Q_t + (x_t - 1)M_t^a$$

Setting $M_t^a = M_t$:

$$\nu P_t w_t h_t = x_t M_t - Q_t.$$

Dividing by $z_t^* P_t$:

$$\nu \tilde{w}_t h_t = x_t m_t - q_t,$$

where the real, scaled monetary base is:

$$m_t = \frac{M_t}{P_t z_t^*}.$$

Log-linearizing the money market clearing condition:

$$\hat{\tilde{w}}_t + \hat{h}_t - \frac{xm\left(\hat{x}_t + \hat{m}_t\right) - q\hat{q}_t}{xm - q} = 0,$$

We adopt the following specification of monetary policy:

$$\hat{x}_t = \hat{x}_{zt} + \hat{x}_{\Upsilon t} + \hat{x}_{Mt},$$

where x_t represents the gross growth rate of high powered money, M_t:

$$M_t = x_{t-1} M_{t-1},$$

or, after dividing by $P_t z_t^*$:

$$m_t = x_{t-1}\frac{P_{t-1} z_{t-1}^*}{P_t z_t^*} m_{t-1} = x_{t-1}\frac{1}{\pi_t \mu_{z^*,t}} m_{t-1}$$

or, after linearizing:

$$\hat{m}_t = \hat{x}_{t-1} - \hat{\pi}_t - \hat{\mu}_{z^*,t} + \hat{m}_{t-1}.$$

We model $\hat{x}_{zt}$ and $\hat{x}_{\Upsilon t}$ as follows:

$$\begin{aligned}
\hat{x}_{M,t} &= \rho_M \hat{x}_{M,t-1} + \varepsilon_{M,t} + \theta_M \varepsilon_{M,t-1} \\
\hat{x}_{z,t} &= \rho_{xz} \hat{x}_{z,t-1} + c_z \varepsilon_{z,t} + c_z^p \varepsilon_{z,t-1} \\
\hat{\mu}_{z,t} &= \rho_{\mu_z} \hat{\mu}_{z,t-1} + \varepsilon_{\mu^z,t} + \theta_{\mu^z} \varepsilon_{\mu^z,t-1}
\end{aligned}$$

also

$$\begin{aligned}
\hat{x}_{\Upsilon,t} &= \rho_{x\Upsilon} \hat{x}_{\Upsilon,t-1} + c_\Upsilon \varepsilon_{\Upsilon,t} + c_\Upsilon^p \varepsilon_{\Upsilon,t-1} \\
\hat{\mu}_{\Upsilon,t} &= \rho_{\mu_\Upsilon} \hat{\mu}_{\Upsilon,t-1} + \varepsilon_{\mu_\Upsilon,t} + \theta_{\mu_\Upsilon} \varepsilon_{\mu_\Upsilon,t-1}
\end{aligned}$$

4. Collecting the Equations

Following are the non-linear equations and the corresponding linearized versions.

4.1. The Firm Sector

The index pertaining to individual firms in the case of the nonlinear equations is suppressed.
Non-linear capital Euler equation:

$$\lambda_{z^*,t} = \beta \frac{1}{\mu_{z^*,t+1}} \lambda_{z^*,t+1} \frac{u_{t+1}\tilde{\rho}_{t+1} - a(u_{t+1}) + \tilde{\mu}_{t+1}(1-\delta)}{\mu_{\Upsilon,t+1}\tilde{\mu}_t},$$

linearized (using $\hat{\mu}_{z^*t} = \frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon t} + \hat{\mu}_{zt}$)

$$z_t = \begin{pmatrix} \hat{c}_t 1(p) \\ \widehat{\tilde{w}}_t 2(p) \\ \hat{\lambda}_{z^*t} 3 \\ \hat{m}_t 4(p) \\ \hat{\pi}_t 5(p) \\ \hat{x}_t 6 \\ \hat{s}_t 7 \\ \hat{\imath}_t 8(p) \\ \hat{h}_t 9 \\ \widehat{\bar{k}}_{t+1} 10(p) \\ \hat{q}_t 11 \\ \widehat{\tilde{y}}_t 12 \\ \hat{R}_t 13 \\ \widehat{\tilde{\mu}}_t 14(p) \\ \widehat{\tilde{\rho}}_t 15 \\ \hat{u}_t 16 \ (p) \end{pmatrix}$$

$$(1) E\left[\hat{\lambda}_{z^*,t+1} - \frac{1}{1-\alpha}\hat{\mu}_{\Upsilon t+1} - \hat{\mu}_{zt+1} - \widehat{\tilde{\mu}}_t + \frac{\tilde{\rho}\widehat{\tilde{\rho}}_{t+1} + (1-\delta)\widehat{\tilde{\mu}}_{t+1}}{\tilde{\rho}+1-\delta} - \hat{\lambda}_{z^*,t}|\Omega_t^p\right] = 0$$

Non-linear investment Euler equation:

$$\begin{aligned} \lambda_{z^*,t} &= \lambda_{z^*,t}\tilde{\mu}_t\left[1 - S\left(\frac{i_t}{i_{t-1}}\mu_{\Upsilon,t}\mu_{z^*,t}\right) - S'\left(\frac{i_t}{i_{t-1}}\mu_{\Upsilon,t}\mu_{z^*,t}\right)\frac{i_t}{i_{t-1}}\mu_{\Upsilon,t}\mu_{z^*,t}\right] \\ &\quad + \beta\frac{1}{\mu_{z^*,t+1}}\lambda_{z^*,t+1}\frac{1}{\mu_{\Upsilon,t+1}}\tilde{\mu}_{t+1}(i)S'\left(\frac{i_{t+1}}{i_t}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}\right)\left(\frac{i_{t+1}}{i_t}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}\right)^2. \end{aligned}$$

Linearized:

$$\begin{aligned} &(2)\ E\{[S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_t - \hat{\imath}_{t-1} + \hat{\mu}_{\Upsilon,t} + \frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon t} + \hat{\mu}_{zt}\right] \\ &-\beta[S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_{t+1} - \hat{\imath}_t + \hat{\mu}_{\Upsilon,t+1} + \frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon t+1} + \hat{\mu}_{zt+1}\right] - \widehat{\tilde{\mu}}_t|\Omega_t^p\} = 0 \end{aligned}$$

Nonlinear expression for shadow rental rate on capital:

$$\tilde{\rho}_t = \frac{\alpha}{1-\alpha} R_t(\nu) \tilde{w}_t \left(\frac{\tilde{y}_t + \phi}{\epsilon_t u_t \bar{k}_t} \mu_{z^*,t} \mu_{\Upsilon,t} \right)^{\frac{1}{1-\alpha}}$$

Linearized (this is an exact relation):

$$(3)\ \frac{\nu R}{\nu R + 1 - \nu} \hat{R}_t + \widehat{\tilde{w}}_t + \frac{1}{1-\alpha} \left(\frac{\tilde{y}}{\tilde{y} + \phi} \widehat{\tilde{y}}_t - \hat{\epsilon}_t - \widehat{\bar{k}}_t + \frac{\alpha}{1-\alpha} \hat{\mu}_{\Upsilon t} + \hat{\mu}_{zt} + \hat{\mu}_{\Upsilon,t} \right) - \widehat{\tilde{\rho}}_t - \frac{1}{1-\alpha} \hat{u}_t = 0.$$

The capital evolution equation:

$$\bar{k}_{t+1} = \frac{(1-\delta)}{\mu_{\Upsilon,t} \mu_{z^*,t}} \bar{k}_t + (1 - S\left(\frac{i_t}{i_{t-1}} \mu_{\Upsilon,t} \mu_{z^*,t} \right)) i_t.$$

Linearized (this is an exact relation):

$$(4)\ \left[\mu_\Upsilon \mu_{z^*} - (1-\delta) \right] \hat{\imath}_t - \left\{ \mu_\Upsilon \mu_{z^*} \widehat{\bar{k}}_{t+1} - (1-\delta) \left[\widehat{\bar{k}}_t - \frac{1}{1-\alpha} \hat{\mu}_{\Upsilon t} - \hat{\mu}_{zt} \right] \right\} = 0$$

The inflation equation is:

$$(5)\ E\left[\beta \left(\hat{\pi}_{t+1} - \hat{\pi}_t \right) + \gamma \hat{s}_t - \left(\hat{\pi}_t - \hat{\pi}_{t-1} \right) | \Omega_t^p \right].$$

The marginal cost equation is (this is an exact relation):

$$(6)\ \frac{\nu R}{\nu R + 1 - \nu} \hat{R}_t + \widehat{\tilde{w}}_t - \hat{\epsilon}_t + \frac{\alpha}{1-\alpha} \left[\frac{\tilde{y}}{\tilde{y} + \phi} \widehat{\tilde{y}}_t - \hat{\epsilon}_t - \hat{u}_t - \widehat{\bar{k}}_t + \frac{1}{1-\alpha} \hat{\mu}_{\Upsilon t} + \hat{\mu}_{zt} \right] - \hat{s}_t = 0$$

4.2. Household Sector

Money demand (this is exact)

$$(7)\ \hat{c}_t - \frac{R}{R-1} \frac{1}{2 + \sigma_\eta} \hat{R}_t - \hat{q}_t = 0$$

The consumption Euler equation:

$$
\begin{aligned}
(8)\qquad & E\{ - \left(\frac{1}{c\left(1 - b\mu_{z^*}^{-1}\right)} \right)^2 \left[c\hat{c}_t - \frac{bc}{\mu_{z^*}} \hat{c}_{t-1} + \frac{bc}{\mu_{z^*}} \left(\frac{\alpha}{1-\alpha} \hat{\mu}_{\Upsilon t} + \hat{\mu}_{zt} \right) \right] \\
& + \beta b \left(\frac{1}{\mu_{z^*_{t+1}} c_{t+1} - bc_t} \right)^2 \left[\mu_{z^*} c \left(\left[\frac{\alpha}{1-\alpha} \hat{\mu}_{\Upsilon t+1} + \hat{\mu}_{zt+1} \right] + \hat{c}_{t+1} \right) - bc\hat{c}_t \right] \\
& - \lambda_{z^*} \left[(1 + \eta(V)) + \eta'(V) V \right] \hat{\lambda}_{z^* t} - \lambda_{z^*} \left[2 + \frac{\eta''(V) V}{\eta'(V)} \right] \eta'(V) V \times (\hat{c}_t - \hat{q}_t) | \Omega_t^p \} \\
= \ & 0.
\end{aligned}
$$

The monetary base first order condition:

$$(9)\ E\left[-\hat{\lambda}_{z^*t}+\hat{\lambda}_{z^*t+1}+\hat{R}_{t+1}-\hat{\pi}_{t+1}-\frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon,t+1}-\hat{\mu}_{z,t+1}|\Omega_t\right]=0.$$

The wage first order condition:

$$\begin{aligned}&(10)\ \eta_0\widehat{\bar{w}}_{t-1}+\eta_1\widehat{\bar{w}}_t+\eta_2\widehat{\bar{w}}_{t+1}+\eta_{\bar{3}}\hat{\pi}_{t-1}+\eta_3\hat{\pi}_t+\eta_4\hat{\pi}_{t+1}\\&+\eta_5\hat{H}_t+\eta_6\hat{\lambda}_{z^*t}+\eta_7\frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon,t}+\eta_7\hat{\mu}_{z,t}+\eta_8\frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon,t+1}+\eta_8\hat{\mu}_{z,t+1}\\=\ &0.\end{aligned}$$

where

$$\eta=\begin{pmatrix}b_w\xi_w\\-b_w[1+\beta\xi_w^2]+\sigma_L\lambda_w\\\beta\xi_w b_w\\b_w\xi_w\\-\xi_w b_w\\b_w\beta\xi_w\\-\sigma_L(1-\lambda_w)\\1-\lambda_w\\-b_w\xi_w(1-\vartheta)\\b_w\beta\xi_w(1-\vartheta)\end{pmatrix}=\begin{pmatrix}\eta_0\\\eta_1\\\eta_2\\\eta_{\bar{3}}\\\eta_3\\\eta_4\\\eta_5\\\eta_6\\\eta_7\\\eta_8\end{pmatrix}.$$

and

$$b_w=[\lambda_w\sigma_L-(1-\lambda_w)]\,/\,[(1-\xi_w)(1-\beta\xi_w)]$$

It is useful to write out the entries in the canonical form for the model directly.

$$\begin{aligned}\alpha_0(10,2)&=\eta_2,\ \alpha_0(10,5)=\eta_4\\\alpha_1(10,2)&=\eta_1,\ \alpha_1(10,5)=\eta_3,\ \alpha_1(10,9)=\eta_5,\\\alpha_1(10,3)&=\eta_6,\\\alpha_2(10,2)&=\eta_0,\ \alpha_2(10,5)=\eta_{\bar{3}}\\\beta_0(10,6)&=\eta_8\frac{\alpha}{1-\alpha},\ \beta_0(10,3)=\eta_8\\\beta_1(10,6)&=\eta_7\frac{\alpha}{1-\alpha},\ \beta_1(10,3)=\eta_7\end{aligned}$$

4.3. Aggregate Conditions

The resource constraint is (this is an exact relation):

$$\begin{aligned}(11)\qquad&(1+\eta)c\hat{c}_t+\eta'\frac{c^2}{q}(\hat{c}_t-\hat{q}_t)+i\hat{\imath}_t\\&-(\tilde{y}+\phi)\left[\hat{\epsilon}_t+\alpha\left(\hat{u}_t+\widehat{\bar{k}}_t-\frac{1}{1-\alpha}\hat{\mu}_{\Upsilon t}-\hat{\mu}_{zt}\right)+(1-\alpha)\hat{h}_t\right]+\tilde{\rho}\frac{\bar{k}}{\mu_{z^*}\mu_\Upsilon}\hat{u}_t\\=\ &0\end{aligned}$$

The money market clearing condition is (this is an exact relation):

$$(12)\ \widehat{\tilde{w}}_t + \hat{h}_t - \frac{xm\left(\hat{x}_t + \hat{m}_t\right) - q\hat{q}_t}{xm - q} = 0,$$

The equation governing monetary policy is:

$$(13)\ \hat{x}_{zt} + \hat{x}_{\Upsilon t} + \hat{x}_{Mt} - \hat{x}_t = 0,$$

The equation linking base growth to the base is (this is an exact relation):

$$(14)\ \hat{x}_{t-1} - \hat{\pi}_t - \frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon t} - \hat{\mu}_{zt} + \hat{m}_{t-1} - \hat{m}_t = 0.$$

The production function:

$$(15)\ \tilde{y}\widehat{\tilde{y}}_t = (\tilde{y} + \phi)\left[\hat{\epsilon}_t + \alpha\left(\hat{u}_t + \widehat{\bar{k}}_t - \frac{1}{1-\alpha}\hat{\mu}_{\Upsilon t} - \hat{\mu}_{zt}\right) + (1-\alpha)\hat{h}_t\right] - \tilde{\rho}\frac{\bar{k}}{\mu_{z^*}\mu_{\Upsilon}}\hat{u}_t$$

The equation governing capital utilization:

$$(16)\ E\left[\hat{u}_t - \frac{1}{\sigma_a}\widehat{\tilde{\rho}}_t|\Omega_t^p\right] = 0$$

5. Solving the Model

5.1. Canonical Form

The canonical representation for the above 16 equations is:

$$\mathcal{E}_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] = 0, \qquad (5.1)$$

where $\mathcal{E}_t$ indicates that the different equations have different information sets. Equations 1, 2, 5, 8, 10, 16 are 'partial information set' equations, because the expectation is conditional on all date t variables, except the date t monetary policy shock. Equations 4 and 14 can also be treated as partial information equations, because the variables in these equations all have the property that they are predetermined relative to the monetary policy shock. So, the partial information equations are 1, 2, 4, 5, 8, 10, 14, 16. There are 8 variables which are predetermined relative to the monetary policy shock: $\hat{c}_t$, $\widehat{\tilde{w}}_t$, $\hat{m}_t$, $\hat{\pi}_t$, $\hat{\imath}_t$, $\widehat{\bar{k}}_{t+1}$, $\widehat{\tilde{\mu}}_t$, $\hat{u}_t$. The other equations and variables are functions of all date t variables and shocks. These restrictions will be imposed in the calculations described below.

Let the vector of shocks be denoted s_t. This is assumed to have the following representation:

$$s_t = Ps_{t-1} + \varepsilon_t,$$

where s_t is not to be confused with real marginal cost! Here, s_t, P and ε_t are defined as follows:

$$\begin{pmatrix} \hat{x}_{M,t} \\ \varepsilon_{M,t} \\ \hat{\mu}_{z,t} \\ \varepsilon_{\mu^z,t} \\ \hat{x}_{z,t} \\ \hat{\mu}_{\Upsilon,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ \hat{x}_{\Upsilon,t} \end{pmatrix} = \begin{bmatrix} \rho_M & \theta_M & & & & & & \\ 0 & 0 & & & & & & \\ & & \rho_{\mu_z} & \theta_{\mu^z} & 0 & & & \\ & & 0 & 0 & 0 & & & \\ & & 0 & c_z^p & \rho_{xz} & & & \\ & & & & & \rho_{\mu_\Upsilon} & \theta_{\mu_\Upsilon} & 0 \\ & & & & & 0 & 0 & 0 \\ & & & & & 0 & c_\Upsilon^p & \rho_{x\Upsilon} \end{bmatrix} \begin{pmatrix} \hat{x}_{M,t-1} \\ \varepsilon_{M,t-1} \\ \hat{\mu}_{z,t-1} \\ \varepsilon_{\mu^z,t-1} \\ \hat{x}_{z,t-1} \\ \hat{\mu}_{\Upsilon,t-1} \\ \varepsilon_{\mu_\Upsilon,t-1} \\ \hat{x}_{\Upsilon,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{M,t} \\ \varepsilon_{M,t} \\ \varepsilon_{\mu^z,t} \\ \varepsilon_{\mu^z,t} \\ c_z\varepsilon_{\mu^z,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ c_\Upsilon\varepsilon_{\mu_\Upsilon,t} \end{pmatrix}$$

Also, the vector of endogenous variables determined at time t are:

$$z_t = \begin{pmatrix} \hat{c}_t 1(p) \\ \widehat{\tilde{w}}_t 2(p) \\ \hat{\lambda}_{z^*t} 3 \\ \hat{m}_t 4(p) \\ \hat{\pi}_t 5(p) \\ \hat{x}_t 6 \\ \hat{s}_t 7 \\ \hat{\imath}_t 8(p) \\ \hat{h}_t 9 \\ \widehat{\bar{\bar{k}}}_{t+1} 10(p) \\ \hat{q}_t 11 \\ \widehat{\tilde{y}}_t 12 \\ \hat{R}_t 13 \\ \widehat{\tilde{\mu}}_t 14(p) \\ \widehat{\tilde{\rho}}_t 15 \\ \hat{u}_t 16\ (p) \end{pmatrix}$$

Here, the number after the variable indicates its order in z_t. A variable with a (p) is one that is predetermined relative to the monetary policy shock.

5.2. Solution to Canonical Form

We seek a solution of the following form:

$$z_t = Az_{t-1} + B\theta_t, \tag{5.2}$$

where A and B are to be determined. Substituting into (5.1) we find:

$$\begin{aligned} \alpha_0 A^2 + \alpha_1 A + \alpha_2 &= 0, \\ \mathcal{E}_t F\theta_t &= \tilde{F}\theta_t, \end{aligned}$$

where

$$F = (\tilde{\beta}_0 + \alpha_0 B)\rho + (\tilde{\beta}_1 + \alpha_1 B + \alpha_0 AB),$$

and θ_t is constructed from s_t. Also, the i^{th} row of $\tilde{F}$ has zeros if the corresponding entries in θ_t are not included in the information set for the i^{th} equation in (5.1). Other relations between $\tilde{F}$ and F are discussed below. Also, $\tilde{\beta}_i$ are constructed from β_i, as explained below. We use the algorithm in Anderson and Moore to find A and we use the strategy in Christiano (2003) to find B.

In the 'full information' case, the conditional information in each equation of (5.1) is based on all date t information. The 'partial information' case corresponds to the case of interest, and is defined in the previous section. In the full information case, $\theta_t = s_t$. In the partial information case,

$$\theta_t = \begin{pmatrix} s_t \\ \hat{x}_{M,t-1} \\ \varepsilon_{M,t-1} \end{pmatrix}.$$

Then,

$$\theta_t = \rho\theta_{t-1} + e_t, \tag{5.3}$$

where

$$\rho_{10\times 10} = \begin{bmatrix} P & 0_{8\times 1} & 0_{8\times 1} \\ \tau & 0_{2\times 1} & 0_{2\times 1} \end{bmatrix},\ e_t = \begin{pmatrix} \varepsilon_{M,t} \\ \varepsilon_{M,t} \\ \varepsilon_{\mu^z,t} \\ \varepsilon_{\mu^z,t} \\ c_z\varepsilon_{\mu^z,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ c_\Upsilon\varepsilon_{\mu_\Upsilon,t} \\ 0 \\ 0 \end{pmatrix} \tag{5.4}$$

where

$$\tau = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Also,

$$\tilde{\beta}_i = \left[\beta_i \vdots \underset{\sim}{0} \vdots \underset{\sim}{0}\right],\ i = 0, 1,$$

where $\underset{\sim}{0}$ is a column vector of zeros.

For finding B, the vectorization operator useful. Recall that the vectorization operator, $vec(\cdot)$, takes the columns of a matrix and stacks them into a row vector:

$$vec(X) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ \text{where } X = [x_1, x_2, ..., x_n].$$

In MATLAB, this operation is achieved by $reshape(X, n \times m, 1)$, where m is the number of rows of X. Two properties of the vectorization operator include additivity, $vec(a+b) = vec(a) + vec(b)$, and

$$vec(A_1 A_2 A_3) = (A_3' \otimes A_1)\, vec(A_2).$$

Write

$$F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{16} \end{bmatrix},$$

so that

$$\begin{aligned} vec(F') &= \begin{bmatrix} F_1' \\ F_2' \\ \vdots \\ F_{16}' \end{bmatrix} = vec\left[\rho'\tilde{\beta}_0' + \rho' B'\alpha_0' + \tilde{\beta}_1' + B'\alpha_1' + B'A'\alpha_0'\right] \\ &= vec\left(\rho'\tilde{\beta}_0' + \tilde{\beta}_1'\right) + vec\left(\rho' B'\alpha_0' + B'\alpha_1' + B'A'\alpha_0'\right) \\ &= vec\left(\rho'\tilde{\beta}_0' + \tilde{\beta}_1'\right) + vec\left(\rho' B'\alpha_0'\right) + vec\left(B'\alpha_1'\right) + vec\left(B'A'\alpha_0'\right) \\ &= vec\left(\rho'\tilde{\beta}_0' + \tilde{\beta}_1'\right) + \{(\alpha_0 \otimes \rho') + (\alpha_1 \otimes I_{10}) + (\alpha_0 A \otimes I_{10})\}\, vec(B') \\ &= d + q\delta, \end{aligned}$$

say, where $\otimes$ denotes the Kronecker product and

$$\begin{aligned} d &= vec\left(\rho'\tilde{\beta}_0' + \tilde{\beta}_1'\right) \\ q &= (\alpha_0 \otimes \rho') + (\alpha_1 \otimes I_{10}) + (\alpha_0 A \otimes I_{10}) \\ \delta &= vec(B'). \end{aligned}$$

In the full information case, finding B is straightforward. Simply compute $\delta = -q^{-1}d$ and construct B from δ.

In the partial information case, this procedure must be adapted. In this case, the entries in B corresponding to the first two elements of θ_t are set to zero in the rows of B corresponding to the partial information equations. Since B is 16×10, there are 160 elements in B. The number to be determined is only $160 - 6 \times 2 = 148$, because there are 6 partial information equations. Let $\overline{vec}(\cdot)$ be the vectorization operator in which the 12 entries that are required to be exactly zero are suppressed. Let R be the matrix which satisfies:

$$\begin{aligned} \overline{vec}(\tilde{F}') &= R vec(F') \\ &= \begin{bmatrix} R_1 F_1' \\ R_2 F_2' \\ \vdots \\ R_{16} F_{16}' \end{bmatrix}, \end{aligned}$$

where

$$R = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{16} \end{bmatrix}$$

If the i^{th} equation is a 'full information' equation, then $R_i = I_{10}$. Now suppose i corresponds to a limited information row. Then,

$$\begin{bmatrix} F_{i,3} \\ F_{i,4} \\ \vdots \\ F_{i,9} + \rho_M F_{i,1} \\ F_{i,10} + \theta_M F_{i,2} \end{bmatrix} = R_i \begin{bmatrix} F_{i,1} \\ F_{i,2} \\ F_{i,3} \\ F_{i,4} \\ \vdots \\ F_{i,9} \\ F_{i,10} \end{bmatrix}$$

Thus, R_i is I_{10} with the first two rows removed and with ρ_M in the 9,1 place and θ_M in the 10,2 place of the resulting matrix. In this case, R_i is an 8×10 matrix, and R is 148×160. So

$$\overline{vec}(\tilde{F}') = Rvec(F') = Rd + Rq\delta.$$

Let

$$\tilde{\delta} = \overline{vec}(B').$$

that is, $\tilde{\delta}$ is δ with the entries which are restricted to be zero suppressed. Let $\tilde{q}$ be Rq in which the columns corresponding to entries of δ that are zero suppressed. Let $\tilde{d} = Rd$. Then,

$$\tilde{d} + \tilde{q}\tilde{\delta} = 0.$$

We solve this by computing

$$\tilde{\delta} = -\tilde{q}^{-1}\tilde{d}.$$

Then, B can be constructed using the elements of $\tilde{\delta}$. To see how this is done, note first:

$$vec(B') = vec\left(\begin{bmatrix} b_1' \\ b_2' \\ \vdots \\ b_{16}' \end{bmatrix}\right),$$

where

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{16} \end{bmatrix}.$$

Given a 160 dimensional vector, $vec(B')$, one computes B' as $reshape(vec(B'), 10, 16)$. One can obtain $vec(B')$ by suitably padding $\tilde{\delta}$ with zeros.

A problem with this model is that it is inconsistent with the CEE identification assumption for monetary policy shocks. For one parameterization, for example, we found that $B(12,1) = -0.0263$. What this means is that output falls with a positive monetary policy shock. The reason is that, given the predeterminate nature of consumption and the price level, the monetary policy shock drives velocity down. Because all other components of demand are fixed, the level of output falls. Similarly, $B(9,1) = -0.0342$, so that hours worked falls. It is useful to understand what these magnitudes mean, precisely. Recall that money growth is:

$$\frac{M_{t+1}}{M_t} = x_t,$$

so that

$$\begin{aligned} \hat{x}_t &= \log\left(\frac{x_t}{x}\right) = \log\left(\frac{M_{t+1}/M_t}{x}\right) \\ &= [\log(M_{t+1}) - \log(M_t)] - \log x. \end{aligned}$$

Similarly,

$$\hat{y}_t = \log\left(\frac{y_t}{y}\right),$$

so that

$$B(12,1) = \frac{d\log(y_t)}{d\log(M_{t+1})},$$

i.e., it is the percent change in output associated with a one percent change in the money stock. So, a one percent rise in the money stock induced by a policy shock produces a 0.0263 percent contemporaneous drop in output. Similarly, a one percent rise in the money stock induced by a policy shock induces a 0.03 percent contemporaneous drop in employment. When all variables and equations are 'full information', then output rises 0.57 percent with a one percent rise in money due to policy. The rise in hours is 0.50 percent.

5.3. Steady State

The steady state rental rate on capital can be computed from:

$$\tilde{\rho} = \frac{\mu_\Upsilon \mu_{z^*}}{\beta} - (1-\delta),$$

where

$$\mu_{z^*} = (\mu_\Upsilon)^{\frac{\alpha}{1-\alpha}} \mu_z.$$

Inflation is given by the usual formula

$$\pi = \frac{x}{\mu_{z^*}}.$$

The Fisherian relation determines the nominal rate of interest:

$$\frac{\pi\mu_{z^*}}{\beta} = R$$

Suppose velocity, V, is preset, say to 1.4. Then, the following equation can be solved for η'.

$$R = 1 + \eta' V^2.$$

Solve for σ_η using:

$$\epsilon = \frac{1}{R-1}\frac{1}{2+\sigma_\eta}\frac{1}{4},$$

that is

$$\sigma_\eta = \frac{1}{R-1}\frac{1}{\epsilon}\frac{1}{4} - 2$$

The variable, s, is the reciprocal of the markup:

$$s = \frac{1}{\lambda_f} = \frac{\theta - 1}{\theta}.$$

Consider the following two conditions:

$$\begin{aligned}
\tilde{\rho} &= \frac{\alpha}{1-\alpha} R(\nu)\tilde{w}\left(\frac{\tilde{y}+\phi}{\bar{k}}\mu_{z^*}\mu_\Upsilon\right)^{\frac{1}{1-\alpha}} \\
s &= \frac{R(\nu)\tilde{w}}{(1-\alpha)}\left(\frac{\tilde{y}+\phi}{\bar{k}}\mu_{z^*}\mu_\Upsilon\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}$$

So, after taking the ratio:

$$\frac{\tilde{\rho}}{s} = \alpha\frac{\tilde{y}+\phi}{\bar{k}}\mu_{z^*}\mu_\Upsilon.$$

In steady state,

$$\tilde{y} + \phi = \left(\frac{\bar{k}}{\mu_{z^*}\mu_\Upsilon}\right)^\alpha h^{1-\alpha} \equiv F$$

or,

$$\frac{\tilde{y}+\phi}{\bar{k}}\mu_{z^*}\mu_\Upsilon = \left(\frac{h}{\bar{k}}\mu_{z^*}\mu_\Upsilon\right)^{1-\alpha}.$$

Substitute this into the expression for $\tilde{\rho}/s$,

$$\frac{\tilde{\rho}}{s} = \alpha\left(\frac{h}{\bar{k}}\mu_{z^*}\mu_\Upsilon\right)^{1-\alpha},$$

(which just says that $\tilde{\rho}$ is the marginal product of capital, divided by the markup) so,

$$\frac{h}{\bar{k}} = (\mu_{z^*}\mu_\Upsilon)^{-1}\left(\frac{\tilde{\rho}}{\alpha s}\right)^{\frac{1}{1-\alpha}}.$$

We can solve for the wage rate, $\tilde{w}$, from

$$\begin{aligned} s &= \frac{R(\nu)\tilde{w}}{(1-\alpha)}\left(\frac{\tilde{y}+\phi}{\bar{k}}\mu_{z^*}\mu_{\Upsilon}\right)^{\frac{\alpha}{1-\alpha}} \\ &= \frac{R(\nu)\tilde{w}}{(1-\alpha)}\left(\frac{\tilde{\rho}}{\alpha s}\right)^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

In what follows, we take two different positions on ϕ, the constant term. In the first case we assume it is positive and that firms make zero profits in steady state. In the second, we assume it is zero. In this case, firms make positive profits in steady state. In terms of the algebra necessary for computing the steady state, the differences between these two cases are slight.

The zero profit condition corresponds, in steady state, to:

$$y_t - w_t R_t(\nu) h_t - \rho_t u_t \bar{K}_t = 0.$$

In terms of scaled variables, this is:

$$\tilde{y}_t z_t^* - z_t^* \tilde{w}_t R_t(\nu) h_t - \Upsilon_t^{-1} \tilde{\rho}_t u_t \bar{k}_t z_t^* \left(z_{t-1}^*/z_t^*\right) \Upsilon_t \left(\Upsilon_{t-1}/\Upsilon_t\right) = 0,$$

or, after dividing by z_t^* an rewriting a little:

$$\tilde{y}_t - \tilde{w}_t R_t(\nu) h_t - \tilde{\rho}_t u_t \frac{\bar{k}_t}{\mu_{z^*,t}\mu_{\Upsilon,t}} = 0,$$

so that, in steady state,

$$\tilde{y} = \tilde{w} R(\nu) h + \frac{\tilde{\rho}\bar{k}}{\mu_{z^*}\mu_{\Upsilon}}.$$

At the same time, price markup behavior leads to the result that total factor costs are less than total variable costs by the amount of the markup:

$$\frac{\tilde{\rho}\bar{k}}{\mu_{z^*}\mu_{\Upsilon}} + \tilde{w} R(\nu) h = sF = \frac{1}{\lambda_f} F,$$

where F is gross production, including the fixed cost,

$$F = \left(\bar{k}\frac{1}{\mu_{z^*}\mu_{\Upsilon}}\right)^{\alpha} (h)^{1-\alpha}.$$

That is, $F = \tilde{y} + \phi$, so that F is the Cobb-Douglas part of the production function. Putting this into the zero profit condition,

$$F - \phi - \frac{1}{\lambda_f} F = 0,$$

or,

$$\left(1-\frac{1}{\lambda_f}\right)F=\phi.$$

It is also usefult to have ϕ in terms of y :

$$\begin{aligned}\left(1-\frac{1}{\lambda_f}\right)y+\phi\left(1-\frac{1}{\lambda_f}\right) &= \phi\\ \left(1-\frac{1}{\lambda_f}\right)y &= \frac{1}{\lambda_f}\phi\\ \left(\lambda_f-1\right)y &= \phi\end{aligned}$$

Combining this with the resource constraint, to obtain:

$$\begin{aligned}\left(1+\eta\right)c+\left[1-\frac{(1-\delta)}{\mu_\Upsilon\mu_{z^*}}\right]\bar{k} &= \left(\bar{k}\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^\alpha(h)^{1-\alpha}-\left(1-\frac{1}{\lambda_f}\right)\left(\bar{k}\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^\alpha(h)^{1-\alpha}\\ &= \frac{1}{\lambda_f}\left(\bar{k}\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^\alpha(h)^{1-\alpha},\end{aligned}$$

where steady state investment has been substituted out for the capital stock. When $\phi=0$ and positive profits are permitted, λ_f in the preceding formula is simply replaced by unity. Rewriting this:

$$c=\bar{k}\frac{\left(\frac{1}{\lambda_f}\left(\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^\alpha\left(\frac{h}{\bar{k}}\right)^{1-\alpha}-\left[1-\frac{(1-\delta)}{\mu_\Upsilon\mu_{z^*}}\right]\right)}{1+\eta},$$

where everything to the right of $\bar{k}$ is known. Again, the case $\phi=0$ requires replacing λ_f with unity in the above expression.

From (2.4) the steady state equation for hours worked is:

$$\frac{1}{\lambda_w}\tilde{w}\lambda_{z^*}=h^{\sigma_L}\psi_L.$$

From (2.1) the first order condition for consumption, in steady state, is:

$$\begin{aligned}\frac{\mu_{z^*}}{\mu_{z^*}c-bc} &= \frac{\beta b}{\mu_{z^*}c-bc}+\lambda_{z^*}\left[1+\eta+\eta' V\right]\\ \lambda_{z^*} &= \frac{1}{c}\frac{\mu_{z^*}-\beta b}{\mu_{z^*}-b}\frac{1}{1+\eta+\eta' V}\end{aligned}$$

Substitute out for λ_{z^*} :

$$\frac{1}{\lambda_w}\tilde{w}\frac{1}{c}\frac{\mu_{z^*}-\beta b}{\mu_{z^*}-b}\frac{1}{1+\eta+\eta' V}=h^{\sigma_L}\psi_L,$$

or,

$$\begin{aligned}c &= \frac{1}{h^{\sigma_L}\psi_L}\frac{1}{\lambda_w}\tilde{w}\frac{\mu_{z^*}-\beta b}{\mu_{z^*}-b}\frac{1}{1+\eta+\eta' V}\\ &= \bar{k}^{-\sigma_L}\frac{\tilde{w}}{\left(\frac{h}{\bar{k}}\right)^{\sigma_L}\psi_L}\frac{\mu_{z^*}-\beta b}{\lambda_w\left(\mu_{z^*}-b\right)}\frac{1}{1+\eta+\eta' V}\end{aligned}$$

Use this to substitute out for c in the expression for c in the resource constraint:

$$\begin{aligned}
&(1+\eta)\,\bar{k}^{-\sigma_L}\frac{\tilde{w}}{\left(\frac{h}{\bar{k}}\right)^{\sigma_L}\psi_L}\frac{\mu_{z^*}-\beta b}{\lambda_w\left(\mu_{z^*}-b\right)}\frac{1}{1+\eta+\eta' V}+\left[1-\frac{(1-\delta)}{\mu_\Upsilon \mu_{z^*}}\right]\bar{k} \\
&= \left(\bar{k}\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^{\alpha}(h)^{1-\alpha}-\left(1-\frac{1}{\lambda_f}\right)\left(\bar{k}\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^{\alpha}(h)^{1-\alpha} \\
&= \frac{1}{\lambda_f}\left(\bar{k}\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^{\alpha}(h)^{1-\alpha},
\end{aligned}$$

$$\begin{aligned}
&(1+\eta)\,\bar{k}^{-\sigma_L}\frac{\tilde{w}}{\left(\frac{h}{\bar{k}}\right)^{\sigma_L}\psi_L}\frac{\mu_{z^*}-\beta b}{\lambda_w\left(\mu_{z^*}-b\right)}\frac{1}{1+\eta+\eta' V} \\
&= \frac{1}{\lambda_f}\left(\bar{k}\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^{\alpha}(h)^{1-\alpha}-\left[1-\frac{(1-\delta)}{\mu_\Upsilon \mu_{z^*}}\right]\bar{k} \\
&= \bar{k}\left\{\frac{1}{\lambda_f}\left(\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^{\alpha}\left(\frac{h}{\bar{k}}\right)^{1-\alpha}-\left[1-\frac{(1-\delta)}{\mu_\Upsilon \mu_{z^*}}\right]\right\}
\end{aligned}$$

so,

$$\begin{aligned}
&\bar{k}^{-\sigma_L}\frac{\tilde{w}}{\left(\frac{h}{\bar{k}}\right)^{\sigma_L}\psi_L}\frac{\mu_{z^*}-\beta b}{\lambda_w\left(\mu_{z^*}-b\right)}\frac{1}{1+\eta+\eta' V} \\
&= \bar{k}\frac{\left\{\frac{1}{\lambda_f}\left(\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^{\alpha}\left(\frac{h}{\bar{k}}\right)^{1-\alpha}-\left[1-\frac{(1-\delta)}{\mu_\Upsilon \mu_{z^*}}\right]\right\}}{(1+\eta)}
\end{aligned}$$

or,

$$\begin{aligned}
\bar{k} &= \left[\frac{\frac{\tilde{w}}{\psi_L\left(\frac{h}{\bar{k}}\right)^{\sigma_L}}\frac{(\mu_{z^*}-\beta b)}{\lambda_w(\mu_{z^*}-b)}\frac{1}{1+\eta+\eta' V}}{\frac{\left(\frac{1}{\lambda_f}\left(\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^{\alpha}\left(\frac{h}{\bar{k}}\right)^{1-\alpha}-\left[1-\frac{(1-\delta)}{\mu_\Upsilon \mu_{z^*}}\right]\right)}{1+\eta}}\right]^{\frac{1}{1+\sigma_L}} \\
&= \left[\frac{\frac{\tilde{w}}{\psi_L\left(\frac{h}{\bar{k}}\right)^{\sigma_L}}\frac{(\mu_{z^*}-\beta b)}{\lambda_w(\mu_{z^*}-b)}\frac{1+\eta}{1+\eta+\eta' V}}{\frac{1}{\lambda_f}\left(\frac{1}{\mu_{z^*}\mu_\Upsilon}\right)^{\alpha}\left(\frac{h}{\bar{k}}\right)^{1-\alpha}-\left(1-\frac{(1-\delta)}{\mu_\Upsilon \mu_{z^*}}\right)}\right]^{\frac{1}{1+\sigma_L}}
\end{aligned}$$

Then, hours worked may be obtained from $h=\bar{k}\times(h/\bar{k})$. The case $\phi=0$ requires replacing λ_f with unity in the above expression.

Finally, we obtain q from

$$q=\frac{c}{V},$$

and m is obtained by solving:

$$\nu\tilde{w}h=xm-q.$$

The variable, λ_{z^*}, can be obtained from the scaled first order condition for consumption:

$$\lambda_{z^*},=\frac{1}{c}\frac{\mu_{z^*}-\beta b}{\mu_{z^*}-b}\frac{1}{1+\eta+\eta' V}$$

6. Estimation

The parameters of the 'non-stochastic part' of the model are:

$$\epsilon, \xi_w, \gamma, S'', \sigma_a, b, \lambda_w, \lambda_f, \sigma_L$$

and

$$\psi_L, \eta, \beta, \mu_\Upsilon, \mu_z, \delta, \alpha, \nu, \psi_L, x, V.$$

The first 9 seem natural candidates for estimation based on impulse response functions. The second group should be fixed based on the estimates in sample averages or the like. The parameters of the stochastic part of the model are the following 15:

$$\rho_M, \theta_M, \rho_{xz}, c_z, c_z^p, \rho_{\mu_z}, \theta_{\mu^z}, \rho_{x\Upsilon}, c_\Upsilon, c_\Upsilon^p, \rho_{\mu_\Upsilon}, \theta_{\mu_\Upsilon}, \sigma_{\mu_\Upsilon}, \sigma_{\mu_z}, \sigma_M.$$

We may want to set the moving average parameters, θ_M, c_z^p, θ_{μ^z}, c_Υ^p, θ_{μ_Υ} to zero and use these only for experiments. This leaves 7 for estimation. Thus, the total number of parameters to be estimated based on impulse responses are 24.

We do the estimation by matching up impulse responses in the model and the data. To do this for the model, set initial conditions to zero, i.e., $\theta_0 = z_0 = 0$. Then assign a value to e_1 and simulate a sequence of θ_t's:

$$\theta_t = \rho^{t-1}\theta_1, \ \theta_1 = e_1, \ t = 2, ..., T.$$

Similarly,

$$z_t = Az_{t-1} + B\theta_t, \ t = 1, 2, ..., T.$$

The elements of z_t can be used to uncover the responses. For example, in the case of a monetary policy shock, the response of log, output is computed as the sequence, $z_{12,t}$, $t = 1, 2, ..., T$. This is interpreted as the log, deviation of output from its unshocked path.

Now consider the response of output to one of the technology shocks. In this case, we have to be careful to take into account that the scaling factor, z_t^*, is also affected. What we want in an impulse response function is the response, relative to what would have happened in the absence of a shock. Now output, y_t, in the presence of a shock is written $\tilde{y}_t z_t^*$. Output in the absence of a shock is $\tilde{y} z_t^{*+}$, where $\tilde{y}$ is the steady state value of $\tilde{y}_t$ and z_t^{*+} is what z^* would have been, had there been no shock. What we want is the logarithm of the following ratio:

$$\frac{y_t}{y_t^+} = \frac{\tilde{y}_t}{\tilde{y}} \frac{z_t^*}{z_t^{*+}}.$$

Now,

$$\begin{aligned} z_1^* &= \mu_{z^*,1} z_0^* \\ z_2^* &= \mu_{z^*,2}\mu_{z^*,1} z_0^* \\ &\ldots \\ z_t^* &= \mu_{z^*,t}\cdots\mu_{z^*,1} z_0^*. \end{aligned}$$

What we can recover from simulations of θ_t is:

$$\hat{\mu}_{z^*,t} = \frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon,t} + \hat{\mu}_{z,t}.$$

Then,

$$\begin{aligned} \mu_{z^*,1} &= \mu_{z^*}\left(\hat{\mu}_{z^*,1} + 1\right) \\ &\ldots \\ \mu_{z^*,T} &= \mu_{z^*}\left(\hat{\mu}_{z^*,T} + 1\right), \end{aligned}$$

giving us the $\mu_{z^*,t}$'s. Now,

$$z_t^{*+} = \mu_{z^*}\cdots\mu_{z^*} z_0^*,$$

so that

$$\frac{z_t^*}{z_t^{*+}} = \frac{\mu_{z^*,t}\cdots\mu_{z^*,1}}{\mu_{z^*}\cdots\mu_{z^*}}.$$

Then,

$$\begin{aligned} \log\left(\frac{y_t}{y_t^+}\right) &= \log\left(\frac{\tilde{y}_t}{\tilde{y}}\right) + \log\left(\frac{z_t^*}{z_t^{*+}}\right) \\ &= \widehat{\tilde{y}}_t + \log\left(\frac{\mu_{z^*,t}\cdots\mu_{z^*,1}}{\mu_{z^*}\cdots\mu_{z^*}}\right) \\ &= \widehat{\tilde{y}}_t + \hat{\mu}_{z^*,t} + \hat{\mu}_{z^*,t-1} + \ldots + \hat{\mu}_{z^*,1}, \end{aligned}$$

for $t = 1, \ldots, T$. The response of consumption, real balances, Q_t/P_t, and the real wage are treated in exactly the same way.

Now consider money growth. We have

$$\hat{x}_t = \log\left(\frac{x_t}{x}\right) = \log\left(M_{t+1}/M_t\right) - \log x,$$

which is money growth relative to what it would have been along an unshocked path. We can multiply this by 4 to put it in annual terms. The deviation of the interest rate from its unshocked value is:

$$\begin{aligned} \hat{R}_t &= \frac{R_t - R}{R} \\ R\hat{R}_t &= R_t - R, \end{aligned}$$

which could be multiplied by 4 to express in self terms.

7. Deriving the Reduced Form Inflation Equation

The strategy for deriving the reduced form inflation process is the usual one. First, derive a relationship between the average price set by optimizing firms and the aggregate inflation rate. Then, derive the first order condition for the price set by optimizing firms. This first order condition resembles the one in the standard Calvo literature in that it involves equating price to marginal cost on average. It is more complicated than usual, however, because marginal cost is idiosyncratic to the individual firm.

7.1. Some Results for Prices

We suppose that non-optimizing firms are partially indexed:

$$P_{t+1}(i) = \pi^{1-\varrho}\pi_t^{\varrho}P_t(i),\ 0 \leq \varrho \leq 1.$$

This is the price set by a firm in period $t+1$, whose price in period t is $P_t(i)$. With $\varrho = 1$ they are fully indexed, and with $\varrho = 0$ they just follow the steady state inflation rate, π. Dividing both sides by P_{t+1} :

$$\frac{P_{t+1}(i)}{P_{t+1}} = \pi^{1-\varrho}\pi_t^{\varrho}\frac{P_t}{P_{t+1}}\frac{P_t(i)}{P_t}$$

or,

$$p_{t+1}(i) = \frac{\pi^{1-\varrho}\pi_t^{\varrho}}{\pi_{t+1}}p_t(i). \tag{7.1}$$

As a consequence,

$$\begin{aligned}\hat{p}_{t+1}(i) &= \hat{p}_t(i) - \hat{\pi}_{t+1} + \varrho\hat{\pi}_t \\ &= \hat{p}_t(i) - \Delta_{\varrho}\hat{\pi}_{t+1},\end{aligned}$$

say, where

$$\Delta_{\varrho}\hat{\pi}_{t+1} = \hat{\pi}_{t+1} - \varrho\hat{\pi}_t.$$

Similarly, for a firm that happens not to have the opportunity to reoptimize in periods $t+1$, $t+2, ..., t+j$:

$$\hat{p}_{t+j}(i) = \hat{p}_t(i) - \Delta_{\varrho}\pi_{t+1} - \Delta_{\varrho}\pi_{t+2} - ... - \Delta_{\varrho}\pi_{t+j}.$$

The aggregate price index must satisfy the following condition:

$$\begin{aligned}P_t &= \left[\int P_t(j)^{1-\theta}dj\right]^{\frac{1}{1-\theta}} \\ &= \left[\int_I P_t^*(i)^{1-\theta}di + \int_J P_t(j)^{1-\theta}dj\right]^{\frac{1}{1-\theta}},\end{aligned}$$

where $i \in I$ corresponds to those intermediate good firms that reoptimize and $j \in J$ corresponds to those firms which do not reoptimize. To see why this is so,

$$\begin{aligned}
\int_i \left(\frac{P_t}{P_t(i)}\right)^{\theta} Y_t di &= y_t(i),\ \theta = \frac{\lambda_f}{\lambda_f - 1} \\
\int_I \hat{y}_t^*(i)di + \int_J \hat{y}_t(j)dj &= \hat{Y}_t
\end{aligned}$$

$$(1-\xi_p)\hat{p}_t^* = \xi_p \Delta_\varrho \hat{\pi}_t,$$

$(1-\xi_p)\xi_p$The firms who Simplifying and dividing by P_t :

$$\begin{aligned}
1 &= \left[\int_I p_t^*(i)^{1-\theta}di + \left(\frac{\pi^{1-\varrho}\pi_{t-1}^{\varrho}}{P_t}\right)^{1-\theta}\int_J P_{t-1}^{1-\theta}dj\right]^{\frac{1}{1-\theta}} \\
&= \left[\int_I p_t^*(i)^{1-\theta}di + \left(\frac{\pi^{1-\varrho}\pi_{t-1}^{\varrho}}{P_t}\right)^{1-\theta}\xi_p P_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}} \\
&= \left[\int_I p_t^*(i)^{1-\theta}di + \xi_p\left(\frac{\pi^{1-\varrho}\pi_{t-1}^{\varrho}}{\frac{P_t}{P_{t-1}}}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}} \\
&= \left[\int_I p_t^*(i)^{1-\theta}di + \xi_p\left(\frac{\pi^{1-\varrho}\pi_{t-1}^{\varrho}}{\pi_t}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}
\end{aligned}$$

Then,

$$1 = \int_I p_t^*(i)^{1-\theta}di + \xi_p\left(\frac{\pi^{1-\varrho}\pi_{t-1}^{\varrho}}{\pi_t}\right)^{1-\theta}$$

Differentiating:

$$\begin{aligned}
0 &= (1-\theta)\int_I p_t^*(i)^{1-\theta}\hat{p}_t^*(i)di + \xi_p(1-\theta)\left(\frac{\pi^{1-\varrho}\pi_{t-1}^{\varrho}}{\pi_t}\right)^{-\theta}\left[\frac{\varrho\pi^{1-\varrho}\left(\pi_{t-1}\right)^{\varrho-1}d\pi_{t-1}}{\pi_t} - \frac{\pi^{1-\varrho}\left(\pi_{t-1}\right)^{\varrho}}{\pi_t^2}d\pi_t\right] \\
&= (1-\theta)\int_I p_t^*(i)^{1-\theta}\hat{p}_t^*(i)di + \xi_p(1-\theta)\left(\frac{\pi^{1-\varrho}\pi_{t-1}^{\varrho}}{\pi_t}\right)^{-\theta}\left[\frac{\varrho\pi^{1-\varrho}\left(\pi_{t-1}\right)^{\varrho}\hat{\pi}_{t-1}}{\pi_t} - \frac{\pi^{1-\varrho}\left(\pi_{t-1}\right)^{\varrho}}{\pi_t}\hat{\pi}_t\right] \\
&= (1-\theta)\int_I p_t^*(i)^{1-\theta}\hat{p}_t^*(i)di + \xi_p(1-\theta)\left(\frac{\pi^{1-\varrho}\pi_{t-1}^{\varrho}}{\pi_t}\right)^{1-\theta}\left[\varrho\hat{\pi}_{t-1} - \hat{\pi}_t\right] \\
&= (1-\theta)\int_I \hat{p}_t^*(i)di + \xi_p(1-\theta)\left[\varrho\hat{\pi}_{t-1} - \hat{\pi}_t\right]
\end{aligned}$$

After dividing and rearranging, and taking into account that $p_t^*(i) = 1$ in a symmetric steady state equilibrium,

$$0 = \int_I \hat{p}_t^*(i)di - \xi_p\Delta_\varrho\hat{\pi}_t.$$

We suppose that

$$\hat{p}_t^*(i) = \hat{p}_t^* + g(i),$$

where

$$\int_I g(i)di = 0.$$

Then,

$$\int_I \hat{p}_t^*(i)di = (1 - \xi_p)\hat{p}_t^*.$$

Substituting,

$$(1 - \xi_p)\hat{p}_t^* = \xi_p \Delta_\varrho \hat{\pi}_t,$$

or,

$$\left(\frac{P_t}{P_t(i)}\right)^\theta Y_t = y_t(i),\ \theta = \frac{\lambda_f}{\lambda_f - 1}$$

the ratio of output to a firm that changes its price to the output of a firm that does not:

$$\begin{aligned}\left(\frac{P_t(i)}{P_t(i')}\right)^\theta &= \frac{y_t(i')}{y_t(i)} \\ \left(\frac{p_t(i)}{p_t(i')}\right)^\theta &= \frac{\tilde{y}_t(i')}{\tilde{y}_t(i)},\end{aligned}$$

so that let the period of the shock be called 1. in this period, all prices are the same, so that all outputs are the same. In the next period, a subset of $\left(1 - \xi_p\right)$ firms gets to reoptimize their prices and on average these prices are set to $\hat{p}_2^* = \frac{\xi_p}{1-\xi_p}\left(\hat{\pi}_2 - \varrho\hat{\pi}_1\right)$.

The output of a firm that optimally sets its price to $\hat{p}_t(i)$ is:

$$\left(\frac{1}{p_t^*(i)}\right)^\theta Y_t = y_t(i),$$

so,

$$-\theta\hat{p}_t^*(i) + \hat{Y}_t = \hat{y}_t^*(i).$$

Integrate over all the $\left(1 - \xi_p\right)$ firms which reoptimize:

$$-\theta\hat{p}_t^* + \hat{Y}_t = \hat{y}_t^*.$$

$$\widehat{\tilde{y}}_t(i') - \widehat{\tilde{y}}_t(i) = \theta\left(\hat{p}_t(i) - \hat{p}_t(i')\right)$$

remember that the integral of output is:

$$\begin{aligned}
\hat{Y}_t &= \int_0^1 \hat{y}_t(i)di && (7.2)\\
&= \int_I \hat{y}_t(i)di + \int_J \hat{y}_t(j)dj && (7.3)\\
&= (1-\xi_p)\left(-\theta\hat{p}_t^* + \hat{Y}_t\right) && (7.4)\\
&\quad +\xi_p && (7.5)
\end{aligned}$$

$$\begin{aligned}
\hat{p}_t^* &= \frac{\xi_p}{1-\xi_p}\Delta_\varrho\hat{\pi}_t. && (7.6)\\
\left(1-\xi_p\right)\hat{p}_t^* + \xi_p x &= \hat{\pi}_t - \hat{\pi}_{t-1} && (7.7)\\
& && (7.8)
\end{aligned}$$

7.2. The Capital Euler Equation

The intertemporal Euler equation is (1.4):

$$\hat{\lambda}_{z^*,t} = \hat{\lambda}_{z^*,t+1} - \hat{\mu}_{z^*,t+1} - \hat{\mu}_{\Upsilon,t+1} - \widehat{\bar{\mu}}_t(i) + \frac{\tilde{\rho}\widehat{\bar{\rho}}_{t+1}(i) + (1-\delta)\widehat{\bar{\mu}}_{t+1}(i)}{\tilde{\rho}+1-\delta}$$

$$\begin{aligned}
(***\,*) \qquad \widehat{\bar{\mu}}_t(i) &= [S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_t(i) - \hat{\imath}_{t-1}(i) + \hat{\mu}_{\Upsilon,t} + \hat{\mu}_{z^*,t}\right] && (7.9)\\
&\quad -\beta[S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}\right]
\end{aligned}$$

$$\begin{aligned}
\hat{\lambda}_{z^*,t} &= \hat{\lambda}_{z^*,t+1} - \hat{\mu}_{z^*,t+1} - \hat{\mu}_{\Upsilon,t+1} - [S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_t(i) - \hat{\imath}_{t-1}(i) + \hat{\mu}_{\Upsilon,t} + \hat{\mu}_{z^*,t}\right]\\
&\quad +\beta[S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}\right]\\
&\quad +\frac{(1-\delta)}{\tilde{\rho}+1-\delta}([S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_t(i) - \hat{\imath}_{t-1}(i) + \hat{\mu}_{\Upsilon,t} + \hat{\mu}_{z^*,t}\right]\\
&\quad -\beta[S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}\right])\\
&\quad +\frac{\tilde{\rho}}{\tilde{\rho}+1-\delta}\left[\frac{\hat{R}_{t+1}(\nu) + \widehat{\bar{w}}_{t+1} + \frac{1}{1-\alpha}\left(\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_{t+1}(i) - \hat{\epsilon}_{t+1} - \widehat{\bar{k}}_{t+1}(i) + \hat{\mu}_{z^*,t+1} + \hat{\mu}_{\Upsilon,t+1}\right)}{1+\frac{1}{1-\alpha}\frac{1}{\sigma_a}}\right]
\end{aligned}$$

Substitute out for $\widehat{\bar{\rho}}_{t+1}(i)$ (rom (1.5)) and $\widehat{\bar{\mu}}_{t+1}(i)$ (from (1.3)):

$$\begin{aligned}
\hat{\lambda}_{z^*,t} &= \hat{\lambda}_{z^*,t+1} - \hat{\mu}_{z^*,t+1} - \hat{\mu}_{\Upsilon,t+1}\\
&\quad -[S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_t(i) - \hat{\imath}_{t-1}(i) + \hat{\mu}_{\Upsilon,t} + \hat{\mu}_{z^*,t}\right] + \beta[S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}\right]\\
&\quad +\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\{[S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}\right]\\
&\quad -\beta[S''](\mu_\Upsilon\mu_{z^*})^2\left[\hat{\imath}_{t+2}(i) - \hat{\imath}_{t+1}(i) + \hat{\mu}_{\Upsilon,t+2} + \hat{\mu}_{z^*,t+2}\right]\}\\
&\quad +\frac{\tilde{\rho}}{\tilde{\rho}+1-\delta}\frac{\hat{R}_{t+1}(\nu) + \widehat{\bar{w}}_{t+1} + \frac{1}{1-\alpha}\left(\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_{t+1}(i) - \hat{\epsilon}_{t+1} - \widehat{\bar{k}}_{t+1}(i) + \hat{\mu}_{z^*,t+1} + \hat{\mu}_{\Upsilon,t+1}\right)}{1+\frac{1}{1-\alpha}\frac{1}{\sigma_a}},
\end{aligned}$$

or,

$$
\begin{aligned}
\hat{\lambda}_{z^*,t} &= \hat{\lambda}_{z^*,t+1} - \hat{\mu}_{z^*,t+1} - \hat{\mu}_{\Upsilon,t+1} \\
&- [S''] (\mu_{\Upsilon}\mu_{z^*})^2 \{ [\hat{\imath}_t(i) - \hat{\imath}_{t-1}(i) + \hat{\mu}_{\Upsilon,t} + \hat{\mu}_{z^*,t}] \\
&- \left(\beta + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right) [\hat{\imath}_{t+1}(i) - \hat{\imath}_t(i) + \hat{\mu}_{\Upsilon,t+1} + \hat{\mu}_{z^*,t+1}] \\
&+ \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta [\hat{\imath}_{t+2}(i) - \hat{\imath}_{t+1}(i) + \hat{\mu}_{\Upsilon,t+2} + \hat{\mu}_{z^*,t+2}] \} \\
&+ \frac{\tilde{\rho}}{\tilde{\rho}+1-\delta} \frac{\hat{R}_{t+1}(\nu) + \widehat{\tilde{w}}_{t+1} + \frac{1}{1-\alpha}\left(\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_{t+1}(i) - \hat{\epsilon}_{t+1} - \widehat{\bar{k}}_{t+1}(i) + \hat{\mu}_{z^*,t+1} + \hat{\mu}_{\Upsilon,t+1}\right)}{1 + \frac{1}{1-\alpha}\frac{1}{\sigma_a}},
\end{aligned}
$$

From this equation, subtract the equation that results after aggregating over all i (simply delete the (i) argument wherever it appears):

$$
\begin{aligned}
0 &= -[S''] (\mu_{\Upsilon}\mu_{z^*})^2 \{ [\hat{\imath}_t^+(i) - \hat{\imath}_{t-1}^+(i)] - \left(\beta + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right) [\hat{\imath}_{t+1}^+(i) - \hat{\imath}_t^+(i)] \\
&+ \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta [\hat{\imath}_{t+2}^+(i) - \hat{\imath}_{t+1}^+(i)] \} \\
&+ \frac{\tilde{\rho}}{\tilde{\rho}+1-\delta} \frac{\frac{\tilde{y}}{\tilde{y}+\phi}\widehat{\tilde{y}}_{t+1}^+(i) - \widehat{\bar{k}}_{t+1}^+(i)}{1-\alpha+\frac{1}{\sigma_a}},
\end{aligned}
$$

where a '+' means the i^{th} firm's value of the variable, minus the aggregate.

From the firm's demand curve:

$$
\widehat{\tilde{y}}_t^+(i) = -\theta\hat{p}_t(i)
$$

Substitute this into the preceding expression:

$$
\begin{aligned}
0 &= -[S''] (\mu_{\Upsilon}\mu_{z^*})^2 \{ [\hat{\imath}_t^+(i) - \hat{\imath}_{t-1}^+(i)] - \left(\beta + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right) [\hat{\imath}_{t+1}^+(i) - \hat{\imath}_t^+(i)] \\
&+ \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta [\hat{\imath}_{t+2}^+(i) - \hat{\imath}_{t+1}^+(i)] \} \\
&- \frac{\tilde{\rho}}{\tilde{\rho}+1-\delta} \frac{\frac{\tilde{y}}{\tilde{y}+\phi}\theta\hat{p}_{t+1}(i) + \widehat{\bar{k}}_{t+1}^+(i)}{1-\alpha+\frac{1}{\sigma_a}},
\end{aligned}
$$

From the capital evolution equation, (1.6),

$$
\begin{aligned}
\hat{\imath}_t^+(i) &= \frac{\mu_{\Upsilon}\mu_{z^*}\widehat{\bar{k}}_{t+1}^+(i) - (1-\delta)\widehat{\bar{k}}_t^+(i)}{\mu_{\Upsilon}\mu_{z^*} - (1-\delta)} \\
&= a_i(L)\widehat{\bar{k}}_{t+1}^+(i),
\end{aligned}
$$

where

$$
\begin{aligned}
a_i(L) &= \frac{\mu_\Upsilon \mu_{z^*}}{\mu_\Upsilon \mu_{z^*} - (1-\delta)} - \frac{(1-\delta)L}{\mu_\Upsilon \mu_{z^*} - (1-\delta)} \\
&= \frac{\mu_\Upsilon \mu_{z^*}}{\mu_\Upsilon \mu_{z^*} - (1-\delta)} \left(1 - \frac{1-\delta}{\mu_\Upsilon \mu_{z^*}} L\right).
\end{aligned}
$$

Substitute this into the capital euler equation:

$$
\begin{aligned}
0 &= -\left[S''\right] \left(\mu_\Upsilon \mu_{z^*}\right)^2 \{a_i(L)(1-L)\widehat{\bar{k}}^{+}_{t+1}(i) - \left(\beta + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right) a_i(L)(1-L)\widehat{\bar{k}}^{+}_{t+2}(i) \\
&\quad + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta a_i(L)(1-L)\widehat{\bar{k}}^{+}_{t+3}(i)\} \\
&\quad - \frac{\tilde{\rho}}{\tilde{\rho}+1-\delta} \frac{\frac{\tilde{y}}{\tilde{y}+\phi}\theta \hat{p}_{t+1}(i) + \widehat{\bar{k}}^{+}_{t+1}(i)}{1-\alpha+\frac{1}{\sigma_a}},
\end{aligned}
$$

or, after dividing by $-\left[S''\right] \left(\mu_\Upsilon \mu_{z^*}\right)^2$:

$$
\begin{aligned}
0 &= \{a_i(L)(1-L)L^2 - \left(\beta + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right) a_i(L)(1-L)L \\
&\quad + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta a_i(L)(1-L)\}\widehat{\bar{k}}^{+}_{t+3}(i) \\
&\quad + \frac{\tilde{\rho}}{\tilde{\rho}+1-\delta} \frac{1}{\left[S''\right] \left(\mu_\Upsilon \mu_{z^*}\right)^2} \frac{\frac{\tilde{y}}{\tilde{y}+\phi}\theta \hat{p}_{t+1}(i) + L^2\widehat{\bar{k}}^{+}_{t+3}(i)}{1-\alpha+\frac{1}{\sigma_a}},
\end{aligned}
$$

or,

$$
\begin{aligned}
0 &= \{a_i(L)(1-L)L^2 - \left(\beta + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right) a_i(L)(1-L)L \\
&\quad + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta a_i(L)(1-L) + \frac{1}{\left[S''\right] \left(\mu_\Upsilon \mu_{z^*}\right)^2} \frac{\tilde{\rho}}{\tilde{\rho}+1-\delta} \frac{L^2}{1-\alpha+\frac{1}{\sigma_a}}\}\widehat{\bar{k}}^{+}_{t+3}(i) \\
&\quad + \frac{1}{\left[S''\right] \left(\mu_\Upsilon \mu_{z^*}\right)^2} \frac{\tilde{\rho}}{\tilde{\rho}+1-\delta} \frac{\frac{\tilde{y}}{\tilde{y}+\phi}\theta}{1-\alpha+\frac{1}{\sigma_a}} \hat{p}_{t+1}(i),
\end{aligned}
$$

or,

$$
Q(L)E\left[\widehat{\bar{k}}^{+}_{t+3}(i)|\Omega_t^p\right] = \Phi E\left[\hat{p}_{t+1}(i)|\Omega_t^p\right] \tag{7.10}
$$

where

$$
\begin{aligned}
& Q(L) \\
&= a_i(L)(1-L)\left[L^2 - \left(\beta + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right)L + \frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta\right] \\
&\quad + \frac{1}{\left[S''\right] \left(\mu_\Upsilon \mu_{z^*}\right)^2} \frac{\tilde{\rho}}{\tilde{\rho}+1-\delta} \frac{1}{1-\alpha+\frac{1}{\sigma_a}} L^2 \\
& \Phi = -\frac{1}{\left[S''\right] \left(\mu_\Upsilon \mu_{z^*}\right)^2} \frac{\tilde{\rho}}{\tilde{\rho}+1-\delta} \frac{\frac{\tilde{y}}{\tilde{y}+\phi}\theta}{1-\alpha+\frac{1}{\sigma_a}}.
\end{aligned}
$$

It is useful to write out the coefficients on powers of L in $Q(L)$ explicitly

$$
\begin{aligned}
& a_i(L)(1-L)\left[L^2-\left(\beta+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right)L+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta\right] \\
& +\frac{1}{[S''](\mu_\Upsilon\mu_{z^*})^2}\frac{\tilde{\rho}}{\tilde{\rho}+1-\delta}\frac{1}{1-\alpha+\frac{1}{\sigma_a}}L^2 \\
= \; & \frac{\mu_\Upsilon\mu_{z^*}}{\mu_\Upsilon\mu_{z^*}-(1-\delta)}\left(1-\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}L\right)(1-L)\left[L^2-\left(\beta+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right)L+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta\right] \\
& +\frac{1}{[S''](\mu_\Upsilon\mu_{z^*})^2}\frac{\tilde{\rho}}{\tilde{\rho}+1-\delta}\frac{1}{1-\alpha+\frac{1}{\sigma_a}}L^2 \\
= \; & \frac{\mu_\Upsilon\mu_{z^*}}{\mu_\Upsilon\mu_{z^*}-(1-\delta)}\left(1-\left[1+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\right]L+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}L^2\right)\left[L^2-\left(\beta+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right)L+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta\right] \\
& +\frac{1}{[S''](\mu_\Upsilon\mu_{z^*})^2}\frac{\tilde{\rho}}{\tilde{\rho}+1-\delta}\frac{1}{1-\alpha+\frac{1}{\sigma_a}}L^2 \\
= \; & \frac{\mu_\Upsilon\mu_{z^*}}{\mu_\Upsilon\mu_{z^*}-(1-\delta)}\{L^2-\left(\beta+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right)L+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta \\
& -\left(1+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\right)L^3+\left(1+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\right)\left(\beta+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right)L^2-\left(1+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\right)\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta L \\
& +\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}L^4-\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\left(\beta+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right)L^3+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta L^2\} \\
& +\frac{1}{[S''](\mu_\Upsilon\mu_{z^*})^2}\frac{\tilde{\rho}}{\tilde{\rho}+1-\delta}\frac{1}{1-\alpha+\frac{1}{\sigma_a}}L^2 \\
= \; & \left[\frac{\mu_\Upsilon\mu_{z^*}}{\mu_\Upsilon\mu_{z^*}-(1-\delta)}\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta\right]-\frac{\mu_\Upsilon\mu_{z^*}}{\mu_\Upsilon\mu_{z^*}-(1-\delta)}\left[\beta+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}+\left(1+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\right)\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta\right]L \\
& +\{\frac{\mu_\Upsilon\mu_{z^*}}{\mu_\Upsilon\mu_{z^*}-(1-\delta)}\left[1+\left(1+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\right)\left(\beta+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right)+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\beta\right] \\
& +\frac{1}{[S''](\mu_\Upsilon\mu_{z^*})^2}\frac{\tilde{\rho}}{\tilde{\rho}+1-\delta}\frac{1}{1-\alpha+\frac{1}{\sigma_a}}\}L^2 \\
& -\frac{\mu_\Upsilon\mu_{z^*}}{\mu_\Upsilon\mu_{z^*}-(1-\delta)}\left[1+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}+\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}\left(\beta+\frac{(1-\delta)}{\tilde{\rho}+1-\delta}\right)\right]L^3+\frac{\mu_\Upsilon\mu_{z^*}}{\mu_\Upsilon\mu_{z^*}-(1-\delta)}\frac{1-\delta}{\mu_\Upsilon\mu_{z^*}}L^4 \\
= \; & \gamma_0+\gamma_1 L+\gamma_2 L^2+\gamma_3 L^3+\gamma_4 L^4,
\end{aligned}
$$

say.

We posit (and later verify) that in equilibrium the following relations are satisfied:

$$\hat{\bar{k}}^{+}_{t+1}(i) = \kappa_1\hat{\bar{k}}^{+}_t(i)+\kappa_2\hat{\bar{k}}^{+}_{t-1}(i)+\kappa_3\hat{p}_t(i) \tag{7.11}$$

$$\hat{p}^*_t(i) = \hat{p}^*_t-\psi_0\hat{\bar{k}}^{+}_t(i)-\psi_1\hat{\bar{k}}^{+}_{t-1}(i),\ \hat{\bar{k}}^{+}_t(i)\equiv\hat{\bar{k}}_t(i)-\hat{\bar{k}}_t, \tag{7.12}$$

where κ_1, κ_2, κ_3,ψ_0, ψ_1 are coefficients to be determined.

From the standpoint of period t, in period $t+1$ the i^{th} firm has probability ξ_p of not being able to reoptimize its price, and probability $1-\xi_p$ of being able to reoptimize. The price it sets (relative to the aggregate price) if it is able to reoptimize in $t+1$ is denoted $\hat{p}^*_{t+1}(i)$. Then,

$$\begin{aligned}
E_t\hat{p}_{t+1}(i) &= \xi_p\left[\hat{p}_t(i) - \Delta_\varrho E_t\pi_{t+1}\right] + (1-\xi_p)E_t\hat{p}^*_{t+1}(i) \\
&= \xi_p\left[\hat{p}_t(i) - \Delta_\varrho\pi_{t+1}\right] + (1-\xi_p)\left[\hat{p}^*_{t+1} - \psi_0\widehat{\bar{k}}^+_{t+1}(i) - \psi_1\widehat{\bar{k}}^+_t(i)\right] \\
&= \xi_p\left[\hat{p}_t(i) - \Delta_\varrho\pi_{t+1}\right] + (1-\xi_p)\left[\frac{\xi_p}{1-\xi_p}\Delta_\varrho\pi_{t+1} - \psi_0\widehat{\bar{k}}^+_{t+1}(i) - \psi_1\widehat{\bar{k}}^+_t(i)\right] \\
&= \xi_p\hat{p}_t(i) + (1-\xi_p)\left[-\psi_0\left(\kappa_1\widehat{\bar{k}}^+_t(i) + \kappa_2\widehat{\bar{k}}^+_{t-1}(i) + \kappa_3\hat{p}_t(i)\right) - \psi_1\widehat{\bar{k}}^+_t(i)\right] \\
&= \xi_p\hat{p}_t(i) - (1-\xi_p)\psi_0\kappa_1\widehat{\bar{k}}^+_t(i) - (1-\xi_p)\psi_0\kappa_2\widehat{\bar{k}}^+_{t-1}(i) - (1-\xi_p)\psi_0\kappa_3\hat{p}_t(i) - (1-\xi_p)\psi_1\widehat{\bar{k}}^+_t(i) \\
&= \left[\xi_p - (1-\xi_p)\psi_0\kappa_3\right]\hat{p}_t(i) - (1-\xi_p)\left[\psi_0\kappa_1 + \psi_1\right]\widehat{\bar{k}}^+_t(i) - (1-\xi_p)\psi_0\kappa_2\widehat{\bar{k}}^+_{t-1}(i)
\end{aligned}$$

Substitute this into (7.10):

$$Q(L)E\left[\widehat{\bar{k}}^+_{t+3}(i)|\Omega^p_t\right] = \Phi\left[\xi_p - (1-\xi_p)\psi_0\kappa_3\right]\hat{p}_t(i) - \Phi(1-\xi_p)\left[\psi_0\kappa_1 + \psi_1\right]L^3\widehat{\bar{k}}^+_{t+3}(i) - \Phi(1-\xi_p)\psi_0\kappa_2 L^4\widehat{\bar{k}}^+_{t+3}(i),$$

or,

$$\tilde{Q}(L)E\left[\widehat{\bar{k}}^+_{t+3}(i)|\Omega^p_t\right] = \Phi\left[\xi_p - (1-\xi_p)\psi_0\kappa_3\right]\hat{p}_t(i),$$

where

$$\begin{aligned}
\tilde{Q}(L) &= Q(L) + \Phi(1-\xi_p)\left[\psi_0\kappa_1 + \psi_1\right]L^3 + \Phi(1-\xi_p)\psi_0\kappa_2 L^4 \\
&= \tilde{\gamma}_0 + \tilde{\gamma}_1 L + \tilde{\gamma}_2 L^2 + \tilde{\gamma}_3 L^3 + \tilde{\gamma}_4 L^4,
\end{aligned}$$

say. Then,

$$\tilde{\gamma}_0 E_t\widehat{\bar{k}}^+_{t+3}(i) + \tilde{\gamma}_1 E_t\widehat{\bar{k}}^+_{t+2}(i) + \tilde{\gamma}_2\widehat{\bar{k}}^+_{t+1}(i) + \tilde{\gamma}_3\widehat{\bar{k}}^+_t(i) + \tilde{\gamma}_4\widehat{\bar{k}}^+_{t-1}(i) = \Phi\left[\xi_p - (1-\xi_p)\psi_0\kappa_3\right]\hat{p}_t(i) \tag{7.13}$$

To evaluate this, we require $E_t\widehat{\bar{k}}^+_{t+3}(i)$ and $E_t\widehat{\bar{k}}^+_{t+2}(i)$. Consider the first of these:

$$E_t\widehat{\bar{k}}^+_{t+3}(i) = \kappa_1 E_t\widehat{\bar{k}}^+_{t+2}(i) + \kappa_2\widehat{\bar{k}}^+_{t+1}(i) + \kappa_3 E_t\hat{p}_{t+2}(i)$$

$$
\begin{aligned}
E_t\hat{p}_{t+2}(i) &= \xi_p^2\left[\hat{p}_t(i) - \Delta_\varrho\pi_{t+2} - \Delta_\varrho\pi_{t+1}\right] \text{ (don't change in } t+1 \text{ and } t+2) \\
&\quad +(1-\xi_p)\xi_p\left[\hat{p}^*_{t+1}(i) - \Delta_\varrho\pi_{t+2}\right] \text{ (change in } t+1 \text{ don't change in } t+2) \\
&\quad +\xi_p(1-\xi_p)\hat{p}^*_{t+2}(i) \text{ (don't change in } t+1 \text{ do change in } t+2) \\
&\quad +(1-\xi_p)^2\hat{p}^*_{t+2}(i) \text{ (change in } t+1 \text{ and } t+2) \\
&= \xi_p^2\left[\hat{p}_t(i) - \Delta_\varrho\pi_{t+2} - \Delta_\varrho\pi_{t+1}\right] \\
&\quad +(1-\xi_p)\xi_p\left[\hat{p}^*_{t+1} - \psi_0\hat{\bar{k}}^+_{t+1}(i) - \psi_1\hat{\bar{k}}^+_t(i) - \Delta_\varrho\pi_{t+2}\right] \\
&\quad +\xi_p(1-\xi_p)\left[\hat{p}^*_{t+2} - \psi_0\hat{\bar{k}}^+_{t+2}(i) - \psi_1\hat{\bar{k}}^+_{t+1}(i)\right] \\
&\quad +\left(1-\xi_p\right)^2\left[\hat{p}^*_{t+2} - \psi_0\hat{\bar{k}}^+_{t+2}(i) - \psi_1\hat{\bar{k}}^+_{t+1}(i)\right].
\end{aligned}
$$

To avoid cluttering notation, the last expression does not distinguish between $\hat{\bar{k}}^+_{t+2}(i)$ chosen in a period $t+1$ history when price reoptimization was permitted and a period $t+1$ history when it was not.

$$
\begin{aligned}
E_t\hat{p}_{t+2}(i) &= \xi_p^2\left[\hat{p}_t(i) - \Delta_\varrho\pi_{t+2} - \Delta_\varrho\pi_{t+1}\right] \\
&\quad +(1-\xi_p)\xi_p\left[\hat{p}^*_{t+1} - \psi_0\hat{\bar{k}}^+_{t+1}(i) - \psi_1\hat{\bar{k}}^+_t(i) - \Delta_\varrho\pi_{t+2}\right] \\
&\quad +\xi_p(1-\xi_p)\left[\hat{p}^*_{t+2} - \psi_0\left(\kappa_1\hat{\bar{k}}^+_{t+1}(i) + \kappa_2\hat{\bar{k}}^+_t(i) + \kappa_3\left[\hat{p}_t(i) - \Delta_\varrho\pi_{t+1}\right]\right) - \psi_1\hat{\bar{k}}^+_{t+1}(i)\right] \\
&\quad +\left(1-\xi_p\right)^2\left[\hat{p}^*_{t+2} - \psi_0\left(\kappa_1\hat{\bar{k}}^+_{t+1}(i) + \kappa_2\hat{\bar{k}}^+_t(i) + \kappa_3\left[\hat{p}^*_{t+1} - \psi_0\hat{\bar{k}}^+_{t+1}(i) - \psi_1\hat{\bar{k}}^+_t(i)\right]\right)\right. \\
&\quad \left.-\psi_1\hat{\bar{k}}^+_{t+1}(i)\right]
\end{aligned}
$$

or,

$$
\begin{aligned}
E_t\hat{p}_{t+2}(i) &= \xi_p^2\left[\hat{p}_t(i) - \Delta_\varrho\pi_{t+2} - \Delta_\varrho\pi_{t+1}\right] \\
&\quad +(1-\xi_p)\xi_p\left[\frac{\xi_p}{1-\xi_p}\Delta_\varrho\pi_{t+1} - \psi_0\hat{\bar{k}}^+_{t+1}(i) - \psi_1\hat{\bar{k}}^+_t(i) - \Delta_\varrho\pi_{t+2}\right] \\
&\quad +\xi_p(1-\xi_p)\Big(\frac{\xi_p}{1-\xi_p}\Delta_\varrho\pi_{t+2} - (\psi_0\kappa_1 + \psi_1)\,\hat{\bar{k}}^+_{t+1}(i) \\
&\quad -\psi_0\kappa_2\hat{\bar{k}}^+_t(i) - \psi_0\kappa_3\left[\hat{p}_t(i) - \Delta_\varrho\pi_{t+1}\right]\Big) \\
&\quad +\left(1-\xi_p\right)^2\Big[\frac{\xi_p}{1-\xi_p}\Delta_\varrho\pi_{t+2} - \left(\psi_0\kappa_1 + \psi_1 - \psi_0^2\kappa_3\right)\hat{\bar{k}}^+_{t+1}(i) \\
&\quad -\left(\psi_0\kappa_2 - \psi_0\psi_1\kappa_3\right)\hat{\bar{k}}^+_t(i) - \psi_0\kappa_3\frac{\xi_p}{1-\xi_p}\Delta_\varrho\pi_{t+1}\Big]
\end{aligned}
$$

or,

$$
\begin{aligned}
E_t\hat{p}_{t+2}(i) &= \xi_p^2\left[\hat{p}_t(i)-\Delta_\varrho\pi_{t+2}-\Delta_\varrho\pi_{t+1}\right] \\
&+\xi_p^2\Delta_\varrho\pi_{t+1}-(1-\xi_p)\xi_p\Delta_\varrho\pi_{t+2}-(1-\xi_p)\xi_p\left[\psi_0\widehat{\bar{k}}_{t+1}^{+}(i)+\psi_1\widehat{\bar{k}}_t^{+}(i)\right] \\
&+\xi_p^2\Delta_\varrho\pi_{t+2}+\xi_p(1-\xi_p)\psi_0\kappa_3\Delta_\varrho\pi_{t+1} \\
&-\xi_p(1-\xi_p)\left[\left(\psi_0\kappa_1+\psi_1\right)\widehat{\bar{k}}_{t+1}^{+}(i)+\psi_0\kappa_2\widehat{\bar{k}}_t^{+}(i)+\psi_0\kappa_3\hat{p}_t(i)\right] \\
&+\xi_p\left(1-\xi_p\right)\left[\Delta_\varrho\pi_{t+2}-\psi_0\kappa_3\Delta_\varrho\pi_{t+1}\right]-\left(1-\xi_p\right)^2\left[\left(\psi_0\kappa_1+\psi_1-\psi_0^2\kappa_3\right)\widehat{\bar{k}}_{t+1}^{+}(i)\right. \\
&\left.+\left(\psi_0\kappa_2-\psi_0\psi_1\kappa_3\right)\widehat{\bar{k}}_t^{+}(i)\right]
\end{aligned}
$$

or,

$$
\begin{aligned}
E_t\hat{p}_{t+2}(i) &= \xi_p^2\hat{p}_t(i) \\
&-(1-\xi_p)\xi_p\left[\psi_0\widehat{\bar{k}}_{t+1}^{+}(i)+\psi_1\widehat{\bar{k}}_t^{+}(i)\right] \\
&-\xi_p(1-\xi_p)\left[\left(\psi_0\kappa_1+\psi_1\right)\widehat{\bar{k}}_{t+1}^{+}(i)+\psi_0\kappa_2\widehat{\bar{k}}_t^{+}(i)+\psi_0\kappa_3\hat{p}_t(i)\right] \\
&-\left(1-\xi_p\right)^2\left[\left(\psi_0\kappa_1+\psi_1-\psi_0^2\kappa_3\right)\widehat{\bar{k}}_{t+1}^{+}(i)+\left(\psi_0\kappa_2-\psi_0\psi_1\kappa_3\right)\widehat{\bar{k}}_t^{+}(i)\right]
\end{aligned}
$$

or,

$$
\begin{aligned}
E_t\hat{p}_{t+2}(i) &= \left[\xi_p^2-\xi_p(1-\xi_p)\psi_0\kappa_3\right]\hat{p}_t(i)-\left[(1-\xi_p)\xi_p\left(\psi_0+\psi_0\kappa_1+\psi_1\right)\right. \\
&\left.+\left(1-\xi_p\right)^2\left(\psi_0\kappa_1+\psi_1-\psi_0^2\kappa_3\right)\right]\widehat{\bar{k}}_{t+1}^{+}(i) \\
&-\left[(1-\xi_p)\xi_p\left(\psi_1+\psi_0\kappa_2\right)+\left(1-\xi_p\right)^2\left(\psi_0\kappa_2-\psi_0\psi_1\kappa_3\right)\right]\widehat{\bar{k}}_t^{+}(i)
\end{aligned}
$$

or, substituting out for $\widehat{\bar{k}}_{t+1}^{+}(i)$,

$$
\begin{aligned}
E_t\hat{p}_{t+2}(i) &= \left[\xi_p^2-\xi_p(1-\xi_p)\psi_0\kappa_3\right]\hat{p}_t(i) \\
&-\left[(1-\xi_p)\xi_p\left(\psi_0+\psi_0\kappa_1+\psi_1\right)+\left(1-\xi_p\right)^2\left(\psi_0\kappa_1+\psi_1-\psi_0^2\kappa_3\right)\right] \\
&\times\left(\kappa_1\widehat{\bar{k}}_t^{+}(i)+\kappa_2\widehat{\bar{k}}_{t-1}^{+}(i)+\kappa_3\hat{p}_t(i)\right) \\
&-\left[(1-\xi_p)\xi_p\left(\psi_1+\psi_0\kappa_2\right)+\left(1-\xi_p\right)^2\left(\psi_0\kappa_2-\psi_0\psi_1\kappa_3\right)\right]\widehat{\bar{k}}_t^{+}(i)
\end{aligned}
$$

or,

$$\begin{aligned}
E_t\hat{p}_{t+2}(i) &= \{\xi_p^2 - \xi_p(1-\xi_p)\psi_0\kappa_3 \\
&\quad - \left[(1-\xi_p)\xi_p\left(\psi_0+\psi_0\kappa_1+\psi_1\right) + \left(1-\xi_p\right)^2\left(\psi_0\kappa_1+\psi_1-\psi_0^2\kappa_3\right)\right]\kappa_3\}\hat{p}_t(i) \\
&\quad - \left[(1-\xi_p)\xi_p\left(\psi_0+\psi_0\kappa_1+\psi_1\right) + \left(1-\xi_p\right)^2\left(\psi_0\kappa_1+\psi_1-\psi_0^2\kappa_3\right)\right]\kappa_2\hat{\bar{k}}_{t-1}^{+}(i) \\
&\quad -\{(1-\xi_p)\xi_p\left(\psi_1+\psi_0\kappa_2\right) + \left(1-\xi_p\right)^2\left(\psi_0\kappa_2-\psi_0\psi_1\kappa_3\right) \\
&\quad + \left[(1-\xi_p)\xi_p\left(\psi_0+\psi_0\kappa_1+\psi_1\right) + \left(1-\xi_p\right)^2\left(\psi_0\kappa_1+\psi_1-\psi_0^2\kappa_3\right)\right]\kappa_1\}\hat{\bar{k}}_t^{+}(i) \\
&= a_0^p\hat{p}_t(i) + a_1^p\hat{\bar{k}}_{t-1}^{+}(i) + a_2^p\hat{\bar{k}}_t^{+}(i),
\end{aligned}$$

where

$$\begin{aligned}
a_1^p &= -(1-\xi_p)\left[\xi_p\left(\psi_0+\psi_0\kappa_1+\psi_1\right) + \left(1-\xi_p\right)\left(\psi_0\kappa_1+\psi_1-\psi_0^2\kappa_3\right)\right]\kappa_2 \\
a_0^p &= \xi_p^2 - \xi_p(1-\xi_p)\psi_0\kappa_3 + a_1^p\kappa_3/\kappa_2 \\
a_2^p &= -(1-\xi_p)\left[\xi_p\left(\psi_1+\psi_0\kappa_2\right) + \left(1-\xi_p\right)\left(\psi_0\kappa_2-\psi_0\psi_1\kappa_3\right)\right] + a_1^p\kappa_1/\kappa_2
\end{aligned}$$

Next,

$$\begin{aligned}
E_t\hat{\bar{k}}_{t+2}^{+}(i) &= \xi_p E_t\hat{\bar{k}}_{t+2}^{+}(i) \text{ (don't change price in } t+1) \\
&\quad +(1-\xi_p)E_t\hat{\bar{k}}_{t+2}^{+}(i) \text{ (do change price in } t+1) \\
&= \xi_p\left[\kappa_1\hat{\bar{k}}_{t+1}^{+}(i) + \kappa_2\hat{\bar{k}}_t^{+}(i) + \kappa_3\left(\hat{p}_t(i) - \Delta_\varrho\pi_{t+1}\right)\right] \\
&\quad +(1-\xi_p)\left[\kappa_1\hat{\bar{k}}_{t+1}^{+}(i) + \kappa_2\hat{\bar{k}}_t^{+}(i) + \kappa_3\left(\frac{\xi_p}{1-\xi_p}\Delta_\varrho\pi_{t+1} - \psi_0\hat{\bar{k}}_{t+1}^{+}(i) - \psi_1\hat{\bar{k}}_t^{+}(i)\right)\right]
\end{aligned}$$

or,

$$\begin{aligned}
E_t\hat{\bar{k}}_{t+2}^{+}(i) &= \xi_p\left[\kappa_1\hat{\bar{k}}_{t+1}^{+}(i) + \kappa_2\hat{\bar{k}}_t^{+}(i) + \kappa_3\hat{p}_t(i)\right] \\
&\quad +(1-\xi_p)\left[\kappa_1\hat{\bar{k}}_{t+1}^{+}(i) + \kappa_2\hat{\bar{k}}_t^{+}(i) - \kappa_3\psi_0\hat{\bar{k}}_{t+1}^{+}(i) - \kappa_3\psi_1\hat{\bar{k}}_t^{+}(i)\right] \\
&= \left[\xi_p\kappa_1 + (1-\xi_p)\kappa_1 - (1-\xi_p)\kappa_3\psi_0\right]\hat{\bar{k}}_{t+1}^{+}(i) \\
&\quad + \left[\xi_p\kappa_2 + (1-\xi_p)\kappa_2 - (1-\xi_p)\kappa_3\psi_1\right]\hat{\bar{k}}_t^{+}(i) \\
&\quad +\xi_p\kappa_3\hat{p}_t(i) \\
&= \left[\kappa_1 - (1-\xi_p)\kappa_3\psi_0\right]\left(\kappa_1\hat{\bar{k}}_t^{+}(i) + \kappa_2\hat{\bar{k}}_{t-1}^{+}(i) + \kappa_3\hat{p}_t(i)\right) \\
&\quad + \left[\kappa_2 - (1-\xi_p)\kappa_3\psi_1\right]\hat{\bar{k}}_t^{+}(i) + \xi_p\kappa_3\hat{p}_t(i)
\end{aligned}$$

or,

$$\begin{aligned}
E_t\widehat{\bar{k}}^{+}_{t+2}(i) &= \left\{\left[\kappa_1-(1-\xi_p)\kappa_3\psi_0\right]\kappa_1+\left[\kappa_2-(1-\xi_p)\kappa_3\psi_1\right]\right\}\widehat{\bar{k}}^{+}_t(i)\\
&\quad+\left[\kappa_1-(1-\xi_p)\kappa_3\psi_0\right]\kappa_2\widehat{\bar{k}}^{+}_{t-1}(i)\\
&\quad+\left\{\left[\kappa_1-(1-\xi_p)\kappa_3\psi_0\right]\kappa_3+\xi_p\kappa_3\right\}\hat{p}_t(i)\\
&= a_0^k\hat{p}_t(i)+a_1^k\widehat{\bar{k}}^{+}_{t-1}(i)+a_2^k\widehat{\bar{k}}^{+}_t(i),
\end{aligned}$$

where

$$\begin{aligned}
a_1^k &= \left[\kappa_1-(1-\xi_p)\kappa_3\psi_0\right]\kappa_2\\
a_2^k &= \kappa_2-(1-\xi_p)\kappa_3\psi_1+a_1^k\kappa_1/\kappa_2\\
a_0^k &= \xi_p\kappa_3+a_1^k\kappa_3/\kappa_2
\end{aligned}$$

Let's now substitute all this into (7.13):

$$\begin{aligned}
&\tilde{\gamma}_0\left[\kappa_1E_t\widehat{\bar{k}}^{+}_{t+2}(i)+\kappa_2\widehat{\bar{k}}^{+}_{t+1}(i)+\kappa_3E_t\hat{p}_{t+2}(i)\right]+\tilde{\gamma}_1E_t\widehat{\bar{k}}^{+}_{t+2}(i)+\tilde{\gamma}_2E_t\widehat{\bar{k}}^{+}_{t+1}(i)\\
+\tilde{\gamma}_3\widehat{\bar{k}}^{+}_t(i)+\tilde{\gamma}_4\widehat{\bar{k}}^{+}_{t-1}(i) &= \Phi\left[\xi_p-(1-\xi_p)\psi_0\kappa_3\right]\hat{p}_t(i)
\end{aligned}$$

or,

$$\begin{aligned}
&\left(\tilde{\gamma}_0\kappa_1+\tilde{\gamma}_1\right)E_t\widehat{\bar{k}}^{+}_{t+2}(i)+\tilde{\gamma}_0\kappa_3E_t\hat{p}_{t+2}(i)+\left[\tilde{\gamma}_0\kappa_2+\tilde{\gamma}_2\right]E_t\widehat{\bar{k}}^{+}_{t+1}(i)\\
+\tilde{\gamma}_3\widehat{\bar{k}}^{+}_t(i)+\tilde{\gamma}_4\widehat{\bar{k}}^{+}_{t-1}(i) &= \Phi\left[\xi_p-(1-\xi_p)\psi_0\kappa_3\right]\hat{p}_t(i)
\end{aligned}$$

or,

$$\begin{aligned}
&\left(\tilde{\gamma}_0\kappa_1+\tilde{\gamma}_1\right)\left(a_0^k\hat{p}_t(i)+a_1^k\widehat{\bar{k}}^{+}_{t-1}(i)+a_2^k\widehat{\bar{k}}^{+}_t(i)\right)\\
&+\tilde{\gamma}_0\kappa_3\left(a_0^p\hat{p}_t(i)+a_1^p\widehat{\bar{k}}^{+}_{t-1}(i)+a_2^p\widehat{\bar{k}}^{+}_t(i)\right)\\
&+\left[\tilde{\gamma}_0\kappa_2+\tilde{\gamma}_2\right]\left(\kappa_1\widehat{\bar{k}}^{+}_t(i)+\kappa_2\widehat{\bar{k}}^{+}_{t-1}(i)+\kappa_3\hat{p}_t(i)\right)\\
+\tilde{\gamma}_3\widehat{\bar{k}}^{+}_t(i)+\tilde{\gamma}_4\widehat{\bar{k}}^{+}_{t-1}(i) &= \Phi\left[\xi_p-(1-\xi_p)\psi_0\kappa_3\right]\hat{p}_t(i)
\end{aligned}$$

or,

$$\begin{aligned}
&\left[\left(\tilde{\gamma}_0\kappa_1+\tilde{\gamma}_1\right)a_2^k+\tilde{\gamma}_0\kappa_3a_2^p+\left(\tilde{\gamma}_0\kappa_2+\tilde{\gamma}_2\right)\kappa_1+\tilde{\gamma}_3\right]\widehat{\bar{k}}^{+}_t(i)\\
&+\left[\left(\tilde{\gamma}_0\kappa_1+\tilde{\gamma}_1\right)a_1^k+\tilde{\gamma}_0\kappa_3a_1^p+\left(\tilde{\gamma}_0\kappa_2+\tilde{\gamma}_2\right)\kappa_2+\tilde{\gamma}_4\right]\widehat{\bar{k}}^{+}_{t-1}(i)\\
&+\{\left(\tilde{\gamma}_0\kappa_1+\tilde{\gamma}_1\right)a_0^k+\tilde{\gamma}_0\kappa_3a_0^p+\left(\tilde{\gamma}_0\kappa_2+\tilde{\gamma}_2\right)\kappa_3\\
&-\Phi\left[\xi_p-(1-\xi_p)\psi_0\kappa_3\right]\}\hat{p}_t(i)\\
&= 0.
\end{aligned}$$

This requires that the following three equations be satisfied:

$$\left(\tilde{\gamma}_0\kappa_1+\tilde{\gamma}_1\right)a_2^k+\tilde{\gamma}_0\kappa_3 a_2^p+\left(\tilde{\gamma}_0\kappa_2+\tilde{\gamma}_2\right)\kappa_1+\tilde{\gamma}_3 = 0 \tag{7.14}$$

$$\left(\tilde{\gamma}_0\kappa_1+\tilde{\gamma}_1\right)a_1^k+\tilde{\gamma}_0\kappa_3 a_1^p+\left(\tilde{\gamma}_0\kappa_2+\tilde{\gamma}_2\right)\kappa_2+\tilde{\gamma}_4 = 0 \tag{7.15}$$

$$\left(\tilde{\gamma}_0\kappa_1+\tilde{\gamma}_1\right)a_0^k+\tilde{\gamma}_0\kappa_3 a_0^p+\left(\tilde{\gamma}_0\kappa_2+\tilde{\gamma}_2\right)\kappa_3 = \Phi\left[\xi_p-(1-\xi_p)\psi_0\kappa_3\right] \tag{7.16}$$

7.3. The Price First Order Condition

The intermediate good firm that reoptimizes its price optimizes:

$$E_t\sum_{j=0}^{\infty}\beta^j\lambda_{t+j}\{\left[\frac{P_{t+j}(i)}{P_{t+j}}\right]^{1-\theta}Y_{t+j}-R_{t+j}(\nu)w_{t+j}\frac{u_{t+j}(i)\bar{K}_{t+j}(i)}{z_{t+j}}\left(\frac{\left[\frac{P_{t+j}(i)}{P_{t+j}}\right]^{-\theta}Y_{t+j}+\phi z_{t+j}^*}{\epsilon_{t+j}u_{t+j}(i)\bar{K}_{t+j}(i)}\right)^{\frac{1}{1-\alpha}}$$
$$-\Upsilon_{t+j}^{-1}I_{t+j}(i)-\left[a(u_{t+j}(i))\Upsilon_{t+j}^{-1}\right]\bar{K}_{t+j}(i)\},$$

with respect to $P_t(i)$, subject to

$$\frac{P_{t+1}(i)}{P_{t+1}}=\frac{\pi^{1-\varrho}\pi_t^{\varrho}}{\pi_{t+1}}\frac{P_t(i)}{P_t}$$

for future histories in which it cannot reoptimize. Also,

$$\frac{P_{t+2}(i)}{P_{t+2}} = \frac{\pi^{1-\varrho}\pi_{t+1}^{\varrho}}{\pi_{t+2}}\frac{P_{t+1}(i)}{P_{t+1}}$$
$$= \frac{\pi^{1-\varrho}\pi_{t+1}^{\varrho}}{\pi_{t+2}}\frac{\pi^{1-\varrho}\pi_t^{\varrho}}{\pi_{t+1}}\frac{P_t(i)}{P_t}$$

and so on...

$$\frac{P_{t+j}(i)}{P_{t+j}} = \frac{\pi^{1-\varrho}\pi_{t+j-1}^{\varrho}}{\pi_{t+j}}\times\cdots\times\frac{\pi^{1-\varrho}\pi_t^{\varrho}}{\pi_{t+1}}\frac{P_t(i)}{P_t}$$
$$= X_{t,j}\frac{P_t(i)}{P_t}$$

Writing out the components of the firm's objective which involve price and neglecting future histories in which it reoptimizes its price:

$$E_t \sum_{j=0}^{\infty} \beta^j \lambda_{t+j} \left\{ \left[\frac{P_{t+j}(i)}{P_{t+j}} \right]^{1-\theta} Y_{t+j} - R_{t+j}(\nu) w_{t+j} \frac{u_{t+j}(i)\bar{K}_{t+j}(i)}{z_{t+j}} \left(\frac{\left[\frac{P_{t+j}(i)}{P_{t+j}}\right]^{-\theta} Y_{t+j} + \phi z^*_{t+j}}{\epsilon_{t+j} u_{t+j}(i)\bar{K}_{t+j}(i)} \right)^{\frac{1}{1-\alpha}} \right\}$$

$$= \lambda_t E_t \left\{ \left[\frac{P_t(i)}{P_t} \right]^{1-\theta} Y_t - R_t(\nu) w_t \frac{u_t(i)\bar{K}_t(i)}{z_t} \left(\frac{\left[\frac{P_t(i)}{P_t}\right]^{-\theta} Y_t + \phi z^*_t}{\epsilon_t u_t(i)\bar{K}_t(i)} \right)^{\frac{1}{1-\alpha}} \right\}$$

$$+\beta \lambda_{t+1} \xi_p E_t \left\{ \left[X_{t,1} \frac{P_t(i)}{P_t} \right]^{1-\theta} Y_{t+1} - R_{t+1}(\nu) w_{t+1} \frac{u_{t+1}(i)\bar{K}_{t+1}(i)}{z_{t+1}} \left(\frac{\left[X_{t,1}\frac{P_t(i)}{P_t}\right]^{-\theta} Y_{t+1} + \phi z^*_{t+1}}{\epsilon_{t+1} u_{t+1}(i)\bar{K}_{t+1}(i)} \right)^{\frac{1}{1-\alpha}} \right\}$$

$$+\beta^2 \lambda_{t+2} \xi_p^2 E_t \left\{ \left[X_{t,2} \frac{P_t(i)}{P_t} \right]^{1-\theta} Y_{t+2} - R_{t+2}(\nu) w_{t+2} \frac{u_{t+2}(i)\bar{K}_{t+2}(i)}{z_{t+2}} \left(\frac{\left[X_{t,2}\frac{P_t(i)}{P_t}\right]^{-\theta} Y_{t+2} + \phi z^*_{t+2}}{\epsilon_{t+2} u_{t+2}(i)\bar{K}_{t+2}(i)} \right)^{\frac{1}{1-\alpha}} \right\}$$

$$+ ...$$

$$+ \left(\beta\xi_p\right)^j \lambda_{t+j} E_t \{ \left[X_{t,j} \frac{P_t(i)}{P_t} \right]^{1-\theta} Y_{t+j}$$

$$- \frac{R_{t+j}(\nu) w_{t+j} u_{t+j}(i)\bar{K}_{t+j}(i)}{z_{t+j}} \left(\frac{\left[X_{t,j}\frac{P_t(i)}{P_t}\right]^{-\theta} Y_{t+j} + \phi z^*_{t+j}}{\epsilon_{t+j} u_{t+j}(i)\bar{K}_{t+j}(i)} \right)^{\frac{1}{1-\alpha}} \}$$

$$+ ...$$

Differentiate the j^{th} term with respect to $P_t(i)$:

$$\left(\beta\xi_p\right)^j \lambda_{t+j} E_t \{ (1-\theta) \left[X_{t,j}\right]^{1-\theta} \left[\frac{P_t(i)}{P_t} \right]^{-\theta} Y_{t+j}$$

$$+R_{t+j}(\nu) w_{t+j} \frac{u_{t+j}(i)\bar{K}_{t+j}(i)}{z_{t+j}} \frac{1}{1-\alpha} \left(\frac{\left[X_{t,j}\frac{P_t(i)}{P_t}\right]^{-\theta} Y_{t+j} + \phi z^*_{t+j}}{\epsilon_{t+j} u_{t+j}(i)\bar{K}_{t+j}(i)} \right)^{\frac{\alpha}{1-\alpha}} \frac{\theta X_{t,j}^{-\theta} \left[\frac{P_t(i)}{P_t}\right]^{-\theta-1} Y_{t+j}}{\epsilon_{t+j} u_{t+j}(i)\bar{K}_{t+j}(i)} \} \frac{1}{P_t},$$

or,

$$\left(\beta\xi_p\right)^j \lambda_{t+j} E_t\{X_{t,j}\frac{P_t(i)}{P_t}Y_{t+j}$$

$$+\frac{\theta}{(1-\theta)}\frac{\epsilon_{t+j}u_{t+j}(i)\bar{K}_{t+j}(i)}{1}\frac{R_{t+j}(\nu)w_{t+j}}{\epsilon_{t+j}\left(1-\alpha\right)z_{t+j}}$$

$$\times\left(\frac{\left[X_{t,j}\frac{P_t(i)}{P_t}\right]^{-\theta}Y_{t+j}+\phi z_{t+j}^*}{\epsilon_{t+j}u_{t+j}(i)\bar{K}_{t+j}(i)}\right)^{\frac{\alpha}{1-\alpha}}\frac{Y_{t+j}}{\epsilon_{t+j}u_{t+j}(i)\bar{K}_{t+j}(i)}\}\frac{X_{t,j}^{-\theta}}{P_t}\left[\frac{P_t(i)}{P_t}\right]^{-\theta-1}.$$

Recall that marginal cost is:

$$s_t(i)=\frac{R_t(\nu)w_t}{(1-\alpha)\epsilon_t z_t}\left(\frac{y_t(i)+\phi z_t^*}{\epsilon_t K_t(i)}\right)^{\frac{\alpha}{1-\alpha}}$$

Substituting,

$$\left(\beta\xi_p\right)^j \lambda_{t+j} E_t\left\{\frac{P_t(i)}{P_t}X_{t,j}+\frac{\theta}{(1-\theta)}s_{t+j}(i)\right\}Y_{t+j}\frac{X_{t,j}^{-\theta}}{P_t}\left[\frac{P_t(i)}{P_t}\right]^{-\theta-1}.$$

The derivative of the firm's objective with respect to $P_t(i)$ is:

$$\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j \lambda_{z^*,t+j}\tilde{Y}_{t+j}X_{t,j}^{-\theta}\left[\frac{P_t(i)}{P_t}\right]^{-\theta-1}E_t\left\{\frac{P_t(i)}{P_t}X_{t,j}-\frac{\theta}{\theta-1}s_{t+j}(i)\right\}=0.$$

Expand this about steady state, taking into account that the object in braces is zero in steady state (so that differentiating the objects outside the braces is unnecessary), and take into account that $\lambda_{z^*,t+j}\ \tilde{Y}_{t+j}$ are constant in steady state and $(P_t(i)/P_t)=X_{t,j}=1$ in steady state:

$$\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j E_t\left\{\hat{p}_t(i)+\hat{X}_{t,j}-\hat{s}_{t+j}(i)\right\}=0,$$

since

$$s=\frac{\theta-1}{\theta}.$$

Now,

$$X_{t,1}=\frac{\pi^{1-\varrho}\pi_t^{\varrho}}{\pi_{t+1}}$$

so that,

$$\hat{X}_{t,1}=\varrho\hat{\pi}_t-\hat{\pi}_{t+1}=-\Delta_{\varrho}\hat{\pi}_{t+1}.$$

Also,

$$X_{t,2}=\frac{\pi^{1-\varrho}\pi_{t+1}^{\varrho}}{\pi_{t+2}}\frac{\pi^{1-\varrho}\pi_t^{\varrho}}{\pi_{t+1}},$$

so that,

$$\hat{X}_{t,2} = -\Delta_{\varrho}\hat{\pi}_{t+1} - \Delta_{\varrho}\hat{\pi}_{t+2},$$

and so on. Then,

$$\begin{aligned}
& E_t\left\{\hat{p}_t(i) - \hat{s}_t(i)\right\} \\
& + \left(\beta\xi_p\right)^1 E_t\left\{\hat{p}_t(i) - \Delta_{\varrho}\hat{\pi}_{t+1} - \hat{s}_{t+1}(i)\right\} \\
& + \left(\beta\xi_p\right)^2 E_t\left\{\hat{p}_t(i) - \Delta_{\varrho}\hat{\pi}_{t+1} - \Delta_{\varrho}\hat{\pi}_{t+2} - \hat{s}_{t+2}(i)\right\} \\
& + \left(\beta\xi_p\right)^3 E_t\left\{\hat{p}_t(i) - \Delta_{\varrho}\hat{\pi}_{t+1} - \Delta_{\varrho}\hat{\pi}_{t+2} - \Delta_{\varrho}\hat{\pi}_{t+3} - \hat{s}_{t+3}(i)\right\} \\
& + ...
\end{aligned}$$

or,

$$\begin{aligned}
& \frac{1}{1-\beta\xi_p}\hat{p}_t(i) - \frac{\beta\xi_p}{1-\beta\xi_p}\Delta_{\varrho}\hat{\pi}_{t+1} - \frac{\left(\beta\xi_p\right)^2}{1-\beta\xi_p}\Delta_{\varrho}\hat{\pi}_{t+2} - \frac{\left(\beta\xi_p\right)^3}{1-\beta\xi_p}\Delta_{\varrho}\hat{\pi}_{t+3} - ... \\
& -\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j \hat{s}_{t+j}(i) \\
= \; & 0,
\end{aligned}$$

or,

$$\hat{p}_t^*(i) = \sum_{j=1}^{\infty}\left(\beta\xi_p\right)^j \Delta_{\varrho}\hat{\pi}_{t+j} + \left(1-\beta\xi_p\right)\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j \hat{s}_{t+j}(i)$$

But,

$$\begin{aligned}
\hat{s}_t(i) &= \hat{s}_t + \frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left[\frac{-\theta\tilde{y}}{\tilde{y}+\phi}\hat{p}_t(i) - \widehat{\bar{k}}_t^+(i)\right], \\
\hat{s}_{t+1}(i) &= \hat{s}_{t+1} + \frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left[\frac{-\theta\tilde{y}}{\tilde{y}+\phi}\left(\hat{p}_t(i) - \Delta_{\varrho}\pi_{t+1}\right) - \widehat{\bar{k}}_{t+1}^+(i)\right] \\
\hat{s}_{t+2}(i) &= \hat{s}_{t+2} + \frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left[\frac{-\theta\tilde{y}}{\tilde{y}+\phi}\left(\hat{p}_t(i) - \Delta_{\varrho}\pi_{t+1} - \Delta_{\varrho}\pi_{t+2}\right) - \widehat{\bar{k}}_{t+2}^+(i)\right] \\
\hat{s}_{t+3}(i) &= \hat{s}_{t+3} + \frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left[\frac{-\theta\tilde{y}}{\tilde{y}+\phi}\left(\hat{p}_t(i) - \Delta_{\varrho}\pi_{t+1} - \Delta_{\varrho}\pi_{t+2} - \Delta_{\varrho}\pi_{t+3}\right) - \widehat{\bar{k}}_{t+3}^+(i)\right] \\
& ...
\end{aligned}$$

Then,

$$\begin{aligned}
& \sum_{j=0}^{\infty} \left(\beta\xi_p\right)^j \hat{s}_{t+j}(i) \\
= \; & \hat{s}_t(i) \\
& + \left(\beta\xi_p\right) \hat{s}_{t+1}(i) \\
& + \left(\beta\xi_p\right)^2 \hat{s}_{t+2}(i) \\
& + \left(\beta\xi_p\right)^3 \hat{s}_{t+3}(i) \\
& + ... \\
= \; & \hat{s}_t + \frac{\alpha}{1-\alpha} \frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1} \left[\frac{-\theta\tilde{y}}{\tilde{y}+\phi} \hat{p}_t(i) - \widehat{\bar{k}}_t^+(i) \right] \\
& + \left(\beta\xi_p\right) \left\{ \hat{s}_{t+1} + \frac{\alpha}{1-\alpha} \frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1} \left[\frac{-\theta\tilde{y}}{\tilde{y}+\phi} \left(\hat{p}_t(i) - \Delta_\varrho \pi_{t+1}\right) - \widehat{\bar{k}}_{t+1}^+(i) \right] \right\} \\
& + \left(\beta\xi_p\right)^2 \left\{ \hat{s}_{t+2} + \frac{\alpha}{1-\alpha} \frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1} \left[\frac{-\theta\tilde{y}}{\tilde{y}+\phi} \left(\hat{p}_t(i) - \Delta_\varrho \pi_{t+1} - \Delta_\varrho \pi_{t+2}\right) - \widehat{\bar{k}}_{t+2}^+(i) \right] \right\} \\
& + \left(\beta\xi_p\right)^3 \left\{ \hat{s}_{t+3} + \frac{\alpha}{1-\alpha} \frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1} \left[\frac{-\theta\tilde{y}}{\tilde{y}+\phi} \left(\hat{p}_t(i) - \Delta_\varrho \pi_{t+1} - \Delta_\varrho \pi_{t+2} - \Delta_\varrho \pi_{t+3}\right) - \widehat{\bar{k}}_{t+3}^+(i) \right] \right\} \\
& + ... \\
= \; & \sum_{j=0}^{\infty} \left(\beta\xi_p\right)^j \hat{s}_{t+j} - \frac{\alpha}{1-\alpha} \frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1} \frac{\theta\tilde{y}}{\tilde{y}+\phi} \\
& \times \left[\frac{1}{1-\beta\xi_p} \hat{p}_t(i) - \frac{1}{1-\beta\xi_p} \sum_{j=1}^{\infty} \left(\beta\xi_p\right)^j \Delta_\varrho \pi_{t+j} + \frac{\tilde{y}+\phi}{\theta\tilde{y}} \sum_{j=0}^{\infty} \left(\beta\xi_p\right)^j \widehat{\bar{k}}_{t+j}^+(i) \right].
\end{aligned}$$

We now substitute this into the price equation. Recall,

$$\hat{p}_t^*(i) = \sum_{j=1}^{\infty} \left(\beta\xi_p\right)^j \Delta_\varrho \hat{\pi}_{t+j} + \left(1-\beta\xi_p\right) \sum_{j=0}^{\infty} \left(\beta\xi_p\right)^j \hat{s}_{t+j}(i)$$

so that,

$$\begin{aligned}
\hat{p}_t^*(i) = \; & \sum_{j=1}^{\infty} \left(\beta\xi_p\right)^j \Delta_\varrho \hat{\pi}_{t+j} + \left(1-\beta\xi_p\right) \sum_{j=0}^{\infty} \left(\beta\xi_p\right)^j \hat{s}_{t+j} \\
& - \frac{\alpha}{1-\alpha} \frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1} \frac{\theta\tilde{y}}{\tilde{y}+\phi} \left[\hat{p}_t^*(i) - \sum_{j=1}^{\infty} \left(\beta\xi_p\right)^j \Delta_\varrho \pi_{t+j} + \left(1-\beta\xi_p\right) \frac{\tilde{y}+\phi}{\theta\tilde{y}} \sum_{j=0}^{\infty} \left(\beta\xi_p\right)^j \widehat{\bar{k}}_{t+j}^+(i) \right]
\end{aligned}$$

We must now evaluate the expression involving the present value of $\widehat{\bar{k}}_{t+j}^+(i)$. Recall:

$$\hat{p}_{t+j}(i) = \hat{p}_t(i) - \Delta_\varrho \pi_{t+1} - \Delta_\varrho \pi_{t+2} - ... - \Delta_\varrho \pi_{t+j},$$

and:

$$\begin{aligned}
\widehat{\tilde{k}}^{+}_{t+1}(i) &= \kappa_1 \tilde{k}_t(i) + \kappa_2 \widehat{\tilde{k}}^{+}_{t-1}(i) + \kappa_3 \hat{p}_t(i) \\
\hat{p}^*_t(i) &= \hat{p}^*_t - \psi_0 \tilde{k}_t(i) - \psi_1 \widehat{\tilde{k}}^{+}_{t-1}(i), \;\; \tilde{k}_t(i) \equiv \hat{k}_t(i) - \hat{K}_t,
\end{aligned}$$

Stack the capital decision rule as a first order system:

$$z_t = \begin{pmatrix} \widehat{\tilde{k}}^{+}_{t+1}(i) \\ \widehat{\tilde{k}}^{+}_{t}(i) \end{pmatrix}$$

Then,

$$\begin{aligned}
z_t &= A z_{t-1} + \begin{pmatrix} \kappa_3 \hat{p}^*_t(i) \\ 0 \end{pmatrix} \\
A &= \begin{bmatrix} \kappa_1 & \kappa_2 \\ 1 & 0 \end{bmatrix}.
\end{aligned}$$

Then,

$$\begin{aligned}
\hat{E}^i_t z_t &= A z_{t-1} + \begin{pmatrix} \kappa_3 \hat{p}^*_t(i) \\ 0 \end{pmatrix} \\
\hat{E}^i_t z_{t+1} &= A z_t + \begin{pmatrix} \kappa_3 \left(\hat{p}^*_t(i) - \Delta_\varrho E_t \pi_{t+1} \right) \\ 0 \end{pmatrix} \\
&= A^2 z_{t-1} + A \begin{pmatrix} \kappa_3 \hat{p}^*_t(i) \\ 0 \end{pmatrix} + \begin{pmatrix} \kappa_3 \left(\hat{p}^*_t(i) - \Delta_\varrho E_t \pi_{t+1} \right) \\ 0 \end{pmatrix} \\
&= A^2 z_{t-1} + (A + I) \begin{pmatrix} \kappa_3 \hat{p}^*_t(i) \\ 0 \end{pmatrix} - \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+1} \\ 0 \end{pmatrix}
\end{aligned}$$

Also,

$$\begin{aligned}
\hat{E}_t^i z_{t+2} &= A^3 z_{t-1} + A^2 \begin{pmatrix} \kappa_3 \hat{p}_t^*(i) \\ 0 \end{pmatrix} + A \begin{pmatrix} \kappa_3 \left(\hat{p}_t^*(i) - \Delta_\varrho E_t \pi_{t+1}\right) \\ 0 \end{pmatrix} \\
&+ \begin{pmatrix} \kappa_3 \left(\hat{p}_t^*(i) - \Delta_\varrho E_t \pi_{t+1} - \Delta_\varrho E_t \pi_{t+2}\right) \\ 0 \end{pmatrix} \\
&= A^3 z_{t-1} + \left(A^2 + A + I\right) \begin{pmatrix} \kappa_3 \hat{p}_t^*(i) \\ 0 \end{pmatrix} - (A + I) \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+1} \\ 0 \end{pmatrix} - \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+2} \\ 0 \end{pmatrix} \\
\hat{E}_t^i z_{t+3} &= A^4 z_{t-1} + A^3 \begin{pmatrix} \kappa_3 \hat{p}_t^*(i) \\ 0 \end{pmatrix} + A^2 \begin{pmatrix} \kappa_3 \left(\hat{p}_t^*(i) - \Delta_\varrho E_t \pi_{t+1}\right) \\ 0 \end{pmatrix} \\
&+ A \begin{pmatrix} \kappa_3 \left(\hat{p}_t^*(i) - \Delta_\varrho E_t \pi_{t+1} - \Delta_\varrho E_t \pi_{t+2}\right) \\ 0 \end{pmatrix} \\
&+ \begin{pmatrix} \kappa_3 \left(\hat{p}_t^*(i) - \Delta_\varrho E_t \pi_{t+1} - \Delta_\varrho E_t \pi_{t+2} - \Delta_\varrho E_t \pi_{t+3}\right) \\ 0 \end{pmatrix} \\
&= A^4 z_{t-1} + \left[A^3 + A^2 + A + I\right] \begin{pmatrix} \kappa_3 \hat{p}_t^*(i) \\ 0 \end{pmatrix} - \left[A^2 + A + I\right] \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+1} \\ 0 \end{pmatrix} \\
&- [A + I] \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+2} \\ 0 \end{pmatrix} - I \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+3} \\ 0 \end{pmatrix} \\
\hat{E}_t^i z_{t+4} &= A^5 z_{t-1} + \left[A^4 + A^3 + A^2 + A\right] \begin{pmatrix} \kappa_3 \hat{p}_t^*(i) \\ 0 \end{pmatrix} - \left[A^3 + A^2 + A\right] \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+1} \\ 0 \end{pmatrix} \\
&- \left[A^2 + A\right] \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+2} \\ 0 \end{pmatrix} - A \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+3} \\ 0 \end{pmatrix} \\
&+ \begin{pmatrix} \kappa_3 \left(\hat{p}_t^*(i) - \Delta_\varrho E_t \pi_{t+1} - \Delta_\varrho E_t \pi_{t+2} - \Delta_\varrho E_t \pi_{t+3} - \Delta_\varrho E_t \pi_{t+4}\right) \\ 0 \end{pmatrix}
\end{aligned}$$

or,

$$\begin{aligned}
\hat{E}_t^i z_{t+4} &= A^5 z_{t-1} + \left[A^4 + A^3 + A^2 + A + I\right] \begin{pmatrix} \kappa_3 \hat{p}_t^*(i) \\ 0 \end{pmatrix} - \left[A^3 + A^2 + A + I\right] \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+1} \\ 0 \end{pmatrix} \\
&- \left[A^2 + A + I\right] \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+2} \\ 0 \end{pmatrix} - [A + I] \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+3} \\ 0 \end{pmatrix} - \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+4} \\ 0 \end{pmatrix}
\end{aligned}$$

The geometic sum formula:

$$\begin{aligned}
S &= I + A + A^2 + ... + A^k \\
AS &= A + A^2 + ... + A^{k+1} \\
[I - A]\, S &= I - A^{k+1} \\
S &= [I - A]^{-1} \left[I - A^{k+1}\right]
\end{aligned}$$

Then,

$$\begin{aligned}
\hat{E}_t^i z_{t+k} &= A^{k+1} z_{t-1} + [I-A]^{-1}\left[I-A^{k+1}\right]\begin{pmatrix}\kappa_3 \hat{p}_t^*(i)\\ 0\end{pmatrix} - [I-A]^{-1}\left[I-A^{k}\right]\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+1}\\ 0\end{pmatrix}\\
&\quad - [I-A]^{-1}\left[I-A^{k-1}\right]\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+2}\\ 0\end{pmatrix} - [I-A]^{-1}\left[I-A^{k-2}\right]\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+3}\\ 0\end{pmatrix}\\
&\quad -... - [I-A]^{-1}\left[I-A^{k-(j-1)}\right]\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+j}\\ 0\end{pmatrix} - ... - [I-A]^{-1}[I-A]\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+k}\\ 0\end{pmatrix}
\end{aligned}$$

Now, we want (let $\tau = [1\ 0]$):

$$\begin{aligned}
&\sum_{j=0}^{\infty}\left(\xi_p\beta\right)^j \hat{E}_t^i \tilde{k}_{t+j}(i)\\
=\ & \tilde{k}_t(i) + \tau\xi_p\beta z_t + \tau\left(\xi_p\beta\right)^2 \hat{E}_t^i z_{t+1}\\
& +\tau\left(\xi_p\beta\right)^3 \hat{E}_t^i z_{t+2} + \tau\left(\xi_p\beta\right)^4 \hat{E}_t^i z_{t+3} + \tau\left(\xi_p\beta\right)^5 \hat{E}_t^i z_{t+4} + ...\\
=\ & \tilde{k}_t(i) + \tau\xi_p\beta\left[A z_{t-1} + \begin{pmatrix}\kappa_3 \hat{p}_t^*(i)\\ 0\end{pmatrix}\right]\\
& +\tau\left(\xi_p\beta\right)^2\left[A^2 z_{t-1} + (A+I)\begin{pmatrix}\kappa_3 \hat{p}_t^*(i)\\ 0\end{pmatrix} - \begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+1}\\ 0\end{pmatrix}\right]\\
& +\tau\left(\xi_p\beta\right)^3\left[A^3 z_{t-1} + \left(A^2+A+I\right)\begin{pmatrix}\kappa_3 \hat{p}_t^*(i)\\ 0\end{pmatrix} - (A+I)\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+1}\\ 0\end{pmatrix} - \begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+2}\\ 0\end{pmatrix}\right]\\
& +\tau\left(\xi_p\beta\right)^4 [A^4 z_{t-1} + \left(A^3+A^2+A+I\right)\begin{pmatrix}\kappa_3 \hat{p}_t^*(i)\\ 0\end{pmatrix} - \left(A^2+A+I\right)\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+1}\\ 0\end{pmatrix}\\
& -[A+I]\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+2}\\ 0\end{pmatrix} - I\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+3}\\ 0\end{pmatrix}]\\
& +\tau\left(\xi_p\beta\right)^5 [A^5 z_{t-1} + \left[A^4+A^3+A^2+A+I\right]\begin{pmatrix}\kappa_3 \hat{p}_t^*(i)\\ 0\end{pmatrix} - \left[A^3+A^2+A+I\right]\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+1}\\ 0\end{pmatrix}\\
& -\left[A^2+A+I\right]\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+2}\\ 0\end{pmatrix} - [A+I]\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+3}\\ 0\end{pmatrix} - \begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+4}\\ 0\end{pmatrix}]\\
& +... +\\
& \tau\left(\xi_p\beta\right)^{k+1} [A^{k+1} z_{t-1} + (I-A)^{-1}\left(I-A^{k+1}\right)\begin{pmatrix}\kappa_3 \hat{p}_t^*(i)\\ 0\end{pmatrix} - (I-A)^{-1}\left(I-A^{k}\right)\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+1}\\ 0\end{pmatrix}\\
& -(I-A)^{-1}\left(I-A^{k-1}\right)\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+2}\\ 0\end{pmatrix} - (I-A)^{-1}\left(I-A^{k-2}\right)\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+3}\\ 0\end{pmatrix}\\
& ... - (I-A)^{-1}\left(I-A^{k-(j-1)}\right)\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+j}\\ 0\end{pmatrix} - ... - (I-A)^{-1}(I-A)\begin{pmatrix}\kappa_3 \Delta_\varrho E_t \pi_{t+k}\\ 0\end{pmatrix}]\\
& +...
\end{aligned}$$

Collecting terms:

$$
\begin{aligned}
& \sum_{j=0}^{\infty} \left(\xi_p \beta\right)^j \hat{E}_t^i \tilde{k}_{t+j} \\
= \; & \tilde{k}_t(i) + \tau \xi_p \beta A \left(I - \xi_p \beta A\right)^{-1} z_{t-1} \\
& + \tau \left[\xi_p \beta I + \left(\xi_p \beta\right)^2 (A + I) + \left(\xi_p \beta\right)^3 \left(A^2 + A + I\right) + ... + \left(\xi_p \beta\right)^{k+1} (I - A)^{-1} \left(I - A^{k+1}\right) + ... \right] \\
& \times \begin{pmatrix} \kappa_3 \hat{p}_t^*(i) \\ 0 \end{pmatrix} \\
& - \tau \left[\left(\xi_p \beta\right)^2 I + \left(\xi_p \beta\right)^3 (A + I) + \left(\xi_p \beta\right)^4 \left(A^2 + A + I\right) + ... + \left(\xi_p \beta\right)^{k+1} (I - A)^{-1} \left(I - A^{k}\right) + ... \right] \\
& \times \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+1} \\ 0 \end{pmatrix} \\
& - \tau \left[\left(\xi_p \beta\right)^3 I + \left(\xi_p \beta\right)^4 (A + I) + \left(\xi_p \beta\right)^5 \left(A^2 + A + I\right) + ... + (I - A)^{-1} \left(I - A^{k-1}\right) \right] \\
& \times \begin{pmatrix} \kappa_3 \Delta_\varrho E_t \pi_{t+2} \\ 0 \end{pmatrix} \\
& - ...
\end{aligned}
$$

Simplifying the coefficient on $\kappa_3 \hat{p}_t^*(i)$:

$$
\begin{aligned}
& \xi_p \beta (I - A)^{-1} (I - A) + \left(\xi_p \beta\right)^2 (I - A)^{-1} \left(I - A^2\right) + \left(\xi_p \beta\right)^3 (I - A)^{-1} \left(I - A^3\right) \\
& + ... + \left(\xi_p \beta\right)^{k+1} (I - A)^{-1} \left(I - A^{k+1}\right) + ... \\
= \; & (I - A)^{-1} \left[\xi_p \beta (I - A) + \left(\xi_p \beta\right)^2 \left(I - A^2\right) + \left(\xi_p \beta\right)^3 \left(I - A^3\right) + ... + \left(\xi_p \beta\right)^{k+1} \left(I - A^{k+1}\right) + ... \right] \\
= \; & (I - A)^{-1} \left[\frac{\xi_p \beta}{1 - \xi_p \beta} I - \xi_p \beta A \left(I - \xi_p \beta A\right)^{-1} \right]
\end{aligned}
$$

The coefficient on $\kappa_3 \Delta_\varrho E_t \pi_{t+1}$:

$$
\begin{aligned}
& \left(\xi_p \beta\right)^2 (I - A)^{-1} (I - A) + \left(\xi_p \beta\right)^3 (I - A)^{-1} \left(I - A^2\right) + \left(\xi_p \beta\right)^4 (I - A)^{-1} \left(I - A^3\right) \\
& + ... + \left(\xi_p \beta\right)^{k+1} (I - A)^{-1} \left(I - A^{k}\right) + ... \\
= \; & (I - A)^{-1} \xi_p \beta \left[\xi_p \beta (I - A) + \left(\xi_p \beta\right)^2 \left(I - A^2\right) + \left(\xi_p \beta\right)^3 \left(I - A^3\right) + ... + \left(\xi_p \beta\right)^{k} \left(I - A^{k}\right) + ... \right] \\
= \; & (I - A)^{-1} \xi_p \beta \left[\frac{\xi_p \beta}{1 - \xi_p \beta} I - \xi_p \beta A \left(I - \xi_p \beta A\right)^{-1} \right]
\end{aligned}
$$

The coefficient on $\kappa_3 \Delta_\varrho E_t \pi_{t+2}$:

$$
(I - A)^{-1} \left(\xi_p \beta\right)^2 \left[\frac{\xi_p \beta}{1 - \xi_p \beta} I - \xi_p \beta A \left(I - \xi_p \beta A\right)^{-1} \right]
$$

and so on. Then,

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\left(\xi_p\beta\right)^j \hat{E}_t^i \tilde{k}_{t+j} \\
= \; & \tilde{k}_t(i) + \tau\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1} z_{t-1} + \tau\left(I-A\right)^{-1}\left[\frac{\xi_p\beta}{1-\xi_p\beta}I - \xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\right]\begin{pmatrix}\kappa_3\hat{p}_t^*(i)\\ 0\end{pmatrix} \\
&-\tau\left(I-A\right)^{-1}\xi_p\beta\left[\frac{\xi_p\beta}{1-\xi_p\beta}I - \xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\right]\begin{pmatrix}\kappa_3\Delta_\varrho E_t\pi_{t+1}\\ 0\end{pmatrix} \\
&-\tau\left(I-A\right)^{-1}\left(\xi_p\beta\right)^2\left[\frac{\xi_p\beta}{1-\xi_p\beta}I - \xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\right]\begin{pmatrix}\kappa_3\Delta_\varrho E_t\pi_{t+2}\\ 0\end{pmatrix} \\
&-...
\end{aligned}
$$

So,

$$
\begin{aligned}
\sum_{j=0}^{\infty}\left(\xi_p\beta\right)^j \hat{E}_t^i \tilde{k}_{t+j} \;=\;& \tilde{k}_t(i) + \tau\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1} z_{t-1} \\
&+\left\{\tau\left(I-A\right)^{-1}\left[\frac{\xi_p\beta}{1-\xi_p\beta}I - \xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\right]\tau'\right\}\kappa_3\hat{p}_t^*(i) \\
&-\left\{\tau\left(I-A\right)^{-1}\left[\frac{\xi_p\beta}{1-\xi_p\beta}I - \xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\right]\tau'\right\}\kappa_3\sum_{j=1}^{\infty}\left(\xi_p\beta\right)^j\Delta_\varrho E_t\pi_{t+j}
\end{aligned}
$$

Substitute this into the first order condition for $\hat{p}_t^*(i)$:

$$
\begin{aligned}
\hat{p}_t^*(i) \;=\;& \sum_{j=1}^{\infty}\left(\beta\xi_p\right)^j\Delta_\varrho\hat{\pi}_{t+j} + \left(1-\beta\xi_p\right)\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j\hat{s}_{t+j} \\
&-\frac{\alpha}{1-\alpha}\frac{\sigma_a\left(1-\alpha\right)}{\sigma_a\left(1-\alpha\right)+1}\frac{\theta\tilde{y}}{\tilde{y}+\phi}\left[\hat{p}_t^*(i) - \sum_{j=1}^{\infty}\left(\beta\xi_p\right)^j\Delta_\varrho\pi_{t+j} + \left(1-\beta\xi_p\right)\frac{\tilde{y}+\phi}{\theta\tilde{y}}\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j\widehat{\tilde{k}}_{t+j}^{+}(i)\right]
\end{aligned}
$$

to obtain:

$$
\begin{aligned}
\hat{p}_t^*(i) \;=\;& \sum_{j=1}^{\infty}\left(\beta\xi_p\right)^j\Delta_\varrho\hat{\pi}_{t+j} + \left(1-\beta\xi_p\right)\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j\hat{s}_{t+j} \qquad (7.17)\\
&-\frac{\alpha}{1-\alpha}\frac{\sigma_a\left(1-\alpha\right)}{\sigma_a\left(1-\alpha\right)+1}\frac{\theta\tilde{y}}{\tilde{y}+\phi}\left[\hat{p}_t^*(i) - \sum_{j=1}^{\infty}\left(\beta\xi_p\right)^j\Delta_\varrho\pi_{t+j}\right] \\
&-\frac{\alpha}{1-\alpha}\frac{\sigma_a\left(1-\alpha\right)}{\sigma_a\left(1-\alpha\right)+1}\left(1-\beta\xi_p\right)\{\tilde{k}_t(i) + \tau\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}z_{t-1} \\
&+\left(\tau\left(I-A\right)^{-1}\left[\frac{\xi_p\beta}{1-\xi_p\beta}I - \xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\right]\tau'\right)\kappa_3\hat{p}_t^*(i) \\
&-\left(\tau\left(I-A\right)^{-1}\left[\frac{\xi_p\beta}{1-\xi_p\beta}I - \xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\right]\tau'\right)\kappa_3\sum_{j=1}^{\infty}\left(\xi_p\beta\right)^j\Delta_\varrho E_t\pi_{t+j}\}
\end{aligned}
$$

We now collect terms in this expression. Move terms in $\hat{p}_t^*(i)$ to the left of the equality in (7.17). The coefficient on $\hat{p}_t^*(i)$ then is:

$$
\begin{aligned}
& 1+\frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\frac{\theta\tilde{y}}{\tilde{y}+\phi} \\
& +\frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left(1-\beta\xi_p\right)\left(\tau(I-A)^{-1}\left[\frac{\xi_p\beta}{1-\xi_p\beta}I-\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\right]\tau'\right)\kappa_3 \\
= & \ 1+\frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left\{\frac{\theta\tilde{y}}{\tilde{y}+\phi}+\left(1-\beta\xi_p\right)\left(\tau(I-A)^{-1}\left[\frac{\xi_p\beta}{1-\xi_p\beta}I-\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\right]\tau'\right)\kappa_3\right\} \\
= & \ \zeta^{-1},
\end{aligned}
$$

say. Collect terms in $\sum_{j=1}^{\infty}\left(\beta\xi_p\right)^j\Delta_\varrho\hat{\pi}_{t+j}$ to the right of the equality in (7.17). The coefficient on these terms is ζ^{-1} too. Thus, collecting terms in (7.17) and multiplying the result by ζ, we obtain:

$$
\begin{aligned}
\zeta^{-1}\hat{p}_t^*(i) = & \ \zeta^{-1}\sum_{j=1}^{\infty}\left(\beta\xi_p\right)^j\Delta_\varrho\hat{\pi}_{t+j}+\left(1-\beta\xi_p\right)\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j\hat{s}_{t+j} \\
& -\frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left(1-\beta\xi_p\right)\left\{\tilde{k}_t(i)+\tau\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}z_{t-1}\right\}
\end{aligned}
$$

or, after multiplication by ζ :

$$
\begin{aligned}
\hat{p}_t^*(i) = & \ \sum_{j=1}^{\infty}\left(\beta\xi_p\right)^j\Delta_\varrho\hat{\pi}_{t+j}+\left(1-\beta\xi_p\right)\zeta\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j\hat{s}_{t+j} \\
& -\frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left(1-\beta\xi_p\right)\zeta\tilde{k}_t(i) \\
& -\frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left(1-\beta\xi_p\right)\zeta\left[\tau\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\tau'\right]\tilde{k}_t(i) \\
& -\frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left(1-\beta\xi_p\right)\zeta\left[\tau\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\begin{pmatrix}0\\1\end{pmatrix}\right]\tilde{k}_{t-1}(i)
\end{aligned}
$$

(recall, $\tau\equiv[1\ 0]$), or, collecting terms in $\tilde{k}_t(i)$:

$$
\begin{aligned}
\hat{p}_t^*(i) = & \ \sum_{j=1}^{\infty}\left(\beta\xi_p\right)^j\Delta_\varrho\hat{\pi}_{t+j}+\left(1-\beta\xi_p\right)\zeta\sum_{j=0}^{\infty}\left(\beta\xi_p\right)^j\hat{s}_{t+j} \\
& -\frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left(1-\beta\xi_p\right)\zeta\left\{1+\tau\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\tau'\right\}\tilde{k}_t(i) \\
& -\frac{\alpha}{1-\alpha}\frac{\sigma_a(1-\alpha)}{\sigma_a(1-\alpha)+1}\left(1-\beta\xi_p\right)\zeta\left[\tau\xi_p\beta A\left(I-\xi_p\beta A\right)^{-1}\begin{pmatrix}0\\1\end{pmatrix}\right]\tilde{k}_{t-1}(i)
\end{aligned}
$$

Write this as:

$$
\hat{p}_t^*(i)=\hat{p}_t^*-\psi_0\tilde{k}_t(i)-\psi_1\widehat{\tilde{k}}_{t-1}^{+}(i),
$$

where.

$$\hat{p}_t^* = \sum_{j=1}^{\infty} \left(\beta\xi_p\right)^j \Delta_\varrho \hat{\pi}_{t+j} + \zeta \left(1 - \beta\xi_p\right) \sum_{j=0}^{\infty} \left(\beta\xi_p\right)^j \hat{s}_{t+j} \tag{7.18}$$

Also,

$$\psi_0 = \frac{\alpha}{1-\alpha} \frac{\sigma_a (1-\alpha)}{\sigma_a (1-\alpha) + 1} \left(1 - \beta\xi_p\right) \zeta \left\{1 + \tau\xi_p\beta A \left(I - \xi_p\beta A\right)^{-1} \tau'\right\} \tag{7.19}$$

$$\psi_1 = \frac{\alpha}{1-\alpha} \frac{\sigma_a (1-\alpha)}{\sigma_a (1-\alpha) + 1} \left(1 - \beta\xi_p\right) \zeta \left[\tau\xi_p\beta A \left(I - \xi_p\beta A\right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]. \tag{7.20}$$

7.4. Pulling Everything Together to Get the Reduced Form

Solve for $\hat{p}_t^*$ in (7.18) using (7.6), to obtain:

$$\begin{aligned} \frac{\xi_p}{1-\xi_p} \Delta_\varrho \hat{\pi}_t &= \sum_{j=1}^{\infty} \left(\beta\xi_p\right)^j \Delta_\varrho \hat{\pi}_{t+j} + \zeta \left(1 - \beta\xi_p\right) \sum_{j=0}^{\infty} \left(\beta\xi_p\right)^j \hat{s}_{t+j} \\ &= \frac{\beta\xi_p L^{-1}}{1 - \beta\xi_p L^{-1}} \Delta_\varrho \hat{\pi}_t + \zeta \frac{\left(1 - \beta\xi_p\right)}{1 - \beta\xi_p L^{-1}} \hat{s}_t. \end{aligned}$$

Multiply by $1 - \beta\xi_p L^{-1}$ and rearrange:

$$\Delta_\varrho \hat{\pi}_t = \beta \Delta_\varrho \hat{\pi}_{t+1} + \frac{\left(1 - \xi_p\right)\left(1 - \beta\xi_p\right)}{\xi_p} \zeta \hat{s}_t$$

The key parameter to be solved for is ζ. To do so, first find κ_1, κ_2, κ_3, ψ_0, ψ_1 to solve (7.14), (7.15), (7.16), (7.19), (7.20). Then, evaluate (**??**).

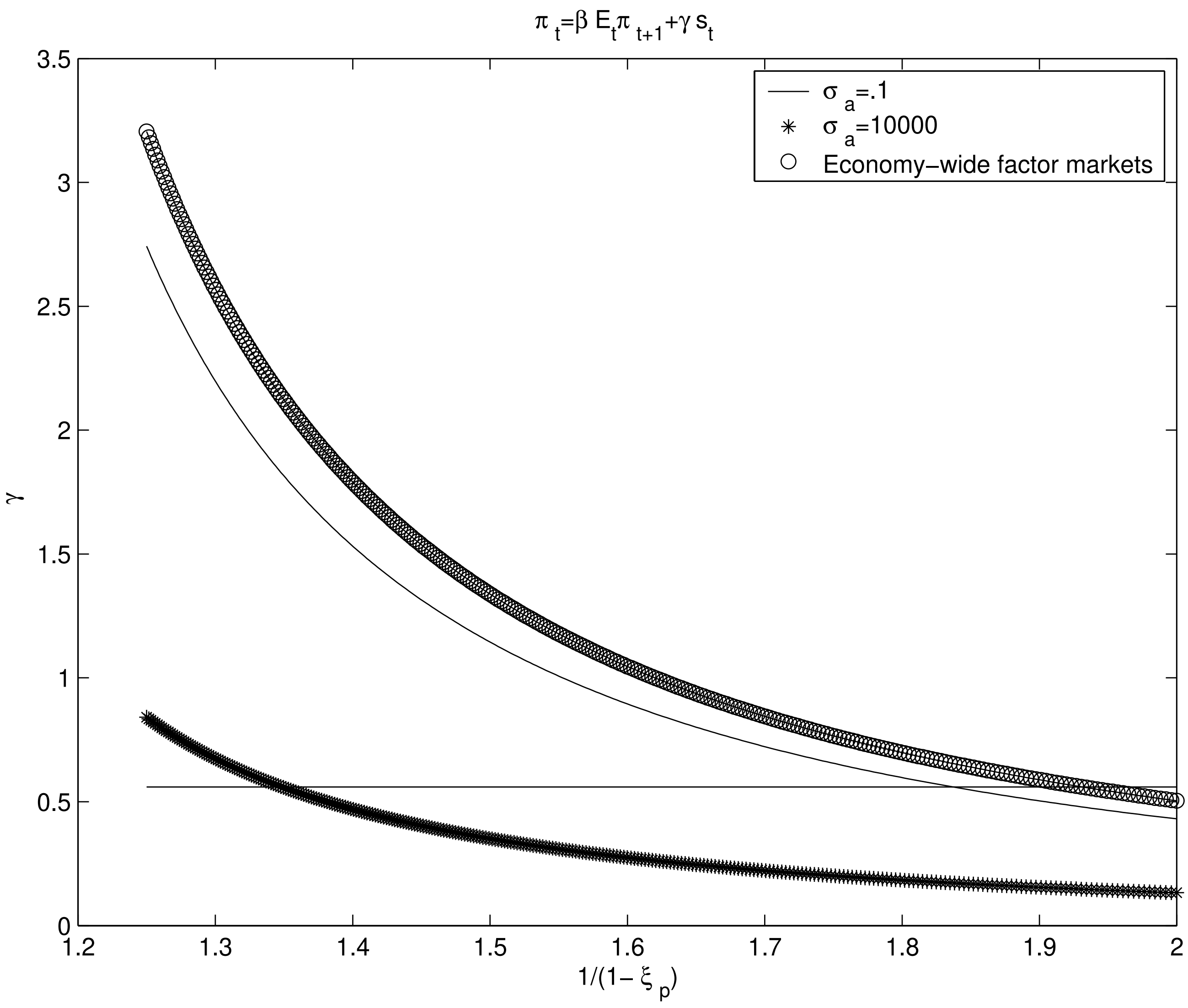
To get a feel for how these formulas work, consider the following example. Here, $\lambda_w = 1.05$, $\lambda_f = 1.2$, $\mu_\Upsilon = 1 + .03/4$, $\alpha = 0.36$, $x = 1.017$, $\beta = 1.03^{-.25}$, $\delta = 0.025$, $\eta = 0.036$, $\mu_z = 1.0001$, $b = 0.73$, $\sigma_L = 1$, $\psi_L = 1$, $V = 1.43$, $\varepsilon = 1.00830983517582$, $S'' = 1.11651914318597$. Steady state consumption to output ratio is $c/\tilde{y} = 0.68$, steady state hours worked are 0.95, and $q = 1.09$, $\phi = 0.42$, $m = 2.50$, $\bar{k} = 19$, $\tilde{w} = 1.52$ (these numbers have been rounded).

The following figure displays γ, where

$$\gamma = \frac{\left(1 - \xi_p\right)\left(1 - \beta\xi_p\right)}{\xi_p} \zeta$$

for $\sigma_a = 0.1$ and $\sigma_a = 10,000$. The former corresponds to variable capital utilization, and the latter, to no variable capital utilization.In addition, the line indicated by circles displays γ in the economy-wide factor market case, when $\zeta = 1$. (The values of γ for the case $\sigma_a = 0.01$ were also computed, but they virtually coincide with the line indicated by circles.) The horizontal axis displays the mean times between reoptimizations, $1/(1 - \xi_p)$. The micro empirical literature suggests that the mean time between reoptimizations may be 1.72 quarters. With this mean time, when there is no variable capital utilization, γ is a bit above 0.2. With economy-wide factor markets, γ is 0.80. Thus, without variable capital utilization, the value of γ is cut by a factor of 4 with the assumption of economy-wide capital markets.

Now, suppose instead that econometric methods produce an estimate $\gamma = 0.56$. What is the implied time between price reoptimizations under economy-wide capital markets and under firm-specific capital? This value of γ is indicated under the horizontal axis. Under economy-wide factor markets, the implied duration between price optimization is 1.93 quarters. Under firm-specific capital the implied duration between price optimization is 1.35 quarters. If the estimate of γ were instead in the range of 0.2, then under firm-specific capital, the estimate of duration would be around 1.7 quarters, while it would be well over 2 quarters for economy-wide capital markets.



7.5. Who'se Doing the Production after a Monetary Shock?

We suppose that the economy is in a steady state up to period 1, when a monetary injection occurs. Because prices are set before the monetary shock, in period 1 all prices are identical, and all production is equal. We now discuss each period in turn. The first part of the discussion is in a sense a failure. It's a laborious discussion of what happens in period 2, 3 and 4. The next subsection covers period N, and is more general and simpler too. Final section discusses what can be done.

7.5.1. Period 2

In Period 2, a fraction of firms, $(1-\xi_p)$ is able to reoptimize its price, and a fraction, ξ_p, is not. From before, we know that aggregate output, $\hat{Y}_2$, is

$$\hat{Y}_2 = \int_I y_2(i)di + \int_J y_2(j)dj,$$

where I denotes the set of firms that can reoptimize and J denotes the others. As discussed above, the ones that can reoptimize their price in period 2 do so according to:

$$\begin{aligned}
\widehat{\bar{k}}^{+}_{t+1}(i) &= \kappa_1 \widehat{\bar{k}}^{+}_{t}(i) + \kappa_2 \widehat{\bar{k}}^{+}_{t-1}(i) + \kappa_3 \hat{p}_t(i) \\
\hat{p}^*_t(i) &= \hat{p}^*_t - \psi_0 \widehat{\bar{k}}^{+}_{t}(i) - \psi_1 \widehat{\bar{k}}^{+}_{t-1}(i),\ \widehat{\bar{k}}^{+}_{t}(i) \equiv \widehat{\bar{k}}_t(i) - \widehat{\bar{k}}_t,
\end{aligned}$$

where ψ_0, ψ_1, κ_1, κ_2, κ_3 are computed as discussed in the previous subsection. The amount that the period 2 optimizers actually produce is determined by their demand curve:

$$-\theta \hat{p}_2(i) + \hat{Y}_2 = \hat{y}_2(i).$$

Substitute the price of the optimizers into this expression:

$$-\theta \left[\hat{p}^*_2 - \psi_0 \widehat{\bar{k}}^{+}_{2}(i) - \psi_1 \widehat{\bar{k}}^{+}_{1}(i) \right] + \hat{Y}_2 = \hat{y}_2(i).$$

In period 1, all firms have the same capital, so that $\widehat{\bar{k}}^{+}_{1}(i) = 0$. In addition, all firms make the same investment decision in period 1, because their situations are symmetric. So, $\widehat{\bar{k}}^{+}_{2}(i) = 0$. Finally, we also have,

$$\hat{p}^*_t = \frac{\xi_p}{(1-\xi_p)} \Delta_\varrho \hat{\pi}_t,$$

in each period. We conclude that the output of the i^{th} price-optimizing firms is given by:

$$-\theta \frac{\xi_p}{(1-\xi_p)} \Delta_\varrho \hat{\pi}_2 + \hat{Y}_2 = \hat{y}^*_2(i).$$

Now consider the j^{th} firm, which cannot optimize in period 2. It sets is price according to:

$$\hat{p}_2(j) = \hat{p}_1(j) - \Delta_\varrho \hat{\pi}_2,$$

since $\hat{p}_1(i) = 0$, due to the fact that all prices are equal in period 1. To determine how much the j^{th} firm produces, substitute its price into the demand curve

$$-\theta \left[\hat{p}_1(j) - \Delta_\varrho \hat{\pi}_2\right] + \hat{Y}_2 = \hat{y}_2(j),$$

or, since $\hat{p}_1(j) = 0$,

$$\hat{y}_2(j) = \theta \Delta_\varrho \hat{\pi}_2 + \hat{Y}_2.$$

Total output of firms that reoptimize their price is:

$$\begin{aligned} \int_I \hat{y}_2(i)di &= (1-\xi_p)\left[-\theta \frac{\xi_p}{(1-\xi_p)} \Delta_\varrho \hat{\pi}_2 + \hat{Y}_2\right] \\ & -\theta \xi_p \Delta_\varrho \hat{\pi}_2 + (1-\xi_p)\hat{Y}_2 \end{aligned}$$

Total output of firms that cannot reoptimize their price is:

$$\begin{aligned} \int_J \hat{y}_2(j)dj &= \xi_p \left[\theta \Delta_\varrho \hat{\pi}_2 + \hat{Y}_2\right] \\ &= \xi_p \theta \Delta_\varrho \hat{\pi}_2 + \xi_p \hat{Y}_2 \end{aligned}$$

The sum of these is obviously $\hat{Y}_2$, aggregate output. The firms that reoptimize their price reduce output and the firms that cannot, must increase their output. A worrisome feature of this result, is that the result seems to have nothing to do with the firm-specificity of capital.

7.5.2. Period 3

Now consider period 3. In this period there are four types of firms:

- (1) the $(1-\xi_p)^2$ those who optimized in period 2 and in period 3
- (2) the $\xi_p(1-\xi_p)$ who did not optimize in period 2 and did in period 3
- (3) the ξ_p^2 who did not optimize in period 2 and period 3
- (4) the $(1-\xi_p)\xi_p$ who optimized in period 2 and did not in period 3.

We now consider the price of the typical firm in each of these four categories. Consider category (1) first. The i^{th} firm in categories (1) and (2) set their price according to:

$$\begin{aligned} \hat{p}_3^*(i) &= \frac{\xi_p}{(1-\xi_p)} \Delta_\varrho \hat{\pi}_3 - \psi_0 \widehat{\bar{k}}_3^{+}(i) - \psi_1 \widehat{\bar{k}}_2^{+}(i) \\ \widehat{\bar{k}}_3^{+}(i) &= \kappa_1 \widehat{\bar{k}}_2^{+}(i) + \kappa_2 \widehat{\bar{k}}_1^{+}(i) + \kappa_3 \hat{p}_2(i). \end{aligned}$$

Actually, for the reasons given above, $\widehat{\bar{k}}_1^+(i) = \widehat{\bar{k}}_2^+(i) = 0$, so that, after substituting,

$$\hat{p}_3^*(i) = \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\hat{p}_2(i).$$

The i^{th} firm in category (1) optimized $\hat{p}_2(i)$, and the price chosen is the same for all i, so that

$$\hat{p}_2^*(i) = \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2.$$

Then,

$$\begin{aligned}\hat{p}_3^*(i) &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2. \\ &= \frac{\xi_p}{(1-\xi_p)}\left[\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2\right]\end{aligned}$$

Given the demand curve:

$$-\theta\hat{p}_2(i) + \hat{Y}_2 = \hat{y}_2(i).$$

the firm in category (1) produces

$$\hat{y}_3(i) = -\theta\left[\frac{\xi_p}{(1-\xi_p)}\left(\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2\right)\right] + \hat{Y}_3.$$

Total production in this category is $(1-\xi_p)^2$ times this much:

$$(1) = -\theta\left[(1-\xi_p)\xi_p\left(\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2\right)\right] + (1-\xi_p)^2\hat{Y}_3.$$

Now consider the firms in category (2). They set their price according to

$$\hat{p}_3^*(i) = \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\hat{p}_2(i),$$

where

$$\begin{aligned}\hat{p}_2(i) &= \hat{p}_1(i) - \Delta_\varrho\hat{\pi}_2 \\ &= -\Delta_\varrho\hat{\pi}_2.\end{aligned}$$

Then,

$$\hat{p}_3^*(i) = \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2.$$

The demand for their product is

$$\hat{y}_3(i) = -\theta\hat{p}_3(i) + \hat{Y}_3,$$

so that total demand for this type of firm's product is:

$$(2)\ -\theta\xi_p(1-\xi_p)\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3+\psi_0\kappa_3\Delta_\varrho\hat{\pi}_2\right]+\xi_p(1-\xi_p)\hat{Y}_3,$$

Now consider the firms in category (3). They set their price according to:

$$\hat{p}_3(i)=\hat{p}_1(i)-\Delta_\varrho\hat{\pi}_2-\Delta_\varrho\hat{\pi}_3$$

The demand curve for their product is:

$$\hat{y}_3(i)=-\theta\hat{p}_3(i)+\hat{Y}_3,$$

so that

$$\hat{y}_3(i)=\theta\left[\Delta_\varrho\hat{\pi}_2+\Delta_\varrho\hat{\pi}_3\right]+\hat{Y}_3.$$

Total production by these firms is:

$$(3)\ \xi_p^2\theta\left[\Delta_\varrho\hat{\pi}_2+\Delta_\varrho\hat{\pi}_3\right]+\xi_p^2\hat{Y}_3.$$

Now consider category (4). They set their price according to:

$$\hat{p}_3(i)=\hat{p}_2^*(i)-\Delta_\varrho\hat{\pi}_3,$$

where

$$\begin{aligned}\hat{p}_2^*(i) &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2-\psi_0\widehat{\bar{k}}_2^+(i)-\psi_1\widehat{\bar{k}}_1^+(i)\\ &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2\end{aligned}$$

where we have used,

$$\widehat{\bar{k}}_2^+(i)=\widehat{\bar{k}}_1^+(i)=0$$

Thus,

$$\hat{p}_3(i)=\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2-\Delta_\varrho\hat{\pi}_3.$$

The demand for their product is:

$$\hat{y}_3(i)=-\theta\hat{p}_3(i)+\hat{Y}_3,$$

so

$$\hat{y}_3(i)=-\theta\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2-\Delta_\varrho\hat{\pi}_3\right]+\hat{Y}_3.$$

Total output of category (4) firms is:

$$(4) \quad -\theta(1-\xi_p)\xi_p\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_3\right] + (1-\xi_p)\xi_p\hat{Y}_3.$$

Total output is just the sum of all four outputs:

$$\begin{aligned}
& -\theta\left[(1-\xi_p)\xi_p\left(\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2\right)\right] + (1-\xi_p)^2\hat{Y}_3 \\
& -\theta\xi_p(1-\xi_p)\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2\right] + \xi_p(1-\xi_p)\hat{Y}_3 \\
& +\xi_p^2\theta\left[\Delta_\varrho\hat{\pi}_2 + \Delta_\varrho\hat{\pi}_3\right] + \xi_p^2\hat{Y}_3 \\
& -\theta(1-\xi_p)\xi_p\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_3\right] + (1-\xi_p)\xi_p\hat{Y}_3 \\
= \; & -\theta\left[(1-\xi_p)\xi_p\left(\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2\right)\right] \text{ (change, change)} \\
& -\theta\xi_p(1-\xi_p)\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2\right] \text{ (don't change, do change)} \\
& +\xi_p^2\theta\left[\Delta_\varrho\hat{\pi}_2 + \Delta_\varrho\hat{\pi}_3\right] \text{ (no change, no change)} \\
& -\theta(1-\xi_p)\xi_p\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_3\right] \text{ (do change, don't change)} \\
& +\hat{Y}_3 \\
= \; & \left[\theta(1-\xi_p)\xi_p\psi_0\kappa_3 - \theta\xi_p(1-\xi_p)\psi_0\kappa_3 + \xi_p^2\theta - \theta\xi_p^2\right]\Delta_\varrho\hat{\pi}_2 \\
& + \left[-\theta(1-\xi_p)\xi_p - \theta\xi_p(1-\xi_p)\frac{\xi_p}{(1-\xi_p)} + \xi_p^2\theta + \theta(1-\xi_p)\xi_p\right]\Delta_\varrho\hat{\pi}_3 \\
& +\hat{Y}_t \\
= \; & \hat{Y}_t
\end{aligned}$$

The case of economy-wide capital rental markets corresponds to these formulas with $\psi_0 = \kappa_3 = 0$.

7.5.3. Period 4

Now consider period 4. In this period there are four types of firms:

- (1) the $(1-\xi_p)^3$ who optimized in periods 2, 3 and 4
- (2) the $\xi_p(1-\xi_p)^2$ who did not optimize in period 2, but did in periods 3 and 4
- (3) the $\xi_p^2(1-\xi_p)$ who did not optimize in periods 2 and 3, but did in period 4
- (4) the ξ_p^3 who did not optimize in periods 2, 3 and 4
- (5) the $(1-\xi_p)^2\xi_p$ who optimized in periods 2, 3, but did not in period 4

- (6) the $(1-\xi_p)\xi_p(1-\xi_p)$ who did not optimize in periods 2 and 4, but did in period 3
- (7) the $(1-\xi_p)\xi_p^2$ who did optimize in period 2, but did not in periods 3 and 4
- (8) the $(1-\xi_p)\xi_p(1-\xi_p)$ who optimized in periods 2 and 4, but did not in period 3

Firms setting prices in period 4, satisfy the following equations:

$$\begin{aligned}\hat{p}_4^*(i) &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - \psi_0\widehat{\bar{k}}_4^+(i) - \psi_1\widehat{\bar{k}}_3^+(i) \\ \widehat{\bar{k}}_4^+(i) &= \kappa_1\widehat{\bar{k}}_3^+(i) + \kappa_2\widehat{\bar{k}}_2^+(i) + \kappa_3\hat{p}_3(i)\end{aligned}$$

$$\hat{p}_3^*(i) = \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\hat{p}_2(i).$$

$$\hat{p}_2^*(i) = \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2.$$

$$\begin{aligned}\hat{p}_3^*(i) &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\widehat{\bar{k}}_3^+(i) - \psi_1\widehat{\bar{k}}_2^+(i) \\ \widehat{\bar{k}}_3^+(i) &= \kappa_1\widehat{\bar{k}}_2^+(i) + \kappa_2\widehat{\bar{k}}_1^+(i) + \kappa_3\hat{p}_2(i).\end{aligned}$$

As noted before, $\widehat{\bar{k}}_1^+(i) = \widehat{\bar{k}}_2^+(i) = 0$. It is useful to have an expression relating the price set by optimizers in period 4, to the prices they set in periods 2 and 3:

$$\begin{aligned}\hat{p}_4^*(i) &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - \psi_0\widehat{\bar{k}}_4^+(i) - \psi_1\widehat{\bar{k}}_3^+(i) \\ &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - \psi_0\left[\kappa_1\widehat{\bar{k}}_3^+(i) + \kappa_2\widehat{\bar{k}}_2^+(i) + \kappa_3\hat{p}_3(i)\right] - \psi_1\widehat{\bar{k}}_3^+(i) \\ &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - \left[\psi_0\kappa_1 + \psi_1\right]\widehat{\bar{k}}_3^+(i) - \psi_0\kappa_3\hat{p}_3(i) \\ &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - \left[\psi_0\kappa_1 + \psi_1\right]\kappa_3\hat{p}_2(i) - \psi_0\kappa_3\hat{p}_3(i).\end{aligned}$$

So, to summarize. Optimizers in each of the three periods set price according to:

$$\hat{p}_4^*(i) = \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - \left[\psi_0\kappa_1 + \psi_1\right]\kappa_3\hat{p}_2(i) - \psi_0\kappa_3\hat{p}_3(i) \tag{7.21}$$

$$\hat{p}_3^*(i) = \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\hat{p}_2(i) \tag{7.22}$$

$$\hat{p}_2^*(i) = \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 \tag{7.23}$$

Firms that do not optimize in a given period set price according to:

$$\hat{p}_t(i) = \hat{p}_{t-1}(i) - \Delta_\varrho \hat{\pi}_t$$

Consider firms of type (1), who optimize in all three periods. To get their price, simply substitute (7.22) and (7.23) into (7.21):

$$\begin{aligned}
\hat{p}_4^*(i) &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - [\psi_0\kappa_1 + \psi_1]\kappa_3\hat{p}_2(i) - \psi_0\kappa_3\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\hat{p}_2(i)\right] \\
&= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - \left[\psi_0\kappa_1 + \psi_1 - \psi_0^2\kappa_3\right]\kappa_3\hat{p}_2(i) - \psi_0\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 \\
&= \frac{\xi_p}{(1-\xi_p)}\left\{\Delta_\varrho\hat{\pi}_4 - \left[\psi_0\kappa_1 + \psi_1 - \psi_0^2\kappa_3\right]\kappa_3\Delta_\varrho\hat{\pi}_2 - \psi_0\kappa_3\Delta_\varrho\hat{\pi}_3\right\}
\end{aligned}$$

The demand for their product is:

$$\hat{y}_4(i) = -\theta\hat{p}_4(i) + \hat{Y}_4. \tag{7.24}$$

the total output of this type of firm is:

$$-(1-\xi_p)^3\theta\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - \left[\psi_0\kappa_1 + \psi_1 - \psi_0^2\kappa_3\right]\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 - \psi_0\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3\right] + (1-\xi_p)^3\hat{Y}_4.$$

Now consider firms of type (2): no, yes, yes. Substitute (7.22) into (7.21)

$$\begin{aligned}
\hat{p}_4^*(i) &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - [\psi_0\kappa_1 + \psi_1]\kappa_3\hat{p}_2(i) - \psi_0\kappa_3\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\hat{p}_2(i)\right] \\
&= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 + \left[\psi_0\kappa_1 + \psi_1 - \psi_0^2\kappa_3\right]\kappa_3\Delta_\varrho\hat{\pi}_2 - \psi_0\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3.
\end{aligned}$$

then, their total output is:

$$(2)\ -\theta\xi_p(1-\xi_p)^2\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 + \left[\psi_0\kappa_1 + \psi_1 - \psi_0^2\kappa_3\right]\kappa_3\Delta_\varrho\hat{\pi}_2 - \psi_0\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3\right] + \xi_p(1-\xi_p)^2\hat{Y}_4$$

Now consider the $\xi_p^2(1-\xi_p)$ firms of type (3), no, no, yes. Their price in period 4 is:

$$\begin{aligned}
\hat{p}_4^*(i) &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - [\psi_0\kappa_1 + \psi_1]\kappa_3\hat{p}_2(i) - \psi_0\kappa_3\left[\hat{p}_2(i) - \Delta_\varrho\hat{\pi}_3\right] \\
&= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - [\psi_0\kappa_1 + \psi_1 + \psi_0]\kappa_3\hat{p}_2(i) + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_3 \\
&= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 + [\psi_0\kappa_1 + \psi_1 + \psi_0]\kappa_3\Delta_\varrho\hat{\pi}_2 + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_3.
\end{aligned}$$

so, their total output is:

$$(3)\ -\theta\xi_p^2(1-\xi_p)\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 + [\psi_0\kappa_1 + \psi_1 + \psi_0]\kappa_3\Delta_\varrho\hat{\pi}_2 + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_3\right] + \xi_p^2(1-\xi_p)\hat{Y}_4$$

Now consider the ξ_p^3 firms of type (4), no, no, no. Their price in period 4 is:

$$\hat{p}_4(i) = -\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_3 - \Delta_\varrho\hat{\pi}_4,$$

so that their total output is:

$$\hat{y}_4(i) = \theta\xi_p^3\left[\Delta_\varrho\hat{\pi}_2 + \Delta_\varrho\hat{\pi}_3 + \Delta_\varrho\hat{\pi}_4\right] + \xi_p^3\hat{Y}_4.$$

Now consider the $(1-\xi_p)^2\xi_p$ firms of type (5), yes, yes, no. Their price in period 4 is:

$$\begin{aligned}\hat{p}_4(i) &= \hat{p}_3(i) - \Delta_\varrho\hat{\pi}_4 \\ &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\hat{p}_2(i) - \Delta_\varrho\hat{\pi}_4 \\ &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_4.\end{aligned}$$

$$(5)\ \ -\theta(1-\xi_p)^2\xi_p\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_4\right] + (1-\xi_p)^2\xi_p\hat{Y}_4.$$

Now consider the $(1-\xi_p)\xi_p(1-\xi_p)$ firms, no, yes, no. Their period 4 price is:

$$\begin{aligned}\hat{p}_4(i) &= \hat{p}_3(i) - \Delta_\varrho\hat{\pi}_4 \\ &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 - \psi_0\kappa_3\hat{p}_2(i) - \Delta_\varrho\hat{\pi}_4 \\ &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_4.\end{aligned}$$

Their total output in period 4 is:

$$(6)\ \ -\theta(1-\xi_p)\xi_p(1-\xi_p)\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_3 + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_4\right] + (1-\xi_p)\xi_p(1-\xi_p)\hat{Y}_4.$$

Now consider the $(1-\xi_p)\xi_p^2$ firms of type (7), yes, no, no:

$$\begin{aligned}\hat{p}_4(i) &= \hat{p}_3(i) - \Delta_\varrho\hat{\pi}_4 \\ &= \hat{p}_2(i) - \Delta_\varrho\hat{\pi}_3 - \Delta_\varrho\hat{\pi}_4 \\ &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_3 - \Delta_\varrho\hat{\pi}_4.\end{aligned}$$

Their total output is:

$$(7)\ \hat{y}_4(i) = -\theta\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 - \Delta_\varrho\hat{\pi}_3 - \Delta_\varrho\hat{\pi}_4\right] + \hat{Y}_4$$

Finally, consider the $(1-\xi_p)\xi_p(1-\xi_p)$ type (8) firms, yes, no, yes:

$$\begin{aligned}
\hat{p}_4^*(i) &= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - [\psi_0\kappa_1+\psi_1]\,\kappa_3\hat{p}_2(i) - \psi_0\kappa_3\,[\hat{p}_2(i)-\Delta_\varrho\hat{\pi}_3] \\
&= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - [\psi_0\kappa_1+\psi_1+\psi_0]\,\kappa_3\hat{p}_2(i) + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_3 \\
&= \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - [\psi_0\kappa_1+\psi_1+\psi_0]\,\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_3
\end{aligned}$$

their total output is:

$$(8)\ -\theta\xi_p(1-\xi_p)\xi_p\left[\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_4 - [\psi_0\kappa_1+\psi_1+\psi_0]\,\kappa_3\frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_2 + \psi_0\kappa_3\Delta_\varrho\hat{\pi}_3\right]+\xi_p(1-\xi_p)\xi_p\hat{Y}_4$$

7.5.4. Period N

Let the state of nature for firm i in time t be $s_t^i \in (0,1)$, where 0 means the firm cannot optimize and 1 means it can. A history of firm i is $s^{i,N} = (s_2^i, ..., s_N^i)$. In period t, the firm inherits $\widehat{\bar{k}}_t^+(i)$ and $\widehat{\bar{k}}_{t-1}^+(i)$. We have that $\widehat{\bar{k}}_1^+(i) = \widehat{\bar{k}}_2^+(i) = \hat{p}_1(i) = 0$. Then,

$$\hat{p}_t(i) = \begin{cases} \frac{\xi_p}{(1-\xi_p)}\Delta_\varrho\hat{\pi}_t - \psi_0\widehat{\bar{k}}_t^+(i) - \psi_1\widehat{\bar{k}}_{t-1}^+(i) & \text{if } s_t^i = 1 \\ \hat{p}_{t-1}(i) - \Delta_\varrho\hat{\pi}_t & \text{if } s_t^i = 0. \end{cases},$$

for $t = 2, 3, ...\ N$. The demand for this firm's product is:

$$\hat{y}_t(i) = -\theta\hat{p}_t(i) + \hat{Y}_t.$$

It's capital decision can be computed too:

$$\widehat{\bar{k}}_{t+1}^+(i) = \kappa_1\widehat{\bar{k}}_t^+(i) + \kappa_2\widehat{\bar{k}}_{t-1}^+(i) + \kappa_3\hat{p}_t(i).$$

Let

$$\hat{p}\left(s^{i,N}\right),\ \hat{y}\left(s^{i,N}\right),\ \widehat{\bar{k}}^+\left(s^{i,N}\right)$$

denote the relative price, output and beginning of period capital choice of a firm with history $s^{i,N}$, in period N. Let $prob\left(s^{i,N}\right)$ denote the probability of history $s^{i,N}$. To be concrete, suppose $N = 4$. In this case, the eight possible $s^{i,4}$ are given by the rows of the following matrix:

$$\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{array}$$

In this case, for a parameterization with $\xi_p = 0.2$, we obtain the following 8 possible period 3 outputs:

$$1.2504,\ 1.1584,\ 1.2087,\ 1.1584,\ 1.2549,\ 1.1584,\ 1.2087,\ 1.1584$$

This is the output of the typical firm in period 4 of each history, with the first corresponding to the first row in the above matrix, the second to the second row, etc. Here the first output is the output of a firm with history $0, 0, 0$, (don't optimize in period 2, don't optimize in period 3, don't optimize in period 4) and the last output is the output of the firm in period 4 with history, 1,1,1. Notice that the output of the firm in the last state is the lowest. This is not surprising, since this firm has the highest price. These are the various possible prices in period 4:

$$-0.0134,\ 0.0020,\ -0.0064,\ 0.0020,\ -0.0141,\ 0.0020,\ -0.0064,\ 0.0020$$

Note that several of these are identical. (The ones that are identical are identical up to all 14 digits after the decimal that MATLAB displays.) The associated probabilities are:

$$0.0080,\ 0.0320,\ 0.0320,\ 0.1280,\ 0.0320,\ 0.1280,\ 0.1280,\ 0.5120.$$

These add up to unity, as they should. The probability of any history corresponds to the number of firms that experience that history.

The total number of firms is unity, and total production in period 4 is 1.17 (i.e., this is the product of each history's probability and the production of the individual firm in that category.). This is the average production across each individual firm. Note that the average production of the firms that reoptimize in period 4, 1.1584, is less than the economy-wide average.

There are 0.8 (=$0.0320 + 0.1280 + 0.1280 + 0.5120$) firms that optimize in period 4, so if each firm in this category produced the economy-wide average, the group as a whole would produce 0.9362 units of output. The histories in which optimization occurs in period 4 are 2, 4, 6, 8. They produce

$$0.92672 = 0.0320 \times 1.1584 + 0.1280 \times 1.1584 + 0.1280 \times 1.1584 + 0.5120 \times 1.1584,$$

which is less than their share, as expected.

Now consider the firms that did not optimize in period 4, and also did not optimize in period 3. These correspond to histories 1 and 5. In period 4, there are .0040 of these firms, and they produce a total of:

$$0.05016 = 0.0080 \times 1.2504 + 0.0320 \times 1.2549.$$

The average output of firms in these categories is 1.254 (=0.05016/(.0080+.0320)). This is higher than the economy-wide average of 1.17.

Now consider the one type of firm that did not reoptimize price in period 2. There are 0.008 of these firms and each one produces 1.2504 units of output. The total output they produce is

$$0.0100 = 0.008 \times 1.2504.$$

7.5.5. Price Dispersion

It is generally thought that different models have different implications for the reallocation of resources in the wake of a demand shock, such as a monetary shock. Here, we discuss various indicators of this. One statistic that would be of interest would be the fraction of total output produced by firms that optimize price in the current period; firms that do not optimize in the current period, but did optimize in the previous period; firms that did not optimize in the current and previous period, but did optimize in the period before that, etc. In addition, it would be useful to know not only the total output of these firms, but also the average output of firms in each category.

This should be done for the model with firm-specific capital, and for the model without firm-specific capital. In the case of the latter, the cross-sectional distribution of resources and prices is obtained by simulations with $\psi_0 = \psi_1 = \kappa_1 = \kappa_2 = \kappa_3 = 0$. The model without firm-specific capital should be simulated both for the case of full indexation and no indexation.

8. Kalman Filter

The idea is to estimate the model using data on:

$$\underbrace{X_t}_{10\times 1} = \left(\begin{array}{c} \Delta \ln(GDP_t/\text{Hours}_t) \\ \Delta \ln(GDP \text{ deflator}_t) \\ \ln(GDP_t/\text{Hours}_t) - \ln(W_t/P_t) \\ \ln(\text{Hours}_t) \\ \ln(C_t/GDP_t) \\ \ln(I_t/GDP_t) \\ \text{Federal Funds Rate}_t \\ \ln(GDP \text{ deflator}_t) + \ln(GDP_t) - \ln(M2_t) \\ \Delta \ln \text{Investment Price} \\ \text{Capacity Utilization}_t \end{array} \right).$$

The first step is to express the time series model for X_t implied by our model. Recall, the law of motion for z_t is:

$$z_t = Az_{t-1} + B\theta_t,$$

where

$$\theta_t = \rho\theta_{t-1} + e_t,\ \ Ee_t e_t' = V. \tag{8.1}$$

Here,

$$e_t = \begin{pmatrix} \varepsilon_{M,t} \\ \varepsilon_{M,t} \\ \varepsilon_{\mu^z,t} \\ \varepsilon_{\mu^z,t} \\ c_z \varepsilon_{\mu^z,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ c_\Upsilon \varepsilon_{\mu_\Upsilon,t} \\ 0 \\ 0 \end{pmatrix},$$

so that

$$V = \begin{bmatrix} \sigma_M^2 & \sigma_M^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_M^2 & \sigma_M^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{\mu^z}^2 & \sigma_{\mu^z}^2 & c_z\sigma_{\mu^z}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{\mu^z}^2 & \sigma_{\mu^z}^2 & c_z\sigma_{\mu^z}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_z\sigma_{\mu^z}^2 & c_z\sigma_{\mu^z}^2 & c_z^2\sigma_{\mu^z}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\mu_\Upsilon}^2 & \sigma_{\mu_\Upsilon}^2 & c_\Upsilon\sigma_{\mu_\Upsilon}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\mu_\Upsilon}^2 & \sigma_{\mu_\Upsilon}^2 & c_\Upsilon\sigma_{\mu_\Upsilon}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_\Upsilon\sigma_{\mu_\Upsilon}^2 & c_\Upsilon\sigma_{\mu_\Upsilon}^2 & c_\Upsilon^2\sigma_{\mu_\Upsilon}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8.2}$$

Write

$$X_t = \alpha + \tau z_t + \bar{\tau} z_{t-1} + \tau^\theta \theta_t, \tag{8.3}$$

where α, τ, $\bar{\tau}$, τ^s are described in the first subsection below. Note that the law of motion for z_t can be written

$$z_t = Az_{t-1} + B\rho\theta_{t-1} + Be_t.$$

Let,

$$\xi_t = \begin{pmatrix} z_t \\ z_{t-1} \\ \theta_t \end{pmatrix},$$

so that the whole system can be written,

$$\begin{pmatrix} z_{t+1} \\ z_t \\ \theta_{t+1} \end{pmatrix} = \begin{bmatrix} A & 0 & B\rho \\ I & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{pmatrix} z_t \\ z_{t-1} \\ \theta_t \end{pmatrix} + \begin{pmatrix} B \\ 0 \\ I \end{pmatrix} e_{t+1},$$

or,

$$\xi_{t+1} = F\xi_t + v_{t+1},$$

where

$$Q \equiv Ev_t v_t' = \begin{pmatrix} B \\ 0 \\ I \end{pmatrix} Ee_t e_t' \begin{pmatrix} B & 0 & I \end{pmatrix}$$
$$= \begin{bmatrix} BVB' & 0 & BV \\ 0 & 0 & 0 \\ VB' & 0 & V \end{bmatrix}.$$

The observed data are a linear combination of ξ_t, plus noise:

$$y_t = H\xi_t + w_t,$$

where $R = Ew_t w_t'$ is a diagonal matrix (sorry for the potentially confusing notation for the variance-covariance matrix of the measurement error).

$$H = \begin{bmatrix} \tau & \bar{\tau} & \tau^{\theta} \end{bmatrix}.$$

The problem of estimating this system is described in the second subsection below.

Notice that the Kalman Filter system is completely characterized by (F, H, R, Q). These in turn can be constructed from the model parameters (including the variances of the stochastic shocks in V, as well as the measurement error variances.) Additional inputs required are the initial state vector $(\hat{\xi}_{1|0} = E(\xi_1))$ and the initial state covariance $(P_{1|0})$. Following Hamilton p. 378, we set $P_{1|0} = \Sigma$, where Σ satisfies the following Riccati equation[1]:

$$\Sigma = F\Sigma F' + Q. \tag{8.4}$$

In case this takes too much time to compute, we can also use $\Sigma_{\bar{r}}$, where $\Sigma_{\bar{r}}$ satisfies

$$\Sigma_r = F\Sigma_{r-1}F' + Q,$$

$r = 1, 2, ..., \bar{r}$, and $\Sigma_0 = 0$, for small $\bar{r}$, say $\bar{r} = 10$.

8.1. The Reduced Form

Consider

$$\ln\frac{y_t}{h_t} = \ln\frac{\tilde{y}_t z_t^*}{h_t} = \ln\tilde{y}_t - \ln h_t + \ln z_t^*,$$

so that

$$\Delta\ln\frac{y_t}{h_t} = (\ln\tilde{y}_t - \ln h_t) - (\ln\tilde{y}_{t-1} - \ln h_{t-1}) + \ln\mu_{z^*,t}.$$

[1] In Matlab, the command `dare` is a more efficient way of computing Σ then a straightforward implementation of the solution, i.e. $\Sigma = [I - (F \otimes F)]^{-1} Q$.

Now, the 'normal' interpretation of a hat over a variable is:

$$\widehat{\tilde{y}}_t = \frac{\tilde{y}_t - \tilde{y}}{\tilde{y}},$$

so that

$$\tilde{y}_t = \tilde{y}\left(\widehat{\tilde{y}}_t + 1\right),$$

and

$$\begin{aligned} \ln \tilde{y}_t &= \ln \tilde{y} + \ln\left(\widehat{\tilde{y}}_t + 1\right) \\ &\approx \ln \tilde{y} + \widehat{\tilde{y}}_t, \end{aligned}$$

for $\widehat{\tilde{y}}_t$ small enough. The latter gives us an alternative interpretation of a variable with a hat. We call this the log interpretation of a variable with a hat. Similarly,

$$\begin{aligned} \ln h_t &= \ln h + \hat{h}_t, \\ \ln \mu_{z^*,t} &= \ln \mu_{z^*} + \hat{\mu}_{z^*,t} \end{aligned}$$

Substituting,

$$\Delta \ln \frac{y_t}{h_t} = \left(\widehat{\tilde{y}}_t - \hat{h}_t\right) - \left(\widehat{\tilde{y}}_{t-1} - \hat{h}_{t-1}\right) + \ln \mu_{z^*} + \hat{\mu}_{z^*,t}.$$

Using $\hat{\mu}_{z^*t} = \frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon t} + \hat{\mu}_{zt}$, this reduces to:

$$\Delta \ln \frac{y_t}{h_t} = \left(\widehat{\tilde{y}}_t - \hat{h}_t\right) - \left(\widehat{\tilde{y}}_{t-1} - \hat{h}_{t-1}\right) + \ln \mu_{z^*} + \frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon t} + \hat{\mu}_{zt}.$$

but,

$$\begin{aligned} \widehat{\tilde{y}}_t &= \tau_y z_t, \\ \hat{h}_t &= \tau_h z_t, \\ \hat{\mu}_{zt} &= \tau_{\mu_z} s_t, \\ \hat{\mu}_{\Upsilon t} &= \tau_{\mu_\Upsilon} s_t \end{aligned}$$

where τ_y, τ_l are 16 dimensional row vectors with zeros everywhere except unity in one location. For τ_y the location is the 12^{th} location; for τ_l the location is the 9^{th}. Also, τ_{μ_z} and τ_{μ_Υ} are 10 dimensional row vectors with zeros everywhere, except unity in one location. For

τ_{μ_z} the location is 3 and for τ_{μ_Υ} the location is 6. Here are the z_t and θ_t vectors:

$$z_t = \begin{pmatrix} \hat{c}_t 1(p) \\ \widehat{\bar{w}}_t 2(p) \\ \hat{\lambda}_{z^*t} 3 \\ \hat{m}_t 4(p) \\ \hat{\pi}_t 5(p) \\ \hat{x}_t 6 \\ \hat{s}_t 7 \\ \hat{\imath}_t 8(p) \\ \hat{h}_t 9 \\ \widehat{\bar{k}}_{t+1} 10(p) \\ \hat{q}_t 11 \\ \widehat{\bar{y}}_t 12 \\ \hat{R}_t 13 \\ \widehat{\bar{\mu}}_t 14(p) \\ \widehat{\bar{\rho}}_t 15 \\ \hat{u}_t 16\ (p) \end{pmatrix}, \ \theta_t = \begin{pmatrix} \hat{x}_{M,t} \\ \varepsilon_{M,t} \\ \hat{\mu}_{z,t} \\ \varepsilon_{\mu^z,t} \\ \hat{x}_{z,t} \\ \hat{\mu}_{\Upsilon,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ \hat{x}_{\Upsilon,t} \\ \hat{x}_{M,t-1} \\ \varepsilon_{M,t-1} \end{pmatrix}$$

Then,

$$\Delta \ln \frac{y_t}{h_t} = (\tau_y - \tau_h) z_t - (\tau_y - \tau_h) z_{t-1} + \ln \mu_{z^*} + \left(\frac{\alpha}{1-\alpha} \tau_{\mu_\Upsilon} + \tau_z \right) \theta_t.$$

Now consider inflation:

$$\begin{aligned} \ln \frac{P_t}{P_{t-1}} &= \ln \pi_t = \ln \pi + \hat{\pi}_t \\ &= \ln \pi + \tau_\pi z_t, \end{aligned}$$

where τ_π is a 16 dimensional row vector with zeros everwhere except unity in the 5^{th} location. Note that this is the net inflation rate. This is converted to annualized terms by multiplying by 4. Another way to compute this is based on the normal approximation of a hat:

$$\hat{\pi}_t = \frac{\pi_t - \pi}{\pi}.$$

Consider:

$$\pi_t - \pi = \pi \hat{\pi}_t.$$

This is the deviation of the inflation rate (or, the net inflation rate) from its population mean. Suppose we want the net inflation rate, $\pi_t - 1$, expressed in annual terms:

$$4 (\pi_t - \pi) + 4 (\pi - 1) = 4 \pi \hat{\pi}_t + 4 (\pi - 1).$$

Now consider the excess of productivity over the real wage, all in logs:

$$\begin{aligned}
\ln \frac{y_t}{h_t} - \ln w_t &= \ln \frac{\tilde{y}_t z_t^*}{h_t} - \ln \tilde{w}_t z_t^* \\
&= \ln \tilde{y}_t - \ln h_t - \ln \tilde{w}_t \\
&= \ln \tilde{y} + \widehat{\tilde{y}}_t - \ln h - \hat{h}_t - \ln \tilde{w} - \widehat{\tilde{w}}_t \\
&= (\ln \tilde{y} - \ln h - \ln \tilde{w}) + (\tau_y - \tau_h - \tau_w) z_t,
\end{aligned}$$

where τ_w is a 16 dimensional row vector with zeros everywhere and unity in the 2^{nd} location.

Now consider the log of the consumption to output ratio:

$$\begin{aligned}
\ln \frac{C_t}{y_t} &= \ln \frac{c_t z_t^*}{\tilde{y}_t z_t^*} \\
&= \ln c_t - \ln \tilde{y}_t \\
&= \ln c - \ln \tilde{y} + (\tau_c - \tau_y) z_t,
\end{aligned}$$

where τ_c is a 16 dimensional row vector with zeros everywhere and unity in the first location.

The log of the investment to output ratio is:

$$\begin{aligned}
\ln \frac{\Upsilon_t^{-1} I_t}{y_t} &= \ln i_t - \ln \tilde{y}_t \\
&= \ln i - \ln \tilde{y} + (\tau_i - \tau_y) z_t,
\end{aligned}$$

where τ_i is a 16 dimensional row vector with zeros everywhere and unity in the 8^{th} location. Note here that investment must be valued in consumption units, just as output is, for this ratio to be stationary.

Now consider the interest rate, R_t. Using the log approximation:

$$\log R_t = \log R + \hat{R}_t = \log R + \tau_R z_t,$$

where τ_R is a 16-dimensional row vector with unity in the 13th location. Since R_t is the *gross* nominal rate of interest, $\log R_t$ is approximately the *net* rate, $R_t - 1$. Then,

$$R_t - 1 \approx \log R + \tau_R z_t,$$

and the annualized rate is:

$$4 (R_t - 1) \approx 4 \log R + 4 \tau_R z_t.$$

Now consider how one proceeds under the normal approximation. In this case, $\hat{R}_t = (R_t - R)/R$, so that

$$R_t = R \left(\hat{R}_t + 1 \right),$$

and the annualized net rate is:

$$4 (R_t - 1) = 4 [R (\tau_R z_t + 1) - 1]$$

Now consider the log of velocity:

$$\begin{aligned} & \ln y_t - \ln \frac{Q_t}{P_t} \\ = & \ln \tilde{y}_t - \ln q_t, \end{aligned}$$

where

$$q_t = \frac{Q_t}{z_t^* P_t}.$$

Then,

$$\ln y_t - \ln \frac{Q_t}{P_t} = \ln \tilde{y} - \ln q + (\tau_y - \tau_q) z_t,$$

where τ_q is a 16 dimensional row vector with zeros everywhere and unity in the 11^{th} location.

Finally,

$$\begin{aligned} \Delta \ln P_t^I &= \ln \frac{\Upsilon_{t-1}}{\Upsilon_t} \\ &= -\ln \mu_{\Upsilon,t} \\ &= -\ln \mu_\Upsilon - \hat{\mu}_{\Upsilon,t} \\ &= -\ln \mu_\Upsilon - \tau_{\mu_\Upsilon} \theta_t, \end{aligned}$$

where τ_{μ_Υ} is a 10-dimensional row vector with all zeros except unity in the 6^{th} location.

We now consider capacity utilization, u_t. We have

$$\hat{u}_t = \log \left(\frac{u_t}{u} \right) = \log u_t = \tau_u z_t,$$

where τ_u is a 16-dimensional row vector with zeros everywhere except a unity in the last location.

Pulling all this together, in the following representation:

$$X_t = \alpha + \tau z_t + \bar{\tau} z_{t-1} + \tau^\theta \theta_t, \tag{8.5}$$

we have:

$$X_t = \begin{pmatrix} \Delta \ln \frac{y_t}{h_t} \\ \ln \frac{P_t}{P_{t-1}} \\ \ln \frac{y_t}{h_t} - \ln w_t \\ \ln h_t \\ \ln \frac{C_t}{y_t} \\ \ln \frac{I_t}{y_t} \\ \ln R_t \\ \ln y_t - \ln \frac{Q_t}{P_t} \\ \Delta \ln P_t^I \end{pmatrix} = \begin{pmatrix} \ln \mu_{z^*} \\ \ln \pi \\ \ln \tilde{y} - \ln h - \ln \tilde{w} \\ \ln h \\ \ln c - \ln \tilde{y} \\ \ln i - \ln \tilde{y} \\ \ln R \\ \ln \tilde{y} - \ln q \\ -\ln \mu_\Upsilon \end{pmatrix} + \begin{pmatrix} \tau_y - \tau_h \\ \tau_\pi \\ \tau_y - \tau_h - \tau_w \\ \tau_h \\ \tau_c - \tau_y \\ \tau_i - \tau_y \\ \tau_R \\ \tau_y - \tau_q \\ 0 \end{pmatrix} z_t \quad (8.6)$$

$$+ \begin{pmatrix} -(\tau_y - \tau_h) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} z_{t-1} + \begin{pmatrix} \frac{\alpha}{1-\alpha}\tau_{\mu_\Upsilon} + \tau_z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\tau_{\mu_\Upsilon} \end{pmatrix} \theta_t$$

so,

$$\tau = \begin{pmatrix} \tau_y - \tau_h \\ \tau_\pi \\ \tau_y - \tau_h - \tau_w \\ \tau_h \\ \tau_c - \tau_y \\ \tau_i - \tau_y \\ \tau_R \\ \tau_y - \tau_q \\ 0 \end{pmatrix}, \ \bar{\tau} = \begin{pmatrix} -(\tau_y - \tau_h) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\tau^\theta = \begin{pmatrix} \frac{\alpha}{1-\alpha}\tau_{\mu_\Upsilon} + \tau_z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\tau_{\mu_\Upsilon} \end{pmatrix}$$

8.2. Estimation

Our system is completely characterized by (F, H, R, V). We could think of F and H as being functions of the parameters governing the exogenous shocks, which we would like to estimate.

Denote these by the vector, β. There is obviously a mapping from β (and the other model parameters, which we here hold fixed) to F, H. So, we can also think of the system as being characterized by (β, R, V).

In Hamilton's section 13.4, he displays the likelihood function for this system. Let

$$\begin{aligned} f_t &= \left(\frac{1}{2\pi}\right)^{\frac{-n}{2}} \left|HP_{t|t-1}H' + R\right|^{-1/2} \\ &\times \exp\left\{-\frac{1}{2}\left(y_t - H\xi_{t|t-1}\right)'\left(HP_{t|t-1}H' + R\right)^{-1}\left(y_t - H\xi_{t|t-1}\right)\right\}, \end{aligned}$$

for $t = 1, 2, ..., T$. Here, n is the dimension of ξ_t, and

$$\xi_{t|t-1} = E\left[\xi_t | y_{t-1}, ..., y_1\right],$$

$t = 1, 2, ...,$ with $\xi_{1|0} = E\left(\xi_t\right)$, the unconditional expectation of ξ_t. Also,

$$\begin{aligned} P_{t+1|t} &\equiv E\left[\left(\xi_{t+1} - \xi_{t+1|t}\right)\left(\xi_{t+1} - \xi_{t+1|t}\right)' | y_t, ..., y_1\right] \\ &= F\left[P_{t|t-1} - P_{t|t-1}H'\left(HP_{t|t-1}H' + R\right)^{-1} HP_{t|t-1}\right] F' + Q, \end{aligned}$$

for $t = 1, 2, ..., T$, with

$$P_{1|0} = E\left(\xi_t - E\xi_t\right)\left(\xi_t - E\xi_t\right)'.$$

Finally,

$$\xi_{t+1|t} = F\xi_{t|t-1} + FP_{t|t-1}H'\left(HP_{t|t-1}H' + R\right)^{-1}\left(y - H\xi_{t|t-1}\right).$$

Then, the log likelihood function is:

$$\sum_{t=1}^{T} \ln f_t.$$

Consider first the log of the exponential term here (suppose $E\left(\xi_t\right) = 0$) :

$$\begin{aligned} &\left(y_1\right)'\left(HP_{1|0}H' + R\right)^{-1}\left(y_1\right) \\ &+ \left(y_2 - H\xi_{2|1}\right)'\left(HP_{2|1}H' + R\right)^{-1}\left(y_2 - H\xi_{2|1}\right) \\ &+ \left(y_3 - H\xi_{3|2}\right)'\left(HP_{3|2}H' + R\right)^{-1}\left(y_3 - H\xi_{3|2}\right) \\ &+ ... + \\ &+ \left(y_T - H\xi_{T|T-1}\right)'\left(HP_{T|T-1}H' + R\right)^{-1}\left(y_T - H\xi_{T|T-1}\right) \end{aligned}$$

Consider the derivative of this expression with respect to the matrix, R. Note that R enters the first term only directly, in the expression being inverted. The matrix R enters in several places in the second term, via $\xi_{2|1}$ and via $P_{2|1}$.

In Hamilton's section 13.6, he shows how to use this system to compute things like

$$\hat{\xi}_{t|T} \equiv E\left[\xi_t | \Omega_T\right], \; t = 1, 2, ..., T,$$

where the observations correspond to periods t = 1,2,...,T, and the information set is the whole data set:

$$\Omega_T = \{y_T, ..., y_1\}.$$

Note that a subset of the elements in $\hat{\xi}_{t|T}$ correspond to the estimates of the shocks. In addition, the estimate of the 'true' value of the data is given by

$$\hat{X}_{t|T} = H'\hat{\xi}_{t|T}.$$

We now derive the Kalman filter algorithm for solving the problem:

$$\hat{\xi}_{t|t-1} \equiv E\left[\xi_t|\Omega_{t-1}\right],\ t = 1, 2, ..., T.$$

We begin with $\hat{\xi}_{1|0}$

9. Reduced Form Vector Autoregression

We are interested in the VAR representation for (possibly a subset) of the variables in the 9 by 1 vector, X_t, in (8.6). Let $J(L)$ be an n by 9 matrix, which selects the subset of variables that interest us. If the matrix, $J(L)$, is the identity matrix, then the vector of variables is just X_t itself. We seek the model's implied VAR representation for $J(L)X_t$. We do this by solving the Yule-Walker equations. We have to confront one problem, which is that the fundamental shocks in our model may be smaller in number than the number of variables, n. The first subsection below discusses how to proceed when the number of shocks is equal to n (i.e., $n = 3$). We then discuss what to do in the other case.

9.1. Full Rank System

From the previous section, we have (the objects in the following representation are computed in kalman_matrices.m, please verify that the elements of α, τ, $\bar{\tau}$, τ^θ in the code correspond to what is in (8.6)):

$$X_t = \alpha + \tau z_t + \bar{\tau} z_{t-1} + \tau^\theta \theta_t,$$

and

$$z_t = Az_{t-1} + B\theta_t,$$

and

$$\theta_t = \rho\theta_{t-1} + Q\eta_t,$$

where θ_t is as in (8.1), so that

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & c_z & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & c_\Upsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \eta_t = \begin{pmatrix} \varepsilon_{M,t} \\ \varepsilon_{\mu^z,t} \\ \varepsilon_{\mu_\Upsilon,t} \end{pmatrix},$$

and

$$E\eta_t\eta_t' = V_\eta = \begin{bmatrix} \sigma_M^2 & 0 & 0 \\ 0 & \sigma_{\mu^z}^2 & 0 \\ 0 & 0 & \sigma_{\mu_\Upsilon}^2 \end{bmatrix}.$$

The variables in X_t are as defined in (8.6). We now write out the moving average representation of X_t. First,

$$\begin{aligned} z_t &= (I - AL)^{-1}B\theta_t \\ &= (I - AL)^{-1}B(I - \rho L)^{-1}Q\eta_t. \end{aligned}$$

Then,

$$\begin{aligned} X_t &= \alpha + \tau z_t + \bar{\tau} z_{t-1} + \tau^\theta \theta_t \\ &= \alpha + (\tau + \bar{\tau}L)(I - AL)^{-1}B(I - \rho L)^{-1}Q\eta_t + \tau^\theta(I - \rho L)^{-1}Q\eta_t \\ &= \alpha + \left[(\tau + \bar{\tau}L)(I - AL)^{-1}B(I - \rho L)^{-1} + \tau^\theta(I - \rho L)^{-1}\right]Q\eta_t \\ &= \alpha + \left[(\tau + \bar{\tau}L)(I - AL)^{-1}B + \tau^\theta\right](I - \rho L)^{-1}Q\eta_t \\ &= \alpha + D(L)\eta_t, \end{aligned} \tag{9.1}$$

say, where

$$D(L) = \left[(\tau + \bar{\tau}L)(I - AL)^{-1}B + \tau^\theta\right](I - \rho L)^{-1}Q$$

Let $Y_t = J(L)X_t$. Then, the spectral density of Y_t is:

$$S_Y(\omega) = \tilde{D}(e^{-i\omega})V_\eta\tilde{D}(e^{i\omega})', \tag{9.2}$$

where

$$\tilde{D}(e^{-i\omega}) = J\left(e^{-i\omega}\right)D(e^{-i\omega}).$$

Let the covariance function of Y_t be defined as:

$$C(\tau) \equiv EY_tY_{t-\tau}',\ \tau = 0, \pm 1, \pm 2, \ldots.$$

The following ('inverse Fourier transform') relationship is easy to establish:

$$C(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_Y(\omega)e^{i\omega\tau}d\omega.$$

This can be approximated using a Riemann sum:

$$C(\tau) = lim_{N\to\infty}\frac{1}{N}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} S_Y(\omega_k)e^{i\omega_k\tau},$$

where $\omega_k = \frac{2\pi k}{N}$ for $k = -N/2, ..., N/2$ (see Sargent (1987, ch. 11, equation (20))). This sum can be further simplified by taking into account the following property:

$$S_{\tilde{y}}(\omega_k)e^{i\omega_1\tau} = conj\left[S_{\tilde{y}}(-\omega_k)e^{-i\omega_1\tau}\right],$$

where $conj$ denotes complex conjugation. As a result, $S_{\tilde{y}}(\omega_k)e^{i\omega_1\tau} + S_{\tilde{y}}(-\omega_k)e^{-i\omega_1\tau} = 2re\left[S_{\tilde{y}}(\omega_k)e^{i\omega_1\tau}\right]$, where $re\left[x\right]$ denotes the real part of the complex variable, x. Then,

$$\begin{aligned}
C(\tau) &= lim_{N\to\infty}\frac{1}{N}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} S_Y(\omega_k)e^{i\omega_k\tau}, \\
&= \frac{1}{N}S_Y(\omega_0) + \frac{1}{N}[S_Y(\omega_1)e^{i\omega_1\tau} + S_Y(\omega_2)e^{i\omega_2\tau} + ... + S_Y(\omega_{N/2})e^{i\omega_{N/2}\tau} \\
&\quad + S_Y(\omega_{-1})e^{i\omega_{-1}\tau} + S_Y(\omega_{-2})e^{i\omega_{-2}\tau} + ... + S_Y(\omega_{-N/2+1})e^{i\omega_{-N/2+1}\tau}] \\
&= \frac{1}{N}S_Y(\omega_0) + \frac{1}{N}[S_Y(\omega_1)e^{i\omega_1\tau} + S_Y(\omega_2)e^{i\omega_2\tau} + ... + S_Y(\omega_{N/2})e^{i\omega_{N/2}\tau} \\
&\quad + S_Y(-\omega_1)e^{-i\omega_1\tau} + S_Y(-\omega_2)e^{-i\omega_2\tau} + ... + S_Y(-\omega_{N/2-1})e^{-i\omega_{N/2-1}\tau}] \\
&= \frac{1}{N}S_Y(\omega_0) + \frac{2}{N}\sum_{k=1}^{\frac{N}{2}-1} re\left(S_Y(\omega_k)e^{i\omega_k\tau}\right) + \frac{1}{N}S_Y(\omega_{N/2})e^{i\omega_{N/2}\tau},
\end{aligned}$$

where $re(X)$ denotes the real part of X. In practice, a fairly small value of N will suffice for this sum to converge.

Write the VAR representation of Y_t (after removing the constant term) as follows:

$$Y_t = A_1Y_{t-1} + ... + A_pY_{t-p} + u_t,$$

where $A_1, ..., A_p$ remain to be determined. Note:

$$EY_tY'_{t-\tau} = A_1EY_{t-1}Y'_{t-\tau} + ... + A_pEY_{t-p}Y'_{t-\tau},$$

for $\tau = 1, 2, ...$. (These are the Yule-Walker equations.) Then, for $\tau = 1$:

$$C(1) = A_1C(0) + A_2C(-1) + A_3C(-2) + ... + A_pC(1-p).$$

Then, using the fact, $C(-\tau) = C(\tau)'$, we obtain:

$$C(1) = A_1 C(0) + A_2 C(1)' + A_3 C(2)' + ... + A_p C(p-1)',$$

since $EY_{t-2}Y_{t-1}' = \left(EY_{t-1}Y_{t-2}'\right)' = C(1)'$. For $\tau = 2:$

$$C(2) = A_1 C(1) + A_2 C(0) + A_3 C(1)' + ... + A_p C(p-2)'.$$

Finally, for $\tau = p:$

$$C(p) = A_1 C(p-1) + A_2 C(p-2) + A_3 C(p-3) + ... + A_p C(0).$$

It is convenient to write the Yule-Walker equations in matrix form. Let

$$d = \begin{pmatrix} C(1) & \cdots & C(p) \end{pmatrix},\ X = \begin{bmatrix} C(0) & & C(p-1) \\ & \ddots & \\ C(p-1)' & & C(0) \end{bmatrix},\ \beta = \begin{pmatrix} A_1 & \cdots & A_p \end{pmatrix}$$

We solve the Yule-Walker equations as follows:

$$\beta = dX^{-1}$$

The elements of β give us the VAR coefficient matrices for the time series representation of Y_t. The correct value of p is $p = \infty$. In practice, A_p is small for small p. I suspect that p about 3 or 4 is right. However, this has to be 'tested' by examining the magnitude of A_{p+1}, A_{p+2}, etc.

To complete the computation of the VAR, we require the variance covariance matrix of the disturbances, u_t, and the constant term. Call the variance-covariance matrix, $V = Eu_t u_t'$. Here is one way to compute V. Note:

$$C(0) = EY_t Y_t' = A_1 C(1)' + ... + A_p C(p)' + Eu_t Y_t'.$$

but,

$$\begin{aligned} & Eu_t Y_t' \\ = \; & Eu_t \left[A_1 Y_{t-1}' + ... + A_p Y_{t-p}' + u_t'\right] \\ = \; & Eu_t u_t' = W. \end{aligned}$$

Here, we have taken into account that $Eu_t Y_{t-\tau}' = 0$ for $\tau = 1, 2, ...$, if p is large enough and the eigenvalues of $[I - A_1 z - ... - A_p z^p]$ lie inside the unit circle. So, we find W as the solution to:

$$W = C(0) - \left[A_1 C(1)' + ... + A_p C(p)'\right].$$

The constant term in the VAR representation for Y_t is γ, where

$$\gamma = [I - A_1 - A_2 - ... - A_p] \, J(1) \, \alpha.$$

There is a question as to what the right choice of p is. In principle, $p = \infty$ with this setup, but presumably p in fact only has to be quite small in order to get a 'good' VAR representation. Still, it's not clear what a 'good' representation is. Here is one idea. The VAR representation itself implies a spectral density:

$$S(\omega; p) = \left[I - A_1 e^{-i\omega} - ... - A_p e^{-i\omega p}\right]^{-1} W \left[I - A_1' e^{i\omega} - ... - A_p' e^{i\omega p}\right]^{-1}.$$

Note that this spectrum can be integrated to compute the implied covariance function, $C(\tau; p)$, from

$$C(\tau; p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega; p) e^{i\omega\tau} d\omega.$$

If p is well-chosen, then $C(\tau; p)$ is similar in size to $C(\tau)$ for various τ. Similarly, if p is properly chosen, then $S(\omega; p)$ should be similar to $S_Y(\omega)$ for a range of $\omega \in (0, \pi)$. It would be useful to see a graph of the diagonal elements of $C(\tau; p)$ and $C(\tau)$ for $\tau = 0, 1, 2, ..., 10$. Similarly, it would be useful to see a graph of the diagonal elements (which are real) of $S(\omega_k; p)$ and $S_Y(\omega_k)$ for $\omega_k = \frac{2\pi k}{N}$ and $k = 0, ..., N/2$. Perhaps two sets of graphs could be constructed, one with $p = 4$ and the other with $p = 10$.

9.2. Singular System

The calculations above will lead to invertibility problems when $n > 3$, because there are not enough shocks in the model. However, in this case, the VAR analysis itself provides the rest of the shocks. In particular, the VAR analysis implies:

$$Y_t = Y_t^{Identified} + Y_t^{Other},$$

where the two components are orthogonal and $Y_t^{Identified}$ corresponds to $J(L)X_t$. The spectral density of this component is provided in (9.2). We will take two approaches to Y_t^{Other}. In the first, Y_t^{Other} will be an iid process, so that its spectral density is simply a constant. In the second, we will consider a more general time series representation.

9.2.1. Independent Noise

We suppose that Y_t^{Other} is iid over time and

$$EY_t^{Other} Y_t^{Other\prime} = F.$$

Here, F may be quite simple, including having zeros everywhere except a scalar on one of its diagonal elements. Obviously, The spectral density of Y_t^{Other}, $S(\omega)$, is just $S(\omega) = F$.

9.2.2. Dependent Noise

To obtain the time series representation of the other component, consider:

$$X_t^{Other} = B(L)X_{t-1}^{Other} + C\varepsilon_t,$$

where ε_t has a variance-covariance matrix equal to the identity matrix and X_t^{Other} is composed of the variables in the vector autoregression:

$$\underbrace{X_t^{Other}}_{10\times 1} = \left(\begin{array}{c} \Delta \ln(\text{relative price of investment}_t) \\ \Delta \ln(GDP_t/\text{Hours}_t) \\ \Delta \ln(GDP \text{ deflator}_t) \\ \text{Capacity Utilization}_t \\ \ln(\text{Hours}_t) \\ \ln(GDP_t/\text{Hours}_t) - \ln(W_t/P_t) \\ \ln(C_t/GDP_t) \\ \ln(I_t/GDP_t) \\ \text{Federal Funds Rate}_t \\ \ln(GDP \text{ deflator}_t) + \ln(GDP_t) - \ln(MZM_t) \end{array}\right).$$

To recover $B(L)$ and C, it is useful to recall the structural form of our VAR

$$A_0 X_t^{Other} = A(L)X_{t-1}^{Other} + \tilde{\varepsilon}_t,$$

where $\tilde{\varepsilon}_t$ has diagonal variance-covariance matrix, D. Then, the reduced form is:

$$X_t^{Other} = A_0^{-1}A(L)X_{t-1}^{Other} + A_0^{-1}\sqrt{D}\varepsilon_t,$$

where ε_t has variance-covariance matrix equal to the identity matrix, and $\sqrt{D}$ is the diagonal matrix formed by computing the square root of the diagonal elements of D.[2] Then,

$$X_t^{Other} = B(L)X_{t-1}^{Other} + C_2\varepsilon_{2t},$$

where[3]

$$B(L) = A_0^{-1}A(L),\ C = A_0^{-1}\sqrt{D}.$$

Now, the matrix, C, is 10 by 10. The object, C_2, is C with its first, second and ninth columns removed and ε_{2t} is ε_t with the first, second and ninth elements removed. The moving average representation of X_t^{Other} is:

$$X_t^{Other} = [I - B(L)]^{-1} C_2\varepsilon_{2t}.$$

[2] The matrix D can be found by applying the MATLAB file getV.m to the fitted VAR disturbances, erzout, produced by the call to mkimplrnew.m. To see exactly how this is done, see lines 32 and 34 in spectdecomp.m.

[3] Our benchmark estimate sets $B(L) = B_0 + B_1L + B_2L^2 + B_3L^3$. The B's may be obtained from the output of mkimplrnew.m. In particular, azeroout= $A_0^{-1}A(L)$, where azeroout is a 10 by 4*10 matrix. Here, B_0 is the first 10 by 10 block of this matrix, B_1 is the second one, and so on. Also, a0betazout corresponds to A_0.

Define $\tilde{J}$ to be the 9 by 10 matrix which makes the elements of X_t^{Other} conformable with the elements of X_t. In particular, if I is the 10 by 10 identity matrix and

$$\zeta = [2, 3, 6, 5, 7, 8, 9, 10, 1], \tag{9.3}$$

then

$$\tilde{J} = I(\zeta, :), \tag{9.4}$$

using MATLAB notation. Thus, $\tilde{J}$ is a 9 by 10 matrix, which is constructed from the into the elements of X_t that interest us. Then, the moving average representation of $J(L)\tilde{J}X_t^{Other}$ is:

$$Y_t^{Other} = J(L)\tilde{J}\left[I - B(L)\right]^{-1} C_2 \varepsilon_{2t}.$$

The spectral density of Y_t^{Other} is:

$$S(\omega) = J(e^{-i\omega})\tilde{J}\left[I - B(e^{-i\omega})e^{-i\omega}\right]^{-1} C_2 C_2' \left[I - B(e^{i\omega})'e^{i\omega}\right]^{-1} \tilde{J}' J(e^{i\omega})'.$$

9.2.3. Spectrum of the Data

The spectrum of $Y_t = Y_t^{Identified} + Y_t^{Other}$ is:

$$S_Y(\omega) = S_{\tilde{X}}(\omega) + S(\omega),$$

where $S_{\tilde{X}}(\omega)$ is given in (9.2). The VAR representation of Y_t is formed by solving the Yule-Walker equations based on the covariance function obtained by integrating (inverse Fourier-transforming) $S_Y(\omega)$.

9.3. Invertibility

We now ask whether the fundamental shocks exist in the space of Y_{t-j}, $j = 1, 2, \ldots$. If they do not, then we cannot hope to recover them using a VAR, regardless of the lag length, p. To determine invertibility, consider the nonsingular case first. From (9.1):

$$X_t = \alpha + D(L)\eta_t,$$

so that (ignoring the constant term):

$$Y_t = \tilde{D}(L)\eta_t,$$

where $\tilde{D}(L) = J(L)D(L)$. Solving this, we obtain that the shocks, η_t, can be represented as linear combination of current and past Y_t as follows:

$$\begin{aligned} \eta_t &= \left[\tilde{D}(L)\right]^{-1} Y_t \\ &= \bar{D}_0 Y_t + \bar{D}_1 Y_{t-1} + \bar{D}_2 Y_{t-2} + \bar{D}_3 Y_{t-3} + \ldots, \end{aligned}$$

where

$$\bar{D}(L) = \bar{D}_0 + \bar{D}_1 L + ... = \left[\tilde{D}(L)\right]^{-1}.$$

We can obtain $\bar{D}_j$, $j = 0, 1, 2, ...$ by:

$$\bar{D}_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\tilde{D}(e^{-i\omega})\right]^{-1} e^{i\omega j} d\omega.$$

This sum can be evaluated using the Riemann approximation discussed above, although we do not have any symmetry we can appeal to here. The question of invertibility corresponds to whether $\bar{D}_j \to 0$ as $j \to \infty$. We can determine this numerically.

If, in the calculation of the VAR representation of Y_t discussed above, p is large enough, then the VAR representation here and the one above should be virtually identical. The VAR representation computed here is:

$$Y_t = \left[-\bar{D}_0^{-1}\bar{D}_1\right] Y_{t-1} + \left[-\bar{D}_0^{-1}\bar{D}_2\right] Y_{t-2} + \left[-\bar{D}_0^{-1}\bar{D}_3\right] Y_{t-3} + ... + u_t,$$

where

$$\begin{aligned} u_t &= \bar{D}_0^{-1}\eta_t \\ E u_t u_t' &= \bar{D}_0^{-1} V_\eta \left[\bar{D}_0^{-1}\right]'. \end{aligned}$$

We now consider the singular case. The moving average representation of Y_t now is:

$$Y_t = \left[\tilde{D}(L)^{-1} \vdots J(L) \left[I - B(L)\right]^{-1} C_2\right] \begin{pmatrix} \eta_t \\ \varepsilon_{2t} \end{pmatrix}.$$

What follows can be done easily only if $J(L)$ is square, so that the matrix in square brackets is square. Inverting this:

$$\begin{pmatrix} \eta_t \\ \varepsilon_{2t} \end{pmatrix} = \left[\tilde{D}(L)^{-1} \vdots J(L) \left[I - B(L)\right]^{-1} C_2\right]^{-1} Y_t.$$

Let

$$\bar{D}_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\tilde{D}(e^{-i\omega})^{-1} \vdots J(e^{-i\omega}) \left[I - B(e^{-i\omega})\right]^{-1} C_2\right]^{-1} e^{i\omega j} d\omega.$$

Let $\bar{D}_j^1$ denote the upper 3×3 block of $\bar{D}_j$. The proposition that η_t lies in the space of current and past Y_t corresponds to

$$\bar{D}_j^1 \to 0, \; j \to \infty.$$

10. Forecasting Using the Kalman Filter and Non-Identified VAR Disturbances

Let the 10×1 vector of non-identified VAR disturbances be denoted w_t, where

$$w_t = B_1 w_{t-1} + ... + B_q w_{t-q} + C_2 \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2,t} \end{pmatrix}.\ E\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2,t} \end{pmatrix}\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2,t} \end{pmatrix}' = I,$$

using notation taken from the ACEL manuscript. (The matrices, $B_1, ..., B_4$, in the ACEL project can be recovered from a0betazout, which is produced by mkimplrnew.m, in the program, main.m. The first column of a0betazout is the constant term in the VAR, and the next 10 by 10 block is B_1, the following 10 by 10 block is B_2, etc.) Here, C_2 is a 10×7 matrix. It is the columns of the C matrix discussed in ACEL, which correspond to the non-identified shocks. (To find C_2, first compute $C = inv(azeroout) * sqrt(getV(erzout))$, then, C_2 is columns 3-8 and 10 of C.) We add w_t to the state equation in the Kalman filter. The other part of our stochastic process comes from the solution to the model, (5.2), and the law of motion for the exogenous shocks, (5.3):

$$\begin{aligned} z_t &= A z_{t-1} + B\theta_t \\ \theta_t &= \rho\theta_{t-1} + e_t, \end{aligned}$$

or,

$$z_t = A z_{t-1} + B\rho\theta_{t-1} + Be_t.$$

Let,

$$\xi_t = \begin{pmatrix} z_t \\ z_{t-1} \\ \theta_t \\ w_t \\ \vdots \\ w_{t-q+1} \end{pmatrix}$$

and

$$F = \begin{bmatrix} \underset{16\times16}{A} & \underset{16\times16}{0} & \underset{16\times10}{B} \times \underset{10\times10}{\rho} & 0 & \cdots & 0 & 0 \\ \underset{16\times16}{I} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \rho & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & B_1 & \cdots & B_{q-1} & B_q \\ 0 & 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

where ρ is defined in (5.4), so that the state equation can be written,

$$\xi_t = F\xi_{t-1} + v_t,\ v_t = \begin{pmatrix} Be_t \\ 0 \\ e_t \\ C_2 \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2,t} \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\begin{aligned} Q &\equiv Ev_t v_t' = \begin{pmatrix} Be_t \\ 0 \\ e_t \\ C_2 \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2,t} \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} e_t'B' & 0 & e_t' & \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2,t} \end{pmatrix}' C_2' & 0 & \cdots & 0 \end{pmatrix} \\ &= \begin{bmatrix} BVB' & 0 & BV & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ VB' & 0 & V & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & C_2C_2' & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \end{aligned}$$

where V is defined in (8.2). The observer equation is written:

$$y_t = H\xi_t$$

where

$$\underset{9\times 1}{H} = \left[\begin{array}{ccccccc} J\tau & J\bar{\tau} & J\tau^{\theta} & J\tilde{J} & 0 & \cdots & 0 \end{array} \right],$$

where $\tilde{J}$ is defined in (9.4). Also, J is a matrix that selects which variables we want to work with. If J is the 9-dimensional identity matrix, then we work with all variables in X_t (see (8.6)). These are also the variables in the ACEL var (see (11.1) below), except that capacity utilitzation is excluded. In case we want to work with a system that does not include the i^{th} variable in X_t, then make J the 9 dimensional identity matrix, with the i^{th} row deleted. If we don't want the i^{th} or j^{th} elements of X_t, then make J the 9 dimensional identity matrix with the i^{th} and j^{th} rows deleted, etc.

We now have all the necessary inputs for the Kalman filter, with two exceptions. We need the matrix called P by forecastkalman.m. It corresponds to Σ in (8.4). There are two ways we can get P. We can find P by iterating in the manner described right after (8.4), starting with $P = Q$. Alternatively, we can execute the following MATLAB command...[P] = dare(F',zeros(size(F)),Q). It would be good to verify that dare is doing what it should, by verifying that the output of dare satisfies the equation to be solved, namely (8.4).

Finally, the Kalman filter also requires the data. For this, load aceldat.mat, and the data are in the 171 by 10 matrix, vardata. To proceed type in MATLAB,

data=vardata(:,ζ)';

where ζ is the vector in (9.3). In addition, if there is an element of ζ that is not desired in the analysis (i.e., it is excluded by J above), then it should be deleted from ζ.

We will also be interested in forecasts using the VAR alone. The easiest way to do this is to simply replace C_2C_2' in the construction of Q, with CC'. In addition, H should be replaced with

$$H = \left[\begin{array}{cccccccc} 0 & 0 & 0 & J\tilde{J} & 0 & \cdots & 0 \end{array}\right].$$

That is, where $J\tau$, $J\bar{\tau}$, $J\tau^{\theta}$ were, there should be zeros instead. This is very inefficient computationally, but the computations go so quickly, that we shouldn't worry about this.

For checking purposes there are two issues. One is whether the data have been imported correctly. The other is whether the various model/VAR parameters have been imported correctly and whether the state space/observer system has been put together properly. We can check the latter by computing impulse response functions and comparing them to ACEL.

Our system is:

$$\begin{aligned} \xi_t &= F\xi_{t-1} + v_t \\ y_t &= H\xi_t, \end{aligned}$$

where

$$v_t = \begin{pmatrix} Be_t \\ 0 \\ e_t \\ C_2 \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2,t} \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; e_t = \begin{pmatrix} \varepsilon_{M,t} \\ \varepsilon_{M,t} \\ \varepsilon_{\mu^z,t} \\ \varepsilon_{\mu^z,t} \\ c_z \varepsilon_{\mu^z,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ \varepsilon_{\mu_\Upsilon,t} \\ c_\Upsilon \varepsilon_{\mu_\Upsilon,t} \\ 0 \\ 0 \end{pmatrix}.$$

We can write e_t as

$$e_t = D \begin{pmatrix} \varepsilon_{M,t} \\ \varepsilon_{\mu^z,t} \\ \varepsilon_{\mu_\Upsilon,t} \end{pmatrix},$$

where D is 10 by 3:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & c_z & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & c_\Upsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We should look at the dynamic response of each element in y_t to a one standard deviation shock in each of $(\varepsilon_{M,t}, \varepsilon_{\mu^z,t}, \varepsilon_{\mu_\Upsilon,t})$. In particular, let the shock occur in period $t = 1$, so that $v_1 \neq 0$. Set $v_t = 0$ for all $t > 0$. Then, compute $\xi_1 = v_1$ and $\xi_t = F\xi_{t-1}$ for $t > 1$. Finally, $y_t = H\xi_t$ for $t \geq 1$. To get impulse responses that are comparable to ACEL, the elements in y_t will have to be 'unwound' appropriately. For example, $ACEL$ reports the response of output, while output is not directly one of the elements of y_t.

11. Variance Decompositions

In this section we analyze the residuals from the VAR and we in particular study the percent of the variance in output due to embodied, neutral and policy shocks. The first subsection discusses technicalities. The second, the results.

11.1. Technicalities

The data in the VAR are, in logs:

$$Y_t = \begin{pmatrix} (1-L)p_t^I \\ (1-L)(y_t - h_t) \\ (1-L)p_t \\ u_t \\ h_t \\ y_t - h_t - w_t \\ c_t - y_t \\ p_t^I + I_t - y_t \\ R_t \\ y_t + p_t - m_t \end{pmatrix} \tag{11.1}$$

Consider

$$\tilde{Y}_t = \begin{pmatrix} y_t \\ 4(1-L)m_t \\ 4(1-L)p_t \\ R_t \\ u_t \\ h_t \\ w_t \\ c_t \\ I_t \\ p_t^I \end{pmatrix}$$

so that $Y_t = F(L)\tilde{Y}_t$, where $F(L)$ is defined as follows:

$$\begin{pmatrix} (1-L)p_t^I \\ (1-L)(y_t - h_t) \\ (1-L)p_t \\ u_t \\ h_t \\ y_t - h_t - w_t \\ c_t - y_t \\ p_t^I + I_t - y_t \\ R_t \\ y_t + p_t - m_t \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-L \\ 1-L & 0 & 0 & 0 & 0 & -(1-L) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{1}{4(1-L)} & \frac{1}{4(1-L)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} y_t \\ 4(1-L)m_t \\ 4(1-L)p_t \\ R_t \\ u_t \\ h_t \\ w_t \\ c_t \\ I_t \\ p_t^I \end{pmatrix}$$

Also, note $\tilde{Y}_t = F(L)^{-1}Y_t$.

Now, we have that

$$\begin{aligned} Y_t &= A(L)Y_{t-1} + C\varepsilon_t, \\ Y_t &= [I - A(L)]^{-1} C\varepsilon_t, \\ \tilde{Y}_t &= F(L)^{-1} [I - A(L)]^{-1} C\varepsilon_t \end{aligned}$$

where ε_t is a 10×1 vector of shocks with variance-covariance matrix equal to the identity matrix. Now, we actually are interested in properties of velocity, $y_t + p_t - m_t$, in addition to the other variables in $\tilde{Y}_t$. Thus, let $\bar{Y}_t$ be:

$$\bar{Y}_t \equiv \begin{pmatrix} y_t \\ 4(1-L)m_t \\ 4(1-L)p_t \\ R_t \\ u_t \\ h_t \\ w_t \\ c_t \\ I_t \\ y_t + p_t - m_t \\ p_t^I \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{4(1-L)} & -\frac{1}{4(1-L)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} y_t \\ 4(1-L)m_t \\ 4(1-L)p_t \\ R_t \\ u_t \\ h_t \\ w_t \\ c_t \\ I_t \\ p_t^I \end{pmatrix}$$

$$= G(L)\tilde{Y}_t,$$

say. The spectral density, $S_{\bar{Y}}(e^{-i\omega})$, of $\bar{Y}_t$ is:

$$S_{\bar{Y}}(e^{-i\omega}) = G(e^{-i\omega})F(e^{-i\omega})^{-1}\left[I - A(e^{-i\omega})\right]^{-1} CC'\left[I - A(e^{i\omega})'\right]^{-1}\left[F(e^{i\omega})^{-1}\right]' G(e^{i\omega})'.$$

The identified shocks are the first, second and ninth. Let the 10 by 10 matrix of zeros with only a unity in the j^{th} diagonal element be denoted I_j. The spectral density of $\bar{Y}_t$ assuming only the j^{th} shock is activated is denoted:

$$S_{\bar{Y}}^j(e^{-i\omega}) = G(e^{-i\omega})F(e^{-i\omega})^{-1}\left[I - A(e^{-i\omega})\right]^{-1} CI_jC'\left[I - A(e^{i\omega})'\right]^{-1}\left[F(e^{i\omega})^{-1}\right]' G(e^{i\omega})'.$$

It is easy to verify that

$$\sum_{j=1}^{10} S_{\bar{Y}}^j(e^{-i\omega}) = S_{\bar{Y}}(e^{-i\omega}).$$

This corresponds to the additive decomposition of variance of $\tilde{Y}_t$. Let $diag(X)$ be the diagonal elements of the matrix, X. We can define the fraction of the variance due to shock j at frequency ω by:

$$var(j) = \frac{diag\left(S_{\bar{Y}}^j(e^{-i\omega})\right)}{diag\left(S_{\bar{Y}}(e^{-i\omega})\right)},$$

where the division means element by element division of the two vectors. Thus, the first element of the 10 by 1 vector $var(j)$ is the fraction of variance in the growth rate of p_t^I accounted for by the jth shock.

We can obtain the fraction of variance over a range of frequencies, by using the following formula for a variance:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{-i\omega})d\omega = lim_{N\to\infty}\frac{1}{N}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} f(e^{-i\omega_k}),$$

where f is the spectral density of a scalar random variable, and $\omega_k = \frac{2\pi k}{N}$ for $k = -N/2, ..., N/2$ (see Sargent (1987, ch. 11, equation (20))).

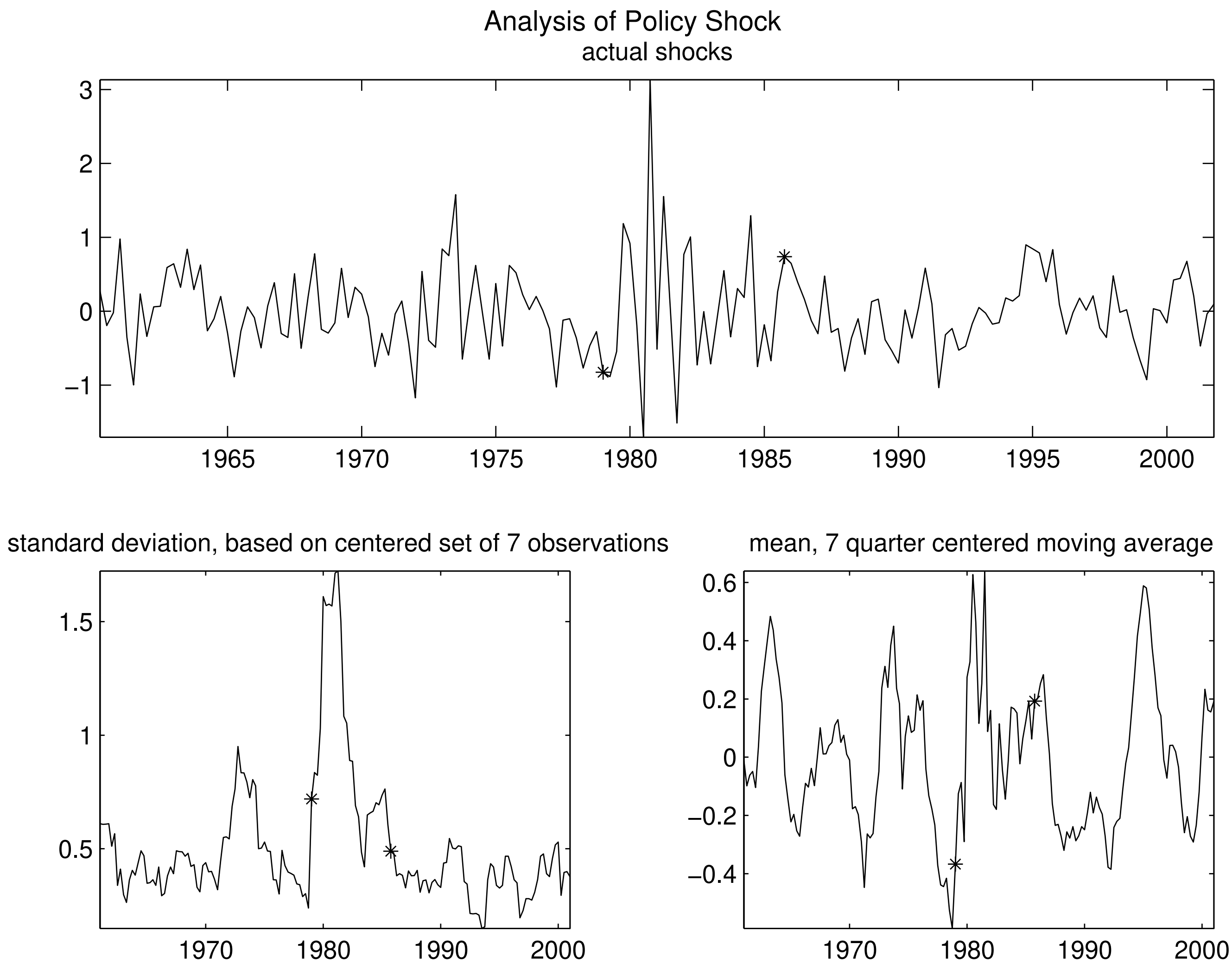
Suppose the range of frequencies that interests us goes from period of fluctuation a to period of fluctuation b. The frequency corresponding to a given period of fluctuation is 2π/period. So, this range of periods (say, a is 8 periods and b is 32 periods, as in the business cycle with quarterly data) corresponds to $k_a = N/a$ and $k_b = N/b$ (these can be rounded to the nearest integer). Note, too, that a spectrum is symmetric about zero. Then, the fraction of variance in the range, a to b, is

$$\frac{\sum_{k=k_b}^{k_a} diag(S_{\tilde{Y}}^{j}(e^{-i\omega_k}))}{\sum_{k=k_b}^{k_a} diag(S_{\tilde{Y}}(e^{-i\omega_k}))}.$$

Here again, the ratio of two column vectors means element by element division. Note that the correct formula should scale the numerator and denominator by $2/N$, which cancel in the ratio.

11.2. Results

The following figure displays results for the estimated policy shocks, after multiplication by 100. The top panel displays the estimated policy shocks themselves. The lower left panel shows the standard deviation of the shocks, computed using a centered set of 7 observations. The bottom right panel displays the centered moving average of the shocks. Note that the standard deviation rises very sharply during the period bracketted by the two stars. These correspond to 1979Q1 and 1985Q4, respectively. The standard deviation of the shocks rises to over 150 basis points in the high variance period. The mean is actually 102 basis points in this period. The standard deviation of the shocks in the early period is on average 52 basis points, and over the later period it is on average 44 basis points. The bottom right panel shows that this high variance is concentrated in the high frequencies. Although it is quite evident from the quarterly shocks observed in the first panel, it is less evidence in the smoothed shocks.



We computed the variance decompositions of the shocks, in two different ways. One was the spectral approach described in the previous subsection. This produced the following results. For the HP filtered data, the fraction of variance due to the disembodied, neutral and all three shocks is:

$$0.16,\ 0.13, 0.14, 0.43$$

Thus, the three shocks account for 43 percent of the HP filtered output data. Of this, 16 percent is due to the disembodied shock, 13 percent to the neutral shock and 14 percent to the monetary policy shock. The results for the bandpass filtered data, allowing components with period 8 quarters to 32 quarters to pass, we obtained the following results:

$$0.15,\ 0.13, 0.15, 0.42.$$

The results are very similar to what was found for the HP filter. The similarity of findings based on the HP and band-pass filters has been noticed before.

We also computed these variance decompositions using a time domain procedure. In one, we generated 1,000 replications of 1,000 artificial data sets each, by bootstrapping the fitted disturbances. For HP filtered data, we obtained the following results:

$$0.16\ (0.029),\ 0.13\ (0.025),\ 0.14\ (0.030),\ 0.43\ (0.069).$$

Numbers in parentheses are standard deviations across replications. The Monte Carlo standard error corresponds to these numbers, divided by $\sqrt{1000} = 32$. Putting the Monte Carlos standard errors in parentheses instead,

$$0.16\ (0.00092),\ 0.13\ (0.00079),\ 0.14\ (0.00095),\ 0.43\ (0.0022).$$

Clearly, these numbers coincide with the ones obtained using the spectral method. The variance decompositions for band pass filtered data are:

$$0.17\ (0.0012), 0.14\ (0.0011), 0.14\ (0.0012), 0.44\ (0.0028).$$

There are differences here with what was reported based on the spectral procedure, and these are greater than what can be accounted for with Monte Carlo standard error. When the number of observations was increased to 4,000 (only one replication), the following results were obtained for the band pass filter:

$$0.18,\ 0.14,\ 0.15,\ 0.51$$

These calculations were then repeated, except that the disturbances were drawn from the Normal distribution:

$$0.17,\ 0.15,\ 0.13,\ 0.41.$$

These results resemble more closely the ones obtained using the bootstrap with 1,000 observations. There is some (slightly) troubling sensitivity evident in the band pass filter calculations.

Turning to the variance decompositions obtained by simulating the model's response to the fitted residuals, we have, for the HP filter:

$$0.210(25.9),\ 0.105(69.0),\ 0.312(3.4),\ 0.644(13),$$

where numbers in parentheses are the percent of times that the simulated statistic (167 observations, 1,000 replications) exceeds the corresponding empirical value. (The simulations were done by bootstrap for this.) Note that all the statistics have reasonable $p-$values, except the one for policy, where the $p-$value is 3.4 percent.

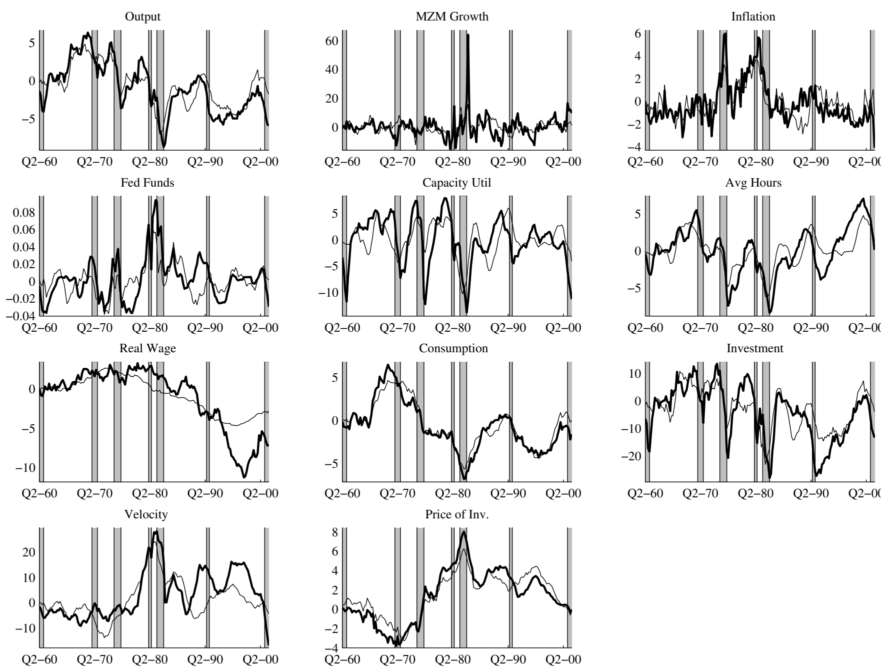
Turning to the band pass filter, we have

$$0.265(20.2),\ 0.099(70.5),\ 0.420(2.6),\ 0.747(11.9).$$

Now the $p-$value for the policy shock is even lower. When the simulations underlying the p-value were done with random numbers generated by the Normal distribution, the $p-$values for the HP filter, policy shock, was 4.6 percent and for the band pass filter it was 3.4 percent. Not much different. The p-values rose somewhat, to 5.4 and 3.8 percent, respectively, when shocks for the early, middle and late period, in terms of variance, were drawn separately.

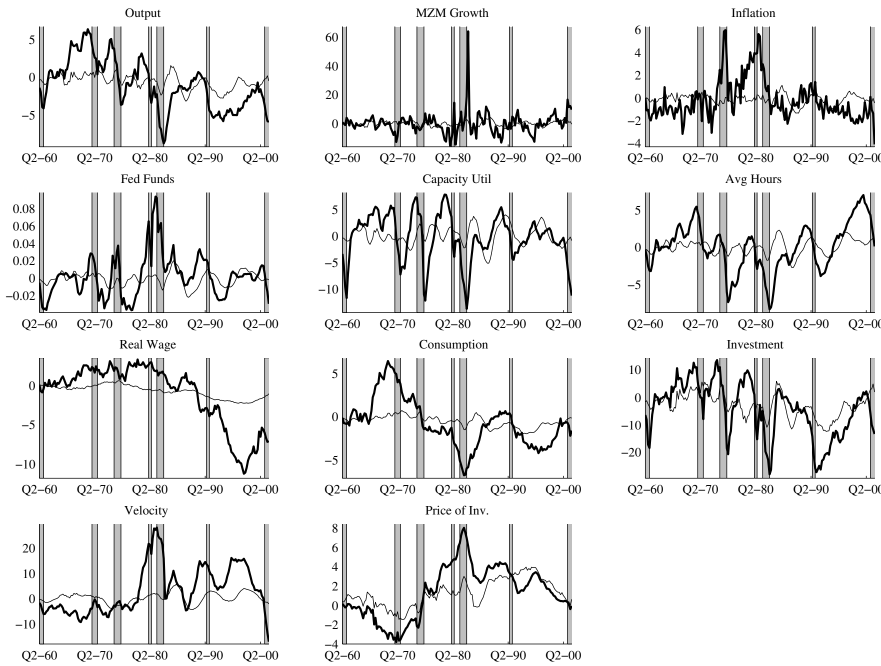
One way to visualize the empirical results is to see what the data would have been like with only the three identified shocks, compared with what it was with all the actual shocks. We can see this in the following figure:

Figure 9: Historical decomposition – monetary policy and technology shocks



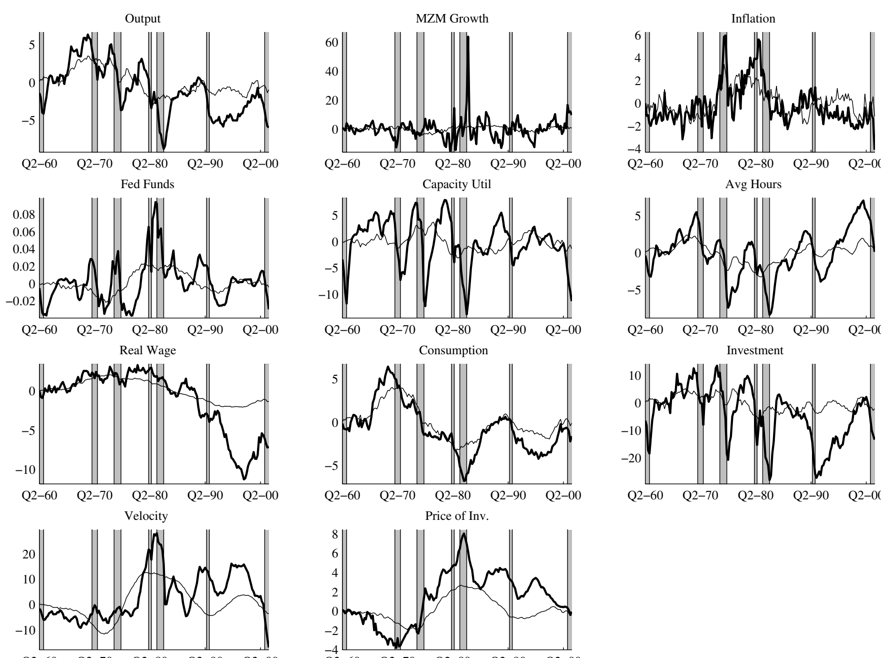
Note how highly correlated the two components are. Now let's have a look at the results for the individual shocks. The results for the embodied technology shock are:

Figure 8: Historical decomposition – embodied technology shocks only



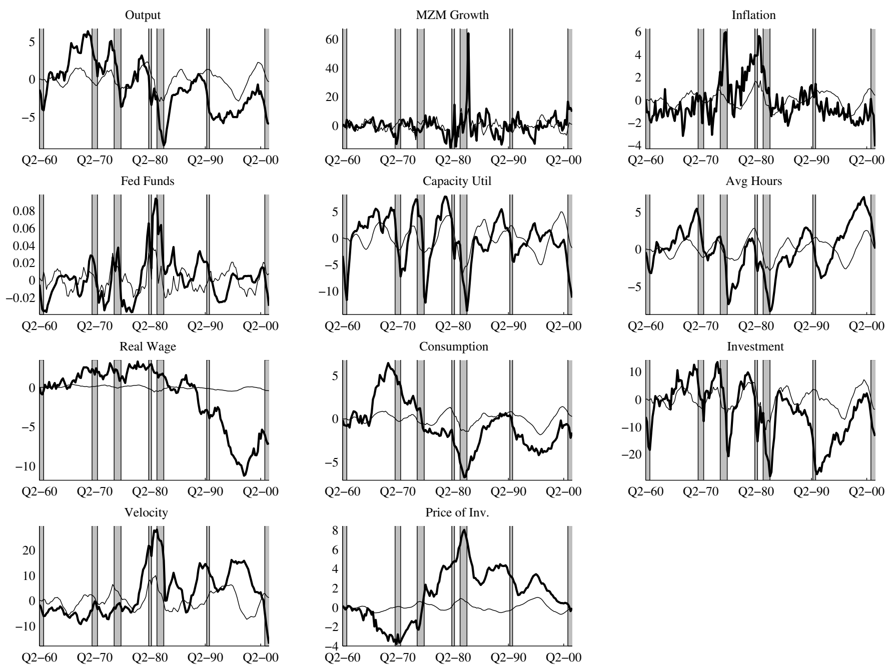
Now consider the neutral technology shocks:

Figure 7: Historical decomposition – neutral technology shocks only

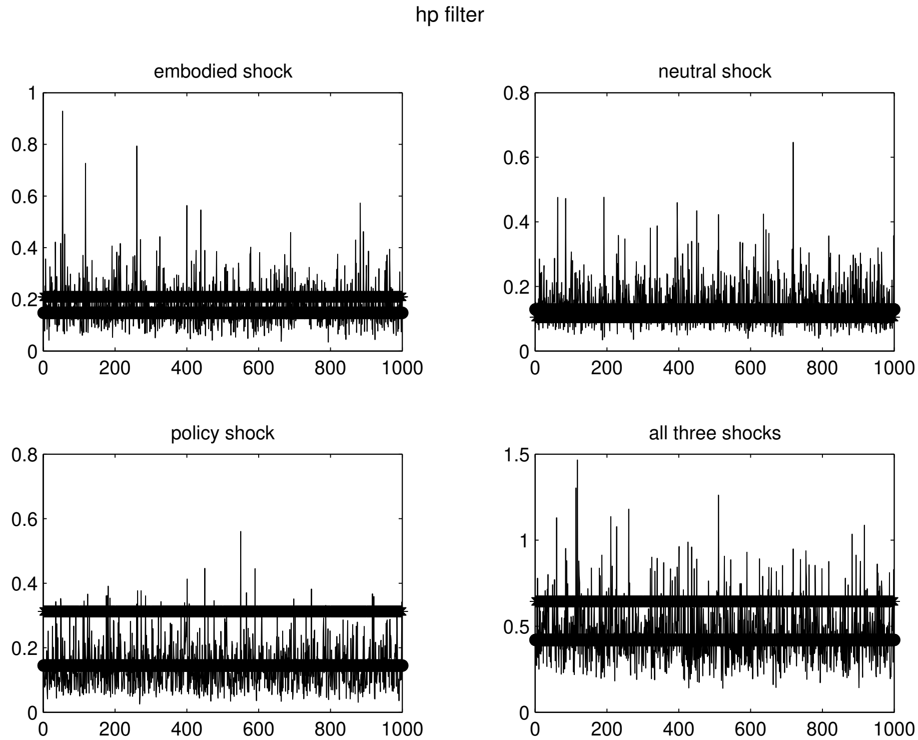


Finally, here are the monetary policy shocks:

Figure 6: Historical decomposition – monetary policy shocks only

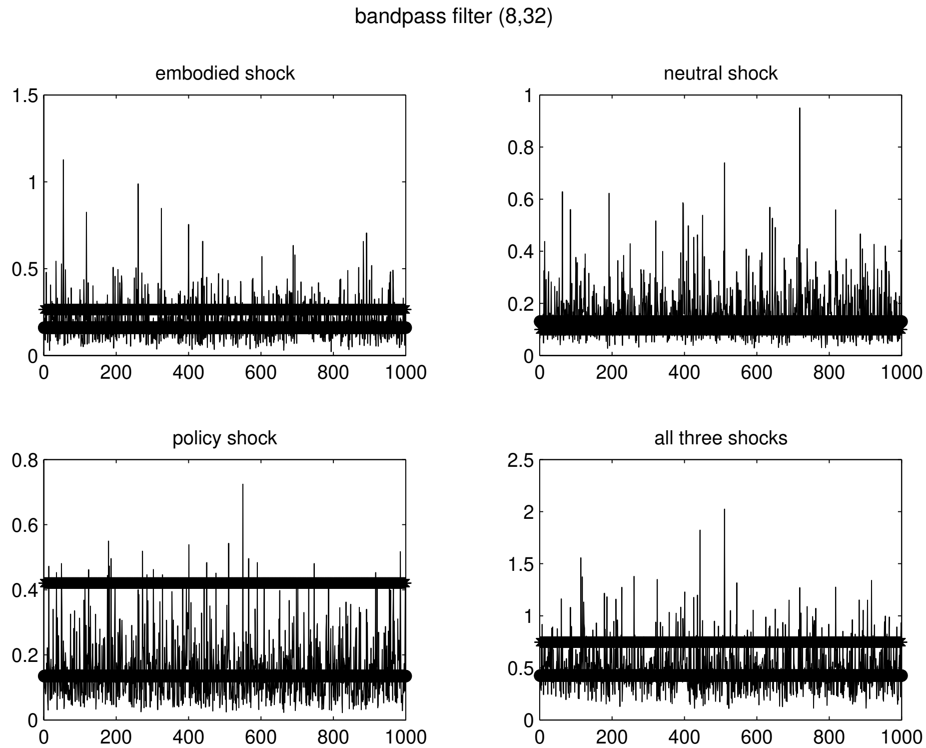


One way to think about the small $p-$values just described is as follows. The 'empirical' variance decompositions were computed by simulating the model's response to the actual fitted disturbances, in the sequence in which they were estimated to occur. This is what gives rise to the high estimated of the fraction of variance due to all shocks and to the policy shock in particular. The lower numbers were obtained by randomly reshuffling these disturbances. The difference in results can be seen in the following two figures. The next figure displays results for the HP filter:



Each figure has two horizontal lines, though in the upper right figure the two lines are hard to distinguish. The lower line is the population value of the variance decomposition, computed using the spectral method. The upper line is the value of the variance decomposition computed for the data. Note how that line is very high for the policy shock.

The results for the band pass filter can be seen in the following figure:



Again, note how uncharacteristically high the contribution of the policy shock is to the variance in output.

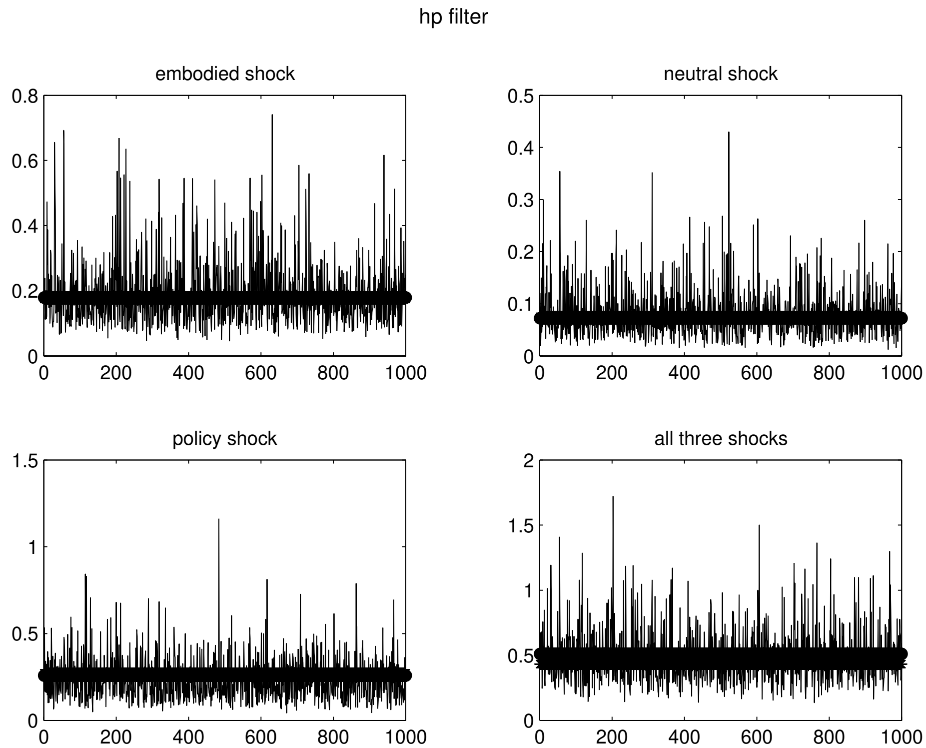
Evidently, one gets one variance decomposition results for the actual sequence of shocks estimated with the fitted VAR and a different one when the shocks are shuffled. This suggests that there may be serial correlation in the shocks. This motivated going to a 6 lag VAR. We now report results based on this. The results are quite different. In particular, the estimate of the variance decomposition based on the fitted residuals is, for the HP filter:

$$0.175(45.5), 0.075(45.1), 0.272(33.2), 0.432(52.9),$$

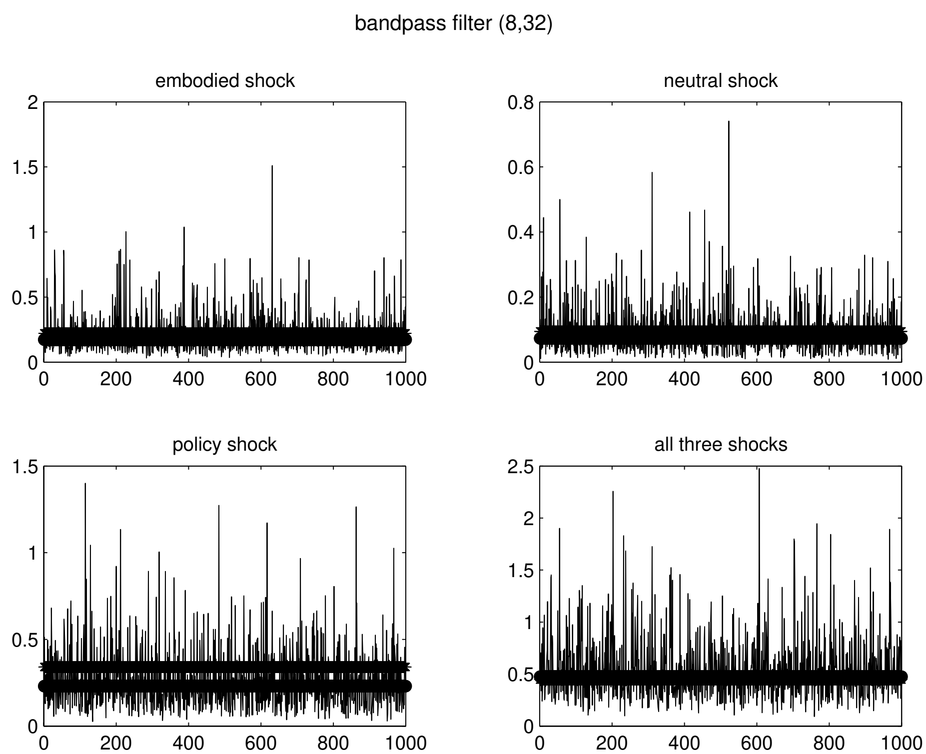
where (as before) numbers in parentheses are the frequency that bootstrapped variance decompositions are bigger than the empirical one. Note how high the empirical p value now is. For the Band Pass filter, the results are:

$$0.221(36.8), 0.094(36.4), 0.341(26.5), 0.447(53.7).$$

Again, $p-$values are quite high. It is interesting to see these results in pictures. For the HP filter, we have:



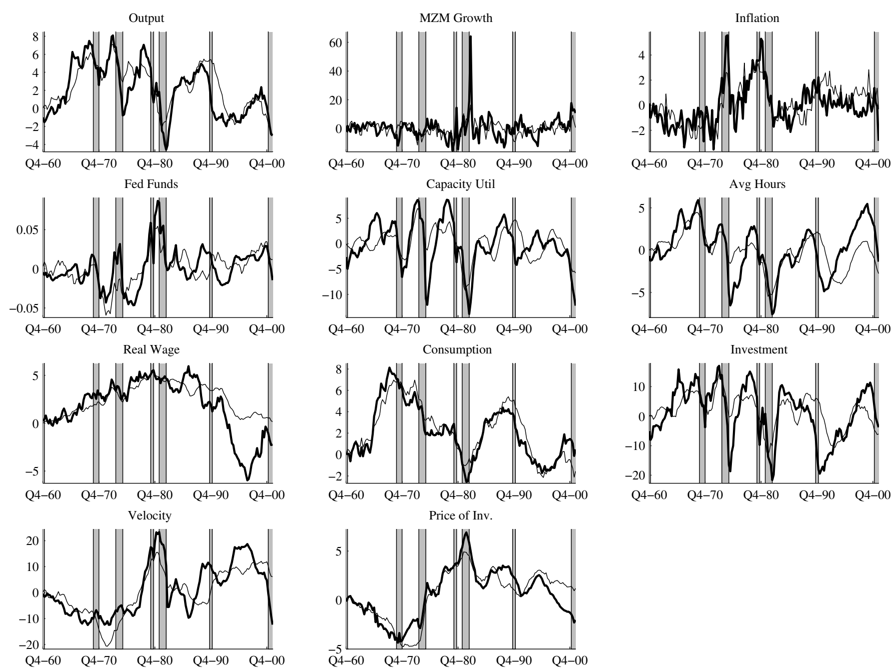
Now, the asymptotic variance decompositions are essentially indistinguishable, and both are in the mean of the simulated variance decompositions. For the Band Pass filter, we have:



Here, the empirical variance decomposition for the policy shock is slightly higher than the corresponding asymptotic estimate, but the difference really isn't very noticeable.

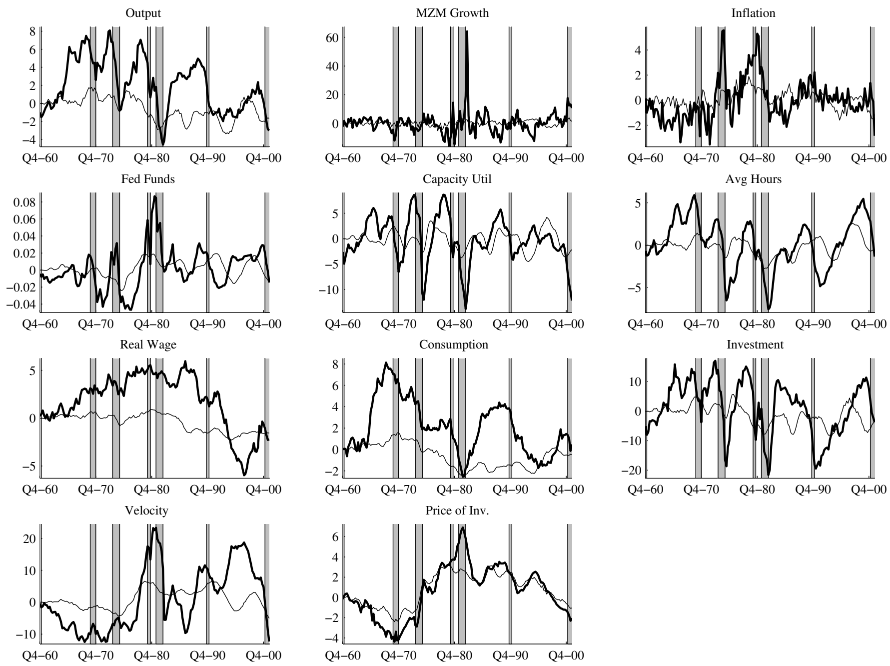
So, the variance of output due to our shocks is now much lower. It is interesting to ask what this does for the picture of the historical decomposition of shocks. Here is the picture for the three shocks together:

Figure 9: Historical decomposition – monetary policy and technology shocks



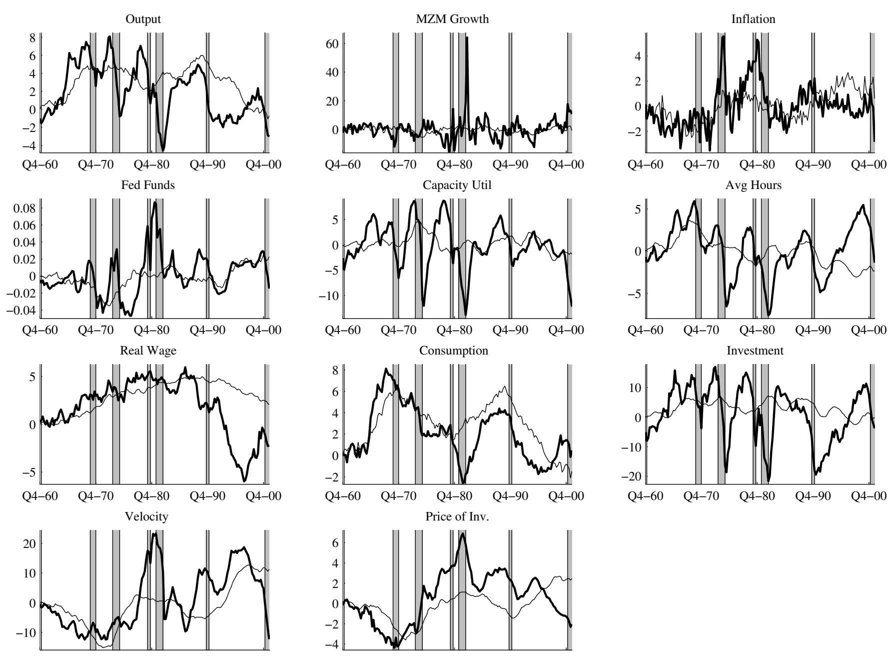
Here are the results for the embodied technology shock:

Figure 8: Historical decomposition – embodied technology shocks only



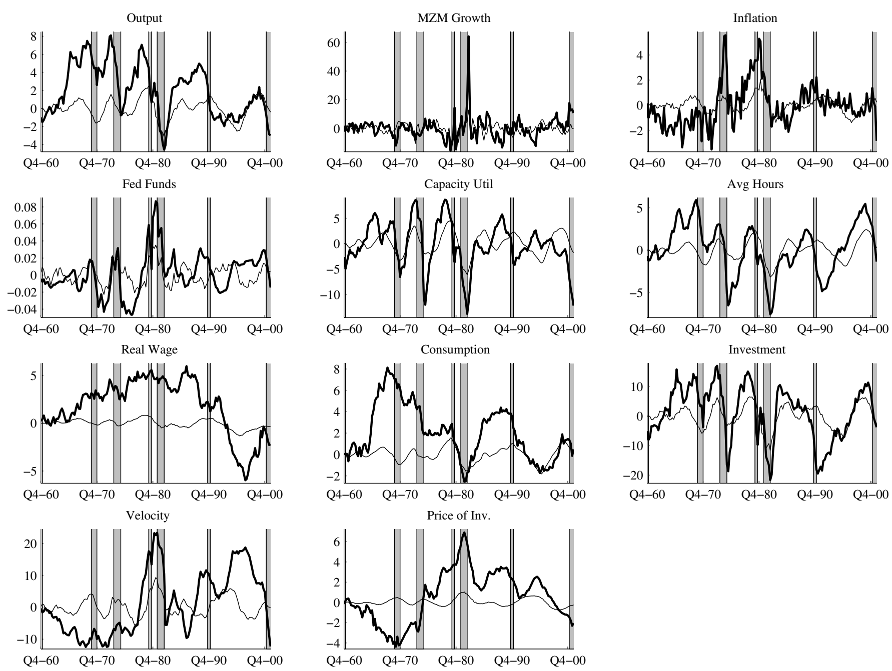
For the neutral shock:

Figure 7: Historical decomposition – neutral technology shocks only



Finally, for the monetary policy shock:

Figure 6: Historical decomposition – monetary policy shocks only



11.3. Conclusion

Our empirical estimates suggest that the three shocks account for a large fraction of the business cycle variation in output. The policy shock is particularly important. However, when we simulate the VAR in small or large samples, we find that the variance of output due to the policy shock is relatively small, and our three shocks account for less than half the variance of output. Why this sharp difference between the empirical estimate and the properties of the VAR? Perhaps the residuals represent an 'unusual' realization, or maybe the model has not been characterized properly. For example, one hypothesis is that there is heteroscedasticity in the results. This is motivated by the above figure. However, when this was modeled, it was found that this hypothesis does not explain the difference between the properties of the estimated VAR and of the fitted residuals.

12. Mapping from z_t, s_t to VAR Variables

The data that go into the VAR are a transformation on the variables in z_t and s_t. There are two transformations possible, and which is used seems to make a difference. Here, we describe in detail what these two transformations are.

12.1. Jesper Transformation

This is the transformation used in Jesper's code. The first step is to take z_t, s_t into unscale.m and produce a transformed series (see GenSimData.m), and in the second step the result is transformed into the data actually used in the VAR. We first discuss unscale.

The first thing that unscale.m does is to recover $\hat{\mu}_{\Upsilon,t}$ and $\hat{\mu}_{z,t}$ from the 6th and 3rd elements of s_t, respectively. Then, $\hat{\mu}_{z^*,t}$ is constructed using the relation discussed previously,

$$\hat{\mu}_{z^*,t} = \hat{\mu}_{z,t} + \frac{\alpha}{1-\alpha}\hat{\mu}_{\Upsilon,t}.$$

The next thing is to recover the level of these variables. For this it is useful to note that there are two interpretations of a variable with a hat. The 'normal' interpretation is that it is a deviation from the steady state, expressed as a fraction of the steady state:

$$\hat{\mu}_{z,t} = \frac{\mu_{z,t} - \mu_z}{\mu_z}.$$

Note that this also be written as

$$\hat{\mu}_{z,t} + 1 = \frac{\mu_{z,t}}{\mu_z}.$$

At the same time, recall that $\log(1+x) \approx x$ for x small, so that since $\hat{\mu}_{z,t}$ is small, it is approximately true that

$$\hat{\mu}_{z,t} = \log\left(\frac{\mu_{z,t}}{\mu_z}\right) = \log\mu_{z,t} - \log\mu_z.$$

We refer to this as the 'log interpretation of $\hat{\mu}_{z,t}$'.From this last approximation, note that (since $\mu_{z,t} = z_t/z_{t-1}$), the cumulative sum of the $\hat{\mu}_{z,t}$'s is:

$$\begin{aligned}
& \hat{\mu}_{z,1} + \hat{\mu}_{z,2} + ... + \hat{\mu}_{z,t} \\
= & \log\left(\frac{\mu_{z,1}}{\mu_z}\right) + \log\left(\frac{\mu_{z,2}}{\mu_z}\right) + ... + \log\left(\frac{\mu_{z,t}}{\mu_z}\right) \\
= & \log\left(\frac{\mu_{z,1}\mu_{z,2}\cdots\mu_{z,t}}{\mu_z^t}\right) \\
= & \log\left(\frac{\frac{z_1}{z_0}\frac{z_2}{z_1}\cdots\frac{z_t}{z_{t-1}}}{\mu_z^t}\right) \\
= & \log\left(\frac{\frac{z_t}{z_0}}{\mu_z^t}\right) \\
= & \log z_t - \log z_0 - t\log(\mu_z).
\end{aligned}$$

This suggests computing $\log z_t$ using

$$\log z_t = \log z_0 + t\log(\mu_z) + \hat{\mu}_{z,1} + \hat{\mu}_{z,2} + ... + \hat{\mu}_{z,t}.$$

There is another way to approximate $\log(z_t)$ based on the 'normal' interpretation of $\hat{\mu}_{z,t}$:

$$\mu_{z,t} = \frac{z_t}{z_{t-1}} = \mu_z \left(\hat{\mu}_{z,t} + 1\right).$$

Here, one computes

$$\mu_{z,1}\mu_{z,2}\cdots\mu_{z,t} = \frac{z_t}{z_0},$$

so that

$$\begin{aligned}
\log z_t &= \log\left(\mu_{z,1}\right) + ... + \log\left(\mu_{z,t}\right) + \log z_0 \\
&= t\log(\mu_z) + \log\left(\hat{\mu}_{z,1}+1\right) + \log\left(\hat{\mu}_{z,2}+1\right) + ... + \log\left(\hat{\mu}_{z,t}+1\right).
\end{aligned}$$

Note that we could apply a second order Taylor series expansion, to obtain:

$$\begin{aligned}
\log z_t &= t\log(\mu_z) + \hat{\mu}_{z,1} - \frac{1}{2}\left(\hat{\mu}_{z,1}\right)^2 \\
&\quad + \hat{\mu}_{z,2} - \frac{1}{2}\left(\hat{\mu}_{z,2}\right)^2 + ... + \hat{\mu}_{z,t} - \frac{1}{2}\left(\hat{\mu}_{z,1t}\right)^2
\end{aligned}$$

These different ways of computing $\log(z_t)$ will give the same answer if $\hat{\mu}_{z,t}$ is close zero. The time series representation of $\hat{\mu}_{zt}$ is given by:

$$\hat{\mu}_{zt} = \rho_{\mu z}\hat{\mu}_{zt-1} + \varepsilon_{\mu^z,t},$$

where $\sigma_{\mu_z} = 0.06$, and σ_{μ_z} is the standard deviation of $\varepsilon_{\mu^z,t}$. Let's adopt the log interpertation of the hat, so that:

$$\log\mu_{zt} = (1-\rho)\log(\mu_z) + \rho_{\mu z}\log\mu_{zt-1} + \varepsilon_{\mu^z,t},$$

or,

$$\log z_t - \log z_{t-1} = (1-\rho)\log(\mu_z) + \rho_{\mu z}\left(\log z_{t-1} - \log z_{t-2}\right) + \varepsilon_{\mu^z,t}.$$

Thus, $\varepsilon_{\mu^z,t}$ is a shock to $\log(z_t)$. Suppose we get a one-standard deviation positive shock to $\varepsilon_{\mu^z,t}$. This induces a move in $\log z_t$ by σ_{μ_z}, i.e., $\Delta\log z_t = \sigma_{\mu_z}$, where Δ means the difference between what $\log(z_t)$ is with the shock and what it would have been in the absence of a shock. To get this into percent terms, multiply σ_{μ_z} by 100. With $\sigma_{\mu_z} = 0.06$, this means that a one-standard deviation (i.e., a shock of 'typical' magnitude) disturbance in $\varepsilon_{\mu^z,t}$ moves z_t by 6 percent. This is too big to make any sense. For example, the first draft of ACEL reports that the standard deviation of $\varepsilon_{\mu^z,t}$ estimated by Prescott is 1 percent. It also reports our estimate of 0.12 percent. A sensible interpretation of what we have here is that the standard deviation of the shock to neutral technology is 0.06 percent.

In unscale.m, the level of technology is computed using the log approximation (see the cumulative sum in the code). After computing the level of technology, the program computes money growth. (Implicitly, it sets $\hat{q}_0 = 0$.) It does so by evaluating:

$$\hat{q}_t - \hat{q}_{t-1} + \hat{\pi}_t + \hat{\mu}_{z^*,t},$$

for $t = 1, ..., T$. Writing this out more carefully (using the log approximation),

$$\begin{aligned}
& \log\frac{q_t}{q} - \log\frac{q_{t-1}}{q} + \log\frac{\pi_t}{\pi} + \log\frac{\mu_{z^*,t}}{\mu_{z^*}} \\
= & \log Q_t - \log P_t - \log z_t^* - \left[\log Q_{t-1} - \log P_{t-1} - \log z_{t-1}^*\right] \\
& + \log\pi_t - \log\pi + \log\mu_{z^*,t} - \log\mu_{z^*} \\
= & \log Q_t - \log Q_{t-1} - \log\pi - \log\mu_{z^*}.
\end{aligned}$$

(Because the object on the left of the equality is zero in steady state, this says that the growth rate of transactions balances is equal to inflation plus the growth rate of the economy, i.e., the growth rate of z_t^*.) The program multiplies the above by 4 and calls the result mgrowth. This is clearly an annualized, decimal, growth rate.

Next unscale.m computes 'output', which is $\widehat{\tilde{y}}_t = \widehat{\frac{y_t}{z_t^*}}$. The program then adds to this, the quantity $\hat{\mu}_{z^*,t}$:

$$\widehat{\tilde{y}}_t + \hat{\mu}_{z^*,t}.$$

Using the log approximation, this is (recall, $\tilde{y}_t = y_t/z_t^*$),

$$\begin{aligned}
& \log\left(\frac{y_t}{z_t^*\tilde{y}}\right) + \log z_t^* - \log z_0^* - t\log(\mu_{z^*}) \\
& \log(y_t) - \log\tilde{y} - \log z_0^* - t\log(\mu_{z^*}).
\end{aligned}$$

Consumption and hours are handled in the same way. Capital utilization ('capa') is $\hat{u}_t$, which we interpret as $\log u_t$, which is 'like' $u_t - 1$.

In the case of R_t ('fedf'), unscale.m computes $4R\hat{R}_t$, which is $4(R_t - R)$ under the normal interpretation of $\hat{R}_t$. Inflation is handled in the same way. The factor, 4, converts to annual. Unfortunately, neither of these transformations is correct. Both the interest rate and the inflation rate are expressed in annual, decimal terms.

Velocity is

$$\begin{aligned}
& \log(y_t) - \log\tilde{y} - \log z_0^* - t\log(\mu_{z^*}) - \hat{q}_t - [\log z_t^* - \log z_0^* - t\log(\mu_{z^*})] \\
= & \log(y_t) - \log\tilde{y} - \log z_0^* - t\log(\mu_{z^*}) - \log\left(\frac{Q_t}{z_t^* P_t q}\right) - [\log z_t^* - \log z_0^* - t\log(\mu_{z^*})] \\
= & \log(y_t) - \log\left(\frac{Q_t}{P_t}\right) - \log\tilde{y} + \log q.
\end{aligned}$$

Consider *pinv*. The cumulative sum of $\hat{\mu}_{\Upsilon,t}$ is

$$\log \Upsilon_t - \log \Upsilon_0 - t \log(\mu_\Upsilon).$$

These data are loaded into a matrix, SimData.

In summary, unscale produces as output,

[output, mgrowth, infl, fedf, capa, hours, rwage, cons, invest, vel, pinv]

The variables here computed using the log approximation are output, mgrowth, capa, hours, rwage, cons, invest, vel, pinv. Variables computed using the normal approximation are infl, fedf. In the calculations, the shocks have been multiplied by 100.

12.2. Riccardo's Approximation

This approximation uses the linearized mapping from z_t, θ_t to X_t in (8.3). This mapping is described in detail in section 8.1.

13. Estimation and Identification of VAR Impulse Response Functions

Following is the *structural form* representation of our VAR system:

$$A_0 Y_t = A(L) Y_{t-1} + e_t. \tag{13.1}$$

The parameters of the reduced form are related to those of the structural form by:

$$C = A_0^{-1}, \; B(L) = A_0^{-1} A(L). \tag{13.2}$$

We obtain impulse responses by first estimating the parameters of the structural form, mapping these into the reduced form, and then simulating (**??**).

13.0.1. Monetary Policy Shocks

We assume that policy makers manipulate the monetary instruments under their control in order to ensure that the following interest rate targeting rule is satisfied:

$$R_t = f(\Omega_t) + \varepsilon_{Rt}, \tag{13.3}$$

where ε_{Rt} is the monetary policy shock. We interpret (13.3) as a reduced form Taylor rule. To ensure identification of the monetary policy shock, we assume f is linear, Ω_t contains $Y_{t-1}, ..., Y_{t-q}$ and the only date t variables in Ω_t are $\{\Delta a_t, \Delta p_{It}, Y_{1t}\}$. Finally, we assume that

ε_{Rt} is orthogonal with Ω_t. It is easy to verify that these identifying assumptions correspond to the following restrictions on A_0:

$$A_0 = \begin{bmatrix} \underset{1\times1}{A_0^{1,1}} & \underset{1\times1}{A_0^{1,2}} & \underset{6\times6}{A_0^{1,3}} & \underset{1\times1}{0} & \underset{1\times1}{0} \\ \underset{1\times1}{A_0^{2,1}} & \underset{1\times1}{A_0^{2,2}} & \underset{1\times6}{A_0^{2,3}} & \underset{1\times1}{0} & \underset{1\times1}{0} \\ \underset{6\times1}{A_0^{3,1}} & \underset{6\times1}{A_0^{3,2}} & \underset{6\times6}{A_0^{3,2}} & \underset{6\times1}{0} & \underset{6\times1}{0} \\ \underset{1\times1}{A_0^{4,1}} & \underset{1\times1}{A_0^{4,2}} & \underset{1\times6}{A_0^{4,3}} & \underset{1\times1}{A_0^{4,4}} & \underset{1\times1}{0} \\ \underset{1\times1}{A_0^{5,1}} & \underset{1\times1}{A_0^{5,2}} & \underset{1\times6}{A_0^{5,3}} & \underset{1\times1}{A_0^{5,4}} & \underset{1\times1}{A_0^{5,5}} \end{bmatrix}. \tag{13.4}$$

The second to last row of A_0 corresponds to the monetary policy rule, (13.3). The zero in this row reflects our assumption that Ω_t does not include the last variable in Y_t. The right two columns of zeros in the first 8 rows of A_0 reflect our identifying assumption that a monetary policy shock has no contemporaneous impact on Δa_t, Δp_{It} or Y_{1t}. Suppose there were a non-zero term somewhere in the first 8 rows of column 9. Since the interest rate is affected by the monetary policy shock, this would imply that a variable in the first 8 rows of column 9 is affected by a policy shock, contradicting our identification assumption. Now suppose that there were a non-zero term in at least one of the eight rows of column 10 in A_0. Since the money supply is affected by the monetary policy shock, this would imply that a variable in the first 8 rows of column 10 is affected by a monetary policy shoc, contradicting our identification assumption.

13.0.2. Technology Shocks

As stated above, we assume that the only shocks which have a non-zero impact on the long-run level of productivity are innovations to neutral and capital-embodied technology. The only shock that has an effect on the price of investment in the long run is a shock to capital-embodied technology. Like the monetary policy shocks, the identification assumptions on the technology shocks imply a set of zero restrictions on an expression that combines the autoregressive parameters in the VAR and A_0^{-1}. We do not exhibit these restrictions here, because it turns out to be more convenient to pursue a variant of the approach advocated by Shapiro and Watson.

13.1. Estimation of Impulse Responses

To discuss our estimation strategy, it is useful to write out the equations of the structural system explicitly, taking into account the restrictions implied by our assumptions about long-run effects of shocks and our assumptions about the effects of a monetary policy shock.

Apart from a constant, the first equation in (13.1) can be written as follows:

$$\Delta p_{It} = a_{11}(L)\Delta p_{It-1}+a_{12}(L)\Delta^2 a_t+a_{13}(L)\Delta Y_{1t}+a_{14}(L)\Delta R_{t-1}+a_{15}(L)\Delta Y_{2,t-1}+\frac{e_{\Upsilon,t}}{A_0^{1,1}}, \quad (13.5)$$

where $\Delta \equiv (1-L)$. The presence of Δ in front of each of Δa_t, Y_{1t}, R_{t-1}, $Y_{2,t-1}$ reflects our identification assumption that shocks other than $e_{\Upsilon,t}$ have no impact on p_{It} in the long run. The polynomial lag operators, correspond to the relevant entries of the first row of $A_0 - A(L)L$, scaled by $A_0^{1,1}$. The restriction that only capital embodied technology shocks have a non-zero impact on the relative price of investment at infinity is equivalent to imposing a unit root in each of the lag polynomials associated with Δa_t, Y_{1t}, R_{t-1} and $Y_{2,t-1}$. Also note that we exclude the contemporaneous values of R_t and Y_{2t} from the right side of (13.5). This reflects our assumption that monetary policy shocks do not have a contemporaneous impact on the price of investment (see the discussion about A_0 above).

We cannot use ordinary least squares to obtain a consistent estimate of the coefficients in (13.5) because $\Delta^2 a_t$ and ΔY_{1t} are in general correlated with $e_{\Upsilon,t}$. We apply two stage least squares to estimate the parameters using as instruments a constant, Δa_{t-i}, Δp_{It-i}, Y_{1t-i}, R_{t-i}, and Y_{2t-i}, $i = 1,2,3,4$. The coefficients in the first row of the structural form can then be obtained by scaling the instrumental variables estimates up by $A_0^{1,1}$, where $A_0^{1,1}$ is estimated as the (positive) square root of the variance of the fitted disturbance in the instrumental variables relation.

The second equation in (13.1) can be written as:

$$\Delta a_t = a_{22}(L)\Delta a_{t-1}+a_{21}(L)\Delta p_{It}+a_{23}(L)\Delta Y_{1t}+a_{24}(L)\Delta R_{t-1}+a_{25}(L)\Delta Y_{2,t-1}+\frac{e_{zt}}{A_0^{2,2}}, \quad (13.6)$$

where the polynomial lag operators correspond to the relevant entries of the second row of $A_0 - A(L)L$, scaled by $A_0^{2,2}$. The presence of a unit root in the polynomial lag operators multiplying Y_{1t}, R_{t-1} and $Y_{2,t-1}$ reflects our assumption that non-technology shocks have no impact on a_t at infinity[4]. Our assumptions do not imply a similar unit root restriction on the polynomial lag operator multiplying Δp_{It}. This is because, by assumption, the moving average relating non capital-embodied technology shocks to Δp_{It} already has a unit root. The fact that the contemporaneous values of R_t and Y_{2t} are excluded from (13.6) reflects our assumption that monetary policy shocks do not have a contemporaneous impact on labor productivity (see the discussion about A_0 above).

We cannot use ordinary least squares to obtain a consistent estimate of the coefficients in (13.6), because e_{zt} is, in general, correlated with Δp_{It} and ΔY_{1t}. Instead, we apply two-stage least squares using as instruments a constant, $\hat{e}_{\Upsilon,t}$, Δa_{t-i}, Δp_{It-i}, Y_{1t-i}, R_{t-i}, and $Y_{2,t-i}$, for

[4]For further discussion, see Shapiro and Watson (1988), and the more recent papers by Christiano, Eichenbaum and Vigfusson (2003, 2003a, 2003b) and Fisher (2003).

$i = 1, 2, 3, 4$. Here, $\hat{e}_{\Upsilon,t}$ is the fitted disturbance from (13.5). By including this disturbance as an instrument, we are imposing our assumption that neutral and capital-embodied technology shocks are orthogonal. The coefficients in the second row of the structural form can be obtained by scaling the instrumental variables estimates up by $A_0^{2,2}$. Here, $A_0^{2,2}$ is estimated as the (positive) square root of the variance of the fitted disturbances in the instrumental variables relation.

The next set of 6 equations in (13.1) can be written as follows:

$$A_0^{3,1}\Delta a_t + A_0^{32}\Delta p_{It} + A_0^{3,3}Y_{1t} = b(L)Y_{t-1} + e_{1t} \tag{13.7}$$

The ninth equation in (13.1) is just the policy rule:

$$R_t + \frac{A_0^{4,1}}{A_0^{4,4}}\Delta p_{It} + \frac{A_0^{4,2}}{A_0^{4,4}}\Delta a_t + \frac{A_0^{4,3}}{A_0^{4,4}}Y_{1t} = c(L)Y_{t-1} + \frac{e_{Mt}}{A_0^{4,4}}. \tag{13.8}$$

Consistent estimates of the parameters in (13.8) can be obtained by ordinary least squares with R_t as the dependent variable. This is because, by assumption, e_{Mt} is not correlated with Δa_t, Δp_{It} and Y_{1t}. The fitted e_{Mt}'s are orthogonal to e_{zt}'s and $e_{\Upsilon t}$'s. This is e_{Mt}'s are orthogonal to the variables that span the space in which the innovations to technology lie. The parameters of the 9^{th} row of the structural form are obtained by scaling the estimates up by $A_0^{3,3}$, where $A_0^{3,3}$ is estimated as the positive square root of the variance of the fitted residuals. Finally, according to the last equation:

$$Y_{2t} + \frac{A_0^{5,1}}{A_0^{5,5}}\Delta a_t + \frac{A_0^{5,2}}{A_0^{5,5}}\Delta p_{It} + \frac{A_0^{5,3}}{A_0^{5,5}}Y_{1t} + \frac{A_0^{5,4}}{A_0^{5,5}}R_t = d(L)Y_{t-1} + \frac{e_{2t}}{A_0^{5,5}}.$$

The coefficients in this relation can be estimated by ordinary least squares. This is because e_{2t} is not correlated with the other contemporaneous variables in this relation. This reflects that Y_{2t} does not enter any of the other equations. The parameter, $A_0^{5,5}$, can be estimated as the square root of the estimated variance of the disturbances in this relation. The parameters in the last row of the structural form are then suitably scaled up by $A_0^{5,5}$.

The previous argument establishes that rows 1, 2, 9 and 10 of A_0 are identified. The block of 6 rows in the middle is not identified. To see this, let w denote an arbitrary 6×6 orthonormal matrix, $ww' = I_6$. Suppose $\bar{A}_0$ and $\bar{A}(L)$ is some set of structural form parameters that satisfies all our restrictions. Let the orthonormal matrix, W, be defined as follows:

$$W = \begin{bmatrix} \underset{2\times2}{I} & \underset{2\times6}{0} & \underset{2\times2}{0} \\ \underset{6\times2}{0} & \underset{6\times6}{w} & \underset{6\times2}{0} \\ \underset{2\times2}{0} & \underset{2\times6}{0} & \underset{2\times2}{I} \end{bmatrix}. \tag{13.9}$$

It is easy to verify that the reduced form corresponding to the parameters, $W\bar{A}_0$, $W\bar{A}(L)$ also satisfies our restrictions, and leads to the same reduced form:

$$Y_t = \left(W\bar{A}_0\right)^{-1} W\bar{A}(L)Y_{t-1} + \left(W\bar{A}_0\right)^{-1} We_t.$$

To see this, note:

$$\begin{aligned} \left(W\bar{A}_0\right)^{-1} W\bar{A}(L) &= \bar{A}_0^{-1}W'W\bar{A}(L) = \bar{A}_0^{-1}\bar{A}(L) \\ E\left(W\bar{A}_0\right)^{-1} Wu_tu_t'W'\left[\left(W\bar{A}_0\right)^{-1}\right]' &= E\bar{A}_0^{-1}W'We_te_t'W'\left[\bar{A}_0^{-1}W'\right]' \\ &= \bar{A}_0^{-1}\left(\bar{A}_0^{-1}\right)'. \end{aligned}$$

Recall that impulse response functions can be computed using the matrices in $B(L)$ and the columns of A_0^{-1}. It is easy to see that the impulse responses to e_{Mt}, e_{zt} and $e_{\Upsilon t}$ are invariant to w. This is because:

$$\left(W\bar{A}_0\right)^{-1} = \bar{A}_0^{-1}W'.$$

It can be verified that columns 1, 2, 9 and 10 of $\bar{A}_0^{-1}W'$ coincide with those of $\bar{A}_0^{-1}$.

We conclude that there is a family of observational equivalent parameterizations of the structural form, which is consistent with our identifying assumptions on the monetary policy shock and the technology shocks. We arbitrarily select an element in this family as follows. Let Q and R be orthonormal and lower triangular (with positive diagonal terms) matrices, respectively, in the QR decomposition of A_0^{33}. That is, $A_0^{33} = QR$. This decomposition is unique and guaranteed to exist given that A_0^{33} is non-singular, a property implied by our assumption that A_0 is invertible. Now, suppose we have a particular parameterization in hand in which A_0^{33} is not lower triangular. Then, the QR decomposition guarantees that we can find an orthonormal matrix, w, such that wA_0^{33} is lower triangular. Suppose that A_0^{33} is already lower triangular. How many orthonormal matrices have the property that premultiplication of A_0^{33} preserves lower triangularity of the result? There is only one. The fact that wA_0^{33} and A_0^{33} are both lower triangular implies that w is too. But orthonormality of w under these circumstances implies that it is the Choleski decomposition of the identity matrix, which known to be unique and equal to the identity matrix itself. We conclude that we may, without loss of generality, restrict A_0^{33} to be lower triangular. This restriction does not restrict the reduced form in any way, nor does it restrict the set of possible impulse response functions associated with e_{Mt}, e_{zt}, $e_{\Upsilon,t}$ or e_{2t}.

Thus, in (13.7) A_0^{33} is lower triangular. We seek consistent estimates of the parameters of (13.7), with this restriction imposed. Ordinary least squares will not work as an estimation procedure here because of simultaneity. To see this, consider the first equation in (13.7). Suppose the left hand variable is the first element in Y_{1t}. The only current period explanatory variables are Δa_t and Δp_{It}. But, note from the first and second equations in the structural form that Δa_t and Δp_{It} respond to Y_{1t} and, hence, to the innovations in Y_{1t}. That is, Δa_t and Δp_{It} is correlated with the first element in e_{1t}. We can instrument for Δa_t using e_{zt}, the (scaled) residual from the first structural equation, and for Δp_{It} using $e_{\Upsilon,t}$, the (scaled) residual from the second structural equation.

Now consider the second equation in (13.7). Think of the left hand variable as being the second variable in Y_{1t}. The current period explanatory variables in that equation are Δa_t, Δp_{It} and the first variable in Y_{1t}. All of these variables are correlated with the second element in e_{1t}. To see this, note that a disturbance in the second element of e_{1t} ends up in Δa_t and Δp_{It} via the first and second equations in the structural form, because Y_{1t} appears in those equations. This explains why Δa_t and Δp_{It} are correlated with the second element of e_{1t}. But, the first element in Y_{1t} is also correlated with this variable because Δa_t and Δp_{It} are 'explanatory' variables in the equation determining the first element in Y_{1t}, i.e., the first equation in (13.7). So, we need an instrument for Δa_t, Δp_{It} and the first element of Y_{1t}. For this, use e_{zt}, $e_{\Upsilon,t}$ and the residual from the first equation in (13.7). Thus, moving down the equations in (13.7), we use as instruments e_{zt}, $e_{\Upsilon,t}$ and the disturbances in the previous equations in (13.7).

With A_0 and $A(L)$ in hand, we are now in a position to compute the reduced form, using (13.2). The dynamic responses of Y_t to technology and monetary policy shocks may be computed by simulating (**??**) with $i = 1, 2, 9$, respectively.