REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 7, número 1: 1994
http://dx.doi.org/10.5209/rev_REMA.1994.v7.n1.17776

On The Locally Uniformly Weak Star Rotundity of Orlicz Spaces *

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ABSTRACT. In the paper, a sufficient and necessary condition is given for the locally uniformly weak star rotundity of Orlicz spaces with Orlicz norms.

A Banach space X is said to be locally uniformly rotund (LUR), locally weakly uniformly rotund $(LWUR)$, locally uniformly weak star rotund (LW^*UR) provided that $\|x_n\| = 1$ $(n = 0, 1, 2, \ldots)$, $\|x_n + x_0\| \to 2$ imply $\|x_n - x_0\| \to 0$, $x_n - x_0 \xrightarrow{w} 0$, $x_n - x_0 \xrightarrow{w^*} 0$, respectively. X is said to be uniformly weak star rotund (W^*UR) provided that $\|x_n\| = \|y_n\| = 1$, $\|x_n + y_n\| \to 2$ imply $x_n - y_n \xrightarrow{w^*} 0$. At a glance we know that

$$LUR \Rightarrow LWUR \Rightarrow LW^*UR \Rightarrow R$$

$$W^*UR \Rightarrow LW^*UR$$

where $'R'$ stands for the rotundity.

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* This subject supported by NSFC and NSFH.
1991 Mathematics Subject Classification: 46E30.
Editorial Complutense. Madrid, 1994.

In the sequel (G, Σ, μ) denotes a finite non-atomic measurable space, M and N denote a pair of complemented N-functions, p and q denote their right-hand derivatives, respectively. For a measurable function $x(t)$ we denote the modular of x by $R_M(x) = \int_G M(x(t))\, d\mu$. $L_M(G, \Sigma, \mu)$ denotes an Orlicz space generated by M, that is

$$L_M(G,\Sigma,\mu) = \{x(t): \text{ for some } a > 0,\ R_M(ax) < \infty\}$$

and endowed with the Orlicz norm

$$\|x\| = \sup_{R_M(y)\leq 1} \int_G x(t)y(t)\, d\mu = \inf_{k>0} \frac{1}{k}(1 + R_M(kx)).$$

$M \in \Delta_2$ stands for that M which satisfies the condition Δ_2 for large u, $M \in \nabla_2$ stands for $N \in \Delta_2$, $M \in SC$ stands for that M which is strictly convex on the whole axis i.e. for $0 < \lambda < 1, u, v,\ u \neq v$,

$$M(\lambda u + (1-\lambda)v) < \lambda M(u) + (1-\lambda)M(v).$$

(cf [1] and [3]).

In Orlicz spaces, for Luxemburg norm, it was obtained in [2] that $LUR \Leftrightarrow LWUR \Leftrightarrow LW^*UR \Leftrightarrow R \Leftrightarrow M \in SC \cap \Delta_2$; for the Orlicz norm, it is more complicated, for instance, $LUR \Leftrightarrow LWUR \Leftrightarrow M \in \Delta_2 \cap \nabla_2 \cap SC$(cf[3]), $W^*UR \Leftrightarrow M \in SC \cap UC$(cf[4]) and $R \Leftrightarrow M \in SC$(cf[5]). But so far it has not been discussed for LW^*UR. The goal of this paper is to fill this gap, we will find a criterion for Orlicz space equipped with the Orlicz norm to be LW^*UR. For the sake of convenience, we first establish several lemmas.

Lemma 1. *For arbitrary $0 \leq \lambda,\ \delta,\ \lambda' < 1$, there exists $0 < \delta' \leq \delta$ such that for all $u, v > 0$ if $M(\lambda u + (1-\lambda)v) \leq (1-\delta)(\lambda M(u) + (1-\lambda)M(v))$, then*

$$M(\lambda' u + (1-\lambda')v) \leq (1-\delta')(\lambda' M(u) + (1-\lambda')M(v))$$

Proof. Without loos of generality, we assume that $\lambda' > \lambda$. Take $\delta' = min\left\{1, \frac{\lambda(1-\lambda')}{\lambda'(1-\lambda)}\right\}\delta$.

Hence $0 < \delta' \leq \delta$ and

$$\begin{aligned}
M(\lambda' u + (1-\lambda')v) &= M\left[\frac{1-\lambda'}{1-\lambda}(\lambda u + (1-\lambda)v)) + \frac{\lambda'-\lambda}{1-\lambda}u\right] \\
&\leq \frac{1-\lambda'}{1-\lambda}M(\lambda u + (1-\lambda)v) + \frac{\lambda'-\lambda}{1-\lambda}\, M(u) \\
&\leq \frac{1-\lambda'}{1-\lambda}(1-\delta)[\lambda M(u) + (1-\lambda)M(v)] + \frac{\lambda'-\lambda}{1-\lambda}M(u) \\
&= \lambda' M(u) - \frac{1-\lambda'}{1-\lambda}\lambda\delta M(u) + (1-\delta)(1-\lambda')M(v) \\
&= \left(1 - \frac{\lambda(1-\lambda')}{\lambda'(1-\lambda)}\delta\right)\lambda' M(u) + (1-\delta)(1-\lambda')M(v) \\
&\leq (1-\delta')\lambda' M(u) + (1-\delta')(1-\lambda')M(v) \\
&= (1-\delta')(\lambda' M(u) + (1-\lambda')M(v)). \quad \blacksquare
\end{aligned}$$

Remark. Notice that for fixed $\lambda, \delta, \delta'(\lambda') = min\left\{1, \frac{\lambda(1-\lambda')}{\lambda'(1-\lambda)}\right\}\delta$ is continuous over the interval $(0,1)$. We deduce that for any $[\alpha,\beta] \subset (0,1)$, in Lemma 1, there is a common δ_0' such that for all $\lambda' \in [\alpha,\beta]$, and $u, v > 0,\ u \neq v$,

$$M(\lambda' u + (1-\lambda')v) \leq (1-\delta_0')(\lambda' M(u) + (1-\lambda')M(v)).$$

Lemma 2. *For* $x \in L_M$, *if for some* $k > 0$, $R_N(p(kx)) = \int_G N(p(kx(t)))\, d\mu \leq 1$ *and for all* $\lambda > 1$, $R_N(p(\lambda kx)) > 1$, *then*

$$||x|| = \frac{1}{k}\Big(1 + R_M(kx)\Big). \qquad (cf[3])$$

Lemma 3. *For $x \in L_M$, there is $k > 0$ satisfying*

$$\|x\| = \frac{1}{k}\Big(1 + R_M(kx)\Big). \qquad (cf[3])$$

Lemma 4. *If $M \in SC$, $\|x_n\| = \frac{1}{k_n}(1 + R_M(k_n x_n))$ $(n = 0, 1, 2, \ldots)$ with bounded $\{k_n\}_{n=0}^{\infty}$ and $\|x_n + x_0\| \to 2$, then*

$$k_n x_n - k_0 x_0 \xrightarrow{\mu} 0. \qquad (cf[3])$$

Lemma 5. *Under the same assumption as in Lemma 4, let $y_n \in L_N$, $R_N(y_n) \le 1$ with $\int_G (x_n(t) + x_0(t)) y_n(t)\, d\mu \to 2$. Then for every $e_n \subset G$,*

$$\lim_{n\to\infty} \int_{e_n} [k_n x_n(t) y_n(t) - M(k_n x_n(t)) - N(y_n(t))]\, d\mu = 0,$$

$$\lim_{n\to\infty} \int_{e_n} [k_0 x_0(t) y_n(t) - M(k_0 x_0(t)) - N(y_n(t))]\, d\mu = 0,$$

$$\lim_{n\to\infty} \int_{e_n} (k_n x_n(t) - k_0 x_0(t)) y_n(t)\, d\mu =$$

$$= \lim_{n\to\infty} \int_{e_n} M(k_n x_n(t)) - M(k_0 x_0(t))\, d\mu.$$

As n tends to ∞, the above limits hold uniformly with respect to subsets e_n.

Proof. We have the following

$$1 \leftarrow \frac{1}{k_n} \int_G k_n x_n(t) y_n(t)\, d\mu \le \frac{1}{k_n}\Big(R_N(y_n) + R_M(k_n x_n)\Big)$$

$$\le \frac{1}{k_n}(1 + R_M(k_n x_n)) = \|x_n\| = 1.$$

Hence

$$\int_G \left[M(k_n x_n(t)) + N(y_n(t)) - k_n x_n(t) y_n(t)\right] d\mu \to 0.$$

Since the integrand is nonnegative, we inmediately get the first and the second identity. The third one is a simple consequence of the others. ■

Lemma 6. *In Orlicz space $L_M(G, \Sigma, \mu)$ endowed with Orlicz norm, the set*

$$A = \left\{ x(t) \in L_M : R_N(p(kx)) = 1 \text{ where } \|x\| = \frac{1}{k}(1 + R_M(kx)) \right\}$$

is dense in L_M.

Proof. It is enough to show that for any $x \in L_M$ with $R_N(p(kx)) > 1$ or < 1 where $\|x\| = \frac{1}{k}(1 + R_M(kx))$, and for any $\varepsilon > 0$, there is $x' \in A$, such that $\|x - x'\| < \varepsilon$ and $R_N(p(kx')) = 1$.

Let $R_N(p(kx)) > 1$.

Notice that for any $\varepsilon > 0$, $R_N(p((1-\varepsilon)kx) \leq 1$. When $R_N(p((1-\varepsilon)kx)) = 1$, set $x'(t) = (1-\varepsilon)x(t)$. Then $R_N(p(kx')) = 1$. Now by Theorem 10.5 in [1], we get that $\|x'\| = \frac{1}{k}(1 + R_M(kx'))$, i.e., $x' \in A$. Clearly $\|x - x'\| \leq \varepsilon$.

When $R_N(p((1-\varepsilon)kx)) < 1$, since (G, Σ, μ) is nonatomic, there is $G' \subset G$

$$\int_{G'} N(p((1-\varepsilon)kx(t)))\, d\mu + \int_{G \setminus G'} N(p(kx(t)))\, d\mu = 1$$

Setting $x'(t) = (1-\varepsilon)x(t)\chi_{G'}(t) + x(t)\chi_{G\setminus G'}(t)$, we get $R_N(p(kx')) = 1$. Also by [1], $\|x'\| = \frac{1}{k}(1 + R_M(kx'))$. Clearly $\|x - x'\| \leq \varepsilon$.

The argument is analogous for the case $R_N(p(kx)) < 1$. ■

Theorem. *Endowed with the Orlicz norm, Orlicz space* $L_M(G,\Sigma,\mu)$ *is* LW^*UR *if and only if*

(i) $M \in SC$,

(ii) $M \in \nabla_2$,

(iii) for any $\varepsilon > 0$, *there exist* $\delta,\ a > 0$ *such that for* u, v *satisfying* $\varepsilon^2 \le \varepsilon u \le v < u$, $M(u) \ge \varepsilon u p(u)$, *and* $M\left(\frac{u+v}{2}\right) > (1-\delta)\frac{M(u)+M(v)}{2}$, *we have*

$$p((1-\varepsilon)u) \le a p((1-\delta)v).$$

Proof. Sufficiency. Suppose $\|x_n\| = \frac{1}{k_n}(1+R_M(k_n x_n)) = 1$ $(n = 0,1,2,\ldots)$ and $\|x_n + x_0\| \to 2$. By Lemma 6, assume $R_N(p(k_n x_n)) = 1$ $(n = 1,2,\ldots)$.

In view of $M \in \nabla_2$, we know from [3] that $\{k_n\}_{n=1}^{\infty}$ is bounded. Denote $\bar{k} = \sup_n k_n$. In the following we shall show that $x_n \xrightarrow{w^*} x_0$, i.e., any subsequence of $\{x_n\}_{n=1}$ has its subsequence w^*-convergent to x_0. So we can assume that $k_n \to k$. On the other hand, by Lemma 4, it yields $k_n x_n - k_0 x_0 \xrightarrow{\mu} 0$. Therefore by Theorem 14.6 in [1], $k_n x_n - k_0 x_0 \xrightarrow{E_N} 0$.

At first we claim that $k \ge k_0$. Indeed, for any $\eta > 0$, take $y \in E_N, R_N(y) \le 1$ such that $\int_G x_0(t)y(t)\,d\mu > \|x_0\| - \eta = 1 - \eta$. Since $\int_G k_n x_n y\,d\mu \to \int_G k_0 x_0 y\,d\mu$, we get that for n large enough $\int_G k_n x_n y\,d\mu > \int_G k_0 x_0 y\,d\mu - \eta > k_0(1-\eta) - \eta$. So $k \leftarrow k_n = \|k_n x_n\| \ge \int_G k_n x_n y\,d\mu > \int_G k_0 x_0 y\,d\mu - \eta > k_0(1-\eta) - \eta$.

Now we only need to show that $k = k_0$, so $x_n - x_0 \xrightarrow{\mu} 0$. Then by Theorem 14.6 in [1], we get that $x_n - x_0 \xrightarrow{E_N} 0$ i.e., $x_n - x_0 \xrightarrow{w^*} 0$.

Take $y_n \in E_N,\ R_N(y_n) \le 1$ satisfying $\int_G (x_n(t) + x_0(t))y_n(t)\,d\mu \to 2$. Then $\int_G x_n(t)y_n(t)\,d\mu \to 1$, and $\int_G x_0(t)y_n(t)\,d\mu \to 1$. Therefore we have

$$k - k_0 = \lim_{n\to\infty} \int_G (k_n x_n(t) - k_0 x_0(t))y_n(t)\,d\mu \tag{1}$$

Let $\varepsilon > 0$ be arbitrary. By $M \in \nabla_2$, there exists $\varepsilon \ge \eta'(\varepsilon) > 0$ (cf[6,3]) such that for all $|u| \ge \varepsilon$, and for all λ, $\frac{1}{1+\bar{k}} \le \lambda \le \frac{2\bar{k}+k_0}{2(k_0+\bar{k})}$, it holds

$$M(\lambda u) \le (1-\eta')\lambda M(u) \tag{2}$$

Denote $m = 1 + \bar{k}$. For $\eta = \frac{\eta'}{m}$, by (iii), there exist δ, $a > 0$ such that for u, v, $0 < \eta^2 \leq \eta u \leq v < u$, if $M(u) \geq \eta u p(u)$, and $M\left(\frac{u+v}{2}\right) > (1-\delta)\frac{M(u)+M(v)}{2}$ then

$$p((1-\eta)u) \leq ap((1-\delta)v) \tag{3}$$

For such δ and $[\alpha, \beta] = \left[\frac{1}{1+\bar{k}}, \frac{\bar{k}}{1+\bar{k}}\right]$, by the remark after Lemma 1, it follows that there exists δ' such that if $M\left(\frac{u+v}{2}\right) \leq (1-\delta)\frac{M(u)+M(v)}{2}$, and $\lambda' \in \left[\frac{1}{1+\bar{k}}, \frac{\bar{k}}{1+\bar{k}}\right]$, then

$$M(\lambda' u + (1-\lambda')v) \leq (1-\delta')(\lambda' M(u) + (1-\lambda')M(v). \tag{4}$$

Since $\int_G |k_0 x_0(t)| p((1-\delta)k_0 x_0(t))\, d\mu \leq R_M(k_0 x_0) + R_N(p((1-\delta)k_0 x_0)) \leq k_0$ we can find $G_0 \subset G$ such that $\mu(G \backslash G_0)$ is small enough to get the following

$$\int_{G \backslash G_0} |k_0 x_0(t)| p((1-\delta)k_0 x_0(t))\, d\mu < \frac{\eta\varepsilon}{a}$$

$$\int_{G \backslash G_0} M(k_0 x_0(t))\, d\mu < \varepsilon \tag{5}$$

and

$$k_n x_n(t) - k_0 x_0(t) \xrightarrow{U} 0$$

uniformly over G_0.

For each n, we split $G \backslash G_0$ into the five parts:

$$A_n = \{t \in G \backslash G_0 : \ |k_n x_n(t)| < |k_0 x_0(t)|\}$$

$$B_n = \{t \in G \backslash G_0 \backslash A_n : \ max(|k_n x_n(t)|, |k_0 x_0(t)|) < \varepsilon\}$$

$$C_n = \{t \in G \backslash G_0 \backslash A_n \backslash B_n : \ M(k_n x_n(t)) < \eta |k_n x_n(t)| p(|k_n x_n(t)|)\},$$

$$D_n = \Big\{t \in G\backslash G_0\backslash A_n\backslash B_n\backslash C_n : |k_0x_0(t)| < \eta|k_nx_n(t)|;\ \text{or } x_n(t)x_0(t) < 0;$$

$$\text{or } M\left(\frac{k_nx_n(t)+k_0x_0(t)}{2}\right) \leq (1-\delta)\frac{M(k_nx_n(t))+M(k_0x_0(t))}{2}\Big\},$$

$$E_n = G\backslash G_0\backslash A_n\backslash B_n\backslash C_n\backslash D_n$$

$$= \Big\{t \in G\backslash G_0 : x_n(t)x_0(t) \geq 0;\ \eta\varepsilon \leq \eta|k_nx_n(t)| \leq |k_0x_0(t)| \leq |k_nx_n(t)|$$

$$M(k_nx_n(t)) \geq \eta k_n|x_n(t)|p(k_n|x_n(t)|);\ \text{and}$$

$$M\left(\frac{k_nx_n(t)+k_0x_0(t)}{2}\right) \geq (1-\delta)\frac{M(k_nx_n(t))+M(k_0x_0(t))}{2}\Big\}.$$

In the following, one by one, we estimate the integrals of the integrand $(k_nx_n - k_0x_0)y_n$ over G_0, A_n, B_n, C_n, D_n, and E_n.

From (6), for n large enough

$$\left|\int_{G_0}(k_nx_n - k_0x_0)y_n\, d\mu\right| < \varepsilon\|y_n\|_{(N)} \tag{7}$$

From the structure of A_n, by Lemma 5, it follows that for n large enough

$$\int_{A_n}(k_nx_n - k_0x_0)y_n\, d\mu \leq \int_{A_n}(M(k_nx_n(t)) - M(k_0x_0(t)))\, d\mu + \varepsilon \leq \varepsilon. \tag{8}$$

From the structure of B_n,

$$\left|\int_{B_n}(k_nx_n - k_0x_0)y_n\, d\mu\right| \leq 2\varepsilon\|y_n\|_{(N)} \leq 2\varepsilon. \tag{9}$$

Since $\|x_n\| = 1$, $R_M(x_n) \le 1$. From $R_N(p(k_n x_n)) = 1$, it yields that for n large enough

$$\int_{C_n} (k_n x_n - k_0 x_0) y_n \, d\mu \le \int_{C_n} \Big(M(k_n x_n(t)) - M(k_0 x_0(t)) \Big) \, d\mu + \varepsilon$$
$$\le \int_{C_n} M(k_n x_n(t)) \, d\mu + 2\varepsilon \le \eta \int_{C_n} |k_n x_n(t)| p(k_n x_n(t)) \, d\mu + 2\varepsilon$$
$$\le \eta \bar{k} \Big(\int_G M(x_n(t)) \, d\mu + \int_G N(p(k_n x_n(t))) \, d\mu \Big) + 2\varepsilon$$
$$\le 2\bar{k}\eta + 2\varepsilon \le 2(1+\bar{k})\varepsilon.$$

When $t \in D_n$, $|k_0 x_0(t)| < \eta |k_n x_n(t)|$, since $t \notin A_n \cup B_n$. So $|k_n x_n(t)| > \varepsilon$, and from (2) it follows

$$M\left(\tfrac{k_n k_0}{k_n + k_0}(x_n(t) + x_0(t))\right) \le M\left(\tfrac{k_0 + \eta k_n}{k_n + k_0} k_n x_n(t)\right)$$
$$\le (1 - m\eta) \tfrac{k_0 + \eta k_n}{k_n + k_0} M(k_n x_n(t)) = (1 - m\eta) \tfrac{k_0 + \eta k_n}{k_0} \tfrac{k_0}{k_n + k_0} M(k_n x_n(t))$$
$$\le (1 - \eta) \tfrac{k_0}{k_n + k_0} M(k_n x_n(t))$$
$$\le (1 - \eta) \left[\tfrac{k_0}{k_n + k_0} M(k_n x_n(t)) + \tfrac{k_n}{k_n + k_0} M(k_0 x_0(t)) \right] \qquad (*)$$

When $t \in D_n$, $x_n(t) x_0(t) < 0$, since $t \notin A_n \cup B_n$. While $|x_n(t)| \ge |x_0(t)|$,

$$M\left(\tfrac{k_n k_0}{k_n + k_0}(x_n(t) + x_0(t))\right) \le M\left(\tfrac{k_n k_0}{k_n + k_0} x_n(t)\right)$$
$$\le (1 - \eta') \tfrac{k_0}{k_n + k_0} M(k_n x_n(t))$$
$$\le (1 - \eta') \left[\tfrac{k_0}{k_n + k_0} M(k_n x_n(t)) + \tfrac{k_n}{k_n + k_0} M(k_0 x_0(t)) \right] \qquad (**)$$

While $|x_n(t)| < |x_0(t)|$,

$$M\left(\tfrac{k_n k_0}{k_n + k_0}(x_n(t) + x_0(t))\right) \le M\left(\tfrac{k_n k_0}{k_n + k_0} x_0(t)\right)$$
$$\le \tfrac{k_n}{k_n + k_0} M(k_0 x_0(t))$$

$$\leq \left(1-\tfrac{k_0}{k_n+k_0}\right)\left[\tfrac{k_0}{k_n+k_0}\ M(k_nx_n(t))+\tfrac{k_n}{k_n+k_0}\ M(k_0x_0(t))\right]. \qquad (***)$$

Taking $\delta'' = min(\delta', \eta, \frac{1}{1+k})$, and applying (4), we have
$0 \leftarrow 2 - \|x_n + x_0\|$

$$\geq \tfrac{1}{k_n}(1 + R_M(k_nx_n)) + \tfrac{1}{k_0}(1 + R_M(k_0x_0))-$$

$$-\tfrac{k_n+k_0}{k_nk_0}\ \left(1 + R_M\left(\tfrac{k_nk_0}{k_n+k_0}\ (x_n + x_0)\right)\right)$$

$$= \tfrac{k_n+k_0}{k_nk_0}\ \int_G \left[\tfrac{k_0}{k_n+k_0}\ M(k_nx_n(t)) + \tfrac{k_n}{k_n+k_0}\ M(k_0x_0(t))-\right.$$

$$\left.-M\ \left(\tfrac{k_nk_0}{k_n+k_0}(x_n + x_0)(t)\right)\right]\ d\mu$$

$$\geq \tfrac{k_n+k_0}{k_nk_0}\ \int_{D_n} \left[\tfrac{k_0}{k_n+k_0}\ M(k_nx_n(t)) + \tfrac{k_n}{k_n+k_0}\ M(k_0x_0(t))-\right.$$

$$\left.-M\ \left(\tfrac{k_nk_0}{k_n+k_0}\ (x_n + x_0)(t)\right)\right]\ d\mu$$

$$\geq \tfrac{k_n+k_0}{k_nk_0}\ \int_{D_n} \delta''\left[\tfrac{k_0}{k_n+k_0}\ M(k_nx_n(t)) + \tfrac{k_n}{k_n+k_0}\ M(k_0x_0(t))\right]\ d\mu$$

$$\geq \tfrac{\delta''}{k}\ \int_{D_n}(M(k_nx_n(t)) + M(k_0x_0(t)))\ d\mu.$$

Obviously, for n large enough

$$\int_{D_n}(k_nx_n - k_0x_0)y_n\ d\mu \leq \int_{D_n} M(k_nx_n(t)) - M(k_0x_0(t))\ d\mu + \varepsilon \leq 2\varepsilon. \qquad (11)$$

When $t \in E_n$, then $|\eta k_nx_n(t)| \leq |k_0x_0(t)| \leq |k_nx_n(t)|$,
and $M(k_nx_n(t)) \geq \eta k_n|x_n(t)|p(k_n|x_n(t)|)$, and

$$M\ \left(\frac{k_nx_n(t) + k_0x_0(t)}{2}\right) > (1-\delta)\ \frac{M(k_nx_n(t)) + M(k_0x_0(t))}{2}.$$

Hence

$$p((1-\eta)|k_nx_n(t)|) \leq ap((1-\delta)|k_0x_0(t)|).$$

Moreover, from Lemma 4 and condition (5) we get for n large enough that

$\int_{E_n}(k_n x_n - k_0 x_0) y_n \, d\mu$

$= \eta \int_{E_n} k_n x_n y_n \, d\mu + \int_{E_n} (1-\eta) k_n x_n y_n \, d\mu - \int_{E_n} k_0 x_0 y_n \, d\mu$

$\leq \eta \bar{k} + \int_{E_n} M((1-\eta)k_n x_n(t)) \, d\mu + \int_{E_n} N(y_n(t)) \, d\mu - \int_{E_n} M(k_0 x_0(t)) \, d\mu$

$- \int_{E_n} N(y_n(t)) \, d\mu + \varepsilon$

$\leq \eta \bar{k} + \varepsilon + \int_{E_n} (1-\eta) k_n |x_n(t)| p((1-\eta) k_n |x_n(t)|) \, d\mu$

$\leq \eta \bar{k} + \varepsilon + (1-\eta) \frac{a}{\eta} \int_{E_N} k_0 |x_0(t)| p((1-\delta)|k_0 x_0(t)|) \, d\mu$

$\leq \eta \bar{k} + \varepsilon + \frac{a\eta\varepsilon}{\eta a} = \eta \bar{k} + 2\varepsilon \leq (2+\bar{k})\varepsilon.$

Combining (7) - (12), and (1), we deduce that

$$0 \leq k - k_0 < 0(\varepsilon),$$

where $0(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Hence $k = k_0$, which completes the proof of the sufficiency.

Necessity.
$LW^*UR \Rightarrow M \in SC$. Since $LW^*UR \Rightarrow R$, it follows (i), by [5].
$LW^*UR \Rightarrow M \in \nabla_2$. Indeed, if we suppose that it is not true, then there exist $u_n \nearrow \infty$, satisfying $\frac{u_n p(u_n)}{N(p(u_n))} > 2^n$ $(n = 1,2,\ldots)$ (cf [6,3]). Choose $c > 0$, $G_0 \subset G$, with $\mu(G \setminus G_0) > 0$ and $N(p(c))\mu G_0 = 1$. Moreover, choose $G_n \subset G \setminus G_0$, with $u_n p(u_n)\mu G_n = 1$. Hence $N(p(u_n))\mu G_n < \frac{1}{2^n}$. Then take $T_n \subset G_0$, such that $N(p(c))\mu T_n + N(p(u_n))\mu G_n = 1$. Hence $\mu T_n \to \mu G_0$. Now set

$$k_0 = cp(c)\mu G_0; \quad k_n = cp(c)\mu T_n + u_n p(u_n)\mu G_n. \quad (n = 1,2,\ldots)$$

Obviously, $k_n \to k_0 + 1$. Define

$$x_0(t) = \frac{c}{k_0}\chi_{G_0}(t); \quad x_n(t) = \frac{1}{k_n}(c\chi_{T_n}(t) + u_n \chi_{G_n}(t)). \; (n = 1,2,\ldots)$$

Since $R_N(p(k_0x_0)) = R_N(p(k_nx_n)) = 1$, by Theorem 10.5 of [1], it follows that

$$\|x_0\| = \frac{1}{k_0} cp(c)\mu G_0 = 1;$$

$$\|x_n\| = \frac{1}{k_n} (cp(c)\mu T_n + u_np(u_n)\mu G_n) = 1 \ (n = 1, 2, \dots)$$

and

$$\|x_n + x_0\| \geq (\frac{1}{k_n} + \frac{1}{k_0})cp(c)\mu T_n + \frac{1}{k_n} (u_np(u_n)\mu G_n \to 2.$$

But on the other hand, we have

$$x_0 - x_n = \left(\frac{1}{k_0} - \frac{1}{k_n}\right)c\chi_{T_n}(t) + \frac{c}{k_0}\chi_{G_0\setminus T_n}(t) - \frac{u_n}{k_n}\chi_{G_n}(t)$$

Since $\mu(G_0\setminus T_n) \to 0$, $\mu G_n \to 0$, $T_n \nearrow G_0$, by Theorem 14.6 in [1], we derive that

$$x_0 - x_n \overset{w^*}{\to} (\frac{1}{k_0} - \frac{1}{1+k_0})c\chi_{G_0} \neq 0.$$

This contradicts to the fact that L_M is LW^*UR, which show that $M \in \nabla_2$.

$LW^*UR \Rightarrow (iii)$. Otherwise, suppose there exist $\varepsilon > 0$, u_n, $v_n \nearrow \infty$ such that $\varepsilon^2 \leq \varepsilon u_n \leq v_n < u_n$, $M(u_n) \geq \varepsilon u_n p(u_n)$,

$$M(\frac{u_n + v_n}{2}) > (1 - \frac{1}{n}) \frac{M(u_n) + M(v_n)}{2}, \text{ and}$$

$$p((1-\varepsilon)u_n) > 2^n p((1 - \frac{1}{n})v_n).$$

In view of the continuity of $M(u)$, we select Θ_n, $0 < \Theta_n < 1$ with $\Theta_n \nearrow 1$ and

$$M(\frac{u_n + \Theta_n v_n}{2}) \geq (1 - \frac{1}{n}) \frac{M(u_n) + M(\Theta_n v_n)}{2}. \tag{13}$$

We first construct two sequences $\{w_n\}_{n=1}^{\infty}$ and $\{\tau_n\}_{n=1}^{\infty}$ satisfying

$$\tau_n \searrow 1,\ \Theta_n v_n \le w_n \le \big(1-\frac{\varepsilon}{3}\big)u_n, \text{ and } p(\tau_n w_n) > 2^n p(w_n). \qquad (14)$$

Since $1 \le \frac{u_n}{v_n} \le \frac{1}{\varepsilon}$, without loss of generality, if necessary we can pass to a subsequence, we assume that $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = b \ge 1$. Denote

$$\xi = sup\{\xi' > 0 : \overline{\lim_{n\to\infty}} \frac{p((1-\frac{\varepsilon}{2})bv_n)}{p(\xi' v_n)} = \infty\}.$$

Obviously, $1 \le \xi \le \big(1-\frac{\varepsilon}{2}\big)b$. In the following we discuss two cases.

$$(I) \quad \text{Let } \overline{\lim_{n\to\infty}} \frac{p((1-\frac{\varepsilon}{2})bv_n)}{p(\xi v_n)} = \infty.$$

For any $\lambda > 1$,

$$\infty \leftarrow \frac{p((1-\frac{\varepsilon}{2})bv_n)}{p(\xi v_n)} = \frac{p((1-\frac{\varepsilon}{2})bv_n)}{p(\lambda\xi v_n)} \cdot \frac{p(\lambda\xi v_n)}{p(\xi v_n)}$$

Since on the right side of the identity the first quotient formula is bounded, $\overline{\lim\limits_{n\to\infty}} \frac{p(\lambda\xi v_n)}{p(\xi v_n)} = \infty$. Passing to a subsequence if necessary, we assume that $p\big((1+\frac{1}{n})\big)\xi v_n\big) > 2^n p(\xi v_n)$. Easily we know that for n large enough, $v_n \le \xi v_n \le \big(1-\frac{\varepsilon}{2}\big)bv_n \le \big(1-\frac{\varepsilon}{3}\big)u_n$. For $w_n = \xi v_n$, $\tau_n = 1+\frac{1}{n}$, condition (14) is satisfied.

$$(II) \quad \text{Let } \overline{\lim_{n\to\infty}} \frac{p((1-\frac{\varepsilon}{2}bv_n)}{p(\xi v_n)} < \infty.$$

For any $\Theta_n < 1$,

$$\infty \leftarrow \frac{p((1-\frac{\varepsilon}{2})bv_i)}{p(\Theta_n\xi v_i)} = \frac{p((1-\frac{\varepsilon}{2})bv_i)}{p(\xi v_i)} \cdot \frac{p(\xi v_i)}{p(\Theta_n\xi v_i)} \quad \text{as} \quad i \to \infty.$$

Hence $\overline{\lim_{i\to\infty}} \frac{p(\xi v_i)}{p(\Theta_n v_i \xi)} = \infty$. Passing to a subsequence if necessary, we get $p(\xi v_n) > 2^n p(\Theta_n \xi v_n)$. Obviously $\Theta_n v_n \leq \Theta_n \xi v_n \leq \xi v_n \leq (1-\frac{\varepsilon}{2}) b v_n < (1-\frac{\varepsilon}{3}) u_n$. If we take $w_n = \Theta_n \xi v_n$, and $\tau_n = \frac{1}{\Theta_n}$, then (14) is satisfied.

By (14), we can choose disjoint subsets $G_n \subset G$, $G_n \cap G_m = \emptyset$ $(n \neq m)$ such that

$$N(p(w_n))\mu G_n = \frac{1}{2^{n+1}} \qquad (n = 1, 2, \ldots)$$

For n large enough,

$$N(p(u_n))\mu G_n > N(p(\tau_n w_n))\mu G_n > 2^n N(p(w_n))\mu G_n = \frac{1}{2}.$$

Pick out $\bar{G}_n \subset G_n$ satisfying

$$N(p(u_n))\mu \bar{G}_n = \frac{1}{2} \qquad (n = 1, 2, \ldots)$$

Now set

$$k_0 = 1 + \sum_{i=1}^{\infty} M(w_i)\mu G_i,$$

$$k_n = 1 + \sum_{i \neq n}^{\infty} M(w_i)\mu G_i + M(u_n)\mu \bar{G}_n \qquad (n = 1, 2, \ldots)$$

By $M \in \nabla_2$, it yields that $\frac{u p(u)}{N(p(u))} \leq d$ $(u \geq u_0)$ $(cf[6,3])$. Then

$$\sum_{i=1}^{\infty} M(w_i)\mu G_i \leq \sum_{i=1}^{\infty} w_i p(w_i)\mu G_i \leq d \sum_{i=1}^{\infty} N(p(w_i))\mu G_i = \frac{d}{2},$$

$$M(u_n)\mu \bar{G}_n \leq u_n p(u_n)\mu \bar{G}_n \leq d N(p(u_n))\mu \bar{G}_n = \frac{d}{2}.$$

So $\{k_n\}_{n=1}^{\infty}$ is bounded. Passing to a subsequence if necessary, we assume that $k_n \to k$. From

$$M(u_n)\mu\bar{G}_n \geq \varepsilon u_n p(u_n)\mu\bar{G}_n \geq \varepsilon N(p(u_n))\mu\bar{G}_n = \frac{\varepsilon}{2}$$

we get that $k - k_0 \geq \frac{\varepsilon}{2}$. Define

$$x_0(t) = \frac{1}{k_0}\sum_{i=1}^{\infty} w_i\chi_{G_i}(t);$$

$$x_n(t) = \frac{1}{k_n}\Big(\sum_{i\neq n} w_i\chi_{G_i}(t) + u_n\chi_{\bar{G}_n}(t)\Big) \qquad (n = 1,2,\ldots)$$

We have

$$\int_G N(p(k_n x_n(t)))\, d\mu = \sum_{i\neq n} N(p(w_i))\mu G_i + N(p(u_n))\mu\bar{G}_n < 1.$$

In addition, for any $\lambda > 1$, take $i_0 > n$ such that $\lambda > \tau_{i_0}$. Then

$$\int_G N(p(\lambda k_n x_n(t)))\, d\mu = \sum_{i\neq n} N(p(\lambda w_i))\mu G_i + N(p(\lambda u_n))\mu\bar{G}_n$$

$$> \sum_{i=i_0}^{\infty} N(p(\tau_i w_i))\mu G_i > \sum_{i=i_0}^{\infty} 2^i N(p(w_i))\mu G_i = \infty.$$

By Lemma 2, it follows that

$$\|x_n\| = \frac{1}{k_n}\,(1+R_M(k_n x_n)) = \frac{1}{k_n}\,\big(1+\sum_{i\neq n} M(w_i)\mu G_i + M(u_n)\mu\bar{G}_n\big) =$$

$$= 1. \qquad (n = 1,2,\ldots)$$

Similarly,

$$\|x_0\| = \frac{1}{k_0}\left(1 + R_M(k_0 x_0)\right) = \frac{1}{k_0}\left(1 + \sum_{i=1}^{\infty} M(w_i)\mu G_i\right) = 1.$$

Since

$$\frac{k_n k_0}{k_n + k_0}\,(x_n(t) + x_0(t)) = \begin{cases} w_i & t \in G \qquad (i \neq n) \\ \frac{k_n}{k_n+k_0}\, w_n & t \in G_n \setminus \bar{G}_n \\ \cdots \qquad \cdots & \\ \frac{k_n}{k_n+k_0}\, w_n + \frac{k_0}{k_n+k_0}\, u_n & t \in \bar{G}_n \\ 0 & \text{otherwise} \end{cases}$$

we derive that

$$\int_G N\Big(p\frac{k_n k_0}{k_n + k_0}\,(x_n + x_0)\Big)\, d\mu < \sum_{i \neq n} N(p(w_i))\mu G_i +$$
$$+ N(p(w_n))\mu(G_n \setminus \bar{G}_n) + N(p(u_n))\mu \bar{G}_n \leq 1$$

But for any $\lambda > 1$,

$$\int_G N\Big(p\big(\lambda\frac{k_n k_0}{k_n + k_0}\,(x_n + x_0)\big)\Big)\, d\mu > \sum_{i \neq n} N(p(\lambda w_i))\mu G_i = \infty.$$

Hence, by Lemma 2, it yields that

$$\|x_n + x_0\| = \frac{k_n + k_0}{k_n k_0}\left(1 + R_M\left(\frac{k_n k_0}{k_n + k_0}\,(x_n + x_0)\right)\right).$$

From (13), we get

$$M\left(\frac{u_n + w_n}{2}\right) \geq \left(1 - \frac{1}{n}\right) \frac{M(u_n) + M(w_n)}{2}$$

By the remark after Lemma 1, we deduce that there exist $\delta_n \searrow 0$ with

$$M\Big(\frac{k_n w_n + k_0 u_n}{k_n + k_0}\Big) \geq (1-\delta_n)\Big(\frac{k_n}{k_n+k_0}\, M(w_n) + \frac{k_0}{k_n+k_0}\, M(u_n)\Big).$$

Therefore,

$$\begin{aligned}
\|x_n + x_0\| &= \tfrac{k_n+k_0}{k_n k_0}\Big[1 + \sum_{i\neq n} M(w_i)\mu G_i + M\big(\tfrac{k_n}{k_n+k_0} w_n\big)\mu(G_n \backslash \bar{G}_n) + \\
&\qquad + M\big(\tfrac{k_n w_n + k_0 u_n}{k_n+k_0}\big)\mu \bar{G}_n\Big] \\
&> \tfrac{k_n+k_0}{k_n k_0}\Big[1 + \sum_{i\neq n} M(w_i)\mu G_i + (1-\delta_n)\big(\tfrac{k_n}{k_n+k_0} M(w_n) + \\
&\qquad + \tfrac{k_0}{k_n+k_0} M(u_n)\big)\mu \bar{G}_n\Big] \\
&> (1-\delta_n)\Big[\tfrac{1}{k_0}\big(1 + \sum_{i\neq n} M(w_i)\mu G_i\big) + \\
&\qquad + \tfrac{1}{k_n}\big(1 + \sum_{i\neq n} M(w_i)\mu G_i + M(u_n)\mu \bar{G}_n\big)\Big] \\
&\to 2
\end{aligned}$$

Since $\mu G_n \to 0$, we have that

$$x_0(t) - x_n(t) \xrightarrow{w^*} \Big(\frac{1}{k_0} - \frac{1}{k}\Big) \sum_{i=1}^{\infty} w_i \chi_{G_i}(t)$$

which contradicts with the fact L_M is LW^*UR. ■

Finally we give an example of an N-function M that satisfies (i) and (ii), but not (iii). So L_N is separable and L_M is rotund, but not LW^*UR.

Let

$$p(t) = \begin{cases} t & 0 \le t < 1 \\ (k+1)^{k+1} + \frac{t-2^k}{2^{2k}} & 2^k \le t < 2^{k+1} \qquad (k = 0,1,2,\ldots) \end{cases}$$

and

$$M(u) = \int_0^{|u|} p(t)\, dt.$$

Then $M \in SC$, since $p(t)$ is strictly increasing on the whole axis. And $M \in \nabla_2$. Indeed, from $q(s) = \sup_{p(t) \le s} t$ (cf § 2 of [1]), it yields that

$$g(s) = \begin{cases} s & 0 \le s < 1 \\ 2^k & k^k + \frac{1}{2^{k-1}} \le s \le (k+1)^{k+1} \\ \text{linear} & (k+1)^{k+1} \le s \le (k+1)^{k+1} + \frac{1}{2^k} \\ 2^{k+1} & (k+1)^{k+1} + \frac{1}{2^k} \le s \le (k+2)^{k+2} \qquad (k = 1,2,\ldots) \end{cases}$$

For any s, there is k such that $k^k + \frac{1}{2^{k-1}} < s \le (k+1)^{k+1} + \frac{1}{2^k}$, so $2s \le (k+2)^{k+2}$. Hence $\frac{q(2s)}{q(s)} \le \frac{2^{k+1}}{2^k} = 2$. By the Young inequality, it yields that $N(2s) \le 2sq(2s) \le 4sq(s) \le 8sq(\frac{s}{2}) = 16\frac{s}{2}q(\frac{s}{2}) \le 16N(s)$, i.e., $M \in \nabla_2$.

But M does not satisfy (iii). In fact for $v_k = 2^k$, $u_k = 2^{k+1}$ $(k = 1,2,\ldots)$, we have

1) $u_k = 2v_k > v_k = \frac{u_k}{2} \ge 2 > \left(\frac{1}{2}\right)^2$.

2) $\frac{M(u_k)}{u_k p(u_k)} \ge \frac{p(v_k)(u_k - v_k)}{u_k p(u_k)} = \frac{p(2^k)(2^{k+1}-2^k)}{p(2_-^{k+1})2^{k+1}} = \frac{(k+1)^{k+1}\frac{1}{2}}{(k+1)^{k+1} + \frac{2^{k+1}-2^k}{2^{2k}}} \to \frac{1}{2}$

(where $2_-^{k+1} = 2^{k+1} - 0$).

3) $\frac{\frac{M(u_k)+M(v_k)}{2}}{M\left(\frac{u_k+v_k}{2}\right)} \to 1$. Indeed,

$$M(u_k) + M(v_k) - 2M\left(\frac{u_k + v_k}{2}\right) = \int_{\frac{u_k+v_k}{2}}^{u_k} p(t)\,dt - \int_{v_k}^{\frac{u_k+v_k}{2}} p(t)\,dt$$

$$= \int_{\frac{u_k+v_k}{2}}^{u_k} p(t) - p\left(t - \frac{u_k - v_k}{2}\right) dt \leq \frac{u_k - v_k}{2}(p(u_k) - p(v_k))$$

$$= \frac{2^{k+1} - 2^k}{2}\left[(k+1)^{k+1} + \frac{2^{k+1} - 2^k}{2^{2k}} - (k+1)^{k+1}\right] = \frac{2^k}{2}\frac{2^k}{2^{2k}} = \frac{1}{2}.$$

Since $M\left(\frac{u_k+v_k}{2}\right) \to \infty$. It follows that 3) holds.

$$4)\ p\left(\left(1-\tfrac{1}{2}\right)u_k\right) = p\left(\tfrac{u_k}{2}\right) = p(v_k) = (k+1)^{k+1} > (k+1)\left(k^k + \tfrac{2^k - 2^{k-1}}{2^{2(k-1)}}\right)$$

$$> (k+1)p\left(\left(1 - \tfrac{1}{k}\right)v_k\right) \quad k = 1, 2, \ldots$$

Combining 1) - 4), we see that M does not satisfy (iii).

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Recibido: 4 de julio de 1991
Revisado:20 de octubre de 1993