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Czechoslovak Mathematical Journal, Vol. 23 (1973), No. 1, 15–23

Persistent URL: http://dml.cz/dmlcz/101140

Czechoslovak Mathematical Journal, 23 (98) 1973, Praha

OPEN AND PROPER MAPS BETWEEN CONVERGENCE SPACES

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(Received September 6, 1971)

INTRODUCTION

This paper is to some extent a continuation of [2], and we therefore ask the reader to refer to this source for the notation and terminology not specified herein. In our study of open and proper maps between convergence spaces, we are interested primarily in results pertaining to the theory of convergence spaces rather than in generalizations of well-known topological theorems. We show, for example, that open maps extend from a convergence space to its decomposition series whereas proper maps do not; furthermore, open maps preserve the property of pretopological coherence of pairs of spaces. Alternate characterizations are given for both types of maps. In the last section we study the behavior of the extension map from a convergence space to its Stone-Čech compactification.

1. PRELIMINARIES

All spaces considered in this paper are assumed to be convergence spaces. We denote by $C(S)$ the set of all convergence structures on a set S.

By a *map* we shall mean a continuous onto function. Let $f : (S, q) \to (T, p)$ be a map; f is a *convergence quotient map* (*biquotient map*) if, whenever a filter (u.f. (ultrafilter)) $\mathscr{F}$ p-converges to y, there is x in $f^{-1}(y)$ and $\mathscr{G}$ mapping on $\mathscr{F}$ such that $\mathscr{G}$ q-converges to x.

For the map f above, let R be the associated equivalence relation $\{(x, y) : f(x) = f(y)\}$. Let θ be the canonical map of S onto S/R defined for all x in S by $\theta(x) = [x]$, the R-equivalence class containing x. Let q/R be the finest convergence structure on S/R which makes θ continuous. Finally, let $f_R : S/R \to T$ be defined by $f_R([x]) = f(x)$.

Proposition 1.1. *$(S/R, q/R)$ is Hausdorff if and only if R is closed in the product space.*

Proof. We prove only the sufficency, since this is not valid in the topological case. Let $\mathscr{F}$ and $\mathscr{G}$ be filters on S/R which q/R-converge to the distinct elements $[x]$, $[y]$, respectively. Since θ is a convergence quotient map, there is $\mathscr{F}_1$ which maps on $\mathscr{F}$ and q-converges to x_1 in $[x]$ and $\mathscr{G}_1$ which maps on $\mathscr{G}$ and q-converges to y_1 in $[y]$. Since $(x_1, y_1) \notin R$ and R is closed, there is F_1 in $\mathscr{F}_1$ and G_1 in $\mathscr{G}_1$ such that $(F_1 \times G_1) \cap R = \emptyset$. Thus $\theta(F_1) \cap \theta(G_1) = \emptyset$, and the filters $\mathscr{F}$ and $\mathscr{G}$ are disjoint.

Proposition 1.2. *The following statements are equivalent:*

(a) *f is a convergence quotient map.*
(b) *f_R is a convergence quotient map.*
(c) *f_R is a homeomorphism.*

If (S, q) and (T, p) are pseudo-topological spaces and q/R the finest pseudo-topology relative to which θ is continuous, then Proposition 1.1 remains valid if "convergence quotient map" is replaced by "biquotient map".

Let $\{(S_\alpha, q_\alpha)\}$ be a family of convergence spaces, and let S be the disjoint union of the S_α's. The *disjoint sum* (S, q) of the given family is defined as follows: $x \in q(\mathscr{F})$ means that $\mathscr{F}$ contains one of the sets S_α, x is in S_α, and the restriction of $\mathscr{F}$ to S_α q_α-converges to x; we write $(S, q) = \sum_\alpha (S_\alpha, q_\alpha)$.

Proposition 1.3. *Let $q \in C(S)$, $\{q_\alpha\} \subset C(S)$; let $(T, p) = \sum_\alpha (S, q_\alpha)$. Then the canonical map h of (T, p) onto (S, q) is a convergence quotient map iff $q = \inf\{q_\alpha\}$.*

Given a convergence space (S, q), there is a finest pretopology $\pi(q)$ on S coarser than q and a finest pseudo-topology $\sigma(q)$ coarser than q; the former is called the *pretopological modification* and the latter the *pseudo-topological modification* of q. If p and q are both in $C(S)$, then $\pi(q) = \pi(p)$ means that p and q have the same neighborhood filters, while $\sigma(q) = \sigma(p)$ means that p and q coincide on u.f.'s.

If x is any point in a set S and $\mathscr{G}$ any filter on S, then we denote by $\tau = (x, \mathscr{F})$ the finest topology on S relative to which $\mathscr{F}$ converges to x. The neighborhood filters for this topology are: $v_\tau(x) = \mathscr{F} \cap \dot{x}$; $v_\tau(y) = \dot{y}$ for $y \neq x$.

Proposition 1.4. *Let $f : (S, q) \to (T, p)$. Then f is biquotient if an only if there is $\{p_\alpha\} \subset C(T)$ such that $\sigma(p) = \sigma(\inf\{p_\alpha\})$ and a map $h : \sum_\alpha (T, p_\alpha) \to (S/R, q/R)$ such that $f_R \circ h$ is the canonical map.*

Proof. Assume the given condition and let $\mathscr{F}$ be a u.f. which p-converges to y. Then $\mathscr{F}$ p_α-converges to y for some α, and so $h(\mathscr{F})$ q/R-converges to $h(y)$. Note that $f_R(h(\mathscr{F})) = \mathscr{F}$, $f_R(h(y)) = y$, and $h(\mathscr{F})$ is a u.f. on S/R. Since θ is a convergence quotient map, there is a u.f. $\mathscr{H}$ q-converging to y such that $\theta(\mathscr{H}) = h(\mathscr{F})$ and $\theta(x) = h(y) = [x]$. Thus $f(\mathscr{H}) = \mathscr{F}$, and f is biquotient.

Conversely, assume that f is biquotient. One can take for the set $\{p_\alpha\}$ the collection of all topologies on T of the form $(x, \mathscr{F})$, where $\mathscr{F}$ is a u.f. which p-converges to x.

Definition 2.1. An onto function $f : (S, q) \to (T, p)$ is *open* if it satisfies the following condition: (A) whenever a u.f. $\mathscr{F}$ on T p-converges to y, then for each x in $f^{-1}(y)$ there is a filter $\mathscr{I}$ which maps on $\mathscr{F}$ and q-converges to x.

Continuous open functions will be called *open maps.*

Proposition 2.2. *If $f : (S, q) \to (T, p)$ is an open function and (S, q) a pretopological space, then f satisfies: (A′) whenever a filter $\mathscr{I}$ on T p-converges to y, then for each x in $f^{-1}(y)$ there is a filter $\mathscr{I}$ which maps on $\mathscr{F}$ and q-converges to x.*

Proof. Let $\mathscr{F}$ p-converge to x. For a given y in $f^{-1}(x)$ and for each u. f. $\mathscr{I}$ finer than $\mathscr{F}$, a u.f. $\mathscr{H}$ maps on $\mathscr{I}$ and q-converges to y; the intersection of these $\mathscr{H}$'s maps on $\mathscr{F}$ and (since q is a pretopology) q-converges to y.

In particular, every open map with a pretopological domain is a convergence quotient map. The next proposition, when compared with Proposition 1.3., exhibits a certain duality between open functions and convergence quotient maps.

Proposition 2.3. *Let $\{q_\alpha\} \subset C(S)$; let $(T, p) = \sum_\alpha (S, q_\alpha)$. If $q = \sup q_\alpha$, then the canonical function from (T, p) onto (S, q) is open. Conversely, if the canonical function is an open map and each q_α is a pseudo-topology, then $q = q_\alpha$, for each α.*

Proposition 2.4. *If $f : (S, q) \to (T, p)$ is an open function, then, for each x in S, $v_p(f(x)) \geqq f(v_q(x))$. If the spaces involved are topological, then f carries open sets into open sets.*

Proof. Let $v \in f(v_q(x))$ and let $\mathscr{F}$ be a u.f. which p-converges to $f(x)$. Then there is a u.f. $\mathscr{I}$ which q-converges to x and maps on $\mathscr{F}$. Since $v = f(u)$ for some $u \in v_q(x)$ and $u \in \mathscr{I}$, $v \in f(\mathscr{I}) = \mathscr{F}$. Thus v is in each u.f. which p-converges to $f(x)$, and $v \in v_p(f(x))$.

Proposition 2.5. *An open map is neighborhood-preserving. A neighborhood-preserving map with a pretopological domain is open.*

Proof. If $f : (S, q) \to (T, p)$ is an open map, then $f(v_q(x)) = v_p(f(x))$ for all x in S by Proposition 2.4 and the continuity of f, and so f is neighborhood-preserving. Next, assume that f is neighborhood-preserving and (S, q) a pretopological space; let $\mathscr{F}$ be a u.f. which p-converges to $f(x)$. Then $\mathscr{F} \geqq f(v_q(x))$ and so some u.f. $\mathscr{I}$ finer than $v_q(x)$ maps on $\mathscr{F}$. Since q is a pretopology, $\mathscr{I}$ must q-converge to x.

The next result is an immediate consequence of Proposition 2.5 and Theorem 3, [2]. Following the notation of [2], we denote by $\{\pi^\alpha(q) : 0 \leqq \alpha \leqq \gamma_q\}$ the decomposition series for (S, q).

Corollary 2.6. *Let $f : (S, q) \to (T, p)$ be an open map. Then:*

(1) $\gamma_p \leqq \gamma_q$.

(2) *If (S, q) is a pretopological (topological) space, then (T, p) is a pretopological (topological) space.*

(3) *In the following diagram, where the horizontal arrows are identity maps and the vertical arrows represent f, all vertical maps are open.*

$$\begin{array}{ccccccccc} (S, q) & \to & (S, \pi(q)) & \to \dots \to & (S, \pi^{\alpha}(q)) & \to \dots \to & (S, \lambda(q)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (T, p) & \to & (T, \pi(p)) & \to \dots \to & (T, \pi^{\alpha}(p)) & \to \dots \to & (T, \lambda(p)) . \end{array}$$

Let $f_i : (S_i, q_i) \to (T_i, p_i)$, i in some index set I, be a family of maps. Let (S, q) and (T, p) be the product spaces of the (S_i, q_i)'s and (T_i, p_i)'s, respectively, and let $f : (S, q) \to (T, p)$ be defined as follows: $f(x) = y$ iff $f_i(x_i) = y_i$ for all $i \in I$, where x_i is the projection of x onto S_i. Then f is called the product of the f_i's. Open maps are *productive* in the sense that f is open whenever each of the f_i's is open; indeed, the same can be said for convergence quotient maps and biquotient maps.

A pair (S_1, q_1) and (S_2, q_2) of convergence spaces is said to be *pretopologically coherent* (see [3]) if the product of the pretopological modifications coincides with the pretopological modification of the product space.

Proposition 2.7. *Let (S_1, q_1) and (S_2, q_2) be a pretopologically coherent pair of spaces, and let (T_1, p_1) and (T_2, p_2) be their images under the open maps f_1 and f_2. Then (T_1, p_1) and (T_2, p_2) also form a pretopologically coherent pair.*

Proof. Let $(S, q) = (S_1, q_1) \times (S_2, q_2)$, $(T, p) = (T_1, p_1) \times (T_2, p_2)$, $(S, r) = (S_1, \pi(q_1)) \times (S_2, \pi(q_2))$, $(T, s) = (T_1, \pi(p_1)) \times (T_2, \pi(p_2))$. By Corollary 2.6, $f_i : (S_i, \pi(q_i)) \to (T_i, \pi(p_i))$ is an open map for $i = 1, 2$. Thus $f : (S, r) \to (T, s)$ is an open map by the aforementioned productivity of open maps. But $\pi(q) = r$ by assumption, so it follows that $\pi(p) = s$.

Proposition 2.8. *Let $f : (S, q) \to (T, p)$. Then f is an open map if and only if there is a set $\{p_\alpha\}$ of convergence structures on T such that $\sigma(p) = \sigma(\inf \{p_\alpha\})$ and a map $h : \sum_\alpha (T, p_\alpha) \to (S/R, r)$ such that $f_R \circ h$ is the canonical map, where r is the pseudo-topology on S/R defined as follows: a u.f. $\mathscr{I}$ r-converges to $[x]$ if, for each $x_1 \in [x]$, there is a u.f. H mapping on $\mathscr{I}$ which q-converges to x_1.*

Proof. Assume the given condition, and let $\mathscr{F}$ be a u.f. which p-converges to y. Then $\mathscr{F}$ p_α-converges to y for some α. Thus $h(\mathscr{F})$ r-converges to $h(y)$, and since $h(\mathscr{F})$ is a u.f. on S/R, there is for a given x in $f^{-1}(y)$ a u.f. $\mathscr{H}$ which q-converges to x such that $\theta(\mathscr{H}) = h(\mathscr{F})$. Thus $f(\mathscr{H}) = \mathscr{F}$, and f is an open map.

For the converse argument one can use the same construction as in the proof of Proposition 1.4.

From the discussion of proper maps in Section 10, [1], we select a characterization which is especially suitable for convergence spaces.

Definition 3.1. A map $f : (S, q) \to (T, p)$ is *proper* if $f(q(\mathscr{F})) = p(f(\mathscr{F}))$ for all u.f.'s $\mathscr{F}$ on S.

In other words, f is proper if, whenever a u.f. $\mathscr{F}$ p-converges to y, each u.f. $\mathscr{I}$ which maps on $\mathscr{F}$ q-converges to some point x in $f^{-1}(y)$. Thus proper maps somewhat resemble open maps, but with the roles of points and u.f.'s interchanged. Note that for one-to-one maps, the notions proper map, open map, and biquotient map are all equivalent.

Given a convergence space (S, q) and $A \subset S$, let $\Gamma_q^0(A) = A$ and let $\Gamma_q^{\prime}(A)$ be the closure of A. If α is an ordinal number and $\alpha - 1$ exists, let $\Gamma_q^{\alpha}(A)$ be the closure of $\Gamma_q^{\alpha-1}(A)$; if α is a limit ordinal, let $\Gamma_q^{\alpha}(A)$ be the union of $\Gamma_q^{\beta}(A)$ for $\beta < \alpha$. Note that the least ordinal α such that $\Gamma_q^{\alpha}(A) = \Gamma_q^{\alpha+1}(A)$ for all $A \subset S$ is what we have been calling γ_q, the length of the decomposition series for (S, q). A set is *closed* if it coincides with its own closure.

Proposition 3.2. *Let $f : (S, q) \to (T, p)$ be a proper map, $A \subset S$, σ an ordinal number. Then $f(\Gamma_q^{\sigma}(A)) = \Gamma_p^{\sigma}(f(A))$. Proper maps preserve closed sets.*

Proof. Transfinite induction. $\sigma = 1$. It is always true for continuous maps that $f(\Gamma_q(A)) \subset \Gamma_p(f(A))$. Let $y \in \Gamma_p(f(A))$; then there is a u.f. $\mathscr{F}$ p-converging to y which contains $f(A)$. Now $A \cap f^{-1}(\mathscr{F})$ has a u.f. refinement $\mathscr{I}$, and $\mathscr{I}$ q-converges to x in $f^{-1}(y)$. Thus x is in $\Gamma_q(A)$, and $y \in f(\Gamma_q(A))$.

Next assume the statement true for all $\varrho < \sigma$. If $\sigma - 1$ exists, then $f(\Gamma_q^{\sigma}(F)) = f(\Gamma_q(\Gamma_q^{\sigma-1}(F))) = \Gamma_p(f(\Gamma_q^{\sigma-1}(F))) = \Gamma_p(\Gamma_p^{\sigma-1}(f(F))) = \Gamma_p^{\sigma}(f(F))$. If σ is a limit ordinal, then $f(\Gamma_q^{\sigma}(F)) = f(\bigcup\{\Gamma_q^{\varrho}(F) : \varrho < \alpha\}) = \bigcup \Gamma_p^{\varrho}(f(F)) = \Gamma_p^{\sigma}(f(F))$.

The last assertion follows by taking σ large enough so that $\Gamma_q^{\sigma}(F) = \Gamma_q^{\sigma+1}(F)$.

Given a filter $\mathscr{F}$ on (S, q), let $\Gamma_q^0(\mathscr{F}) = \mathscr{F}$. If α is a limit ordinal and $\alpha - 1$ exists, let $\Gamma_q^{\alpha}(\mathscr{F})$ be the filter generated by the collection of closures of members of $\Gamma_q^{\alpha-1}(\mathscr{F})$; if α is a limit ordinal, let $\Gamma_q^{\alpha}(\mathscr{F}) = \bigcap\{\Gamma_q^{\beta}(\mathscr{F}) : \beta < \alpha\}$.

Proposition 3.3. *The following are consequences of f being a proper map.*

(1) *For each filter $\mathscr{F}$ on S and each ordinal number σ, $f(\Gamma_q^{\sigma}(\mathscr{F})) = \Gamma_p^{\sigma}(f(\mathscr{F}))$.*

(2) $\gamma_p \leqq \gamma_q$.

(3) *If q is a topology, then $\pi(p)$ is a topology.*

Proof. (1) Follows from Proposition 3.2 by a simple transfinite induction argument.

(2) If $\Gamma_q^{\sigma+1}(B) = \Gamma_q^{\sigma}(B)$ for all $B \subset S$, then it follows from Proposition 3.2 that $\Gamma_p^{\sigma+1}(A) = \Gamma_p^{\sigma}(A)$ for all $A \subset T$. Thus (2) follows.

(3) If q is a topology, then $\gamma_q \leqq 1$ which implies $\gamma_p \leqq 1$. But $\gamma_p \leqq 1$ if and only if $\pi(p)$ is a topology.

By comparing Proposition 3.3 with Corollary 2.6, we see that proper maps have some properties in common with open maps. However the following example shows that the image of a pretopological space under a proper map need not be pretopological; from this it follows that a proper map need not be proper relative to the decomposition series of the domain and range spaces.

Example 3.4. Let T be an infinite set, $x \in T$, and $\mathscr{F}$ a free u.f. on T. Let p be the pseudo-topology on T whose convergence is described on u.f.'s as follows: $\dot{y}$ p-converges to y, all $y \in T$; each free u.f. other than $\mathscr{F}$ p-converges only to x; $\mathscr{F}$ fails to p-converge. With each free u.f. $\mathscr{I}$ on T distinct from $\mathscr{F}$, associate an element $a_{\mathscr{I}}$ not in T and let $A = \{a_{\mathscr{I}}\}$ be the set of all such elements. Let $S = T \cup A$, and let q be the pretopology on S whose neighborhood filters are defined as follows: $v_q(y) = \dot{y}$ for $y \in T$, $y \neq x$; $v_q(x) = \dot{x} \cap \{B \subset S : B \supset A\}$; $v_q(a_{\mathscr{I}}) = \dot{a}_{\mathscr{I}} \cap \mathscr{I}'$ for $a_{\mathscr{I}} \in A$, where $\mathscr{I}'$ is the filter on S generated by the filter $\mathscr{I}$ on T. Finally, let $f : (S, q) \to \to (T, p)$ be defined by: $f(y) = y$ for $y \in T$; $f(y) = x$ for $y \in A$. It is easy to see that f is a proper map. Note that $\mathscr{F}$ $\pi(p)$-converges to x, but no filter which maps on $\mathscr{F}$ q-converges (i.e., $\pi(q)$-converges) to a point in $f^{-1}(x)$, so that $f : (S, \pi(q)) \to (T, \pi(p))$ is not even a biquotient map.

In the preceding example, let $\mathscr{H}$ be any free non-u.f. which does not have $\mathscr{F}$ as a refinement. Then $\mathscr{H}$ p-converges to x, but no filter which maps on $\mathscr{H}$ can q-converge to any preimage of x, and it follows that the map f described above is not a converge quotient map, even though the domain space is pretopological.

We omit the straightforward proofs of the next two results.

Proposition 3.5. *$f : (S, q) \to (T, p)$ is biquotient if and only if $f_R : (S/R, q/R) \to \to (T, p)$ is proper.*

Let $f : (S, q) \to (T, p)$, and let δ be the pseudo-topology on S/R defined by specifying u.f. convergence as follows: a u.f. $\mathscr{I}$ δ-converges to $[x]$ if either $\mathscr{I} = [\dot{x}]$ or, for each u.f. $\mathscr{H}$ on S such that $\theta(\mathscr{H}) = \mathscr{I}$, $\mathscr{H}$ q-converges to some x_1 in $[x]$.

Proposition 3.6. *A map $f : (S, q) \to (T, p)$ is proper if and only if there is a set $\{p_\alpha\} \subset C(T)$ such that $\sigma(p) = \sigma(\inf\{p_\alpha\})$ and a map $h : \sum_\alpha (T, p_\alpha)$ onto $(S/R, \delta)$ such that $f_R \circ h$ is the canonical map onto (T, p).*

It can be shown by a direct argument that proper maps are productive. Indeed the good behavior of proper maps relative to products was used to define proper maps in [1], and in the next proposition we generalize this characterization to convergence spaces.

Proposition 3.7. *Let $f : (S, q) \to (T, p)$. Then f is proper if and only if, for each convergence space (Z, r) and each $A \subset S \times Z$, $(f \times i_z)(\Gamma_s^{\sigma}(A)) = \Gamma_{\tau}^{\sigma}((f \times i_z)(A))$ for all ordinal numbers σ, where $s = q \times r$ and $\tau = p \times r$ are the product convergence structures.*

Proof. If f is proper, then the given condition is established with the help of Proposition 3.2 and productivity of proper maps. Conversely, let $\mathscr{F}$ be a u.f. on S such that $f(\mathscr{F})$ p-converges to y. Let $Z = S \cup \{w\}$, where w is some point not in S; let r be the topology $(w, \mathscr{F}')$ (see Proposition 1.4) where $\mathscr{F}'$ is the filter on Z generated by $\mathscr{F}$. Let $A = \{(x, x) : x \in S\}$; by hypothesis $(f \times i_z)(\Gamma_s(A)) = \Gamma_{\tau}(\{(f(x), x) : x \in S\})$. Also $f(\mathscr{F})$ p-converges to y, $\mathscr{F}$ r-converges to w, and $[f(F) \times F) \cap (f \times i_z) . (A) \neq \emptyset$ for all $F \in \mathscr{F}$, implying $(y, w) \in \Gamma_{\tau}((f \times i_z)(A))$. Thus there is x in S such that $(x, w) \in \Gamma_s(A)$ with x in $f^{-1}(y)$, and there are filters $\mathscr{I}_1$ and $\mathscr{I}_2$ on S such that $\mathscr{I}_1$ q-converges to x and $\mathscr{I}_2$ (regarded as a filter on Z) r-converges to w, with $(G_1 \times G_2) \cap A \neq \emptyset$ for all $G_1 \in \mathscr{I}_1$ and $G_2 \in \mathscr{I}_2$. Thus $\mathscr{I}_1 \vee \mathscr{I}_2$ exists and q-converges to x. But $\mathscr{I}_2 \geqq \mathscr{F}$, which implies $\mathscr{I}_1 \vee \mathscr{I}_2 = \mathscr{F}$, and so $\mathscr{F}$ q-converges to x, as desired.

We conclude this section by listing, without proofs, a few propositions about proper maps from [1] which extend without difficulty to the convergence space setting. We say that a convergence space is *locally compact* if each neighborhood filter contains a compact set. Let f be a map from (S, q) onto (T, p).

(a) If f is proper, then a subset A of T is compact if and only if $f^{-1}(A)$ is compact. In particular, inverse images of singletons are compact.

(b) If (T, p) is locally compact and Hausdorff, then f is proper if and only if $f^{-1}(A)$ is compact in S whenever A is compact in T.

(c) If (S, q) is compact and (T, p) is Hausdorff, then f is proper.

(d) If (S, q) is Hausdorff and f is proper, then (T, p) is Hausdorff.

(e) If f is a proper convergence quotient map and (S, q) is λ-regular (see [4]) for a given cardinal number λ, then (T, p) is also λ-regular.

4. EXTENSIONS TO THE STONE-ČECH COMPACTIFICATION

It is shown in [5] that every Hausdorff convergence space (S, q) can be embedded in a compact Hausdorff space (S^*, q^*) where S^* is a set of u.f.'s from S; this space, along with the natural embedding $x \to \dot{x}$ is called the *Stone-Čech compactification* of (S, q). This terminology is justified by the fact that a map $f : (S, q) \to (T, p)$ has a unique continuous extension $f^* : (S^*, q^*) \to (T, p)$ if the range space is regular, Hausdorff, and compact. Indeed, f^* is well-defined whenever (T, p) is compact and Hausdorff; regularity is needed to secure continuity. The reader is referred to [5] for the details of this construction.

Lemma 4.1. *Let $f:(S, q) \to (T, p)$ be a map, with (T, p) compact and Hausdorff. If f^* is continuous, then $f^*(\Gamma_{q^*}(A)) = \Gamma_p(f(A)$ for all $A \subset S$.*

Proof. Since f^* is continuous, $f^*(\Gamma_{q^*}(A)) \subset \Gamma_p(f(A))$. If $y \in \Gamma_p(f(A))$, then there is a u.f. $\mathscr{H}$ p-converging to y such that $f(A)$ is in $\mathscr{H}$; thus each member of $f^{-1}(\mathscr{H})$ intersects A. Let $\mathscr{K}$ be a u.f. finer than $f^{-1}(\mathscr{H})$ which contains A. If $\mathscr{K}$ q-converges to some x in S, then $x \in \Gamma_q(A)$ and $y \in f^*(\Gamma_{q^*}(A))$. If $\mathscr{K}$ does not q-converge, then (in the notation of [5]) $[\mathscr{K}]$ q^*-converges to $\mathscr{K}$ and A in $\mathscr{K}$ implies $\mathscr{K} \in \Gamma_{q^*}(A)$. Furthermore, $f^*(\mathscr{K}) = y$, implying $y \in f^*(\Gamma_{q^*}(A))$.

Proposition 4.2. *Let $f:(S, q) \to (T, p)$ be a convergence quotient map with (S, q) regular and (T, p) compact Hausdorff. Then f^* is a convergence quotient map if and only if (T, p) is regular.*

Proof. Assume that f^* is a convergence quotient map and let $\mathscr{F}$ p-converge to y. Then there is $\mathscr{I}$ q-converging to x in $f^{-1}(y)$ such that $f(\mathscr{I}) = \mathscr{F}$. By Lemma 4.1, $f^*(\Gamma_{q^*}(\mathscr{I})) = \Gamma_p(\mathscr{F})$. But (in the notation of [5]) $\Gamma_{q^*}(\mathscr{I}) \geqq (\Gamma_q^\wedge(\mathscr{I}))$, and, since q is regular, $\Gamma_q^\wedge(\mathscr{I})$ q^*-converges to x. Since $f^*(\Gamma_q^\wedge(\mathscr{I})) \leqq \Gamma_p(\mathscr{F})$ and f^* is continuous, $\Gamma_p(\mathscr{F})$ p-converges to y and it follows that p is regular.

Conversely, the regularity of p implies that f^* is continuous, and the fact that f^* is a convergence quotient map follows immediately from the fact that f^* is the extension of a convergence quotient map.

If $f:(S, q) \to (T, p)$ is a proper map and (T, p) is compact, then (S, q) is necessarily compact. In this case we have the trivial results $(S^*, q^*) = (S, q)$ and $f^* = f$.

In the next example we show that an open map f can have an extension f^* which is not neighborhood-preserving and hence not open.

Example 4.3. Let $S = (0, 2]$, $T = [0, 1]$ be intervals on the real line, with q and p the usual topologies on S and T, respectively. Define $f: S \to T$ as follows: $f(x) = 1 - x$ for x in $(0, 1]$; $f(x) = x - 1$ for x in $[1, 2]$. Clearly f is an open map from S onto T. Let $A_n = \{1/n, 1/(n+1), \ldots\}$ and let $\mathscr{I}$ be a u.f. finer than the filter base $\{A_n : n = 1, 2, \ldots\}$. Since $\mathscr{I}$ does not q-converge, $\mathscr{I} \in S^* - S$ and $f^*(\mathscr{I}) = \lim f(\mathscr{I}) = 1$. Moreover, each $\mathscr{H} \in S^* - S$ must converge to 0 in the usual sense with $\lim f(\mathscr{H}) = 1$. Hence $f^*(A_n^\wedge)$ does not belong to $v_p(1)$ and thus $f^*(v_{q^*}(\mathscr{I}))$ is not equal to $v_p(1)$.

Definition 4.4. A map $f:(S, q) \to (T, p)$ is *almost open* if, for each x in T, there is y in $f^{-1}(x)$ such that $f(v_q(y)) = v_p(x)$.

An open map is almost open; for pretopological spaces, almost open maps coincide with convergence quotient maps.

The proof of our concluding proposition is straightforward.

Proposition 4.5. *If $f:(S, q) \to (T, p)$ is almost open and (T, p) is compact, Hausdorff, and regular, then f^* is almost open.*

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